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A NOTE ON EXISTENCE OF $\Psi-$BOUNDED SOLUTIONS FOR A NONHOMOGENEOUS MATRIX DIFFERENCE EQUATION ON $\mathbb{N}$

(COMMUNICATED BY DOUGLAS R. ANDERSON)

AUREL DIAMANDESCU

Abstract. In this paper, we give a necessary and sufficient condition for the existence of at least one $\Psi-$bounded solution on $\mathbb{N}$ for the non-homogeneous matrix difference equation $X(n+1) = A(n)X(n)B(n) + F(n)$, where F is a $\Psi-$summable matrix function on $\mathbb{N}$. In addition, we give a result in connection with the asymptotic behavior on $\mathbb{N}$ of the $\Psi-$bounded solutions of this equation.

1. Introduction

The difference equations play an important role in many scientific fields, such as numerical analysis of ordinary and partial differential equations, scientific computing, control theory and computer science (see outstanding monographs of S. Elaydi [9] and R.P. Agarval [1] and references therein). On the other hand, the boundedness of solutions of difference equations play an important role in stability theory.

The aim of this paper is to give a necessary and sufficient condition so that the non-homogeneous linear matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n) \tag{1}$$

has at least one $\Psi-$bounded solution on $\mathbb{N}$ for every $\Psi-$summable matrix function F on $\mathbb{N}$.

Here, Ψ is a matrix function. The introduction of the matrix function Ψ in the study of solutions allows to obtain a mixed asymptotic behavior of the components of the solutions.

The problem of boundedness of the solutions for the systems of ordinary differential equations $x' = A(t)x + f(t)$ was studied by Coppel in [3]. In [5], the author proposes a novel concept, $\Psi-$boundedness of solutions (Ψ being a matrix function), which is interesting and useful in some practical cases and presents the existence condition for such solutions. Also, in [4], the author associates this problem with the concept of $\Psi-$dichotomy on $\mathbb{R}$ of the system $x' = A(t)x$.

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In [10], the authors extend the concept of $\Psi-$boundedness to the solutions of difference equation (via $\Psi-$bounded sequence) and establish a necessary and sufficient condition for existence of $\Psi-$bounded solutions for non-homogeneous linear difference equation

$$x(n+1) = A(n)x(n) + f(n) \tag{2}$$

in case f is a $\Psi-$summable sequence on $\mathbb{N}$. Similarly, in [6], the author solves the same problem in case f is a $\Psi-$bounded sequence on $\mathbb{N}$.

In the last years, in [11], [12], the authors extend the concept of $\Psi-$boundedness to the solutions of linear matrix difference equation (1) and establish a necessary and sufficient condition for existence of $\Psi-$bounded solutions for the equation (1) in case F is $\Psi-$summable on $\mathbb{Z}$ or $\Psi-$bounded on $\mathbb{N}$ respectively. Similarly, in [8], the author solves the same problem in case F is $\Psi-$bounded $\mathbb{Z}$.

The approach used in our paper is essentially based on the technique of Kronecker product of matrices (which has been successfully applied in similar problems - see, e.g. [11], [12]) and on a decomposition of the underlying space at the initial moment (see, e.g. [6], [7], [10], [11], [12] for finite-dimensional spaces and [13], [14] in general case of Banach spaces).

Thus, we obtain results which contain and extend the recent results regarding the boundedness of solutions of the equation (1) (see [10], [12]).

2. Preliminaries

Let $\mathbb{R}^d$ be the Euclidean d-space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $\| x \|$ be the norm of x defined by $\| x \| = \max\{| x_1 |, | x_2 |, ... | x_d |\}$. (^{T} denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all $m \times n$ real valued matrices.

For a matrix $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by $| A |= \sup\limits_{\|x\|\leq 1} \| Ax \|$.

It is well-known that $| A | = \max\limits_{1\leq i\leq d}\{\sum\limits_{j=1}^{d} | a_{ij} |\}$.

Let $\Psi_i : \mathbb{N} \longrightarrow (0, \infty), i = 1, 2, ...d$ and let the matrix function

$$\Psi = \text{diag } [\Psi_1,\Psi_2,...\Psi_d].$$

Then, $\Psi(n)$ is invertible for each $n \in \mathbb{N}$.

Definition 2.1. *([7]). A matrix function $M : N \longrightarrow \mathbb{M}_{d\times d}$ is said to be $\Psi-$ bounded on $\mathbb{N}$ if the function ΨM is bounded on $\mathbb{N}$.*

In equation (1), we consider that A, B, F are $d \times d$ real matrix-valued functions on $\mathbb{N}$. Suppose that $A(n)$ and $B(n)$ are invertible for each $n \in \mathbb{N}$.

By a solution of (1) we mean a matrix function $X(n)$ satisfying (1) :

$$X(n + 1) = A(n)X(n)B(n) + F(n),\ \forall n \in \mathbb{N}.$$

Definition 2.2. *([10]). A function $f : \mathbb{N} \longrightarrow \mathbb{R}^d$ is said to be $\Psi-$ summable on $\mathbb{N}$ if $\sum\limits_{n=1}^{\infty} \| \Psi(n)f(n) \|$ is convergent.*

Extend this Definition for matrix-functions.

Definition 2.3. *A matrix function $F : \mathbb{N} \longrightarrow \mathbb{M}_{d\times d}$ is said to be $\Psi-$ summable on $\mathbb{N}$ if $\sum\limits_{n=1}^{\infty} | \Psi(n)F(n) |$ is convergent.*

Consider the nonautonomous matrix difference linear equations

$$Y(n+1) = A(n)Y(n), \tag{3}$$

$$Z(n+1) = Z(n)B(n). \tag{4}$$

Let U be the fundamental matrix of (3) with $U(0) = I_d$ (identity $d \times d$ matrix). It is well-known that

i). $U(n+1) = A(n)U(n)$, for all $n \in \mathbb{N}$;

ii). the solution of (3) with the initial condition $Y(0) = Y_0$ is

$$Y(n) = U(n)Y_0,\ n \in \mathbb{N};$$

iii). $U(n) = \begin{cases} A(n-1)A(n-2)\cdots A(1)A(0), & n \geq 1 \\ I_d, & n = 0 \end{cases};$

iv). $U(n)$ is invertible for each $n \in \mathbb{N}$ and

$$U^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\cdots A^{-1}(n-1), & n \geq 1 \\ I_d, & n = 0 \end{cases}.$$

Similarly, let V be a fundamental matrix of (4) with $V(0) = I_d$ (i.e. V^T is a fundamental matrix for the equation (3) with $A = B^T$).

Definition 2.4. *([2]). Let $A = (a_{ij}) \in \mathbb{M}_{m\times n}$ and $B = (b_{ij}) \in \mathbb{M}_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 2.5. *The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions are defined:*

1). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;
2). $(A \otimes B)^T = A^T \otimes B^T$;
3). $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;
4). $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5). $A \otimes (B + C) = A \otimes B + A \otimes C$;
6). $(A + B) \otimes C = A \otimes C + B \otimes C$;
7). $I_p \otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;

Proof. See in [2]. □

Definition 2.6. *The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by*

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2.7. *([7]). The vectorization operator* $\mathcal{V}ec : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}^{n^2}$, *is a linear and one-to-one operator. In addition,* Vec *and* Vec^{-1} *are continuous operators.*

Proof. See Lemma 2, ([7]). □

Remark 2.8. *Obviously, if* F *is a matrix function on* $\mathbb{N}$, *then* $f = \mathcal{V}ec(F)$ *is a vector function on* $\mathbb{N}$ *and vice-versa.*

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations:

Lemma 2.9. *([7]). If* $A, B, M \in \mathbb{M}_{n\times n}$, *then:*

1). $\mathcal{V}ec(AMB) = (B^T \otimes A) \cdot \mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T \otimes I_n) \cdot \mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n \otimes A) \cdot \mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T \otimes A) \cdot \mathcal{V}ec(I_n)$.

Proof. It is a simple exercise. □

The following lemmas play a vital role in the proofs of the main results.

Lemma 2.10. *The matrix valued function* $X(n)$ *is a solution of (1) on* $\mathbb{N}$ *iff the vector valued function* $x(n) = Vec(X(n))$ *is a solution on* $\mathbb{N}$ *of the difference equation*

$$x(n+1) = (B^T(n) \otimes A(n))x(n) + \mathcal{V}ec(F(n)) \tag{5}$$

Proof. If the matrix valued function $X(n)$ is a solution of (1) on $\mathbb{N}$, then

$$X(n+1) = A(n)X(n)B(n) + F(n),\ \forall n \in \mathbb{N}.$$

Now, by applying the $\mathcal{V}ec$ operator to this equality, we have

$$\mathcal{V}ec(X(n+1)) = (B^T(n) \otimes A(n))\mathcal{V}ec(X(n)) + \mathcal{V}ec(F(n)),\ \forall n \in \mathbb{N}.$$

Thus, the vector valued function $x(n) = \mathcal{V}ec(X(n))$ is a solution on $\mathbb{N}$ of the difference equation (5).

Conversely, if the vector valued function $x(n)$ is a solution on $\mathbb{N}$ of the difference equation (5), then

$$x(n+1) = (B^T(n) \otimes A(n))x(n) + \mathcal{V}ec(F(n)),\ \forall n \in \mathbb{N}.$$

Now, by applying the $\mathcal{V}ec^{-1}$ operator to this equality, we have

$$X(n+1) = A(n)X(n)B(n) + F(n),\ \forall n \in \mathbb{N},$$

where $X(n) = \mathcal{V}ec^{-1}(x(n))$, $n \in \mathbb{N}$.

Thus, the matrix valued function $X(n)$ is a solution on $\mathbb{N}$ of the equation (1). □

Definition 2.11. *The equation (5) is called the Kronecker product difference equation associated with (1).*

Lemma 2.12. *The matrix function* $M(n)$ *is* $\Psi-$*bounded on* $\mathbb{N}$ *iff the vector function* $Vec(M(n))$ *is* $I_d \otimes \Psi-$*bounded on* $\mathbb{N}$.

Proof. From the proof of Lemma 2.7, it results that

$$\frac{1}{d} \mid M \mid \leq \parallel \mathcal{V}ec(M) \parallel_{\mathbb{R}^{d^2}} \leq \mid M \mid, \forall M \in \mathbb{M}_{d\times d}.$$

Setting $M = \Psi(n)M(n), n \in \mathbb{N}$, and using Lemma 2.9, we have the inequality

$$\frac{1}{d} \mid \Psi(n)M(n) \mid \leq \parallel (I_d \otimes \Psi(n))\mathcal{V}ec(M(n)) \parallel_{\mathbb{R}^{d^2}} \leq \mid \Psi(n)M(n) \mid, \tag{6}$$

for all matrix function M and for all $n \in \mathbb{N}$.

Now, the Lemma follows immediately. □

Lemma 2.13. *The matrix function $M(n)$ is $\Psi-$summable on $\mathbb{N}$ iff the vector function $Vec(M(n))$ is $I_d \otimes \Psi-$summable on $\mathbb{N}$. Then, the matrix*

Proof. It results from the Definitions 2.2 and 2.3 and inequality (6). □

Lemma 2.14. *Let $U(n)$ and $V(n)$ be the fundamental matrices for the matrix equations (3) and (4) respectively.*

Then, the matrix

$$W(n) = V^T(n) \otimes U(n), n \in \mathbb{N},$$

is a fundamental matrix of the equation

$$x(n+1) = (B^T(n) \otimes A(n))x(n). \tag{7}$$

If, in addition, $U(0) = I_d, V(0) = I_d$, then $W(0) = I_{d^2}$.

Proof. For $n \in \mathbb{N}$, we have

$$\begin{aligned} W(n+1) &= V^T(n+1) \otimes U(n+1) \\ &= \left(B^T(n)V^T(n)\right) \otimes (A(n)U(n)) \\ &= \left(B^T(n) \otimes A(n)\right) \cdot \left(V^T(n) \otimes U(n)\right) \\ &= \left(B^T(n) \otimes A(n)\right) W(n). \end{aligned}$$

On the other hand, the matrix $W(n)$ is an invertible matrix for all $n \in \mathbb{N}$, because $U(n)$ and $V(n)$ are invertible matrices for all $n \in \mathbb{N}$.

Thus, the matrix W is a fundamental matrix of (7).

Obviously, $W(0) = I_d \otimes I_d = I_{d^2}$. □

Let X_0 denote the subspace of $\mathbb{M}_{d\times d}$ consisting of all matrices which are values for $n = 0$ of $\Psi-$bounded solutions of the homogeneous matrix difference equation associated with (1),

$$X(n+1) = A(n)X(n)B(n) \tag{8}$$

and let X_1 be an arbitrary fixed subspace of $\mathbb{M}_{d\times d}$ supplementary to X_0. Let P_0, P_1 denote the corresponding projections of $\mathbb{M}_{d\times d}$ onto X_0, X_1 respectively.

Then, the vector space $\mathbb{R}^{d^2}$ is represented as a direct sum of two subspaces S_0, S_1 defined as follows: a solution $x(n)$ of (7) is $I_d \otimes \Psi-$bounded on $\mathbb{N}$ iff $t(0) \in S_0$; S_1 is an arbitrary fixed subspace of $\mathbb{R}^{d^2}$ supplementary to S_0. Let Q_0, Q_1 denote the corresponding projections of $\mathbb{R}^{d^2}$ onto S_0, S_1 respectively.

The following theorems are useful in the proofs of our main results.

Theorem 2.15. *([10]). Let $A(n) \in \mathbb{M}_{d\times d}$, invertible for all $n \in \mathbb{N}$. Then, the difference equation*

$$x(n+1) = A(n)x(n) + f(n) \tag{9}$$

has at least one $\Psi-$bounded solution on $\mathbb{N}$ for every $\Psi-$summable function f on $\mathbb{N}$ if and only if for a fundamental matrix $Y(n)$ of equation

$$x(n+1) = A(n)x(n), \tag{10}$$

there is a positive constant K such that

$$\begin{array}{l} \mid \Psi(n)Y(n)P_0Y^{-1}(k+1)\Psi^{-1}(k) \mid \leq K, \text{ for } 0 \leq k+1 \leq n \\ \mid \Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k) \mid \leq K, \text{ for } 0 \leq n < k+1 \end{array} . \tag{11}$$

Theorem 2.16. *([10]). Suppose that:*

1. the fundamental matrix $Y(n)$ for the equation (10) satisfies:
a). the condition (11), for some $K > 0$;
b). the condition $\lim_{n\to\infty} \mid \Psi(n)Y(n)P_0 \mid = 0$;
2. the function $f : N \longrightarrow R^d$ is $\Psi-$summable on $\mathbb{N}$.
Then, every $\Psi-$bounded solution $x(n)$ of (9) satisfies the condition

$$\lim_{n\to\infty} \parallel \Psi(n)x(n) \parallel = 0.$$

Remark 2.17. *In these Theorems, the projections P_0, P_1 refer to the equation (10) in $\mathbb{R}^d$.*

3. Main results

The main results of this paper are presented in this section.

Theorem 3.1. *Let $A(n)$ and $B(n)$ be invertible $d\times d$ matrices for $n \in \mathbb{N}$. Then, the linear non-homogeneous matrix difference equation (1) has at least one $\Psi-$bounded solution on $\mathbb{N}$ for every $\Psi-$summable matrix function F on $\mathbb{N}$ if and only if for the fundamental matrices $U(n)$ and $V(n)$ for the equations (3) and (4) respectively, there is a positive constant K such that*

$$\begin{array}{l} \mid [V^T(n) \otimes (\Psi(n)U(n))]Q_0[(V^T)^{-1}(k+1) \otimes (U^{-1}(k+1)\Psi^{-1}(k))] \mid \leq K, \\ \hfill \text{for } 0 \leq k+1 \leq n \\ \mid [V^T(n) \otimes (\Psi(n)U(n))]Q_1[(V^T)^{-1}(k+1) \otimes (U^{-1}(k+1)\Psi^{-1}(k))] \mid \leq K, \\ \hfill \text{for } 0 \leq n < k+1 \end{array} \tag{12}$$

Proof. Proof. First, we prove the "only if" part.

Suppose that the equation (1) has at least one $\Psi-$bounded solution on $\mathbb{N}$ for every $\Psi-$bounded function $F : \mathbb{N} \longrightarrow \mathbb{M}_{d\times d}$ on $\mathbb{N}$.

Let $f : \mathbb{N} \longrightarrow \mathbb{R}^{d^2}$ be a $I_d \otimes \Psi-$summable function on $\mathbb{N}$. From Lemma 2.13, it follows that the matrix function $F(n) = \mathcal{V}ec^{-1}(f(n))$ is $\Psi-$summable on $\mathbb{N}$. From the hypothesis, the equation (1) has at least one $\Psi-$bounded solution $T(n)$ on $\mathbb{N}$. From Lemma 2.10 and Lemma 2.12, it follows that the vector valued function $t(n) = \mathcal{V}ec\,(T(n))$ is a $I_d \otimes \Psi-$bounded solution of the difference equation (5) on $\mathbb{N}$. Thus, the difference equation

$$x(n+1) = (B^T(n) \otimes A(n))x(n) + f(n)$$

has at least one $I_d \otimes \Psi-$bounded solution $t(n)$ on $\mathbb{N}$.

Now, from Theorem 2.15, for a fundamental matrix $W(n)$ of (7), there is a positive constant K such that (11) holds, with $I_d \otimes \Psi$ instead of Ψ and W instead of Y :

$$\begin{array}{l} \mid [(I_d \otimes \Psi(n))W(n)]Q_0[W^{-1}(k+1)(I_d \otimes \Psi)^{-1}(k))] \mid \leq K, \text{ for } 0 \leq k+1 \leq n \\ \mid [(I_d \otimes \Psi(n))W(n)]Q_1[W^{-1}(k+1)(I_d \otimes \Psi)^{-1}(k))] \mid \leq K, \text{ for } 0 \leq n < k+1 \end{array} .$$

Putting $W(n) = V^T(n) \otimes U(n)$ (Lemma 2.14) and using Kronecker product properties (Lemma 2.5), (12) holds.

Conversely, we prove the "if" part.

Suppose that (12) holds for some $K > 0$.

Let $F : \mathbb{N} \longrightarrow \mathbb{M}_{d\times d}$ be a $\Psi-$summable matrix function on $\mathbb{N}$. From Lemma 2.13, it follows that the vector valued function $f(n) = \mathcal{V}ec(F(n))$ is a $I_d \otimes \Psi-$summable function on $\mathbb{N}$. From hypothesis and from Theorem 2.15, it follows that the difference equation (5) has at least one $I_d \otimes \Psi-$bounded solution $t(n)$ on $\mathbb{N}$ (because condition (12) is the condition (11), Theorem 2.15, for the difference equation (5), with $I_d \otimes \Psi(n)$ in role of $\Psi(n)$ and $W(n) = V^T(n) \otimes U(n)$ in role of $Y(n)$). From Lemma 2.10 and Lemma 2.12, it follows that the matrix function $T(n) = \mathcal{V}ec^{-1}(t(n))$ is a $\Psi-$bounded solution of the equation (1) on $\mathbb{N}$ (because $F(n) = \mathcal{V}ec^{-1}(f(n))$).

Thus, the matrix difference equation (1) has at least one $\Psi-$bounded solution on $\mathbb{N}$ for every $\Psi-$bounded matrix function F on $\mathbb{N}$. $\square$

Remark 3.2. *Theorem 3.1 extends Theorem 3.1, [12].*

Indeed, in Theorem 3.1, [12], the authors provide necessary and sufficient condition for the existence of at least one $\Psi-$bounded solution on $\mathbb{N}$ for the equation (1) in case $F(n)$ is a $\Psi-$bounded matrix valued function on $\mathbb{N}$.

Remark 3.3. *Theorem 3.1 generalizes Theorem 1, [10].*

Indeed, in particular case

$$B(n) = I_d \text{ and } F(n) = \begin{pmatrix} f_1(n) & f_1(n) & \cdots & f_1(n) \\ f_2(n) & f_2(n) & \cdots & f_2(n) \\ \vdots & \vdots & \vdots & \vdots \\ f_d(n) & f_d(n) & \cdots & f_d(n) \end{pmatrix},$$

it is easy to see that a solution of the equation (1) is

$$X(n) = \begin{pmatrix} x_1(n) & x_1(n) & \cdots & x_1(n) \\ x_2(n) & x_2(n) & \cdots & x_2(n) \\ \vdots & \vdots & \vdots & \vdots \\ x_d(n) & x_d(n) & \cdots & x_d(n) \end{pmatrix},$$

where $x(n) = (x_1(n), x_2(n), \cdots, x_d(n))^T$ is a solution of (9) ((1) from [10])

$$x(n+1) = A(n)x(n) + f(n)$$

with $f(n) = (f_1(n), f_2(n), \cdots, f_d(n))^T$ and vice-versa.

In this case, the condition (12) becomes the condition (11), because the solution X(n) is $\Psi-$bounded on $\mathbb{N}$ iff the solution x(n) is $\Psi-$bounded on $\mathbb{N}$.

Thus, the affirmation from above is correct.

Finally, we give a result in which we will see that the asymptotic behavior of $\Psi-$bounded solutions of (1) is completely determined by the asymptotic behavior of fundamental matrices U and V.

Theorem 3.4. *Suppose that:*

1. the fundamental matrices $U(n)$ and $V(n)$ for the equations (3) and (4) respectively, satisfies

i. the condition (12), for some $K > 0$;

ii. the condition

$$\lim_{n\longrightarrow\infty} \mid \left[V^T(n)\otimes(\Psi(n)U(n))\right] Q_0 \mid = 0;$$

2. *the matrix function* $F:\mathbb{N}\longrightarrow\mathbb{M}_{d\times d}$ *is* $\Psi-$*summable.*
Then, every $\Psi-$*bounded solution* $X(n)$ *of (1) satisfies the condition*

$$\lim_{n\longrightarrow\infty} \mid \Psi(n)X(n) \mid = 0.$$

Proof. Let $X(n)$ a $\Psi-$bounded solution on $\mathbb{N}$ of (1). From Lemma 2.10 and Lemma 2.12, it follows that the vector function $x(n)=\mathcal{V}ec(X(n))$ is a $I_d\otimes\Psi-$bounded solution on $\mathbb{N}$ of the difference equation (5). From Lemma 2.13, we have that the function $\mathcal{V}ec(F(n))$ is $I_d\otimes\Psi-$summable on $\mathbb{N}$.

From Theorem 2.16, it follows that
$$\lim_{n\to\infty} \parallel (I_d\otimes\Psi(n))x(n) \parallel = 0.$$
From inequality (6), we have that
$$\lim_{n\to\infty} \mid \Psi(n)X(n) \mid = 0.$$ □

Remark 3.5. *Theorem 3.4 generalizes Theorem 2, [10] and extends Theorem 3.2, [12].*

Note that the Theorem 3.4 is no longer true if we require that the function F be $\Psi-$bounded on $\mathbb{N}$, instead of the condition 2 of the Theorem. This is shown by the next simple example.

Example 3.1. Consider the equation (1) with

$$A(n)=I_2, B(n)=B=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, F(n)=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, n\in\mathbb{N}.$$

A fundamental matrix for the equation (3) is $U(n)=I_2$, $n\in\mathbb{N}$.
Similarly, a fundamental matrix for the equation (4) is

$$V(n)=B^n=\begin{pmatrix} 2^{-n} & 0 \\ 0 & 2^n \end{pmatrix}, n\in\mathbb{N}.$$

Consider

$$\Psi(n)=\begin{pmatrix} 1 & 0 \\ 0 & 3^{-n} \end{pmatrix}, n\in\mathbb{N}.$$

The solution of (8) with the initial condition

$$X(0)=X_0=\begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}\in\mathbb{M}_{2\times 2}$$

is unique:

$$X(n)=\begin{pmatrix} c_1 2^{-n} & c_3 2^n \\ c_2 2^{-n} & c_4 2^n \end{pmatrix}, n\in\mathbb{N}.$$

Because

$$\Psi(n)X(n)=\begin{pmatrix} c_1 2^{-n} & c_3 2^n \\ c_2 6^{-n} & c_4 (\frac{2}{3})^n \end{pmatrix}, n\in\mathbb{N},$$

we have that a solution $X(n)$ of (8) is $\Psi-$bounded on $\mathbb{N}$ iff $X_0=\begin{pmatrix} c_1 & 0 \\ c_2 & c_4 \end{pmatrix}$.

Hence,
$S_0=Q_0\mathbb{R}^4$, with $Q_0=$diag $[1,1,0,1]$ and
$S_1=Q_1\mathbb{R}^4$, with $Q_1=$diag $[0,0,1,0]$.

Now, the condition (12) becomes

$$\mid \quad (B^n \otimes \Psi(n))Q_0(B^{-k-1} \otimes \Psi^{-1}(k)) \mid \leq K, 0 \leq k+1 \leq n,$$

$$\mid \quad (B^n \otimes \Psi(n))Q_1(B^{-k-1} \otimes \Psi^{-1}(k)) \mid \leq K, 0 \leq n < k+1.$$

We have

$$B^n \otimes \Psi(n) = \begin{pmatrix} 2^{-n} & 0 & 0 & 0 \\ 0 & 6^{-n} & 0 & 0 \\ 0 & 0 & 2^n & 0 \\ 0 & 0 & 0 & (\frac{2}{3})^n \end{pmatrix}$$

and

$$B^{-k-1} \otimes \Psi^{-1}(k) = \begin{pmatrix} 2^{k+1} & 0 & 0 & 0 \\ 0 & 2 \cdot 6^k & 0 & 0 \\ 0 & 0 & 2^{-k-1} & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\frac{3}{2})^k \end{pmatrix}.$$

Now, for $0 \leq k+1 \leq n$, we have

$$(B^n \otimes \Psi(n))Q_0(B^{-k-1} \otimes \Psi^{-1}(k)) = \begin{pmatrix} 2^{k+1-n} & 0 & 0 & 0 \\ 0 & 2 \cdot 6^{k-n} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\frac{3}{2})^{k-n} \end{pmatrix},$$

and for $0 \leq n < k+1$,

$$(B^n \otimes \Psi(n))Q_1(B^{-k-1} \otimes \Psi^{-1}(k)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2^{n-k-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From these, it is easy to see that the condition (12) is satisfied with $K = 1$.

In addition, the condition

$$\lim_{n\to\infty} \mid (B^n \otimes \Psi(n))Q_0) \mid = 0$$

is satisfied.

Because

$$\Psi(n)F(n) = \begin{pmatrix} 1 & 0 \\ 3^{-n} & 0 \end{pmatrix},$$

the function F is $\Psi-$bounded on $\mathbb{N}$, but is not $\Psi-$summable on $\mathbb{N}$.

On the other hand, the solutions on $\mathbb{N}$ of the equation (1) are

$$X(n) = \begin{pmatrix} 2 + c_1 2^{-n} & c_2 2^n \\ 2 + c_1 2^{-n} & c_2 2^n \end{pmatrix},$$

for $n \in \mathbb{N}$, c_1, c_2 being real constants.

We have

$$\Psi(n)X(n) = \begin{pmatrix} 2 + c_1 2^{-n} & c_2 2^n \\ 2 \cdot 3^{-n} + c_1 6^{-n} & c_2 (\frac{2}{3})^n \end{pmatrix}, \ n \in \mathbb{N}.$$

It is easy to see that the $\Psi-$bounded solutions of (1) on $\mathbb{N}$ are

$$X(n) = \begin{pmatrix} 2 + c_1 2^{-n} & 0 \\ 2 + c_1 2^{-n} & 0 \end{pmatrix}, \ n \in \mathbb{N}.$$

For these solutions, we have

$$\lim_{n\to\infty} \mid \Psi(n)X(n) \mid = 2.$$

This shows that there is no $\Psi-$bounded solutions $X(n)$ of (1) such that

$$\lim_{n\to\infty} \mid \Psi(n)X(n) \mid = 0.$$

Remark 3.6. *We can observe that the asymptotic properties of the components of the solutions are not the same. On the other hand, we see that the asymptotic properties of the components of the solutions are the same, via matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.*

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Aurel Diamandescu

Department of Applied Mathematics, University of Craiova, 13, "Al. I. Cuza" st., 200585, Craiova, Romania

E-mail address: `adiamandescu@central.ucv.ro`