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FINITE NUMBER SUMS IN HIGHER ORDER POWERS OF HARMONIC NUMBERS

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Abstract. We develop a set of identities for finite sums of products of harmonic numbers in higher order and reciprocal binomial coefficients. The results are analogous to some identities of Euler type.

1. Introduction and preliminaries

Let, as usual,

$$H_n = \gamma + \psi(n+1) = \sum_{r=1}^{n} \frac{1}{r} = \int_0^\infty \frac{1-t^n}{1-t} dt$$

be the n^{th} harmonic number and γ denotes the Euler–Mascheroni constant. Let, also, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote, respectively the sets of real, complex and natural numbers. A generalized binomial coefficient $\binom{w}{z}$ may be defined by

$$\binom{w}{z} := \frac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)};\ w,\ z \in \mathbb{C}$$

and in the special case when $z = n,\ n \in \mathbb{N} \cup \{0\},\ \mathbb{N} := \{1, 2, 3...\}$ we have

$$\binom{w}{n} := \frac{w(w-1)...(w-n+1)}{n!} = \frac{(-1)^n (-w)_n}{n!}.$$

The familiar gamma function

$$\Gamma(w) = \begin{cases} \int\limits_0^\infty e^{-y} y^{w-1} dy,\ \Re(y) > 0, \\ \frac{\Gamma(w+n)}{\prod\limits_{j=0}^{n-1}(w+j)},\ w \in \mathbb{C}\backslash\{0, -1, -2, -3, ...\},\ n \in \mathbb{N} \end{cases}$$

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and

$$(w)_\lambda := \frac{\Gamma(w+\lambda)}{\Gamma(w)} = \begin{cases} 1, \ \lambda = 0; \ w \in \mathbb{C}\setminus\{0\} \\ w(w+1)\dots(w+n-1), \ w \in \mathbb{C}, \ n \in \mathbb{N} \end{cases}, \ \text{with } (0)_0 := 1$$

is known as the Pochhammer symbol. The *generalized n^{th} harmonic number* in power r, $H_n^{(r)}$, is defined for positive integers n and r as

$$H_n^{(r)} := \sum_{m=1}^{n} \frac{1}{m^r} = \frac{(-1)^{r-1}}{(r-1)!}\Big(\psi^{(r-1)}(n+1) - \psi^{(r-1)}(1)\Big),$$

where

$$\psi^{(m)}(z) = (-1)^{m+1} m! \sum_{r=0}^{\infty} \frac{1}{(z+r)^{m+1}} = (-1)^{m+1} m! \, \zeta(m+1, z)$$

are the polygamma functions of order m which are defined by $\psi^{(0)}(z) \equiv \psi(z)$ and $\psi^{(m)}(z) := d^m \psi(z)/dz^m$, $m \in \mathbb{N}$ and $z \neq 0, -1, -2, \ldots$. Here $\psi(z)$ is the psi, or digamma, function, given as the logarithmic derivative of the well-known gamma function $\Gamma(z)$, *i.e.* $\psi(z) := d \log \Gamma(z)/dz$. $\zeta(\alpha, z)$ denotes the Hurwitz zeta function. There also exists the very useful recurrence relation $\psi^{(m)}(z+1) = \frac{(-1)^m m!}{z^{m+1}} + \psi^{(m)}(z)$. We shall provide, in this paper, identities for the general finite sums

$$\sum_{n=1}^{m} \frac{H_n^{(p)}}{n(n+a)}, \qquad \sum_{n=1}^{m} \frac{H_n^{(p)}}{(n+1)(n+a)}, \qquad \sum_{n=1}^{m} \frac{H_n^{(p)}}{\binom{n+k}{k}} \ \text{and} \ \sum_{n=1}^{m} \frac{H_n^{(p)}}{n\binom{n+k}{k}}.$$

Analogous results of Euler type for infinite series have been developed by many authors, see for example [3], [4], [17] and references therein. Many finite versions of higher order harmonic number sum identities also exist in the literature, for example [15]

$$\sum_{n=0}^{p} (-1)^{n+1} \binom{p}{n} \left[(H_n)^3 + 3H_n H_n^{(2)} + 2H_n^{(3)}\right] = \frac{6}{p^3}$$

and by the inversion formula

$$\sum_{n=1}^{p} \frac{6(-1)^{n+1}}{n^3} \binom{p}{n} = (H_p)^3 + 3H_p H_p^{(2)} + 2H_p^{(3)}.$$

Also, after a minor modification, [11]

$$\sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r}^3 \left(1 + 3r^2\left(H_{k-r}^{(2)} + H_{r-1}^{(2)}\right) + \left(3r\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right) - 1\right)^2\right) = 2.$$

Other finite sum identities can be seen in [1], [18] or [19] where they obtain results like

$$\sum_{k=0}^{2m} (-1)^k \binom{n}{k}^3 H_k = \frac{(-1)^m (3m)!}{2(m!)^3} \left(H_m + 2H_{2m} - H_{3m}\right).$$

Further work in the summation of harmonic numbers and binomial coefficients has also been done by Sofo [14]. The works of, [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13],[16], [20], and references therein, also investigate various representations of binomial sums and zeta functions in simpler form by the use of the Beta function and by means of certain summation theorems for hypergeometric series, [17]. The following results will be useful for later analysis.

Lemma 1.1. *Let m and p be a positive integers and $a>0$ then*

$$\sum_{n=1}^{m}\frac{a\ H_n^{(p)}}{n\,(n+a)} = H_m^{(p+1)}+H_m^{(p)}\left(H_{a-1}+H_m-H_{m+a}\right)+(-1)^{p+1}H_mH_{a-1}^{(p)} + \sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\left(H_j-H_{m+j}\right)+\sum_{s=2}^{p-1}(-1)^{p-s}H_m^{(s)}H_{a-1}^{(p-s+1)} \tag{1.1}$$

Proof. Consider

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{n\,(n+a)}=\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{1}{k^p\ n\ (n+a)}$$

since these sums are absolutely convergent we can write

$$\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{1}{k^p\ n\ (n+a)} = \sum_{k=1}^{m}\sum_{n=k}^{m}\frac{1}{k^p\ n\ (n+a)}$$

$$=\sum_{k=1}^{m}\frac{1}{a\ k^p}\left[\psi(a+k)-\psi(k)-\left(\psi(m+a+1)-\psi(m+1)\right)\right].$$

Now

$$\sum_{k=1}^{m}\frac{1}{a\ k^p}\left[\psi(a+k)-\psi(k)-\left(\psi(m+a+1)-\psi(m+1)\right)\right]$$

$$= \sum_{k=1}^{m}\frac{1}{ak^p}\left(\sum_{j=0}^{a-1}\frac{1}{k+j}-\sum_{j=0}^{a-1}\frac{1}{m+1+j}\right)$$

$$= \sum_{k=1}^{m}\frac{1}{ak^p}\left(\frac{1}{k}-\frac{1}{m+1}+\sum_{j=1}^{a-1}\left(\frac{1}{k+j}-\frac{1}{m+1+j}\right)\right)$$

$$= \frac{H_m^{(p+1)}}{a}-\frac{H_m^{(p)}}{a\,(m+1)}+\frac{1}{a}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\left(\frac{1}{k^p\,(k+j)}-\frac{1}{k^p\,(m+1+j)}\right)$$

$$= \frac{H_m^{(p+1)}}{a}-\frac{H_m^{(p)}}{a\,(m+1)}-\frac{1}{a}\sum_{j=1}^{a-1}\frac{H_m^{(p)}}{(m+1+j)}+\frac{1}{a}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\frac{1}{k^p\,(k+j)}$$

$$=\frac{H_m^{(p+1)}}{a}-\frac{H_m^{(p)}}{a\,(m+1)}-\frac{H_m^{(p)}}{a}\left(H_{m+a}-H_{m+1}\right)+\frac{1}{a}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\left(\frac{(-1)^{p+1}}{j^{p-1}k\,(k+j)}+\sum_{s=2}^{p}\frac{(-1)^{p-s}}{k^sj^{p-s+1}}\right)$$

by partial fraction decomposition, hence

$$= \frac{H_m^{(p+1)}}{a} - \frac{H_m^{(p)}}{a\,(m+1)} - \frac{H_m^{(p)}}{a}\,(H_{m+a} - H_{m+1})$$
$$+\frac{1}{a}\sum_{j=1}^{a-1}\left(\frac{(-1)^{p+1}}{j^p}\,(H_j + H_m - H_{m+j}) + \sum_{s=2}^{p}\frac{(-1)^{p-s}\,H_m^{(s)}}{j^{p-s+1}}\right).$$

Therefore

$$\sum_{n=1}^{m}\frac{a\;H_n^{(p)}}{n\,(n+a)} = H_m^{(p+1)} - H_m^{(p)}\left(\frac{1}{m+1} + H_{m+a} - H_{m+1} - H_{a-1}\right)$$

$$+(-1)^{p+1}\,H_m H_{a-1}^{(p)} + \sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j}) + \sum_{s=2}^{p-1}(-1)^{p-s}\,H_m^{(s)} H_{a-1}^{(p-s+1)}$$

$$= H_m^{(p+1)} + H_m^{(p)}\,(H_{a-1} + H_m - H_{m+a}) + (-1)^{p+1}\,H_m H_{a-1}^{(p)}$$
$$+\sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j})\;+\sum_{s=2}^{p-1}(-1)^{p-s}\,H_m^{(s)} H_{a-1}^{(p-s+1)}$$

and the result (1.1) follows. □

Remark. *When $a = 1$, m and p are positive integers, and we define $H_0^{(\alpha)} = 0$, then*

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{n\,(n+1)} = H_m^{(p+1)} - \frac{H_m^{(p)}}{m+1} \tag{1.2}$$

and for $a = 2$

$$\sum_{n=1}^{m}\frac{2\;H_n^{(p)}}{n\,(n+2)} = H_m^{(p+1)} + \frac{\left(m^2+m-1\right)H_m^{(p)}}{(m+1)\,(m+2)} + \frac{m\,(-1)^{p+1}}{m+1} + \sum_{s=2}^{p-1}(-1)^{p-s}\,H_m^{(s)}.$$

A related Lemma which will be useful later is the following.

Lemma 1.2. *Let m and p be a positive integers and $a > 0$ then*

$$\sum_{n=1}^{m}\frac{(a-1)\;H_n^{(p)}}{(n+1)\,(n+a)} = H_m^{(p)}\,(H_{a-1} + H_{m+1} - H_{m+a}) + (-1)^{p+1}\,H_m H_{a-1}^{(p)}$$
$$+\sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j})\;+\sum_{s=2}^{p-1}(-1)^{p-s}\,H_m^{(s)} H_{a-1}^{(p-s+1)} \tag{1.3}$$

Proof. Consider

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{(n+1)(n+a)}=\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{1}{k^p\ (n+1)\ (n+a)}$$

since these sums are absolutely convergent we can write

$$\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{1}{k^p\ (n+1)\ (n+a)} \quad = \quad \sum_{k=1}^{m}\sum_{n=k}^{m}\frac{1}{k^p\ (n+1)\ (n+a)}$$

$$=\sum_{k=1}^{m}\frac{1}{(a-1)\ k^p}\left[\psi(a+k)-\psi(k+1)-(\psi(m+a+1)-\psi(m+2))\right].$$

Now

$$\sum_{k=1}^{m}\frac{1}{(a-1)\ k^p}\left[\psi(a+k)-\psi(k+1)-(\psi(m+a+1)-\psi(m+2))\right]$$

$$= \quad \sum_{k=1}^{m}\frac{1}{(a-1)\,k^p}\ \left(\sum_{j=1}^{a-1}\frac{1}{k+j}-\sum_{j=1}^{a-1}\frac{1}{m+1+j}\right)$$

$$= \quad \sum_{k=1}^{m}\frac{1}{(a-1)\,k^p}\ \left(\sum_{j=1}^{a-1}\left(\frac{1}{k+j}-\frac{1}{m+1+j}\right)\right)$$

$$= \quad \frac{1}{(a-1)}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\left(\frac{1}{k^p\,(k+j)}-\frac{1}{k^p\,(m+1+j)}\right)$$

$$= \quad -\frac{1}{(a-1)}\sum_{j=1}^{a-1}\frac{H_m^{(p)}}{(m+1+j)}+\frac{1}{(a-1)}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\frac{1}{k^p\,(k+j)}$$

$$=-\frac{H_m^{(p)}}{(a-1)}\left(H_{m+a}-H_{m+1}\right)+\frac{1}{(a-1)}\sum_{j=1}^{a-1}\sum_{k=1}^{m}\left(\frac{(-1)^{p+1}}{j^{p-1}k\,(k+j)}+\sum_{s=2}^{p}\frac{(-1)^{p-s}}{k^s j^{p-s+1}}\right)$$

by partial fraction decomposition, hence

$$= \quad -\frac{H_m^{(p)}}{(a-1)}\left(H_{m+a}-H_{m+1}\right)$$

$$+\frac{1}{(a-1)}\sum_{j=1}^{a-1}\left(\frac{(-1)^{p+1}}{j^p}\left(H_j+H_m-H_{m+j}\right)+\sum_{s=2}^{p}\frac{(-1)^{p-s}\,H_m^{(s)}}{j^{p-s+1}}\right).$$

Therefore

$$\sum_{n=1}^{m}\frac{(a-1)\ H_n^{(p)}}{(n+1)\,(n+a)} \quad = \quad -H_m^{(p)}\left(H_{m+a}-H_{m+1}-H_{a-1}\right)$$

$$+(-1)^{p+1}\,H_m H_{a-1}^{(p)}+\sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\left(H_j-H_{m+j}\right)+\sum_{s=2}^{p-1}(-1)^{p-s}\,H_m^{(s)}H_{a-1}^{(p-s+1)}$$

$$= H_m^{(p)}\left(H_{a-1}+H_{m+1}-H_{m+a}\right)+(-1)^{p+1}H_mH_{a-1}^{(p)}$$

$$+\sum_{j=1}^{a-1}\frac{(-1)^{p+1}}{j^p}\left(H_j-H_{m+j}\right)+\sum_{s=2}^{p-1}(-1)^{p-s}H_m^{(s)}H_{a-1}^{(p-s+1)}$$

and the result (1.3) follows. □

Some special cases are noted in the remark.

Remark. *When $a=2$ and p and m positive integers, then*

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{(n+1)(n+2)}=\frac{(m+1)H_m^{(p)}}{(m+2)}+\frac{m(-1)^{p+1}}{m+1}+\sum_{s=2}^{p-1}(-1)^{p-s}H_m^{(s)}. \tag{1.4}$$

2. Two theorems

We now prove the following two theorems.

Theorem 2.1. *Let p be a positive integer and k be a positive integer greater than 1, then*

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{\binom{n+k}{k}}=kH_m^{(p)}\ H_{m+1} \tag{2.1}$$

$$+\sum_{r=2}^{k}(-1)^r r\binom{k}{r}\left[\begin{array}{c}-H_m^{(p)}H_{m+r}+(-1)^{p+1}H_mH_{r-1}^{(p)}+\\ +\sum_{j=1}^{r-1}\frac{(-1)^{p+1}}{j^p}\left(H_j-H_{m+j}\right)+\sum_{s=2}^{p}(-1)^{p-s}H_m^{(s)}H_{r-1}^{(p-s+1)}\end{array}\right].$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{\binom{n+k}{k}}=\sum_{n=1}^{m}\frac{k!\ H_n^{(p)}}{(n+1)\prod_{r=2}^{k}(n+r)}=\sum_{n=1}^{m}\frac{k!\ H_n^{(p)}}{(n+1)(n+2)_{k+1}}$$

where $(\alpha)_r$ is Pochhammer's symbol given by $(\alpha)_r=\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)$, $r>0$, $(\alpha)_0=1$. Now

$$\sum_{n=1}^{m}\frac{H_n^{(p)}}{\binom{n+k}{k}}=\sum_{n=1}^{m}\frac{k!\ H_n^{(p)}}{(n+1)}\sum_{r=2}^{k}\frac{A_r}{(n+r)} \tag{2.2}$$

where

$$A_r=\lim_{n\to -r}\frac{n+r}{\prod_{r=2}^{k}(n+r)}=\frac{2(-1)^r}{k!}\binom{k}{r}\binom{r}{2}.$$

Now from (2.2) and using Lemma 1.2

$$\sum_{n=1}^{m} \frac{k!\ H_n^{(p)}}{(n+1)} \sum_{r=2}^{k} \frac{A_r}{(n+r)} = \sum_{r=2}^{k} (-1)^r\ r\,(r-1) \binom{k}{r} \sum_{n=1}^{m} \frac{H_n^{(p)}}{(n+1)\,(n+r)}$$

$$= \sum_{r=2}^{k} (-1)^r\ r \binom{k}{r} \left[\begin{array}{c} H_m^{(p)}\,(H_{m+1} - H_{m+r}) + (-1)^{p+1}\,H_m H_{r-1}^{(p)} \\ \\ + \sum_{j=1}^{r-1} \frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j})\ + \sum_{s=2}^{p} (-1)^{p-s}\,H_m^{(s)} H_{r-1}^{(p-s+1)} \end{array} \right]$$

$$= kH_m^{(p)}\ H_{m+1} + \sum_{r=2}^{k} (-1)^r\ r \binom{k}{r} \left[-H_m^{(p)} H_{m+r} + (-1)^{p+1}\,H_m H_{r-1}^{(p)} \right]$$

$$+ \sum_{r=2}^{k} (-1)^r\ r \binom{k}{r} \left[\sum_{j=1}^{r-1} \frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j})\ + \sum_{s=2}^{p} (-1)^{p-s}\,H_m^{(s)} H_{r-1}^{(p-s+1)} \right]$$

$$= \sum_{n=1}^{m} \frac{H_n^{(p)}}{\binom{n+k}{k}}$$

which is the result (2.1). □

Now we consider the following extension of Theorem 2.1.

Theorem 2.2. *Let k, m and p be positive integers, then*

$$\sum_{n=1}^{m} \frac{H_n^{(p)}}{n\binom{n+k}{k}} = H_m^{(p+1)} + H_m^{(p)} H_m + \sum_{r=1}^{k} (-1)^r \binom{k}{r} \left[H_m^{(p)} H_{m+r} + (-1)^p\,H_m H_{r-1}^{(p)} \right] \tag{2.3}$$

$$- \sum_{r=1}^{k} (-1)^r \binom{k}{r} \left[\sum_{j=1}^{r-1} \frac{(-1)^{p+1}}{j^p}\,(H_j - H_{m+j})\ + \sum_{s=2}^{p} (-1)^{p-s}\,H_m^{(s)} H_{r-1}^{(p-s+1)} \right].$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{m} \frac{H_n^{(p)}}{n\binom{n+k}{k}} = \sum_{n=1}^{m} \frac{k!\ H_n^{(p)}}{n\prod_{r=1}^{k}(n+r)} = \sum_{n=1}^{m} \frac{k!\ H_n^{(p)}}{n\,(n+1)_{k+1}}$$

Now

$$\sum_{n=1}^{m} \frac{H_n^{(p)}}{\binom{n+k}{k}} = \sum_{n=1}^{m} \frac{k!\ H_n^{(p)}}{n} \sum_{r=1}^{k} \frac{B_r}{(n+r)}$$

where

$$B_r = \lim_{n\to -r} \frac{n+r}{\prod_{r=1}^{k}(n+r)} = \frac{(-1)^{r+1} r}{k!} \binom{k}{r}.$$

Now from (2.3) and using Lemma 1.1

$$\sum_{n=1}^{m} \frac{k!\ H_n^{(p)}}{n} \sum_{r=1}^{k} \frac{B_r}{(n+r)} = \sum_{r=1}^{k} (-1)^{r+1}\ r \binom{k}{r} \sum_{n=1}^{m} \frac{H_n^{(p)}}{n\,(n+r)}$$

$$= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \left[\begin{array}{c} H_m^{(p+1)} + H_m^{(p)} (H_m - H_{m+r}) + (-1)^{p+1} H_m H_{r-1}^{(p)} \\ + \sum_{j=1}^{r-1} \frac{(-1)^{p+1}}{j^p} (H_j - H_{m+j}) \ + \sum_{s=2}^{p} (-1)^{p-s} H_m^{(s)} H_{r-1}^{(p-s+1)} \end{array} \right]$$

$$\begin{aligned} = \ & H_m^{(p+1)} + H_m^{(p)} H_m \\ & + \sum_{r=1}^{k} (-1)^{r} \binom{k}{r} \left[\begin{array}{c} H_m^{(p)} H_{m+r} + (-1)^{p} H_m H_{r-1}^{(p)} \\ - \sum_{j=1}^{r-1} \frac{(-1)^{p+1}}{j^p} (H_j - H_{m+j}) \ - \sum_{s=2}^{p} (-1)^{p-s} H_m^{(s)} H_{r-1}^{(p-s+1)} \end{array} \right] \\ = \ & \sum_{n=1}^{m} \frac{H_n^{(p)}}{n \binom{n+k}{k}} \end{aligned}$$

which is the result (2.3). □

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