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QUATERNIONIC BERTRAND CURVES IN EUCLIDEAN 4-SPACE

(COMMUNICATED BY UDAY CHAND DE)

OSMAN KEÇİLİOĞLU AND KAZIM İLARSLAN

Abstract. In this paper, by using the similar idea of *Matsuda and Yorozu* [12], we prove that if bitorsion of a quatenionic curve α is no vanish, then there is no quaternionic curve in E^4 is a Bertrand curve. Then we define $(1,3)$ type Bertrand curves for quatenionic curve in Euclidean 4-space. We give some characterizations for a $(1,3)$ type quaternionic Bertrand curves in Euclidean 4-space by means of the curvature functions of the curve.

1. Introduction

Characterization of a regular curve is one of the important and interesting problems in the theory of curves in Euclidean space. There are two ways widely used to solve these problems: to figure out the relationship between the Frenet vectors of the curves (see [11]), and to determine the shape and size of a regular curve by using its curvatures. k_1 (or $\varkappa$) and k_2 (or τ), the curvature functions of a regular curve, have an effective role. For example: if k_1 =constant$\neq 0$ and $k_2 = 0$, the curve is a circle with radius $(1/k_1)$, etc.

In 1845, *Saint Venant* (see [14]) proposed the question whether the principal normal of a curve is the principal normal of another's on the surface generated by the principal normal of the given one. Bertrand answered this question in ([3]) published in 1850. He proved that a necessary and sufficient condition for the existence of such a second curve is required in fact a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, if we denote first and second curvatures of a given curve by $k_1$and k_2 respectively, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in \mathbb{R}$. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [11]).

There are many important papers on Bertrand curves in Euclidean space (see: [4],[5],[13]).

When we investigate the properties of Bertrand curves in Euclidean n-space, it is easy to see that either k_2 or k_3 is zero which means that Bertrand curves in

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$\mathbb{E}^n$ $(n > 3)$ are degenerate curves (see [13]). This result is restated by *Matsuda and Yorozu* [12] . They proved that *there was not any special Bertrand curves in* $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1, 3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3- space and Minkowski space-time (see [1, 7, 10]) as well as in Euclidean space.

K. Bharathi and M. Nagaraj (in [2]) studied a quaternionic curve in Euclidean 3-space $\mathbb{E}^3$ and Euclidean 4-space $\mathbb{E}^4$ and gave the Frenet formula for quaternionic curve. For the newest results for quaternioonic curves, we refer the papers [8, 9, 16].

In this paper, by using the similar idea of *Matsuda and Yorozu* [12], we prove that if bitorsion of a quatenionic curve α is no vanish, then there is no quaternionic curves in $\mathbb{E}^4$ is a Bertrand curve. Then we define $(1, 3)-$ type Bertrand curves for quatenionic curves in Euclidean 4-space. We give some characterizations for a $(1, 3)-$ type quaternionic Bertrand curves in Euclidean 4-space by means of the curvature functions of the curve

2. Preliminaries

Let Q_H denote a four dimensional vector space over a field H whose characteristic grater than 2. Let e_i $(1 \leq i \leq 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \leq i \leq 4)$ and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined by $q = ae_1 + be_2 + ce_3 + de_4$ where a, b, c, d are ordinary numbers. Such that

$$\begin{array}{ll} e_4 = 1 & e_1^2 = e_2^2 = e_3^2 = -1 \\ e_1e_2 = -e_2e_1 = e_3, & e_2e_3 = -e_3e_2 = e_1, \\ e_3e_1 = -e_1e_3 = e_2 & \end{array}$$

If we denote $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$, we can rewrite a real quaternion whose basic algebric form is $q = S_q + V_q$ where S_q is scalar part and V_q is vectorial part of q. Using these basic products we can now expand the product of two quatenions as

$$p \times q = S_pS_q - \langle V_p, V_q \rangle + S_pV_q + S_qV_p + V_p \wedge V_q \text{ for every } p, q \in Q_H$$

where we have used the inner and cross products in Euclidean space E^3. There is a uniqe involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -ae_1 - be_2 - ce_3 + de_4 \text{ for every } ae_1 + be_2 + ce_3 + de_4 \in Q_H$$

which is called the "Hamilton conjugation". This defines the symmetric, real valued, non-degenerate, bilinear form h as follows:

$$h(p, q) = \frac{1}{2}\left[p \times \gamma q + q \times \gamma p\right] \text{ for every } p, q \in Q_H.$$

And then, the norm of any q real quaternion denoted

$$\|q\|^2 = h(q, q) = q \times \gamma q.$$

q is called a spatial quaternion whenever $q + \gamma q = 0$ ([6]).

The Serret-Frenet formulae for quaternionic curves in E^3 and E^4 are as follows ([2]):

Theorem 2.1. *The three-dimensional Euclidean space E^3 is identified with the space of spatial quaternions $\{p \in Q_H \, | p + \gamma p = 0\}$ in an obvious manner. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$. Let*

$$\begin{array}{rccl} \alpha : & I \subset \mathbb{R} & \longrightarrow & Q_H \\ & s & \longrightarrow & \alpha(s) = \sum\limits_{i=1}^{3} \alpha_i(s) e_i \quad (1 \leq i \leq 3) \end{array}$$

be an arc-lenghted curve with nonzero curvatures $\{k, r\}$ and $\{t(s), n(s), b(s)\}$ denote the Frenet frame of the curve α. Then Frenet formulas are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where k is principal curvature, r is torsion of α.

Theorem 2.2. *The four-dimensional Euclidean space E^4 are identified with the space of unique quaternions. Let $I = [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and*

$$\begin{array}{rccl} \alpha : & I \subset \mathbb{R} & \longrightarrow & Q_H \\ & s & \longrightarrow & \alpha(s) = \sum\limits_{i=1}^{4} \alpha_i(s) e_i, \end{array}$$

be a smooth curve in E^4 with nonzero curvatures $\{K, k, r - K\}$ and $\{T(s), N(s)$ $, B_1(s), B_2(s)\}$ denotes the Frenet frame of the curve α. Then the frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where K is the principal curvature, k is torsion of β and $(r - K)$ is bitorsion of α.

Definition 2.3. Let $\alpha(s)$ and $\beta(s^*)$ be two quaternionic curves in E^4. $\{T(s),$ $T(s), N(s), B_1(s),\ B_2(s)\}$ and $\{T^*(s^*),\ N^*(s^*),\ B_1^*(s^*),\ B_2^*(s^*)\}$ are Frenet frames, respectively, on these curves. α and β are called Bertrand curves if there exist a bijection

$$\begin{array}{rccl} \varphi : & I & \longrightarrow & I^* \\ & s & \longrightarrow & \varphi(s) = s^*, \ \frac{ds^*}{ds} \neq 0 \end{array}$$

and the principal normal lines of α and β at corresponding points coincide.

3. Main Results

Theorem 3.1. *Let α be a quaternionic curve in E^4. If bitorsion of α is no vanish, then there is no quaternionic curve in E^4 is a Bertrand curve.*

Proof. Let α be a Bertrand curve in E^4 and β a Bertrand mate of β. β distinct from β. Let the pair of $\alpha(s)$ and $\beta(s^*) = \beta(\varphi(s))$ be of corresponding points of α and β. Then the curve β is given by

$$\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda(s) N(s) \tag{1}$$

where λ is a C^{∞} function on I. Differentiating (1) with respect to s, we obtain

$$\varphi'(s)\frac{d\beta(s^*)}{ds^*} = \alpha'(s) + \lambda'(s)N(s) + \lambda(s)N'(s).$$

Here and hereafter, the prime denotes the derivative with respect to s. By the Frenet equations, it holds that

$$\varphi'(s)T^*(\varphi(s)) = (1 - \lambda(s)K(s))T(s) + \lambda'(s)N(s) + \lambda(s)k(s)B_1(s).$$

Since $\langle T^*(\varphi(s)), N^*(\varphi(s))\rangle = 0$ and $N^*(\varphi(s)) = \mp N(s)$, we obtain, for all $s \in I$,

$$\lambda'(s) = 0,$$

that is, λ is a constant function on I. Thus (1) are rewritten as

$$\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda N(s) \tag{2}$$

and we obtain

$$\varphi'(s)T^*(\varphi(s)) = (1 - \lambda K(s))T(s) + \lambda k(s)B_1(s) \tag{3}$$

for all $s \in I$. By (3), we can set

$$T^*(\varphi(s)) = (\cos\theta(s))T(s) + (\sin\theta(s))B_1(s), \tag{4}$$

where θ is a C^{∞}-function on I and

$$\cos\theta(s) = \frac{1 - \lambda K(s)}{\varphi'(s)} \tag{5}$$

$$\sin\theta(s) = \frac{\lambda k(s)}{\varphi'(s)}. \tag{6}$$

Differentiating (4) and using the Frenet equations, we obtain

$$\begin{aligned}\bar{K}(\varphi(s))\varphi'(s)N^*(\varphi(s)) \quad = \quad & \frac{d\cos\theta(s)}{ds}T(s) \\ & + (K(s)\cos\theta(s) - k(s)\sin\theta(s))N(s) \\ & + \frac{d\sin\theta(s)}{ds}B_1(s) \\ & + (r - K)(s)\sin\theta(s)B_2(s).\end{aligned}$$

Since $N^*(\varphi(s)) = \mp N(s)$ for all $s \in I$, we obtain

$$(r - K)(s)\sin\theta(s) = 0. \tag{7}$$

By $(r - K)(s) \neq 0$ and (7), we obtain that $\sin\theta(s) = 0$. Thus, by $k(s) \neq 0$ and (6), we obtain that $\lambda = 0$. This completes the proof of theorem. □

Definition 3.2. Let $\alpha(s)$ and $\beta(s^*)$ be two quaternionic curves in E^4. $\{T(s), N(s), B_1(s), B_2(s)\}$ and $\{T^*(s^*), N^*(s^*), B_1^*(s^*), B_2^*(s^*)\}$ are Frenet frames, respectively, on these curves. α and β are called quaternionic $(1,3)$-Bertrand curves if there exist a bijection

$$\begin{aligned}\varphi: \quad & I \longrightarrow I^* \\ & s \longrightarrow \varphi(s) = s^*, \ \frac{ds^*}{ds} \neq 0\end{aligned}$$

and the plane spanned by $N(s)$,$B_2(s)$ at the each point $\alpha(s)$of α coincides with the plane spanned by $N^*(s^*)$,$B_2^*(s^*)$ corresponding point $\beta(s^*) = \beta(\varphi(s))$ of β.

Theorem 3.3. *Let α be a quaternionic curve in E^4 with curvature functions $K, k, r-K$ and $r-K \neq 0$. Then α is a $(1,3)$-Bertrand curve if and only if there exist constant real numbers $\lambda, \mu, \gamma, \delta$ satisfying*

(*i*) $\lambda k(s) - \mu(r-K)(s) \neq 0$

(*ii*) $\lambda K(s) + \gamma(\lambda k(s) - \mu(r-K)(s)) = 1$

(*iii*) $\gamma K(s) - k(s) = \delta(r-K)(s)$

(*iv*) $\left\{\left(\gamma^2-1\right) k(s) K(s) + \gamma\left[(K(s))^2 - (k(s))^2 - (r-K)^2(s)\right]\right\} \neq 0$

for all $s \in I$.

Proof. We assume that α is a $(1,3)$-Bertrand curve parametrized by arclenght s. The $(1,3)$-Bertrand mate β is given by

$$\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda(s) N(s) + \mu(s) B_2(s) \tag{8}$$

for all $s \in I$. Here λ and μ are C^∞-function on I, and s^* is the arclenght parameter of β. Differentiating (8) with respect to s, and using the Frenet equations, we obtain

$$\begin{aligned} \varphi'(s) T^*(\varphi(s)) &= (1-\lambda(s) K(s)) T(s) + \lambda'(s) N(s) \\ &\quad + (\lambda(s) k(s) - \mu(s)(r-K)(s))) B_1(s) + \mu'(s) B_2(s) \end{aligned}$$

for all $s \in I$.

Since the plane spanned by $N(s)$ and $B_2(s)$ coincides with the plane spanned by $N^*(\varphi(s))$ and $B_2^*(\varphi(s))$, we can put

$$N^*(\varphi(s)) = (\cos\theta(s)) N(s) + (\sin\theta(s)) B_2(s) \tag{9}$$

$$B_2^*(\varphi(s)) = (-\sin\theta(s)) N(s) + (\cos\theta(s)) B_2(s) \tag{10}$$

and we notice that $\sin\theta(s) \neq 0$ for all $s \in I$. By the following facts

$$0 = \langle \varphi'(s) T^*(\varphi(s)), N^*(\varphi(s))\rangle = \lambda'(s)\cos\theta(s) + \mu'(s)\sin\theta(s)$$

$$0 = \langle \varphi'(s) T^*(\varphi(s)), B_2^*(\varphi(s))\rangle = -\lambda'(s)\sin\theta(s) + \mu'(s)\cos\theta(s)$$

we obtain

$$\lambda'(s) = 0, \mu'(s) = 0$$

that is, λ and μ are constant function on I with values λ and μ, respectively. Therefore, for all $s \in I$, (8) is rewritten as

$$\beta(s^*) = \beta(\varphi(s)) = \alpha(s) + \lambda N(s) + \mu B_2(s) \tag{11}$$

and we obtain

$$\varphi'(s) T^*(\varphi(s)) = (1-\lambda K(s)) T(s) + (\lambda k(s) - \mu(r-K)(s)) B_1(s). \tag{12}$$

Here we notice that

$$(\varphi'(s))^2 = (1-\lambda K(s))^2 + (\lambda k(s) - \mu(r-K)(s))^2 \neq 0 \tag{13}$$

for all $s \in I$. Thus we can set

$$T^*(\varphi(s)) = (\cos\tau(s)) T(s) + (\sin\tau(s)) B_1(s) \tag{14}$$

and

$$\cos\tau(s) = \frac{1-\lambda K(s)}{\varphi'(s)} \tag{15}$$

$$\sin\tau(s) = \frac{\lambda k(s) - \mu(r-K)(s)}{\varphi'(s)} \tag{16}$$

where τ is a C^{∞}-function on I. Differentiating (14) with respect to s and using the Frenet equations, we obtain

$$\begin{aligned}\bar{K}(\varphi(s))\varphi'(s)N^*(\varphi(s)) &= \frac{d\cos\tau(s)}{ds}T(s)\\ &+(K(s)\cos\tau(s)-k(s)\sin\tau(s))N(s)\\ &+\frac{d\sin\tau(s)}{ds}B_1(s)+(r-K)(s)\sin\tau(s)B_2(s).\end{aligned}$$

Since $N^*(\varphi(s))$ is expressed by linear combination of $N(s)$ and $B_2(s)$, it holds that

$$\frac{d\cos\tau(s)}{ds}=0, \frac{d\sin\tau(s)}{ds}=0,$$

that is, τ is a constant function on I with value τ_0. Thus we obtain

$$T^*(\varphi(s))=(\cos\tau_0)T(s)+(\sin\tau_0)B_1(s) \tag{17}$$

$$\varphi'(s)\cos\tau_0=1-\lambda K(s) \tag{18}$$

$$\varphi'(s)\sin\tau_0=\lambda k(s)-\mu(r-K)(s) \tag{19}$$

for all $s\in I$. There fore we obtain

$$(1-\lambda K(s))\sin\tau_0=(\lambda k(s)-\mu(r-K)(s))\cos\tau_0 \tag{20}$$

for all $s\in I$.

If $\sin\tau_0=0$, then it holds $\cos\tau_0=\mp 1$. Thus (17) implies that $T^*(\varphi(s))=\mp T(s)$. Differentiating this equality, we obtain

$$\bar{K}(\varphi(s))\varphi'(s)N^*(\varphi(s))=\mp K(s)N(s),$$

that is,

$$N^*(\varphi(s))=\mp N(s),$$

for all $s\in I$. By Theorem 3.1, this fact is a contradiction. Thus we must consider only the case of $\sin\tau_0\neq 0$. Then (19) imlies

$$\lambda k(s)-\mu(r-K)(s)\neq 0$$

that is, we obtain the relation (i).

The fact $\sin\tau_0\neq 0$ and (20) imply

$$\lambda K(s)+\sin^{-1}\tau_0\cos\tau_0(\lambda k(s)-\mu(r-K)(s))=1.$$

From this, we obtain

$$\lambda K(s)+\gamma(\lambda k(s)-\mu(r-K)(s))=1$$

for all $s\in I$, where $\gamma=\sin^{-1}\tau_0\cos\tau_0$ is a constant number. Thus we obtain the relation (ii).

Differentiating (17) with respect to s and using the Frenet equations, we obtain

$$\begin{aligned}\bar{K}(\varphi(s))\varphi'(s)N^*(\varphi(s)) &= (K(s)\cos\tau_0-k(s)\sin\tau_0)N(s)\\ &+(r-K)(s)\sin\tau_0 B_2(s)\end{aligned}$$

for all $s\in I$. From the above equality, (18), (19) and (b), we obtain

$$\begin{aligned}\left[\bar{K}(\varphi(s))\varphi'(s)\right]^2 &= [K(s)\cos\tau_0-k(s)\sin\tau_0]^2+[(r-K)(s)\sin\tau_0]^2\\ &= (\lambda k(s)-\mu(r-K)(s))^2\\ &\times\left[(\gamma K(s)-k(s))^2+((r-K)(s))^2\right](\varphi'(s))^{-2}\end{aligned}$$

for all $s \in I$. From (13) and (ii), it holds

$$\left(\varphi'(s)\right)^2 = \left(\gamma^2+1\right)\left(\lambda k(s) - \mu(r-K)(s)\right)^2.$$

Thus we obtain

$$\left[\bar{K}(\varphi(s))\varphi'(s)\right]^2 = \frac{1}{\gamma^2+1}\left[\left(\gamma K(s)-k(s)\right)^2 + \left((r-K)(s)\right)^2\right]. \tag{21}$$

By (18), (19) and (ii), we can set

$$N^*(\varphi(s)) = (\cos\eta(s))\,N(s) + (\sin\eta(s))\,B_2(s), \tag{22}$$

where

$$\cos\eta(s) = \frac{\left(\lambda k(s) - \mu(r-K)(s)\right)\left(\gamma K(s)-k(s)\right)}{\bar{K}(\varphi(s))\left(\varphi'(s)\right)^2} \tag{23}$$

$$\sin\eta(s) = \frac{(r-K)(s)\left(\lambda k(s) - \mu(r-K)(s)\right)}{\bar{K}(\varphi(s))\left(\varphi'(s)\right)^2} \tag{24}$$

for all $s \in I$. Here, η is a C^∞-function on I.

Differentiating (22) with respect to s and using Frenet equations, we get

$$\begin{aligned}
& -\varphi'(s)\bar{K}(\varphi(s))T^*(\varphi(s)) + \varphi'(s)\bar{k}(\varphi(s))B_1^*(\varphi(s)) \\
= \;& -K(s)\cos\eta(s)T(s) + \frac{d\cos\eta(s)}{ds}N(s) \\
& + \left\{k(s)\cos\eta(s) - (r-K)(s)\sin\eta(s)\right\}B_1(s) \\
& + \frac{d\sin\eta(s)}{ds}B_2(s)
\end{aligned}$$

for all $s \in I$.From the above fact, it holds

$$\frac{d\cos\eta(s)}{ds} = 0, \;\; \frac{d\sin\eta(s)}{ds} = 0$$

that is, η is a constant function on I with value η_0. Let $\delta = (\cos\eta_0)(\sin\eta_0)^{-1}$ be constant number. Then (23) and (24) imply

$$\gamma K(s) - k(s) = \delta(r-K)(s)$$

that is we obtain the relation (iii).

Moreover, we obtain

$$\begin{aligned}
& -\varphi'(s)\bar{K}(\varphi(s))T^*(\varphi(s)) + \varphi'(s)\bar{k}(\varphi(s))B_1^*(\varphi(s)) \\
= \;& -K(s)\cos\eta(s)T(s) \\
& + \left\{k(s)\cos\eta(s) - (r-K)(s)\sin\eta(s)\right\}B_1(s).
\end{aligned}$$

By the above equality and (12), we obtain

$$\begin{aligned}
\varphi'(s)\bar{k}(\varphi(s))B_1^*(\varphi(s)) &= \varphi'(s)\bar{K}(\varphi(s))T^*(\varphi(s)) \\
& \quad -K(s)\cos\eta_0 T(s) \\
& \quad + \left\{k(s)\cos\eta_0 - (r-K)(s)\sin\eta_0\right\}B_1(s) \\
&= \left\{\left(\varphi'(s)\right)^2\bar{K}(\varphi(s))\right\}^{-1} \\
& \quad \times\left\{A(s)T(s) + B(s)B_1(s)\right\},
\end{aligned}$$

where

$$\begin{aligned} A(s) &= \left\{\varphi'(s)\bar{K}(\varphi(s))\right\}^2 (1-\lambda K(s)) \\ &\quad -K(s)(\lambda k(s)-\mu(r-K)(s))(\gamma K(s)-k(s)) \end{aligned}$$

$$\begin{aligned} B(s) &= \left\{\varphi'(s)\bar{K}(\varphi(s))\right\}^2 (\lambda k(s)-\mu(r-K)(s)) \\ &\quad +\left\{\begin{array}{c} k(s)(\lambda k(s)-\mu(r-K)(s))(\gamma K(s)-k(s)) \\ -(r-K)^2(s)(\lambda k(s)-\mu(r-K)(s)) \end{array}\right\} \end{aligned}$$

From (ii) and (21), $A(s)$ and $B(s)$ are rewritten as:

$$\begin{aligned} A(s) &= (\lambda k(s)-\mu(r-K)(s))\left(\gamma^2+1\right)^{-1} \\ &\quad \times\left\{\left(1-\gamma^2\right)K(s)k(s)-\gamma\left[(K(s))^2-(k(s))^2-((r-K)(s))^2\right]\right\} \end{aligned}$$

$$\begin{aligned} B(s) &= \gamma\left(\gamma^2+1\right)^{-1}(\lambda k(s)-\mu(r-K)(s)) \\ &\quad \times\left\{\left(\gamma^2-1\right)k(s)K(s)+\gamma\left[(K(s))^2-(k(s))^2-(r-K)^2(s)\right]\right\}. \end{aligned}$$

Since $\bar{K}(\varphi(s))\varphi'(s)N^*(\varphi(s))\neq 0$ for all $s\in I$, it holds

$$\left\{\left(\gamma^2-1\right)k(s)K(s)+\gamma\left[(K(s))^2-(k(s))^2-(r-K)^2(s)\right]\right\}\neq 0$$

for all $s\in I$. Thus we obtain the relation (iv).

We assume that α $(\alpha : I\to Q_H)$ is a C^∞special Frenet curve in Q_H with curvature functions K, k and $(r-K)$ satisfying the relation $(i),(ii),(iii)$ and (iv) for constant numvers λ,μ,γ and δ. Then we define a $C^\infty-$curve β by

$$\beta(s)=\alpha(s)+\lambda N(s)+\mu B_2(s) \tag{25}$$

for all $s\in I$,where s is arclenght parameter of α. Diffferentiating (25) with recpect to s and using the Frenet equations, we obtain

$$\frac{d\beta(s)}{ds}=(1-\lambda K(s))T(s)+(\lambda k(s)-\mu(r-K))B_1(s)$$

for all $s\in I$. Thus, by the relation (ii), we obtain

$$\frac{d\beta(s)}{ds}=(\lambda k(s)-\mu(r-K)(s))[\gamma T(s)+B_1(s)]$$

for all $s\in I$. Since the relation (i) holds, the curve β is a regular curve. Then there exists a regular map $\varphi : I\to\bar{I}$ defined by

$$s^*=\varphi(s)=\int_0^s\left\|\frac{d\beta(t)}{dt}\right\|dt$$

where s^* denotes the arclenght parameter of β, and we obtain

$$\varphi'(s)=\varepsilon\sqrt{\gamma^2+1}(\lambda k(s)-\mu(r-K)(s))>0, \tag{26}$$

where $\varepsilon=1$ if $\lambda k(s)-\mu(r-K)(s)>0$, and $\varepsilon=-1$ if $\lambda k(s)-\mu(r-K)(s)<0$.Thus the curve β is rewritten as

$$\begin{aligned} \beta(s^*) &= \beta(\varphi(s)) \\ &= \alpha(s)+\lambda N(s)+\mu B_2(s) \end{aligned}$$

for all $s \in I$. Differentiating the above equality with respect to s, we obtain

$$\varphi'(s)\frac{d\beta(s^*)}{ds^*} = (\lambda k(s) - \mu(r-K)(s))\left[\gamma T(s) + B_1(s)\right]. \tag{27}$$

We can define the unit vector field T^* along β by $T^*(s^*) = \frac{d\beta(s^*)}{ds^*}$ for all $s^* \in \bar{I}$. By (26) and (27), we obtain

$$T^*(\varphi(s)) = \varepsilon\left(\gamma^2+1\right)^{-\frac{1}{2}}\left[\gamma T(s) + B_1(s)\right] \tag{28}$$

for all $s \in I$. Differentiating (28) with respect to s and using the Frenet equations, we obtain

$$\varphi'(s)\left.\frac{dT^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)} = \varepsilon\left(\gamma^2+1\right)^{-\frac{1}{2}}\left\{\left[\gamma K(s) - k(s)\right]N(s) + (r-K)(s)B_2(s)\right\}$$

and

$$\left\|\left.\frac{dT^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)}\right\| = \frac{\sqrt{\left[\gamma K(s) - k(s)\right]^2 + ((r-K)(s))^2}}{\varphi'(s)\sqrt{(\gamma^2+1)}}.$$

By the fact that $(r-K)(s) \neq 0$ for all $s \in I$, we obtain

$$\bar{K}(\varphi(s)) = \left\|\left.\frac{dT^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)}\right\| > 0 \tag{29}$$

for all $s \in I$. Then we can define a unit vector field N^* along β by

$$\begin{aligned} N^*(s^*) &= N^*(\varphi(s)) \\ &= \frac{1}{\bar{K}(\varphi(s))}\left.\frac{dT^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)} \\ &= \frac{1}{\sqrt{\left[\gamma K(s) - k(s)\right]^2 + ((r-K)(s))^2}} \\ &\quad \times\left\{\left[\gamma K(s) - k(s)\right]N(s) + (r-K)(s)B_2(s)\right\} \end{aligned}$$

for all $s \in I$. Thus we can put

$$N^*(\varphi(s)) = \cos\xi(s)N(s) + \sin\xi(s)B_2(s) \tag{30}$$

where

$$\cos\xi(s) = \frac{\gamma K(s) - k(s)}{\sqrt{\left[\gamma K(s) - k(s)\right]^2 + ((r-K)(s))^2}} \tag{31}$$

$$\sin\xi(s) = \frac{(r-K)(s)}{\sqrt{\left[\gamma K(s) - k(s)\right]^2 + ((r-K)(s))^2}} \tag{32}$$

for all $s \in I$. Here ξ is a C^∞function on I. Differentiating (30) with respect to s and using the Frenet equations, we obtain

$$\begin{aligned} \varphi'(s)\left.\frac{dN^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)} &= -\cos\xi(s)K(s)T(s) \\ &\quad + \frac{d\cos\xi(s)}{ds}N(s) \\ &\quad + (k(s)\cos\xi(s) - (r-K)(s)\sin\xi(s))B_1(s) \\ &\quad + \frac{d\sin\xi(s)}{ds}B_2(s). \end{aligned}$$

Differentiatin (iii) with respect to s, we obtain

$$\left(\gamma K'(s)-k'(s)\right)(r-K)(s)-\left(\gamma K(s)-k(s)\right)(r-K)'(s)=0. \tag{33}$$

Differentiating (31) and (32) with respect to s and using (33), we obtain

$$\frac{d\cos\xi(s)}{ds}=0,\ \ \frac{d\sin\xi(s)}{ds}=0$$

that is, ξ is a constant function on I with value ξ_0. Thus we obtain

$$\cos\xi_0=\frac{\gamma K(s)-k(s)}{\varepsilon\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}} \tag{34}$$

$$\sin\xi_0=\frac{(r-K)(s)}{\varepsilon\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}}\neq 0. \tag{35}$$

From (30), it holds

$$N^*(\varphi(s))=\cos\xi_0 N(s)+\sin\xi_0 B_2(s). \tag{36}$$

Thus we obtain, by (28) and (29),

$$\bar{K}(\varphi(s))\,T^*(\varphi(s)) = \frac{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}{\varepsilon\varphi'(s)\left(\gamma^2+1\right)\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}} \times\left[\gamma T(s)+B_1(s)\right]$$

and by (34),(35) and (36)

$$\left.\frac{dN^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)} = -\frac{K(s)\left(\gamma K(s)-k(s)\right)}{\varepsilon\varphi'(s)\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}}T(s) + \left(\frac{k(s)\left(\gamma K(s)-k(s)\right)-\left((r-K)(s)\right)^2}{\varepsilon\varphi'(s)\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}}\right)B_1(s)$$

for all $s\in I$. By the above equalities, we obtain

$$\left.\frac{dN^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)}+\bar{K}(\varphi(s))\,T^*(\varphi(s))=\frac{P(s)}{R(s)}T(s)+\frac{Q(s)}{R(s)}B_1(s),$$

where

$$P(s)=-\left[\gamma\left\{(K(s))^2-(k(s))^2-\left((r-K)(s)\right)^2\right\}+\left(\gamma^2-1\right)K(s)\,k(s)\right]$$

$$Q(s)=\gamma\left[\gamma\left\{(K(s))^2-(k(s))^2-\left((r-K)(s)\right)^2\right\}+\left(\gamma^2-1\right)K(s)\,k(s)\right]$$

$$R(s)=\varepsilon\varphi'(s)\left(\gamma^2+1\right)\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}\neq 0$$

for all $s\in I$. We notice that, by (iii), $P(s)\neq 0$ for all $s\in I$. Thus we obtain

$$\bar{k}(\varphi(s)) = \left\|\left.\frac{dN^*(s^*)}{ds^*}\right|_{s^*=\varphi(s)}+\bar{K}(\varphi(s))\,T^*(\varphi(s))\right\| = \frac{\left|\gamma\left\{(K(s))^2-(k(s))^2-\left((r-K)(s)\right)^2\right\}+\left(\gamma^2-1\right)K(s)\,k(s)\right|}{\varphi'(s)\sqrt{\gamma^2+1}\sqrt{\left[\gamma K(s)-k(s)\right]^2+\left((r-K)(s)\right)^2}}$$

for all $s \in I$. Thus we can define a unit vector field $B_1^*(s^*)$ along β by

$$\begin{aligned} B_1^*(s^*) &= B_1^*(\varphi(s)) \\ &= \frac{1}{\bar{k}(\varphi(s))}\left(\frac{dN^*(s^*)}{ds^*}_{s^*=\varphi(s)} + \bar{K}(\varphi(s))\,T^*(\varphi(s))\right) \end{aligned}$$

that is,

$$B_1^*(\varphi(s)) = \frac{1}{\varepsilon\sqrt{\gamma^2+1}}\left(-T(s) + \gamma B_2(s)\right) \tag{37}$$

for all $s \in I$. Next we can define a unit vector field B_2^* along β by

$$\begin{aligned} B_2^*(s^*) &= B_2^*(\varphi(s)) \\ &= \frac{1}{\varepsilon\sqrt{\left[\gamma K(s) - k(s)\right]^2 + \left((r-K)(s)\right)^2}} \\ &\quad \left\{-(r-K)(s)\,N(s) + (\gamma K(s) - k(s))\,B_2(s)\right\} \end{aligned}$$

that is

$$B_2^*(\varphi(s)) = -\sin\xi_0 N(s) + \cos\xi_0 B_2(s) \tag{38}$$

for all $s \in I$. Now we obtain, by (28), (36), (37) and (38),

$$\det\left[T^*(\varphi(s)), N^*(\varphi(s)), B_1^*(\varphi(s)), B_2^*(\varphi(s))\right] = 1$$

and $\{T^*(\varphi(s)), N^*(\varphi(s)), B_1^*(\varphi(s)), B_2^*(\varphi(s))\}$ is orthonormal for all $s \in I$.Thus the frame $\{T^*, N^*, B_1^*, B_2^*\}$ along β is of orthonormal and of positive. And we obtain

$$\begin{aligned} \overline{(r-K)}(s) &= \left\langle \frac{dB_1^*(s^*)}{ds^*}_{s^*=\varphi(s)}, B_2^*(\varphi(s))\right\rangle \\ &= \frac{\sqrt{\gamma^2+1}K(s)\,(r-K)(s)}{\varphi'(s)\sqrt{\left[\gamma K(s) - k(s)\right]^2 + \left((r-K)(s)\right)^2}} \neq 0 \end{aligned}$$

for all $s \in I$. Thus the curve β is a Frenet curve in Q_H. And it is trivial that he plane spanned by $N(s)$,$B_2(s)$ at the each point $\alpha(s)$of α coincides with the plane spanned by $N^*(s^*)$,$B_2^*(s^*)$ corresponding point $\beta(s^*) = \beta(\varphi(s))$ of β. Therefore α is a $(1,3)$-Bertrand curve in Q_H. □

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Osman Keçilioğlu
Department of Statistics, Faculty of Science and Arts, Kirikkale University 71450, Kirikkale, Turkey.

E-mail address: okecilioglu@yahoo.com

Kazim İlarslan
Department of Mathematics, Faculty of Science and Arts, Kirikkale University 71450, Kirikkale, Turkey.

E-mail address: kilarslan@yahoo.com