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FIXED POINT THEOREMS IN N-POLYGONAL CONE METRIC SPACES (COMMUNICATED BY MARK AGRANOVSKY)

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Abstract. In this paper we present a fixed point theorem for mappings defined in an N-polygonal cone metric space. Some generalizations of fixed point theorems for S-Kannan and S-Chatterjea contractive mappings on cone metric spaces will be also shown. In these results the underlying cone metric space is considered over a Banach space ordered by a normal cone.

1. Introduction and Preliminaries

After the extension of Banach Contraction Principle for contractions on vector-valued metric spaces by Perov in 1964 ([21]), the notion of cone metric space over an ordered Banach space (also called K-metric space) has been used as a natural framework for proving useful fixed point theorems in the theory of differential equations (see [1], [7], [9], [19], [20] and [22]). For more information on this topic we refer the reader to [23] and the bibliography therein.

Recently, from a paper by Long-Guang and Xian [15] many authors have taken up the question of the existence of fixed points for mappings satisfying some contractive conditions in cone metric spaces (see [2], [8], [11], [16] and [18]).

Motivated by two recent papers on rectangular cone metric space ([2] and [16]), we unify various results deal with generalizations of the Banach's Contraction Principle. On the other hand, we present a fixed point result for a S-Hardy-Rogers cone contraction following some ideas from [12]. Among other consequences we obtain an extension of the well known Kannan fixed point theorem appeared in [8].

It is worth pointing out that in the above-mention papers it is usually required the cone to be normal with nonempty interior. Last condition on the cone is very restrictive, for example, the positive cone of the space of functions L_1 and the positive cone of the space of sequences ℓ_p $(1 \leq p < \infty)$ have empty interior. We emphasize that in our approach just a normality assumption is imposed on the cone. Furthermore, we deal with a class of contractive mappings for which the constant of contractiveness is replaced by a positive operator in the spirit of the papers [9] and [23].

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Let $(V, ||\cdot||)$ be a Banach space. A set $K \subset V$ is called a **cone** if and only if:

(1) K is nonempty and $K \neq \{0_V\}$.
(2) If $\alpha, \beta \in K$ and $a, b \in \mathbb{R}_0^+$, then $a\alpha + b\beta \in K$.
(3) $K \cap (-K) = \{0_V\}$.

For a given cone $K \subset V$, we can define a partial ordering $\leq$ with respect to K by $\alpha \leq \beta$ if and only if $\beta - \alpha \in K$. We will refer $(V, ||\cdot||, K)$ as an ordered Banach space.

The following definitions relate the norm $||\cdot||$ with the cone K:

- The cone K is called normal if there exists a number $\lambda \geq 1$ such that for all $\alpha, \beta \in V$, $0_V \leq \alpha \leq \beta$ implies $||\alpha|| \leq \lambda ||\beta||$. The least positive number satisfying above is called the normal constant of K.
- The cone K is called closed if K is closed with respect to the topology induced by the norm.

Definition 1.1. *Let X be a set and $d_K : X \times X :\to K$ a mapping. We say that d_K is a cone metric, if for all $x, y, z \in X$, one has*

(1) $d_K(x, y) = 0_V \Leftrightarrow x = y$;
(2) $d_K(x, y) = d_K(y, x)$;
(3) $d_K(x, y) \leq d_K(x, z) + d_K(z, y)$.

The pair (X, d_K) is said to be a cone metric space (CMS).

Example 1.2. *Let $X = V = L_p[0, 1]$ such that*

$$K_p = \{f \in L_p[0, 1] : 0 \leq f \ \ e.c.t\}.$$

We define $d_K : L_p[0, 1] \times L_p[0, 1] \to K_p$ such that

$$d_K(f, g) = |f - g|.$$

It is clear that d_K is a cone metric.

The proof of the following lemma easily follows from the definition of cone metric.

Lemma 1.3. *Let (X, d_K) be a cone metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ. Then the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$ satisfies the following properties:*

(1) $D(x, y) = 0 \Leftrightarrow x = y$;
(2) $D(x, y) = D(y, x)$;
(3) $D(x, y) \leq \lambda[D(x, z) + D(z, y)]$ *for all* $x, y, z \in X$.

Recently, M.A. Khamsi [17] introduced the concept of a metric type space: for an arbitrary set X, the pair (X, D) is called a metric type space if $D : X \times X \to [0, \infty)$ is a function satisfying properties (1), (2) and (3) in the above lemma. Defining a topology on this class of spaces, he obtained some metric and topological fixed point theorems (see [18]). However, it is worth pointing out that this definition of D corresponds to the concept of *quasi-metric* and the pair (X, D) is called a *quasi-metric space* in the literature. If (X, D) is a quasi-metric space then the topology in X induced by D, is canonically defined by means of the theory of *uniform structures.* The balls $B(x, r) = \{y \in X : D(x, y) < r\}$ for $r > 0$ form a basis of neighbourhoods of x for the topology induced by the uniformity of X. This is a metric topology since the uniform structure associated to D has a numerable basis (we refer the reader to Chapter 8 in [10]). Therefore, a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X, if $\lim_n D(x_n, x) = 0$ and X is complete if every Cauchy sequence is convergent in X. This fact suggests that many fixed point results proved in cone metric spaces do not need full structure because they could follow by using the real valued function D. However, this approach does not work in order to prove certain fixed point theorems [8].

Example 1.4. *Let (Ω, Σ, μ) be a positive measure space. For every $1 < p < \infty$ consider the Banach space $X = L_p(\Omega)$.*

Define $D : X \times X \to [0, +\infty)$ by

$$D(f, g) = \int_\Omega |f - g|^p \, d\mu, \quad f, g \in X.$$

Then D trivially satisfies properties (1), (2) in the above Lemma. Now take $f, g, h \in X$. Since $p > 1$, it is well known that

$$\int_\Omega |f - g|^p \, d\mu \leq 2^{p-1} \left(\int_\Omega |f - h|^p \, d\mu + \int_\Omega |h - g|^p \, d\mu \right),$$

which means

$$D(f, g) \leq 2^{p-1}[D(f, h) + D(h, g)].$$

Therefore D is a quasi-metric on X.

Concerning example 1.4 and the concept of "quasi-metric space" (which is called in some paper b-metric space), it is worth to notice that several examples of quasi-metrics (including example 1.4) and some fixed point results in this framework are given in [3], [4], [5], [6], and the references therein.

Following some ideas from [2] next we introduce a class of spaces including cone metric spaces.

Definition 1.5. *Let X be a set and $d_K : X \times X :\to K$ a mapping. We say that d_K is a N-polygonal cone metric, if for all $x, y \in X$ and for all distinct points $z_1, z_2, \ldots, z_N \in X$, each of them different from x and y, one has*

1. $d_K(x, y) = 0_V \Leftrightarrow x = y$;
2. $d_K(x, y) = d_K(y, x)$;
3. $d_K(x, y) \leq d_K(x, z_1) + d_K(z_1, z_2) + \cdots + d_K(z_{N-1}, z_N) + d_K(z_N, y)$.

The pair (X, d_K) is said to be a N-polygonal cone metric space, N_p-CMS for short.

If $N = 1$ the pair (X, d_K) is a cone metric space. The concept of 2-polygonal cone metric space is referred in [2] as cone rectangular metric space. It is clear that a cone metric space is a N-polygonal cone metric space, for all $N \geq 2$.

Example 1.6. *Let $X = \mathbb{N}$, $(V, ||\cdot||) = (\mathbb{R}^2, ||\cdot||)$ with $||\alpha|| = \sqrt{\alpha_1^2 + \alpha_2^2}$ and*

$$K = \{(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \geq 0\}.$$

We define $d_K : X \times X \to K$ as follow:

$$d_K(x, y) = \begin{cases} (0, 0) & : \textit{if} \quad x = y, \\ (3a, 3) & : \textit{if} \quad x \in \{1, 2\} \wedge y \in \{1, 2\} \wedge x \neq y, \\ (a, 1) & : \textit{if} \quad (x \in \{1, 2\}^c \vee y \in \{1, 2\}^c) \wedge x \neq y. \end{cases}$$

with $a \in (0, \infty)$. Then d_K is a rectangular cone metric but it is not a cone metric because it lacks the triangular property:

$$d_K(1, 2) = (3a, 3) > d_K(1, 3) + d_K(3, 2) = (2a, 2).$$

From the definition of N-polygonal cone metric we can easily deduce the following lemma.

Lemma 1.7. *Let (X, d_K) be a N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ. Then the function $D : X \times X \to [0, \infty)$ defined by $D(x, y) = ||d_K(x, y)||$ satisfies the following properties:*

1. *$D(x, y) = 0 \Leftrightarrow x = y$;*
2. *$D(x, y) = D(y, x)$;*
3. *$D(x, y) \leq \lambda[D(x, z_1) + D(z_1, z_2) + \cdots + D(z_{N-1}, z_N) + D(z_N, y)]$ for all distinct points $z_1, z_2, \ldots, z_N \in X$, each of them different from x and y.*

Given a N-polygonal cone metric space (X, d_K), the concept of sequence will be the usual. We next define convergence and completeness.

Definition 1.8. *Let (X, d_K) be a N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}} \subset X$. We say that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in X, if $\lim_n D(x_n, x) = 0$. We write $x_n \to x$ to denote that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to x, that is,*

$$x_n \to x \Leftrightarrow D(x_n, x) \to 0.$$

In general, for a N-polygonal cone metric space with $N \geq 2$, the uniqueness of the limit of a sequence does not hold (see Example 1.6 in [16]). However, the limit is unique for a convergent Cauchy sequence as we show below.

Definition 1.9. *Let (X, d_K) be a N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. We say that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if for every positive real number a, there exists $\hat{N} \in \mathbb{N}$ such that for all $n, m \geq \hat{N}$, we have that $D(x_n, x_m) < a$.*

Next Lemma is a extension of Lemma 1.10 presented in [16].

Lemma 1.10. *Let (X, d_K) be a N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is an Cauchy sequence, such that satisfies the following conditions:*

- **(a):** $(\exists x, y \in X)(x_n \to x \wedge x_n \to y)$,
- **(b):** $(\exists \hat{N} \in \mathbb{N})(n, m \geq \hat{N} \Rightarrow x_n \neq x_m \wedge x_n \neq x \wedge x_n \neq y)$,

then $x = y$.

Proof. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is an Cauchy sequence satisfying conditions (a) and (b). For all $n \geq \hat{N}$ we have

$$D(x, y) \leq \lambda[D(x, x_n) + D(x_{n+1}, x_{n+2}) + \cdots + D(x_{n+N-1}, y)].$$

Taking limit as $n \to \infty$, we obtain $D(x, y) = 0$ and hence $x = y$. □

Definition 1.11. *Let (X, d_K) be a N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone. Then X is called a complete N-polygonal cone metric space, if every Cauchy sequence is convergent in X.*

Finally, we recall some definitions and facts about positive operators.

Definition 1.12. *Let (V, K) be an ordered vector space and $Q : V \to V$ an operator. We say that*

1. *Q is* ***positive*** *if $Q(K) \subseteq K$.*

(2) *Q is* ***monotone non-decreasing*** *if for all $\alpha, \beta \in V$ such that $\beta - \alpha \in K$, then $Q(\beta) - Q(\alpha) \in K$.*

It is clear that every linear positive operator is monotone non-decreasing.

Lemma 1.13. *Let $(V, ||\cdot||, K)$ be an ordered Banach space such that K is a normal cone with normal constant λ and $Q : V \to V$ be a linear positive operator. If $||Q^n(\alpha)|| \to 0$ for $\alpha \in K - \{0_V\}$, then $Q(\alpha) - \alpha \notin K - \{0_V\}$.*

Proof. Consider $\alpha \in K - \{0_V\}$ such that $||Q^n(\alpha)|| \to 0$. Suppose that $Q(\alpha) - \alpha \in K$, that is $\alpha \leq Q(\alpha)$. Since Q is a linear and positive operator we have that $\alpha \leq Q^n(\alpha)$ for all $n \in \mathbb{N}$. Hence $||\alpha|| \leq \lambda ||Q^n(\alpha)||$ for all $n \in \mathbb{N}$. Taking limit as $n \to \infty$ we have $||\alpha|| = 0$, and so $\alpha = 0_V$ which is a contradiction. □

In the sequel we shall use the following notation:

- $B^+(V) = \{Q : V \to V / Q$ is a positive bounded linear operator $\}$.
- $||Q||_o = \sup\{||Q(\alpha)|| : \alpha \in V;\ ||\alpha|| \leq 1\}$.

2. Fixed Point Theorems for generalized contraction

In this section we present an adaptation from classical Banach Contraction Principle for generalized contraction on N-polygonal cone metric spaces.

Theorem 2.1. *Let (X, d_K) be a complete N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$(*) \quad d_K(Tx, Ty) \leq Q(d_K(x, y)),$$

where $Q \in B^+(V)$ and $\sum_{n=0}^{\infty} ||Q^n||_o < \infty$, then T has a unique fixed point x^* in X. Moreover, the iterative sequence $x_n = T^n x_0$ converges to x^* for any initial point $x_0 \in X$ and one has the following estimation:

$$d_K(x_n, x^*) \leq Q^n(I - Q)^{-1}(d_K(x_0, Tx_0)).$$

Proof. Take any point $x_0 \in X$ such that $d_K(x_0, Tx_0) \in K - \{0_V\}$ and consider the sequence $\{x_n\}_{n=0}^{\infty} \subset X$ given by $x_1 = Tx_0$, $x_2 = T^2x_0$,..., $x_n = T^n x_0$. From the assumption $(*)$, we have that

$$d_K(x_1, x_2) = d_K(Tx_0, T^2x_0) \leq Q(d_K(x_0, x_1)),$$
$$d_K(x_2, x_3) = d_K(T^2x_0, T^3x_0) \leq Q(d_K(x_1, x_2)),$$

and

$$d_K(x_n, x_{n+1}) = d_K(T^n x_0, T^{n+1}x_0) \leq Q(d_K(x_{n-1}, x_n)),$$

for all $n \geq 1$. Since Q is monotone non-decreasing it follows that, for any $n \in \mathbb{N}$, $d_K(x_n, Tx_n) \leq Q^n(d_K(x_0, Tx_0))$.

Now, we will show that $x_n \neq x_0$ for all $n \in \mathbb{N}$. Suppose that there exists $\hat{n} \in \mathbb{N}$ such that $x_{\hat{n}} = x_0$, then

$$d_K(x_0, Tx_0) = d_K(x_{\hat{n}}, Tx_{\hat{n}}) \leq Q^{\hat{n}}(d_K(x_0, Tx_0)).$$

Therefore, $r = d_K(x_0, Tx_0) \in K - \{0_V\}$ and $Q^{\hat{n}}(r) - r \in K$. On the other hand, since $\sum_{n=0}^{\infty} ||Q^n||_o < \infty$, we can apply Lemma 1.13 to the operator $Q^{\hat{n}}$ to get a contradiction.

We proceed to prove that T has a fixed point by dividing the proof into two cases.

(Case a): Suppose that $x_n = x_m$ for some $n, m \in \mathbb{N}$ such that $1 \leq n < m$, that is $T^n x_0 = T^m x_0$. If we define $y_0 := T^n x_0$ we have that $y_0 = T^s y_0$ where $s = m - n > 1$. Therefore, we obtain

$$d_K(y_0, Ty_0) = d_K(T^s y_0, T^{s+1} y_0) \leq Q^s(d_K(y_0, Ty_0)).$$

Letting $s \to \infty$, in view of Lemma 1.13 we must have $d_K(y_0, Ty_0) = 0_V$ and y_0 is a fixed point of T.

(Case b): Assume that $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using N-polygonal property (3) and by definition of T we have that,

$$\begin{aligned} d_K(x_0, T^N x_0) &\leq d_K(x_0, Tx_0) + d_K(Tx_0, T^2 x_0) + \cdots + d_K(T^{N-2} x_0, T^{N-1} x_0) \\ &+ d_K(T^{N-1} x_0, T^{N+1} x_0) + d_K(T^{N+1} x_0, T^N x_0) \\ &\leq (I + Q + \cdots + Q^{N-2})(d_K(x_0, Tx_0)) + Q^{N-1}(d_K(x_0, T^2 x_0)) \\ &+ Q^N(d_K(x_0, Tx_0)). \end{aligned}$$

Hence

$$d_K(x_0, x_N) \leq (I + Q + \cdots + Q^N) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right).$$

Similarly we obtain

$$\begin{aligned} d_K(x_0, x_{N+1}) &\leq (I + Q + \cdots + Q^N)(d_K(x_0, x_1)) \\ &\leq (I + Q + \cdots + Q^N) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right), \end{aligned}$$

and

$$\begin{aligned} d_K(x_0, x_{N+l}) &\leq (I + Q + \cdots + Q^{N-1})(d_K(x_0, x_1)) + Q^N(d_K(x_0, x_l)) \\ &\leq (I + Q + \cdots + Q^N) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right), \end{aligned}$$

for any $l \in \{2, \ldots, N\}$.

In the same way, for $l = 1, \cdots, N$ we have that

$$\begin{aligned} d_K(x_0, x_{2N+l}) &\leq (I + Q + \cdots + Q^{2N-1}) d_K(x_0, x_1) + Q^{2N}(d(x_0, x_l)) \\ &\leq (I + Q + \cdots + Q^{2N}) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right). \end{aligned}$$

Continuing this process, we get for each $k \in \mathbb{N}$ and $l \in \{1, \ldots, N\}$

$$d_K(x_0, x_{kN+l}) \leq (I + Q + \cdots + Q^{kN}) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right)$$

.

It is clear that for any $p \in \mathbb{N}$ there exist $k \in \mathbb{N} \cup \{0\}$ and $l \in \{1, 2 \cdot, \cdot, \cdot, N\}$ such that $p = kN + l$. From the above we obtain

$$d_K(x_0, x_p) \leq (I + Q + \cdots + Q^{kN}) \left(\sum_{i=1}^{N} d_K(x_0, x_i) \right).$$

In particular

$$d_K(x_0,x_p) \le (I+Q+\cdots+Q^p)\left(\sum_{i=1}^{N} d_K(x_0,x_i)\right).$$

Let $n,p\in\mathbb{N}$. From above we have

$$\begin{aligned}(**)\quad d_K(x_n,x_{n+p}) &= d_K(T^n x_0, T^{n+p}x_0)\\ &\le Q^n(d_K(x_0,T^p x_0))\\ &\le Q^n\left(\textstyle\sum_{j=0}^{p} Q^j\right)\left(\textstyle\sum_{i=1}^{N} d_K(x_0,x_i)\right).\end{aligned}$$

Now, we apply the normality of the cone to get

$$||d_K(x_n,x_{n+p})|| = D(x_n,x_{n+p}) \le \lambda ||Q^n||_o\left(\sum_{j=0}^{\infty}||Q^j||_o\right)\left(\sum_{i=1}^{N} D(x_0,x_i)\right).$$

Letting $n\to\infty$ we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^*\in X$ such that $T^n x_0 \to x^*$.

We shall now show that $Tx^*=x^*$. Without any loss of generality, we can assume $T^n x_0 \ne x^*$ and $T^n x_0 \ne Tx^*$ for all $n\in\mathbb{N}$. Therefore, since $T^n x_0 \ne T^m x_0$ for all $n,m\in\mathbb{N}$ such that $n\ne m$, we obtain

$$\begin{aligned}d_K(x^*,Tx^*) &\le d_K(x^*,x_{n+1}) + d_K(x_{n+1},x_{n+2}) + \cdots + d_K(x_{n+N-1},x_{n+N})\\ &\quad + d_K(x_{n+N},Tx^*)\\ &\le (Q^{n+1}+Q^{n+2}+\cdots+Q^{n+N-1})(d_K(x_0,x_1)) +\\ &\quad d_K(x^*,x_{n+1}) + Q(d_K(x_{n+N-1},x^*))\\ &= (Q^n\circ W)(d_K(x_0,x_1)) + d_K(x^*,x_{n+1}) + Q(d_K(x_{n+N-1},x^*)),\end{aligned}$$

where $W=(Q+Q^2+\cdots+Q^{N-1})$. Thus,

$$\begin{aligned}D(x^*,Tx^*) &\le \lambda||(Q^n\circ W)(d_K(x_0,x_1)) + d_K(x^*,x_{n+1}) + Q(d_K(x_{n+N-1},x^*))||\\ &\le \lambda(||Q^n||_o||W||_o D(x_0,x_1) + D(x^*,x_{n+1}) + ||Q||_o D(x_{n+N-1},x^*)).\end{aligned}$$

It is clear that $D(x^*,x_n)\to 0$ and $||Q^n||_o\to 0$, therefore $D(x^*,Tx^*)=0$. Hence $Tx^*=x^*$.

In addition, letting $p\to\infty$ in $(**)$ we get

$$d_K(x_n,x^*) \le Q^n\left(\sum_{j=0}^{\infty}Q^j\right)\left(\sum_{i=1}^{N} d_K(x_0,x_i)\right) = Q^n(I-Q)^{-1}(d_K(x_0,Tx_0)).$$

Uniqueness: Suppose that x^* is not a unique fixed point of T. That is, there exists $y^*\in X$ such that $x^*\ne y^*$ and $Ty^*=y^*$. Since $x^*\ne y^*$ then $d_K(x^*,y^*)\in K-\{0_V\}$ and

$$d_K(x^*,y^*) = d_K(Tx^*,Ty^*) \le Q(d_K(x^*,y^*)).$$

Therefore there exists $r=d_K(x^*,y^*)\in K-\{0_V\}$ such that $Q(r)-r\in K$ which is a contradiction by Lemma 1.13. We conclude that $x^*=y^*$.

□

It is easy to prove that $\lim_{n\to\infty}(||Q^n||_o)^{1/n} = \inf\{(||Q^n||_o)^{1/n} : n = 1, 2, ...\}$. Thus the series $\sum_{n=0}^{\infty} ||Q^n||_o$ converges if and only if $\lim_{n\to\infty}(||Q^n||_o)^{1/n} < 1$. As a consequence we can formulate the following corollaries.

Corollary 2.2. *Let (X, d_K) be a complete N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$d_K(Tx, Ty) \leq Q(d_K(x, y)),$$

where $Q \in B^+(V)$ and $\lim_{n\to\infty}(||Q^n||_o)^{1/n} < 1$, then T has a unique fixed point in X.

Corollary 2.3. *Let (X, d_K) be a complete N_p-CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$d_K(Tx, Ty) \leq Q(d_K(x, y)),$$

where $Q \in B^+(V)$ and $||Q||_o < 1$, then T has a unique fixed point in X.

Since a rectangular metric space is a 2-polygonal cone metric space, we obtain the following consequence.

Corollary 2.4. ([2]) *Let (X, d_K) be a complete rectangular cone metric space over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant $\lambda \geq 1$ and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$d_K(Tx, Ty) \leq k \cdot d_K(x, y),$$

where $k \in [0, 1[$ is a real constant. Then T has a unique fixed point in X.

3. Fixed point theorems of S-Kannan and S-Chatterjea cone contractive mappings

We begin this section with a more general definition of S-Hardy-Rogers contraction than that from [12], in order to obtain some news fixed point results on cone metric spaces.

Definition 3.1. *Let (X, d_K) be a CMS over an ordered Banach space $(V, ||\cdot||, K)$ and $S, T : X \to X$ two mappings. We say that T is a **S-Hardy-Rogers cone contraction**, if there exists $\{Q_i\}_{i=1}^5 \subset B^+(V)$ with $\sum_{i=1}^5 ||Q_i||_o < 1$ such that for all $x, y \in X$ we have*

$$\begin{aligned} d_K(STx, STy) \quad \leq \quad & Q_1(d_K(Sx, Sy)) + Q_2(d_K(Sx, STx)) + Q_3(d_K(Sy, STy)) \\ & + Q_4(d_K(Sx, STy)) + Q_5(d_K(Sy, STx)). \end{aligned}$$

In particular, if

(1) $Q_2 \equiv Q_3 \equiv Q_4 \equiv Q_5 \equiv \Theta$, then T is a S-Banach cone contraction;
(2) $Q_4 \equiv Q_5 \equiv \Theta$, then T is a S-Reich cone contraction;
(3) $Q_1 \equiv Q_4 \equiv Q_5 \equiv \Theta$, then T is S-Kannan cone contraction;
(4) $Q_1 \equiv Q_2 \equiv Q_3 \equiv \Theta$, then T is S-Chatterjea cone contraction,

where Θ is the null operator.

If we put in the previous definition $S = Id$, (X, d_K) a metric space, and $Q_i(\alpha) = a_i\alpha$ where $a_i > 0$ such that $\sum_{i=1}^5 a_i < 1$, we have that T is a Hardy-Rogers contraction (see [14]).

Example 3.2. *Let $V = C([0,1])$, $K = \{f \in V : f \geq 0\}$, $X = \mathbb{R}$ and $d_K : X \times X \to K$ defined by $d_K(x,y) = |x-y|e^t$, where $e^t \in V$. Then (X, d_K) is a cone metric space. We consider the functions $S, T : X \to X$ defined by $Sx = \frac{1}{2x} + 2$ and $Tx = 3x$. Obviously, T is not a cone contraction but it is a S-Banach cone contraction from the following:*

$$d_K(STx, STy) = |STx - STy|e^t = |\frac{1}{6x} - \frac{1}{6y}|e^t = \frac{1}{3}|Sx - Sy|e^t = \frac{1}{3}d_K(Sx, Sy).$$

Example 3.3. *Let $V = C([0,1])$, $K = \{f \in V : f \geq 0\}$, $X = \mathbb{R}$ and $d_K : X \times X \to K$ defined by $d_K(x,y) = |x-y|e^t$, where $e^t \in V$. Then (X, d_K) is a cone metric space. We consider the functions $S, T : X \to X$ defined by $Sx = x^2$ and $Tx = \frac{x}{2}$. Then*

$$\begin{aligned} d_K(STx, STy) &= |STx - STy|e^t = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| e^t \\ &\leq \frac{1}{3}\left[|Sx - STx| + |Sy - STy|\right] e^t \\ &\leq \frac{1}{3}\left[d_K(Sx, STx) + d_K(Sy, STy)\right]. \end{aligned}$$

Therefore T is a S-Kannan cone contraction but, it is easy to see that it is not a Kannan contraction. Moreover, it is not difficult to show that T is besides a S-Chatterjea cone contraction.

Definition 3.4. *Let (X, d_K) be a CMS over an ordered Banach space $(V, ||\cdot||, K)$ and $S : X \to X$ a mapping. We say that:*

1. *S is a **sequentially continuous** mapping, if for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, such that $x_n \to x^*$ then $Sx_n \to Sx^*$;*
2. *S is a **sequentially convergent** mapping, if for all sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, such that $Sx_n \to x^*$ then there exists $y^* \in X$ such that $x_n \to y^*$.*

Our main result requires the following lemma.

Lemma 3.5. *Let (X, d_K) be a CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone, $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X and $x^* \in X$. If $d_K(x_n, x^*) \to 0$, then for all $y \in X$ $d_K(x_n, y) \to d_K(x^*, y)$.*

Proof. Suppose that $d_K(x_n, x^*) \to 0$ and let $y \in X$. By triangular inequality for d_K we have

$$-d_K(x_n, x^*) \leq d_K(x_n, y) - d_K(x^*, y) \leq d_K(x_n, x^*), \ \ \forall n \in \mathbb{N}.$$

Since K is a normal cone, we may apply Theorem 1.1.1 of [13] to obtain

$$||d_K(x_n, y) - d_K(x^*, y)|| \to 0,$$

that is, $d_K(x_n, y) \to d_K(x^*, y)$. □

Theorem 3.6. *Let (X, d_K) be a complete CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone with normal constant λ and $S, T : X \to X$ two mappings, such that T is a S-Hardy-Rogers cone contraction. Then*

1. *For all $x_0 \in X$, $\{ST^n x_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence.*
2. *There exists $x^* \in X$ such that $ST^n x_0 \to x^*$.*

(3) *If S is one to one, sequentially continuous and sequentially convergent, then there exists a unique y^* such that $Ty^* = y^*$ and for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to y^*.*

Proof. Let $x_0 \in X$. We consider the sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $x_1 = Tx_0$, $x_2 = T^2x_0$,..., $x_n = T^n x_0$ and $d_K(x_0, Tx_0) \in K - \{0_V\}$. It is clear that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Since T is a S-Hardy-Rogers cone contraction, we have the following

$$\begin{aligned} d_K(Sx_1, Sx_2) &= d_K(STx_0, STx_1) \\ &\leq Q_1(d_K(Sx_0, Sx_1)) + Q_2(d_K(Sx_0, STx_0)) + Q_3(d_K(Sx_1, STx_1)) \\ &\quad + Q_4(d_K(Sx_0, STx_1)) + Q_5(d_K(Sx_1, STx_0)) \\ &\leq Q_1(d_K(Sx_0, Sx_1)) + Q_2(d_K(Sx_0, Sx_1)) + Q_3(d_K(Sx_1, Sx_2)) \\ &\quad + Q_4(d_K(Sx_0, Sx_1)) + Q_4(d_K(Sx_1, Sx_2)), \end{aligned}$$

because $Q_4 \in B^+(V)$ and $Q_5(d_K(Sx_1, Sx_1)) = 0_V$. Therefore

$$[I - Q_3 - Q_4](d_K(Sx_1, Sx_2)) \leq [Q_1 + Q_2 + Q_4](d_K(Sx_0, Sx_1)).$$

Also

$$\begin{aligned} d_K(Sx_2, Sx_1) &= d_K(STx_1, STx_0) \\ &\leq Q_1(d_K(Sx_1, Sx_0)) + Q_2(d_K(Sx_1, STx_1)) + Q_3(d_K(Sx_0, STx_0)) \\ &\quad + Q_4(d_K(Sx_1, STx_0)) + Q_5(d_K(Sx_0, STx_1)) \\ &\leq Q_1(d_K(Sx_1, Sx_0)) + Q_2(d_K(Sx_1, Sx_2)) + Q_3(d_K(Sx_0, Sx_1)) \\ &\quad + Q_5(d_K(Sx_0, Sx_1)) + Q_5(d_K(Sx_1, Sx_2)), \end{aligned}$$

that is

$$[I - Q_2 - Q_5](d_K(Sx_2, Sx_1)) \leq [Q_1 + Q_3 + Q_5](d_K(Sx_1, Sx_0)).$$

Thus

$$[2I - Q_2 - Q_3 - Q_4 - Q_5](d_K(Sx_1, Sx_2)) \leq [2Q_1 + Q_2 + Q_3 + Q_4 + Q_5](d_K(Sx_0, Sx_1)).$$

If we put $W := [2I - Q_2 - Q_3 - Q_4 - Q_5]^{-1} \circ [2Q_1 + Q_2 + Q_3 + Q_4 + Q_5]$, then

$$d_K(Sx_1, Sx_2) \leq W(d_K(Sx_0, Sx_1)),$$

which implies that

$$D(Sx_1, Sx_2) \leq \lambda ||W||_o D(Sx_0, Sx_1).$$

On the other hand,

$$\begin{aligned} d_K(Sx_2, Sx_3) &= d_K(STx_1, STx_2) \\ &\leq Q_1(d_K(Sx_1, Sx_2)) + Q_2(d_K(Sx_1, STx_1)) + Q_3(d_K(Sx_2, STx_2)) \\ &\quad + Q_4(d_K(Sx_1, STx_2)) + Q_5(d_K(Sx_2, STx_1)) \\ &\leq Q_1(d_K(Sx_1, Sx_2)) + Q_2(d_K(Sx_1, Sx_2)) + Q_3(d_K(Sx_2, Sx_3)) \\ &\quad + Q_4(d_K(Sx_1, Sx_2)) + Q_4(d_K(Sx_2, Sx_3)). \end{aligned}$$

Hence

$$[I - Q_3 - Q_4](d_K(Sx_2, Sx_3)) \leq [Q_1 + Q_2 + Q_4](d_K(Sx_1, Sx_2)).$$

Also we have

$$\begin{aligned} d_K(Sx_3, Sx_2) &= d_K(STx_2, STx_1) \\ &\leq Q_1(d_K(Sx_2, Sx_1)) + Q_2(d_K(Sx_2, STx_2)) + Q_3(d_K(Sx_1, STx_1)) \\ &\quad + Q_4(d_K(Sx_2, STx_1)) + Q_5(d_K(Sx_1, STx_2)) \\ &\leq Q_1(d_K(Sx_2, Sx_1)) + Q_2(d_K(Sx_2, Sx_3)) + Q_3(d_K(Sx_1, Sx_2)) \\ &\quad + Q_5(d_K(Sx_1, Sx_2)) + Q_5(d_K(Sx_2, Sx_3)). \end{aligned}$$

By symmetry of d_K, we have

$$[I - Q_2 - Q_5](d_K(Sx_3, Sx_2)) \leq [Q_1 + Q_3 + Q_5](d_K(Sx_2, Sx_1)).$$

Thus

$$d_K(Sx_2, Sx_3) \leq W(d_K(Sx_1, Sx_2)) \leq W^2(d_K(Sx_0, Sx_1)),$$

Following the above reasoning we obtain

$$d_K(Sx_n, Sx_{n+1}) \leq W^n d_K(Sx_0, Sx_1),$$

for all $n \in \mathbb{N}$.

Given $R \in B^+(V)$ notice that if $||R||_o < 1$, then there exists $[I - R]^{-1}$ and $||[I - R]^{-1}||_o \leq \dfrac{1}{1 - ||R||_o}$. Also, if $Q \in B(V)$ such that there exists Q^{-1}, then for all $k \in \mathbb{R}^+$, $[kQ]^{-1} = \dfrac{Q^{-1}}{k}$ and $||[kQ]^{-1}||_o = \dfrac{||Q^{-1}||_o}{k}$. Take in to account these facts we proceed as follows.

Denote by $R := \dfrac{Q_2 + Q_3 + Q_4 + Q_5}{2}$ and $J := 2[Q_1 + R]$. Thus $W = [2[I - R]]^{-1} \circ [J]$. It is clear that $||R||_o < 1$, because $\sum_{i=1}^5 ||Q_i||_o < 1$. Therefore there exists $[I - R]^{-1}$, in consequence there exists $[2[I - R]]^{-1}$ and $||[2[I - R]]^{-1}||_o \leq \dfrac{1}{2 - 2||R||_o}$. Hence

$$\begin{aligned} ||W||_o &\leq ||[2[I - R]]^{-1}||_o ||J||_o \\ &\leq \frac{||J||_o}{2 - 2||R||_o} \\ &= \frac{||2Q_1 + Q_2 + Q_3 + Q_4 + Q_5||_o}{2 - ||Q_2 + Q_3 + Q_4 + Q_5||_o} \\ &\leq \frac{2||Q_1||_o + ||Q_2||_o + ||Q_3||_o + ||Q_4||_o + ||Q_5||_o}{2 - ||Q_2||_o - ||Q_3||_o - ||Q_4||_o - ||Q_5||_o} < 1, \end{aligned}$$

because $\sum_{i=1}^5 ||Q_i||_o < 1$. We next prove that $\{Sx_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. For each $m \geq 1$ we have

$$\begin{aligned} d_K(Sx_n, Sx_{n+m}) &\leq d_K(Sx_n, Sx_{n+1}) + d_K(Sx_{n+1}, Sx_{n+2}) + \cdots \\ &\quad \cdots + d_K(Sx_{n+m-1}, Sx_{n+m})] \\ &\leq (W^n + W^{n+1} + \cdots + W^{n+m-1}) d_K(Sx_0, Sx_1). \end{aligned}$$

Hence

$$
\begin{aligned}
D(Sx_n, Sx_{n+m}) &\le \lambda[(||W||_o^n + ||W||_o^{n+1} + \cdots + ||W||_o^{n+m-1})D(Sx_0, Sx_1)] \\
&\le \lambda||W||_o^n[(1 + ||W||_o + \cdots + ||W||_o^{m-1} + \cdots)D(Sx_0, Sx_1)] \\
&= \frac{\lambda||W||_o^n}{1-||W||_o} D(Sx_0, Sx_1).
\end{aligned}
$$

Therefore, $D(Sx_n, Sx_{n+m}) \to 0$ as $n \to \infty$, hence $\{Sx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $Sx_n \to x^*$.

Proof of (3) Suppose that S is one to one, sequentially continuous and sequentially convergent mapping. Since $Sx_n \to x^*$, there exists $y^* \in X$ such that $x_n \to y^*$ and it must be the case that $Sx_n \to Sy^*$. Therefore $Sy^* = x^*$. We first show that $Ty^* = y^*$.

For each $n \ge 1$

$$
d_K(STy^*, Sy^*) \le d_K(STy^*, Sx_n) + d_K(Sx_n, Sy^*).
$$

We observe that

$$
\begin{aligned}
d_K(STy^*, Sx_n) &= d_K(STy^*, STx_{n-1}) \\
&\le Q_1(d_K(Sy^*, Sx_{n-1})) + Q_2(d_K(Sy^*, STy^*)) \\
&\quad + Q_3(d_K(Sx_{n-1}, Sx_n)) + Q_4(d_K(Sy^*, Sx_n)) \\
&\quad + Q_5(d_K(Sx_{n-1}, STy^*)).
\end{aligned}
$$

Which implies

$$
\begin{aligned}
[I - Q_2 - Q_5](d_K(STy^*, Sy^*)) &\le d_K(Sx_n, Sy^*) + Q_1(d_K(Sy^*, Sx_{n-1})) \\
&\quad + Q_3(d_K(Sx_{n-1}, Sx_n)) + Q_4(d_K(Sy^*, Sx_n)) \\
&\quad + Q_5(d_K(Sx_{n-1}, STy^*) - d_K(STy^*, Sy^*)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d_K(STy^*, Sy^*) &\le [I - Q_2 - Q_5]^{-1}(d_K(Sx_n, Sy^*) + Q_1(d_K(Sy^*, Sx_{n-1})) \\
&\quad + Q_3(d_K(Sx_{n-1}, Sx_n)) + Q_4(d_K(Sy^*, Sx_n)) \\
&\quad + Q_5(d_K(Sx_{n-1}, STy^*) - d_K(STy^*, Sy^*))).
\end{aligned}
$$

$$
\begin{aligned}
||d_K(STy^*, Sy^*)|| &= D(STy^*, Sy^*) \\
&\le \lambda||[I - Q_2 - Q_5]^{-1}||||d_K(Sx_n, Sy^*) + Q_1(d_K(Sy^*, Sx_{n-1})) \\
&\quad + Q_3(d_K(Sx_{n-1}, Sx_n)) + Q_4(d_K(Sy^*, Sx_n)) \\
&\quad + Q_5(d_K(Sx_{n-1}, STy^*) - d_K(STy^*, Sy^*))|| \\
&\le \lambda||[I - Q_2 - Q_5]^{-1}||[D(Sx_n, Sy^*) + ||Q_1||_o D(Sx_{n-1}, Sy^*) \\
&\quad + ||Q_3||_o D(Sx_{n-1}, Sx_n) + ||Q_4||_o D(Sx_n, Sy^*) \\
&\quad + ||Q_5||_o ||d_K(Sx_{n-1}, STy^*) - d_K(Sy^*, STy^*)||].
\end{aligned}
$$

Denote by $\epsilon := D(STy^*, Sy^*)$, $a_n := D(Sx_n, Sy^*)$, $b_n := \lambda||W||^n D(Sx_0, Sx_1)$ and $c_n = ||d_K(Sx_n, STy^*) - d_K(Sy^*, STy^*)||$. Thus

$$
\epsilon \le \lambda||[I - Q_2 - Q_5]^{-1}||[a_n + ||Q_1||_o a_{n-1} + ||Q_3||_o b_{n-1} + ||Q_4||_o a_n + ||Q_5||_o c_{n-1}].
$$

It is clear that $a_n, b_n \to 0$ as $n \to \infty$. On the other hand, by Lemma 3.5 we have $c_n \to 0$ as $n \to \infty$. Thus, $D(STy^*, Sy^*) = 0$. This fact implies $STy^* = Sy^*$, and from the injectivity of S it follows that $Ty^* = y^*$.

We shall show that y^* is the unique fixed point of T. Suppose that there exists another $x^* \in X$ such that $Tx^* = x^*$ and $x^* \neq y^*$. Hence, from the injectivity of

S we get $d_K(Sx^*, Sy^*) \in K - \{0_V\}$. Applying that T is a S-Hardy-Rogers cone contraction, we have that

$$\begin{aligned} d_K(Sx^*, Sy^*) &= d_K(STx^*, STy^*) \\ &\leq Q_1(d_K(Sx^*, Sy^*)) + Q_2(d_K(Sx^*, STx^*)) + Q_3(d_K(Sy^*, STy^*)) \\ &\quad + Q_4(d_K(Sx^*, STy^*)) + Q_5(d_K(Sy^*, STx^*)). \\ &= [Q_1 + Q_4 + Q_5](d_K(Sx^*, Sy^*)), \end{aligned}$$

which is a contradiction by Lemma 1.13. Therefore y^* is the unique fixed point of T and the proof is complete.

□

Taking $Sx = x$ in the above theorem, the conditions $Q_2 \equiv Q_3 \equiv Q_4 \equiv Q_5 \equiv \Theta$ yields Banach's fixed point theorem, while $Q_1 \equiv Q_4 \equiv Q_5 \equiv \Theta$ yields Kannan's fixed point theorem and $Q_1 \equiv Q_2 \equiv Q_3 \equiv \Theta$ yields Chatterjea fixed point theorem in the context of cone metric spaces.

Corollary 3.7. *Let (X, d_K) be a complete CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone and $T : X \to X$ be a mapping. If for all $x, y \in X$ we have that*

$$d_K(Tx, Ty) \leq Q(d_K(x, y)),$$

where $Q \in B^+(V)$ and $||Q||_o < 1$, then T has a unique fixed point in X.

Corollary 3.8. ([8]) *Let (X, d_K) be a complete CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone. If T is a self-mapping on X such that*

$$d_K(Tx, Ty) \leq Q(d_K(x, Tx)) + R(d_K(y, Ty)),$$

for all $x, y \in X$, where $Q, R \in B^+(V)$ and $||Q||_o + ||R||_o < 1$, then T has a unique fixed point in X.

Corollary 3.9. *Let (X, d_K) be a complete CMS over an ordered Banach space $(V, ||\cdot||, K)$ such that K is a normal cone. If T is a self-mapping on X such that*

$$d_K(Tx, Ty) \leq Q(d_K(x, Ty)) + R(d_K(y, Tx)),$$

for all $x, y \in X$, where $Q, R \in B^+(V)$ and $||Q||_o + ||R||_o < 1$, then T has a unique fixed point in X.

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