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POSITIVE SOLUTION OF SYSTEM OF THIRD-ORDER BOUNDARY VALUE PROBLEM WITH THREE-POINT AND INTEGRAL BOUNDARY CONDITIONS

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Abstract. This paper is concerned with the nonlinear third-order boundary value problem

$$u_i'''(t) + f_i(t, u_1(t), ..., u_n(t), u_1'(t), ..., u_n'(t)) = 0,\ 0 < t < 1$$

with three-point and integral conditions

$$\begin{cases} u_i(0) = \int_0^1 h_{1,i}(s, u_1(s), ..., u_n(s))ds \\ u_i'(0) = 0 \\ \alpha_i u_i'(1) = \beta_i u_i(\eta_i) + \int_0^1 h_{2,i}(s, u_1(s), ..., u_n(s))ds \end{cases}$$

where for $i \in \{1, ..., n\}$, $\alpha_i > 0$, $\beta_i > 0$, $1 > \eta_i \geq 0$, $2\alpha_i > \beta_i \eta_i^2$, $f_i : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, for all $k \in \{1, 2\}$, $h_{k,i} : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions. By using Guo-Krasnosel'skii fixed point theorem in cone, we discuss the existence of positive solution of this problem. We also prove nonexistence of positive solution and we give some examples to illustrate our results.

1. Introduction

Third-order ordinary differential equations arise in a variety of different areas of applied mathematics and physics, in the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1]. The main propose of the present paper is to investigate sufficient conditions for the existence of positive solution of following problem:

$$u_i'''(t) + f_i(t, u_1(t), ..., u_n(t), u_1'(t), ..., u_n'(t)) = 0,\ 0 < t < 1 \tag{1.1}$$

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with three-point and integral conditions

$$\begin{cases} u_i(0) = \int_0^1 h_{1,i}(s, u_1(s), ..., u_n(s))ds \\ u_i'(0) = 0 \\ \alpha_i u_i'(1) = \beta_i u_i(\eta_i) + \int_0^1 h_{2,i}(s, u_1(s), ..., u_n(s))ds \end{cases} \tag{1.2}$$

where for $i \in \{1, ..., n\}$, for $k \in \{1, 2\}$, $\alpha_i > 0$, $\beta_i > 0$, $1 > \eta_i \geq 0$, $2\alpha_i > \beta_i \eta_i^2$, $f_i : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $h_{k,i} : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions. The integral conditions has physical significance such as total mass, moment, etc. Sometimes it is better to impose integral conditions to get a more accurate measure than a local condition see [9].

Various types of boundary value problems were studied by many authors using fixed point theorems on cones, fixed point index theory, upper and lower solutions method, differential inequality, topological transversality and Leggett-Williams fixed point theorem [2-8,12]. In [2], Yao and Feng used the upper and lower solutions method to prove some existence results for the following third-order two-point boundary value problem:

$$u'''(t) + f(t, u(t)) = 0, \ 0 < t < 1 \tag{1.3}$$

$$u(0) = u'(0) = u'(1) = 0. \tag{1.4}$$

In [3], Sanyang Liu and Yuqiang Feng used the upper and lower solutions method and a new maximum principle to prove some existence results for the more general third-order two-point boundary value problem:

$$u'''(t) + f(t, u(t), u'(t)) = 0, \ 0 < t < 1 \tag{1.5}$$

$$u(0) = u'(0) = u'(1) = 0. \tag{1.6}$$

In [8] Guo, Sun and Zhao, considered the third-order three-point boundary value problem

$$u'''(t) + a(t)f(u(t)) = 0, \ 0 < t < 1 \tag{1.7}$$

$$u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta) \tag{1.8}$$

where $0 < \eta < 1$ and $1 < \alpha < \frac{1}{\eta}$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is given function. The existence of at least one positive solution for (1.7)-(1.8) was proved when f was superlinear or sublinear. By the use of some fixed point theorem in cones, A. Guezane-Lakoud et al. [12], investigated the existence of positive solutions for the following third-order eigenvalue problem

$$u'''(t) + f(t, u(t), u'(t)) = 0, \ 0 < t < 1 \tag{1.9}$$

$$u(0) = u'(0) = 0, \alpha u'(1) = \beta u(\eta) \tag{1.10}$$

where $\alpha \in \mathbb{R}_+^*, \beta \in \mathbb{R}_+^*$, $0 < \eta < 1$, by Guo-Krasnosel'skii fixed point theorem in cone the author studied the existence of least positive solution of (1.9)-(1.10) when $2\alpha > \beta\eta^2$ and $f(t, u(t), u'(t)) = a(t)g(u(t), u'(t))$. In [7], Sun et al considered the following third-order boundary value problem:

$$u''' + a(t)f(t, u(t)) = 0, \ 0 < t < 1 \tag{1.11}$$

$$u(0) = u'(0) = 0, u'(1) - \alpha u'(\eta) = \lambda \tag{1.12}$$

by employing the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem, the author studied the existence and nonexistence of positive solutions to the third-order three-point nonhomogeneous boundary condition (1.11)-(1.12). For more knowledge about the boundary value problem, we refer the reader to (see [13-21]). Our aim is to use the Guo-Krasnosel'skii fixed point theorem to prove the existence of least one positive solutions of our problem. For this, we formulate the boundary value problem as the fixed point problem. The particularity of our equation (1.1)-(1.2) is that the boundary condition involving three-point boundary condition and integral condition which leads to extra difficulties. To the best of our knowledge, no one has studied the existence and nonexistence of positive solution for nonlinear differential equation (1.1) jointly with conditions (1.2). Our work is new and more general than [2, 3, 12], for example our problem reduces to the problem (1.9)-(1.10) in the case $n = 1$, $h_{1,1} \equiv h_{2,1} \equiv 0$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $f_1(t, u_1(t), u_1'(t)) = f(t, u(t), u'(t))$.

This paper is organized as follows. In Section 2, we present some preliminaries that will be used to prove our results. In Section 3, the study the positivity of solution is based on a Guo-Krasnosel'skii fixed point theorem. In Section 4, the nonexistence of positive solution is studied. Finally, we shall give two examples to illustrate our main results.

2. Preliminaries and lemmas

The cartesian power of $C^1([0,1];\mathbb{R})$ can be defined as:

$$\begin{aligned} E &= C^1([0,1];\mathbb{R})^n \\ &= \underbrace{C^1([0,1];\mathbb{R}) \times ... \times C^1([0,1];\mathbb{R})}_{n \text{ times}} \end{aligned}$$

equipped with the norm $\|u\|_E = \sum_{i=1}^{n} \|u_i\|$ where $\|u_i\| = \max(\|u_i\|_\infty, \|u_i'\|_\infty)$, $\|u_i\|_\infty = \max_{t\in[0,1]} |u_i(t)|$ and $u = (u_1, .., u_n) \in E$. The space E is a Banach space. We denote by for $x \in \mathbb{R}^n$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

Definition 1. *We give the following definitions:*

1. *The function $u = (u_1, .., u_n)$ is called nonnegative solution of the system (1.1)-(1.2) if and only if, u satisfies (1.1)-(1.2) and for all $i \in \{1, ..., n\}$, $u_i(t) \geq 0$ for $t \in [0, 1]$.*
2. *The function $u = (u_1, .., u_n)$ is called positive solution of the system (1.1)-(1.2) if and only if, u satisfies (1.1)-(1.2) and for all $i \in \{1, ..., n\}$, $u_i(t) > 0$ for $t \in]0, 1[$.*

Lemma 2.1. *Let $i \in \{1, ..., n\}$, $k \in \{1, 2\}$, $g_{k,i} \in C([0, 1]; \mathbb{R})$, $h_i \in C([0, 1]; \mathbb{R})$, then the problem*

$$\begin{cases} u_i'''(t) + h_i(t) = 0, \ 0 < t < 1, \ i \in \{1, ..., n\} \\ u_i(0) = \displaystyle\int_0^1 g_{1,i}(s)ds \\ u_i'(0) = 0 \\ \alpha_i u_i'(1) = \beta_i u_i(\eta_i) + \displaystyle\int_0^1 g_{2,i}(s)ds \end{cases} \tag{2.1}$$

has an unique solution

$$u(t) = \int_0^1 H(t,s)h(s)ds + \varphi(t)$$

where $u(t) = (u_1(t), ..., u_n(t))$, $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$,

$$\int_0^1 H(t,s)h(s)ds = (\int_0^1 H_1(t,s)h_1(s)ds, ..., \int_0^1 H_n(t,s)h_n(s)ds)$$

with

$$H_i(t,s) = G(t,s) + t^2 K_i G(\eta_i, s) \tag{2.2}$$

$$G(t,s) = \frac{1}{2}\begin{cases} (1-s)t^2 & \text{if } 0 \leq t \leq s \\ (-s+2t-t^2)s & \text{if } s \leq t \leq 1 \end{cases} \tag{2.3}$$

$$\varphi_i(t) = [K_i \int_0^1 g_{1,i}(s)ds + C_i \int_0^1 g_{2,i}(s)ds]t^2 + \int_0^1 g_{1,i}(s)ds \tag{2.4}$$

$$K_i = \frac{\beta_i}{2\alpha_i - \beta_i \eta_i^2} \tag{2.5}$$

$$C_i = \frac{1}{2\alpha_i - \beta_i \eta_i^2}. \tag{2.6}$$

Proof. Let $i \in \{1, ..., n\}$, integrating the equation (2.1), it yields

$$u_i(t) = -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)ds + \frac{1}{2}C_{1,i}t^2 + C_{2,i}t + C_{3,i}.$$

From the boundary condition $u_i(0) = \int_0^1 g_{1,i}(s)ds$, $u_i'(0) = 0$ we deduce that $C_{3,i} = \int_0^1 g_{1,i}(s)ds$ and $C_{2,i} = 0$. From the condition $\alpha_i u_i'(1) = \beta_i u_i(\eta_i) + \int_0^1 g_{2,i}(s)ds$, we have

$$\begin{aligned} C_{1,i} &= \frac{-\beta_i}{2\alpha_i - \beta_i\eta_i^2}\int_0^{\eta_i}(\eta_i - s)^2 h_i(s)ds + \frac{2\alpha_i}{2\alpha_i - \beta_i\eta_i^2}\int_0^1 (1-s)h_i(s)ds \\ &+ \frac{2\beta_i}{2\alpha_i - \beta_i\eta_i^2}\int_0^1 g_{1,i}(s)ds + \frac{2}{2\alpha_i - \beta_i\eta_i^2}\int_0^1 g_{2,i}(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned}
u_i(t) &= -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)ds + \frac{t^2}{2}[\frac{-\beta_i}{2\alpha_i-\beta_i\eta_i^2}\int_0^{\eta_i}(\eta_i-s)^2 h_i(s)ds \\
&+ \frac{2\alpha_i}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 (1-s)h_i(s)ds + \frac{2\beta_i}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 g_{1,i}(s)ds \\
&+ \frac{2}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 g_{2,i}(s)ds] + \int_0^1 g_{1,i}(s)ds \\
&= -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)ds + \frac{t^2}{2}\int_0^1 (1-s)h_i(s)ds + \frac{\beta_i t^2}{2\alpha_i-\beta_i\eta_i^2} \\
&\times [\frac{\eta_i^2}{2}\int_0^1 (1-s)h(s)ds - \frac{1}{2}\int_0^{\eta_i}(\eta_i-s)^2 h_i(s)ds] \\
&+ [\frac{2\beta_i}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 g_{1,i}(s)ds + \frac{2}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 g_{2,i}(s)ds]\frac{t^2}{2} + \int_0^1 g_{1,i}(s)ds \\
&= \int_0^1 G(t,s)h_i(s)ds + \frac{t^2\beta_i}{2\alpha_i-\beta_i\eta_i^2}\int_0^1 G(\eta_i,s)h_i(s)ds + P_i(t) \\
&= \int_0^1 G(t,s)h_i(s)ds + K_i t^2\int_0^1 G(\eta_i,s)h_i(s)ds + \varphi_i(t) \\
&= \int_0^1 H_i(t,s)h_i(s)ds + \varphi_i(t)
\end{aligned}$$

where $G(t,s)$, $H_i(t,s)$, $P_i(t)$, K_i and C_i are given by (2.2), (2.3), (2.4), (2.5) and (2.6) which achieve the proof of Lemma 1. □

Lemma 2.2. *Let $i \in \{1,...,n\}$, $k \in \{1,2\}$, $h_{k,i} \in C([0,1]\times\mathbb{R}^n;\mathbb{R})$, $f_i \in C([0,1]\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$. Then u is a solution of (1.1)-(1.2) if and only if for all, $T(u)(t) = u(t)$. Where $T(u) = (T_1(u),...,T_n(u))$ and for all $t\in[0,1]$, for all $i\in\{1,...,n\}$*

$$T_i(u)(t) = P_i(t) + \int_0^1 H_i(t,s)f_i(s,u(s),u'(s))ds$$

with $H_i(t,s)$ is given by (2.2)-(2.4) and

$$P_i(t) = [K_i\int_0^1 h_{1,i}(s,u(s))ds + C_i\int_0^1 h_{2,i}(s,u(s))ds]t^2 + \int_0^1 h_{1,i}(s,u(s))ds.$$

3. Existence of positive solution

In this section, we will give some preliminary considerations and some lemmas which are essential to establish sufficient conditions for the existence of least one positive solution for our problem. We make the following additional assumption.

(H1) For all $i\in\{1,...,n\}$, $k\in\{1,2\}$, $f_i\in C([0,1]\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}_+)$, $h_{k,i}\in C([0,1]\times\mathbb{R}^n,\mathbb{R}_+)$.

Lemma 3.1. *For all $t\in[0,1]$, $s\in[0,1]$, we have*

(1) $0\le G(t,s)\le\varphi(s)$
(2) $0\le\frac{\partial G(t,s)}{\partial t}\le 2\varphi(s)$

where $\varphi(s)=\frac{(1-s)s}{2}$.

Proof. It is easy to see that, if $t \leq s$, $G(t,s) = \frac{1}{2}(1-s)t^2 \geq 0$
and $G(t,s) = \frac{1}{2}(1-s)t^2 \leq \frac{(1-s)s}{2}$.
If $s \leq t$, $G(t,s) = \frac{1}{2}(2t-t^2-s)s = \frac{1}{2}[(1-s)-(1-t)^2]s \geq 0$
and $G(t,s) \leq \frac{1}{2}(1-s)(1-1+s) = \frac{1}{2}(1-s)s$
then the proof of (1) is complete.

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-s)t & \text{if } 0 \leq t \leq s \\ (1-t)s & \text{if } s \leq t \leq 1 \end{cases} \tag{3.1}$$

If $t \leq s$, $\frac{\partial G(t,s)}{\partial t} = (1-s)t \leq (1-s)s$.
If $s \leq t$, then $-t \leq -s$ this implies that $\frac{\partial G(t,s)}{\partial t} = (1-t)s \leq (1-s)s$.
We deduce that proof of (2) is complete. □

Lemma 3.2. *Let $a \in]0,1[$ and $b \in]0,1[$, then for all $(t,s) \in [a,b] \times [a,b]$*

(1) $G(t,s) \geq \omega_1(s)$
where $\omega_1(s) = \frac{1}{2}(1-s)a^2$
(2) $\frac{\partial G(t,s)}{\partial t} \geq \omega_2(s)$
where $\omega_2(s) = \frac{1}{2}(1-b)a$.

Proof. Let $a \in]0,1[$ and $b \in]0,1[$, then for all $(t,s) \in [a,b] \times [a,b]$.
If $t \leq s$,

$$\begin{aligned} G(t,s) &= \frac{1}{2}(-s+2t-t^2)s \\ &= \frac{1}{2}((1-s)-(1-t)^2)s \\ &= \frac{1}{2}((1-s)-(1-s)^2)s \\ &\geq \frac{1}{2}(1-s)s^2 \\ &\geq \frac{1}{2}(1-s)a^2. \end{aligned}$$

If $t \geq s$,

$$\begin{aligned} G(t,s) &= \frac{1}{2}(1-s)t^2 \\ &\geq \frac{1}{2}(1-s)a^2. \end{aligned}$$

Then proof of (1) is complete.
If $t \leq s$,

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= (1-s)t \\ &\geq (1-s)a \\ &\geq (1-b)a. \end{aligned}$$

If $t \geq s$,

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t} &= (1-t)s \\ &\geq (1-b)s \\ &\geq (1-b)a.\end{aligned}$$

Then proof of (2) is complete, and the proof of Lemma 3.2 is complete. □

Lemma 3.3. *Suppose that (H 1) holds and let $a \in]0,1[$, $b \in]0,1[$, then the solution $u = (u_1, ..., u_n)$ of the problem (1.1)-(1.2) is nonnegative and satisfies*

$$\min_{t\in[a,b]} \sum_{i=1}^{n} (u_i(t) + u_i'(t)) \geq \gamma(a,b)\|u\|$$

where

$$\gamma(a,b) = \frac{\max\limits_{i\in\{1,\ldots,n\}} \gamma_i(a,b)}{n}$$

$$\gamma_i(a,b) = \frac{\displaystyle\int_a^b (\omega_1(s) + \omega_2(s)) f_i(s, u(s), u'(s))ds}{\max(A,B)}$$

$$A = (1 + \max_{i\in\{1,\ldots,n\}} K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\ldots,n\}} P_i(1) + 1$$

and

$$B = 2(1 + \max_{i\in\{1,\ldots,n\}} K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\ldots,n\}} P_i'(1) + 1.$$

Proof. Suppose that $u = (u_1, ..., u_n)$ is a solution of (1.1)-(1.2), then from lemma 2.2, (H1) and $G(t,s) \geq 0$ for all $(s,t) \in [0,1] \times [0,1]$, it is obvious that for all $i \in \{1, ..., n\}$, for all $t \in [0,1]$, $u_i(t) \geq 0$. For all $i \in \{1, ..., n\}$, for all $t \in [0,1]$, we have

$$\begin{aligned}|u_i(t)| &\leq \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))ds + P_i(t) \\ &\leq (1 + K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + P_i(1) \\ &\leq (1 + \max_{i\in\{1,\ldots,n\}} K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\ldots,n\}} P_i(1).\end{aligned}$$

This implies that, for all $i \in \{1, ..., n\}$

$$\|u_i\|_\infty \leq (1 + \max_{i\in\{1,\ldots,n\}} K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\ldots,n\}} P_i(1).$$

For all $i \in \{1, ..., n\}$, for all $t \in [0,1]$, we have

$$\begin{aligned} |u_i'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s,u(s),u'(s))ds + P_i'(1) \\ &\leq 2(1+K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + P_i'(1) \\ &\leq 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1). \end{aligned}$$

This implies that, for all $i \in \{1, ..., n\}$

$$\|u_i'\|_\infty \leq 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1).$$

We denote by

$$A = (1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i(1) + 1$$

and

$$B = 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1) + 1.$$

Then for all $i \in \{1, ..., n\}$,

$$\|u_i\| = \max(\|u_i\|_\infty, \|u_i'\|_\infty) \leq \max(A,B).$$

Using lemma 3.2-(1) we have, for all $i \in \{1, ..., n\}$, for all $t \in [a,b]$

$$\begin{aligned} u_i(t) &\geq \int_a^b \omega_1(s) f_i(s,u(s),u'(s))ds + P_i(t) \\ &\geq \int_a^b \omega_1(s) f_i(s,u(s),u'(s))ds \\ &\geq \frac{\displaystyle\int_a^b \omega_1(s) f_i(s,u(s),u'(s))ds}{\max(A,B)} \times \max(A,B) \\ &\geq \frac{\displaystyle\int_a^b \omega_1(s) f_i(s,u(s),u'(s))ds}{\max(A,B)} \|u_i\|. \end{aligned}$$

Then for all $i \in \{1, ..., n\}$

$$\min_{t\in[a,b]} u_i(t) \geq \frac{\displaystyle\int_a^b \omega_1(s) f_i(s,u(s),u'(s))ds}{\max(A,B)} \|u_i\|.$$

Similarly, using lemma 3.2-(2) for all $i \in \{1, ..., n\}$, for all $t \in [a, b]$

$$\begin{aligned} u_i'(t) &\geq \int_a^b \omega_2(s) f_i(s, u(s), u'(s)) ds + P'(t) \\ &\geq \int_a^b \omega_2(s) f_i(s, u(s), u'(s)) ds \\ &\geq \frac{\int_a^b \omega_2(s) f_i(s, u(s), u'(s)) ds}{\max(A, B)} \times \max(A, B) \\ &\geq \frac{\int_a^b \omega_2(s) f_i(s, u(s), u'(s)) ds}{\max(A, B)} \|u_i\|. \end{aligned}$$

Then for all $i \in \{1, ..., n\}$,

$$\min_{t \in [a,b]} u_i'(t) \geq \frac{\int_a^b \omega_2(s) f_i(s, u(s), u'(s)) ds}{\max(A, B)} \|u_i\|.$$

Then for all $i \in \{1, ..., n\}$,

$$\begin{aligned} \min_{t \in [a,b]} (u_i(t) + u_i'(t)) &\geq \frac{\int_a^b (\omega_1(s) + \omega_2(s)) f_i(s, u(s), u'(s)) ds}{\max(A, B)} \|u_i\| \\ &= \gamma_i(a, b) \|u_i\|. \end{aligned}$$

We deduce that

$$\begin{aligned} \min_{t \in [a,b]} \sum_{i=1}^n (u_i(t) + u_i'(t)) &\geq \sum_{i=1}^n \gamma_i(a, b) \|u_i\| \\ &\geq \max_{j \in \{1, \dots, n\}} [\gamma_j(a, b) \times \|u_j\|] \\ &\geq \max_{j \in \{1, \dots, n\}} \gamma_j(a, b) \times \max_{j \in \{1, \dots, n\}} \|u_j\|. \end{aligned} \tag{3.2}$$

We deduce that for all $j \in \{1, ..., n\}$,

$$\min_{t \in [a,b]} \sum_{i=1}^n (u_i(t) + u_i'(t)) \geq \max_{i \in \{1, \dots, n\}} \gamma_i(a, b) \times \|u_j\|.$$

Then

$$\begin{aligned} \min_{t \in [a,b]} \sum_{i=1}^n (u_i(t) + u_i'(t)) &\geq \frac{\max_{i \in \{1, \dots, n\}} \gamma_i(a, b)}{n} \sum_{j=1}^n \|u_j\| \\ &= \gamma(a, b) \|u\|_E. \end{aligned}$$

The proof is complete. □

Definition 2. *We denote by* E^+ *the following set*

$$E^+ = \{u = (u_1, ..., u_n) \in E, u_i(t) \geq 0, t \in [0, 1], i \in \{1, ..., n\}\}.$$

Definition 3. *Let* E *be a Banach space,* Con *nonempty closed convex subset* $Con \subset E$*, is called a cone if it satisfies the following two conditions:*

(1) *$x \in Con$, $\lambda \geq 0$ implies $\lambda x \in Con$*
(2) *$x \in Con$, $-x \in Con$ implies $x = 0$.*

Remark. *For all $a, b \in]0,1[$, the set defined by*

$$Con(a,b) = \{u \in E^+, \min_{t\in[a,b]} \sum_{i=1}^{n} (u_i(t) + u_i'(t)) \geq \gamma(a,b)\|u\|_E\}$$

is a cone subset of E.

Theorem 3.4. *(Guo-Krasnosel'skii fixed point theorem)* [10]
Let E be a Banach space, and let $Con \subset E$ be a cone. Assume Ω_1 and Ω_2 be two bounded open subsets in E, with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $A : Con \cap (\overline{\Omega_2} \backslash \Omega_1) \to Con$ by a completely continuous operator such that:

(1) *$\|A(u)\| \leq \|u\|$, $u \in Con \cap \partial\Omega_1$ and $\|A(u)\| \geq \|u\|$, $u \in Con \cap \partial\Omega_2$ or*
(2) *$\|A(u)\| \geq \|u\|$, $u \in Con \cap \partial\Omega_1$ and $\|A(u)\| \leq \|u\|$, $u \in Con \cap \partial\Omega_2$.*

Then A has a fixed point in $Con \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Now, we give the following assumptions

(H2) For all $i \in \{1, ..., n\}$, $k \in \{1,2\}$, $f_i \in C([0,1] \times \mathbb{R}_+^n \times \mathbb{R}^n, \mathbb{R}_+)$, $h_{k,i} \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$.
(H3) There exists $i_0 \in \{1, ..., n\}$ and there exists $t_0 \in]0,1[$ such that $f_{i_0}(t_0, x, y) > 0$, for all $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}^n$.

Theorem 3.5. *Suppose that (H2) and (H3) hold. Then the problem (1.1)-(1.2) has at least one nonnegative solution in the case:*
For all $i \in \{1, ..., n\}$,

$$\lim_{\|u\|_1+\|v\|_1 \to 0} \min_{t\in[0,1]} \frac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = +\infty$$

and

$$\lim_{\|u\|_1 + \|v\|_1 \to +\infty} \max_{t\in[0,1]} \frac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = 0$$

where $u = (u_1, ..., u_n) \in \mathbb{R}^n$ and $v = (v_1, ..., v_n) \in \mathbb{R}^n$.

Proof. **Step 1:** From (H2) and (H3), there exists $[\alpha, \beta] \subset]0,1[$, such that $t_0 \in [\alpha, \beta]$ and for all $t \in [\alpha, \beta]$, $(x, y) \in \mathbb{R}_+^n \times \mathbb{R}^n$ we have $f_{i_0}(t, x, y) > 0$. Then for all $u \in E^+$,

$$\int_\alpha^\beta \omega_1(s) f_{i_0}(s, u(s), u'(s))ds > 0 \text{ and } \int_\alpha^\beta \omega_2(s) f_{i_0}(s, u(s), u'(s))ds > 0.$$

This implies that

$$\gamma_{i_0}(\alpha, \beta) > 0 \text{ and } \gamma(\alpha,\beta) = \frac{\max_{i\in\{1,...,n\}} \gamma_i(\alpha,\beta)}{n} \geq \frac{\gamma_{i_0}(\alpha,\beta)}{n} > 0.$$

From Remark, $Con(\alpha, \beta) = \{u \in E^+, \min_{t\in[\alpha,\beta]} \sum_{i=1}^{n} (u_i(t) + u_i'(t)) \geq \gamma(\alpha, \beta)\|u\|_E\}$ is a cone subset of E. By using Arzela-Ascoli theorem [11], $T : Con(\alpha, \beta) \to E$ is a completely continuous mapping.

We will show that $T(Con(\alpha,\beta)) \subset Con(\alpha,\beta)$. In fact, from (H2) and $G(t,s) \geq 0$ for all $[0,1]\times[0,1]$, it is obvious that for all $i \in \{1,...,n\}$, for all $t \in [0,1]$, $T_i(u)(t) \geq 0$. Using lemma 3.1-(1), for all $i \in \{1,...,n\}$, for all $t \in [0,1]$, we have

$$\begin{aligned} |T_i(u)(t)| &\leq \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))ds + P_i(t) \\ &\leq (1+K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + P_i(1) \\ &\leq (1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i(1). \end{aligned}$$

This implies that for all $i \in \{1,...,n\}$

$$\|T_i(u)\|_\infty \leq (1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i(1).$$

Using lemma 3.1-(2), for all $i \in \{1,...,n\}$, for all $t \in [0,1]$, we have

$$\begin{aligned} |T_i(u)'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s,u_i(s),u_i'(s))ds + P_i'(1) \\ &\leq 2(1+K_i)\int_0^1 \varphi(s) f_i(s,u_i(s),u_i'(s))ds + P_i'(1) \\ &\leq 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1). \end{aligned}$$

This implies that for all $i \in \{1,...,n\}$

$$\|T_i(u)'\|_\infty \leq 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1).$$

We denote by

$$A = (1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i(1) + 1$$

and

$$B = 2(1+\max_{i\in\{1,\dots,n\}} K_i)\int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds + \max_{i\in\{1,\dots,n\}} P_i'(1) + 1.$$

Then for all $i \in \{1,...,n\}$,

$$\|T_i(u)\| = \max(\|T_i(u)\|_\infty, \|T_i(u)'\|_\infty) \leq \max(A,B).$$

Using lemma 3.2-(1), we have for all $i \in \{1, ..., n\}$, for all $t \in [\alpha, \beta]$

$$\begin{aligned}
T_i(u)(t) &\geq \int_\alpha^\beta \omega_1(s) f_i(s, u(s), u'(s))ds + P_i(t) \\
&\geq \int_\alpha^\beta \omega_1(s) f_i(s, u(s), u'(s))ds \\
&\geq \frac{\displaystyle\int_\alpha^\beta \omega_1(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \times \max(A, B) \\
&\geq \frac{\displaystyle\int_\alpha^\beta \omega_1(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \|T_i(u)\|.
\end{aligned}$$

Then for all $i \in \{1, ..., n\}$,

$$\min_{t \in [\alpha, \beta]} T_i(u)(t) \geq \frac{\displaystyle\int_\alpha^\beta \omega_1(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \|T_i(u)\|.$$

Similarly by use lemma 3.2-(2), we have for all $i \in \{1, ..., n\}$, for all $t \in [\alpha, \beta]$

$$\begin{aligned}
T_i(u)'(t) &\geq \int_\alpha^\beta \omega_2(s) f_i(s, u(s), u'(s))ds + P'(t) \\
&\geq \int_\alpha^\beta \omega_2(s) f_i(s, u(s), u'(s))ds \\
&\geq \frac{\displaystyle\int_\alpha^\beta \omega_2(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \times \max(A, B) \\
&\geq \frac{\displaystyle\int_\alpha^\beta \omega_2(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \|T_i(u)\|.
\end{aligned}$$

Then for all $i \in \{1, ..., n\}$,

$$\min_{t \in [\alpha, \beta]} T_i(u)'(t) \geq \frac{\displaystyle\int_\alpha^\beta \omega_2(s) f_i(s, u(s), u'(s))ds}{\max(A, B)} \|T_i(u)\|.$$

We deduce that

$$\begin{aligned}
\min_{t \in [\alpha, \beta]} (T_i(u)(t) + T_i(u)'(t)) &\geq \frac{\displaystyle\int_\alpha^\beta (\omega_1(s) + \omega_2(s)) f_i(s, u(s), u'(s))ds}{\max(A, B)} \|T(u_i)\| \\
&= \gamma_i(\alpha, \beta) \|T_i(u)\|
\end{aligned}$$

where

$$\gamma_i(\alpha, \beta) = \frac{\displaystyle\int_\alpha^\beta (\omega_1(s) + \omega_2(s)) f_i(s, u(s), u'(s))ds}{\max(A, B)}$$

we deduce that

$$\begin{aligned}\min_{t\in[\alpha,\beta]}\sum_{i=1}^{n}(T_i(u)(t)+T_i(u)'(t)) &\geq \sum_{i=1}^{n}\gamma_i(\alpha,\beta)\|T_i(u)\|\\ &\geq \max_{j\in\{1,\dots,n\}}[\gamma_j(\alpha,\beta)\|T_j(u)\|]\\ &\geq \max_{j\in\{1,\dots,n\}}\gamma_j(\alpha,\beta)\times\max_{j\in\{1,\dots,n\}}\|T_j(u)\|.\end{aligned}$$

This implies that for all $j\in\{1,...,n\}$,

$$\min_{t\in[\alpha,\beta]}\sum_{i=1}^{n}(T_i(u)(t)+T_i(u)'(t))\geq \max_{i\in\{1,\dots,n\}}\gamma_i(a,b)\|T_j(u)\|.$$

Then

$$\begin{aligned}\min_{t\in[\alpha,\beta]}\sum_{i=1}^{n}(T_i(u)(t)+T_i(u)'(t)) &\geq \frac{\max_{i\in\{1,\dots,n\}}\gamma_i(\alpha,\beta)}{n}\sum_{j=1}^{n}\|T_j(u)\|\\ &= \gamma(\alpha,\beta)\|T(u)\|_E.\end{aligned}$$

Step 2: Let $i\in\{1,...,n\}$ we have $\lim_{\|u\|_1+\|v\|_1\to 0}\min_{t\in[0,1]}\frac{f_i(t,u,v)}{\|u\|_1+\|v\|_1}=+\infty$. Then for all $M>0$, $\exists R_{i,1}>0$ such that

$$\min_{t\in[0,1]}f_i(t,u,v)\geq M(\|u\|_1+\|v\|_1),\ \text{for } \|u\|_1+\|v\|_1\leq R_{i,1}.$$

This implies that, for all $M>0$ $\exists R_{i,1}>0$ such that for all $t\in[0,1]$,

$$f_i(t,u,v)\geq \min_{t\in[0,1]}f_i(t,u,v)\geq M(\|u\|_1+\|v\|_1)\ \text{for } \|u\|_1+\|v\|_1\leq R_{i,1},$$

we choose

$$M=\frac{1}{n\ \gamma(\alpha,\beta)\ \min(\int_\alpha^\beta\omega_1(s)ds,\int_\alpha^\beta\omega_2(s)ds)}.$$

Let Ω_1 be the bounded open set in E defined by $\Omega_1=\{u\in E,\ \|u\|_E<\min_{i\in\{1,\dots,n\}}R_{i,1}\}$. Then for all $u\in Con(\alpha,\beta)\cap\partial\Omega_1$, it yields, for all $s\in[\alpha,\beta]$,

$$\sum_{i=1}^{n}(u_i(s)+u_i'(s))\geq\gamma(\alpha,\beta)\min_{i\in\{1,\dots,n\}}R_{i,1}.$$

Then for all $i\in\{1,...,n\}$, $s\in[\alpha,\beta]$,

$$f_i(s,u(s),u'(s))\geq M(\|u(s)\|_1+\|u'(s)\|_1)\geq M\ \gamma(\alpha,\beta)\ \min_{i\in\{1,\dots,n\}}R_{i,1}.$$

This implies that, for all $i\in\{1,...,n\}$, $t\in[0,1]$

$$|T_i(u)(t)|\geq M\ \int_\alpha^\beta\omega_1(s)ds\ \gamma(\alpha,\beta)\ \min_{i\in\{1,\dots,n\}}R_{i,1}$$

and for all $i\in\{1,...,n\}$

$$\|T_i(u)\|\geq\|T_i(u)\|_\infty\geq\frac{\min_{i\in\{1,\dots,n\}}R_{i,1}}{n}.$$

We deduce that

$$\|T(u)\|\geq\min_{i\in\{1,\dots,n\}}R_{i,1}=\|u\|_E.$$

Step 3: Now, let $i \in \{1, ..., n\}$ we have $\lim\limits_{\|u\|_1+\|v\|_1 \to +\infty} \max\limits_{t\in[0,1]} \dfrac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = 0$. Then for all $\varepsilon > 0$, $\exists R_0^{f_i} > 0$ such that

$$\max_{t\in[0,1]} f_i(t,u,v) \le \varepsilon(\|u\|_1 + \|v\|_1) \text{ for } \|u\|_1 + \|v\|_1 \ge R_0^{f_i}.$$

This implies that, for all $\varepsilon > 0$, $\exists R_0^{f_i} > 0$ such that for all $t \in [0,1]$,

$$f_i(t,u,v) \le \max_{t\in[0,1]} f_i(t,u,v) \le \varepsilon(\|u\|_1 + \|v\|_1) \text{ for } \|u\|_1 + \|v\|_1 \ge R_0^{f_i}.$$

From $(H2)$, there exist three constants $M_{1,i}, M_{2,i} \ge 0$ such that, for $k \in \{1,2\}$,

$$|h_{k,i}(t,u(t))| \le M_{k,i}, \text{ for each } t \in [0,1].$$

We choose

$$\varepsilon = \frac{1}{4n(1 + \max\limits_{i\in\{1,\dots,n\}} K_i)\displaystyle\int_0^1 \varphi(s)ds}.$$

Let

$$\begin{aligned} R_2 &= \max\{2 \min_{i\in\{1,\dots,n\}} R_{i,1}, \max_{i\in\{1,\dots,n\}} R_0^{f_i}, 2n \max_{i\in\{1,\dots,n\}} [K_i M_{1,i} + C_i M_{2,i} + M_{1,i} + 1] \\ &, \quad 2n \max_{i\in\{1,\dots,n\}} [2(K_i M_{1,i} + C_i M_{2,i}) + 1]\} \end{aligned}$$

and Ω_2 be the bounded open set in E defined by $\Omega_2 = \{u \in E, \|u\|_E < R_2\}$. Then for all $u \in Con(\alpha,\beta) \cap \partial\Omega_2$, we have for all $i \in \{1, ..., n\}$ for all $s \in [0,1]$

$$f_i(s,u(s),u'(s)) \le \max_{s\in[0,1]} f_i(s,u(s),u'(s)) \le \varepsilon(\|u(s)\|_1 + \|u'(s)\|_1) \le \varepsilon R_2.$$

Using lemma 3.1-(1) we have, for all $t \in [0,1]$

$$\begin{aligned} |T_i(u)(t)| &\le \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))ds + P_i(t) + 1 \\ &\le \varepsilon\ R_2\ (1 + \max_{i\in\{1,\dots,n\}} K_i) \int_0^1 \varphi(s)ds \\ &+ \max_{i\in\{1,\dots,n\}} [K_i M_{1,i} + C_i M_{2,i} + M_{1,i} + 1] \\ &\le \frac{R_2}{4n} + \frac{R_2}{2n} \\ &\le \frac{R_2}{n}. \end{aligned}$$

Then, for all $i \in \{1, ..., n\}$,

$$\|T_i(u)\|_\infty \le \frac{R_2}{n}.$$

Using lemma 3.1-(2) we have, for all $t \in [0,1]$

$$\begin{aligned} |T_i(u)'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s,u(s),u'(s))ds + P_i'(t) + 1 \\ &\leq \varepsilon\ R_2\ (1 + 2 \max_{i\in\{1,\ldots,n\}} K_i) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))ds \\ &+ \max_{i\in\{1,\ldots,n\}} [2(K_i M_{1,i} + C_i M_{2,i}) + 1] \\ &\leq \frac{R_2}{2n} + \frac{R_2}{2n} = \frac{R_2}{n}. \end{aligned}$$

Then for all $i \in \{1,...,n\}$,

$$\|T_i(u)'\|_\infty \leq \frac{R_2}{n}.$$

Therefore

$$\|T_i(u)\| \leq \frac{R_2}{n}.$$

We deduce that

$$\|T(u)\|_E \leq R_2 = \|u\|_E.$$

Step 4: Let $u \in \overline{\Omega_1}$ then $\|u\| \leq \min_{i\in\{1,\ldots,n\}} R_{i,1} < 2 \min_{i\in\{1,\ldots,n\}} R_{i,1} \leq R_2$. This implies that $\|u\| < R_2$, then $u \in \Omega_2$. We deduce that $\overline{\Omega_1} \subset \Omega_2$. By theorem 3.4, T has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$. Then (1.1)-(1.2) has a least one nonnegative solution u. □

Theorem 3.6. *Under assumptions of theorem 3.5 and adding the following condition:*

$$\text{For all } i \in \{1,\ldots,n\}, \text{there exist } t_{0,i} \in]0,1[\text{ such that } f_i(t_{0,i},x,y) > 0 \quad (3.3)$$

for all $x \in \mathbb{R}^n_+$, for all $y \in \mathbb{R}^n$.

Then the problem (1.1)-(1.2) has at least one positive solution.

Proof. Consider the nonnegative solution u for problem (1.1)-(1.2) whose existence is guaranteed by theorem 3.5. Notice that u_i satisfied for all $i \in \{1,...,n\}$ $u_i(t) = \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))ds + P_i(t)$. From (H2) and condition (3.3), there exist $[\alpha_i, \beta_i] \subset]0,1[$, such that, for all $t \in [\alpha_i, \beta_i]$ and $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n$, $f_i(t,x,y) > 0$. Then for all $i \in \{1,\ldots,n\}$ and for all $t \in]0,1[$,

$$u_i(t) = \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))ds + P_i(t) \geq \int_{\alpha_i}^{\beta_i} H_i(t,s) f_i(s,u(s),u'(s))ds > 0.$$

The proof is complete. □

4. Nonexistence results

In this section, we give some sufficient conditions for the nonexistence of positive solutions. We define the constants $B_i = (1+K_i)\int_0^1 \varphi(s)ds$ and $C(a,b) = \int_a^b \omega_1(s)ds$.

Theorem 4.1. *Suppose that (H2) holds. There exists $i_0 \in \{1,...,n\}$ such that for all $t \in]0,1[$, for all $x_{i_0} \in]0,+\infty[$, for all $i \in \{1,...,n\} \backslash \{i_0\}$, $x_i \in \mathbb{R}$*

(1) $(K_{i_0}+1)h_{1,i_0}(t,x_1,...,x_{i_0},...,x_n) < \frac{x_{i_0}}{3}$
(2) $C_{i_0}h_{2,i_0}(t,x_1,...,x_{i_0},...,x_n) < \frac{x_{i_0}}{3}$
(3) $B_{i_0}f_{i_0}(t,x_1,...,x_{i_0},...,x_n,y_1,...,y_n) < \frac{x_{i_0}}{3}$.

Then the problem (1.1)-(1.2) has no positive solutions.

Proof. Assume, to the contrary, that $u(t)$ is a positive solution of (1.1)-(1.2). We denote by $u(s) = (u_1(s),...,u_{i_0}(s),...,u_n(s))$ and $u'(s) = (u_1'(s),...,u_{i_0}'(s),...,u_n'(s))$.

Then for all $s \in]0,1[$ we have

$$f_{i_0}(t,u(s),u'(s)) < \frac{u_{i_0}(s)}{3}B_{i_0}^{-1}.$$

Then for all $t\in[0,1]$ and for all $s\in]0,1[$ we have

$$H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s)) < H_{i_0}(t,s)\frac{u_{i_0}(s)B_{i_0}^{-1}}{3}.$$

Multiplying this by $H_{i_0}(t,s)$ and integrating on $[0,1]$ we deduce that

$$\begin{aligned}\int_0^1 H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s))ds &< \frac{B_{i_0}^{-1}\int_0^1 H_{i_0}(t,s)u_{i_0}(s)ds}{3}\\ &\leq \frac{\int_0^1 u_{i_0}(s)ds}{3}.\end{aligned}$$

Since, for all $t\in[0,1]$

$$\begin{aligned}P_{i_0}(t) &\leq C_{i_0}\int_0^1 h_{2,i_0}(s,u(s))ds + (K_{i_0}+1)\int_0^1 h_{1,i_0}(s,u(s))ds\\ &\leq \frac{\int_0^1 u_{i_0}(s)ds}{3} + \frac{\int_0^1 u_{i_0}(s)ds}{3} = \frac{2}{3}\int_0^1 u_{i_0}(s)ds.\end{aligned}$$

Then for all $t\in[0,1]$

$$u_{i_0}(t) < \frac{2}{3}\int_0^1 u_{i_0}(s)ds + \frac{\int_0^1 u_{i_0}(s)ds}{3} = \int_0^1 u_{i_0}(s)ds.$$

By mean value theorem there exists $s_0 \in]0,1[$ such that $\int_0^1 u_{i_0}(s)ds = u_{i_0}(s_0)$ which is a contradiction. The proof is complete. □

Theorem 4.2. *Suppose that (H2) holds. There exists $i_0\in\{1,...,n\}$, there exists $a,b\in]a,b[$ such that for all $t\in[0,1]$, for all $x_{i_0}\in]0,+\infty[$, for all $i\in\{1,...,n\}\backslash\{i_0\}$, $x_i\in\mathbb{R}$*

$$C(a,b)f_{i_0}(t,x_1,...,x_{i_0},...,x_n,y_1,...,y_n) > x_{i_0}. \tag{4.1}$$

Then the problem (1.1)-(1.2) has no positive solutions.

Proof. Assume, to the contrary, that $u(t)$ is a positive solution of (1.1)-(1.2). We denote by $u(s) = (u_1(s),...,u_{i_0}(s),...,u_n(s))$ and $u'(s) = (u_1'(s),...,u_{i_0}'(s),...,u_n'(s))$. Then for all $s\in]0,1[$ we have

$$f_{i_0}(s,u_1(s),...,u_{i_0}(s),...,u_n(s),u_1'(s),...,u_n'(s)) > u_{i_0}(s)C(a,b)^{-1}.$$

Then for all $t \in [0,1]$ for all $s \in]0,1[$ we have

$$H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s)) > H_{i_0}(t,s)u_{i_0}(s)C(a,b)^{-1}.$$

Multiplying this by $H_{i_0}(t,s)$ and integrating on $[0,1]$ we deduce that

$$\begin{aligned}\int_0^1 H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s))ds &> C(a,b)^{-1}\int_0^1 H_{i_0}(t,s)u_{i_0}(s)ds \\ &\geq \int_0^1 u_{i_0}(s)ds.\end{aligned}$$

Then for all $t \in [0,1]$

$$\begin{aligned} u_{i_0}(t) &= \int_0^1 H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s))ds + P_{i_0}(t) \\ &\geq \int_0^1 H_{i_0}(t,s)f_{i_0}(t,u(s),u'(s))ds \\ &> \int_0^1 u_{i_0}(s)ds.\end{aligned}$$

By mean value theorem there exists $s_0 \in]0,1[$ such that $\displaystyle\int_0^1 u_{i_0}(s)ds = u_{i_0}(s_0)$ which is a contradiction. The proof is complete. □

Example 1. Consider the following system of boundary value problem.

$$\begin{cases} u_1'''(t) + \frac{t^3+1}{\sqrt{(u_1'(t))^2+1}} + e^{-u_2(t)} = 0 \\ u_2'''(t) + e^{-t}\sqrt{|u_1(t)| + |u_2'(t)|} + e^{-u_2'(t)} = 0 \\ u_1(0) = u_2(0) = 1 \\ u_1'(0) = u_2'(0) = 0 \\ u_1'(1) = 2u_1(\frac{1}{6}) + 3 \\ u_2'(1) = 2u_2(\frac{1}{8}) + 1. \end{cases} \tag{4.2}$$

We have $f_1(t,x_1,x_2,y_1,y_2) = \frac{t^3+1}{\sqrt{y_1^2+1}} + e^{-x_2}$

$$\min_{t\in[0,1]} \frac{f_1(t,x_1,x_2,x_1,x_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = e^{-x_2}$$

then $\displaystyle\lim_{(|x_1|+|x_2|+|y_2|+|y_2|)\to 0} \min_{t\in[0,1]} \frac{f_1(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = +\infty$

$$\max_{t\in[0,1]} \frac{f_1(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = \frac{2}{\sqrt{y_1^2+1}} + e^{-x_2} \text{ then}$$

$$\lim_{(|x_1|+|x_2|+|y_2|+|y_2|)\to +\infty} \max_{t\in[0,1]} \frac{f_1(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = 0.$$

$f_2(t,x_1,x_2,y_1,y_2) = e^{-t}\sqrt{|x_1|+|x_2|} + e^{-y_2}$

$$\min_{t\in[0,1]} \frac{f_2(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = e^{-1}\sqrt{|x_1|+|y_2|} + e^{-y_2}$$

then $\displaystyle\lim_{(|x_1|+|x_2|+|y_2|+|y_2|)\to 0} \min_{t\in[0,1]} \frac{f_2(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = +\infty$

$$\max_{t\in[0,1]} \frac{f_2(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = \sqrt{|x_1|+|y_2|} + e^{-y_2}$$

then $\lim\limits_{(|x_1|+|x_2|+|y_2|+|y_2|)\to+\infty} \max\limits_{t\in[0,1]} \dfrac{f_2(t,x_1,x_2,y_1,y_2)}{(|x_1|+|x_2|+|y_2|+|y_2|)} = 0.$

By using Theorem 3.6 the problem (4.2) has no positive solution.

Example 2. Consider the following system of boundary value problem.

$$\begin{cases} u_1'''(t) + (1+\frac{1024}{5}e^{u_1(t)})^2 + |u_2(t))| + (u_1'(t))^2 = 0 \\ u_2'''(t) + e^{2u_1(t)}) + |u_2'(t)|^3 = 0 \\ u_1(0) = u_2(0) = \displaystyle\int_0^1 \frac{|u_2(s)|ds}{|u_1(s)|+1} \\ u_1'(0) = u_2'(0) = 0 \\ u_1'(1) = 2u_1(\frac{1}{6}) + \displaystyle\int_0^1 |u_2(s)|ds \\ u_2'(1) = 2u_2(\frac{1}{9}) + \displaystyle\int_0^1 (u_1(s)+1)^2 ds \end{cases} \tag{4.3}$$

we denote by $f_1(t,x_1,x_2,y_1,y_2) = (1+\frac{1024}{5}e^{x_1})^2 + |x_2| + y_1^2$, $a=\frac{1}{4}$, $b=\frac{1}{2}$, $B = \displaystyle\int_0^1 \varphi(s)ds = \frac{1}{12}$ and $C(a,b) = \displaystyle\int_a^b \omega_1(s)ds = \frac{5}{1024}$.

$C(a,b)[f_1(t,x_1,x_2,y_1,y_2)] = \frac{5((1+\frac{1024}{5}e^{x_1})^2+|x_2|+y_1^2}{1024} > x_1$

By using Theorem 4.2 the problem (4.3) has no positive solution.

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References

[1] M. Gregus, *Third order linear differential equations*, in: Math. Appl., Reidel, Dordrecht, 1987.

[2] Yao Q, Feng Y, *The existence of solution for a third-order two-point boundary value problem*, Appl Math Lett **15** (2002), pp.227–232.

[3] Y. Feng, S. Liu, *Solvability of a third-order two-point boundary value problem*, Appl. Math. Lett. **18** (2005), pp.1034–1040.

[4] Y. G. Klaasen, *Differential inequalities and existence theorems for second and third order boundary value problems*, J. Differential Equations **10** (1971), pp.529–537.

[5] L. K. Jackson, *Existence and uniqueness of solutions of boundary value problems for third order differential equations*, J.Differential Equations **13** (1973), pp.432–537.

[6] D. J. O'Regan, *Topological transversality: Application to third order boundary value problems*, SIAM J. Math. Anal. **19** (1987), pp.630–641.

[7] Sun. Y, *Positive solutions for third-order three-point nonhomogeneous boundary value problems*, Appl. Math. Lett. **22** (2009), pp.45–51.

[8] L. J. Guo, J. P. Sun and Y. H. Zhao, *Existence of positive solution for nonlinear third-order three-point boundary value problem*, Nonlinear Anal. **68** (2008), pp.3151–3158.

[9] A. Guezane-Lakoud, D. Belakroum, *Rothe's method for telegraph equation with integral conditions*, Nonlinear Anal. **70**(2009),3842-3853.

[10] D. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Academic Press, San Diego, 1988.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[12] A. G. Lakoud, L. Zenkoufi, *Study of Third-Order Three-Point Boundary value problem with dependence on the first-order derivative*, Advences in Applied Mathematics and Approximation Theory, **41** (2013), pp.421–439.

[13] H. Chen, *Positive solutions for the nonhomogeneous three-point boundary value problem of second order differential equations*, Math, Comput. Modelling **45** (2007), pp.844–852.

[14] J. R. Graef and B. Yang, *Multiple positive solution to a three-point third order boundary value problems*, Discrete and Continuous Dynamical Systems Suppliment Volume (2005), pp.1–8.

[15] R. Ma, *Positive solutions for a second order three-point boundary value problems*, Appl. Math. Lett., **14** (2001), pp.1–5.

[16] Q. L. Yao, *The existence and multiplicity of positive solutions for third-order three-point boundary value problem*, Acta Math. Appl. Sinica, **19**(2003), pp.117–122.

[17] D. Anderson, *Multiple positive solutions for a three-point boundary value problem*, Math. Comput. Modelling,**27**(1998), pp.49–57.

[18] D. Anderson, *Green's function for a third-order generalized right focal problem*, J. Math. Anal. Appl., **288**(2003), pp.1–14.

[19] A. Boucherif, N. Al-Malki, *Nonlinear three-point third order boundary value problems*, Appl. Math. Comput., **190**(2007), pp.1168–1177.

[20] L. Kong, Q. Kong, *Multi-point boundary value problems of second-order differential equations (I)*, Nonlinear. Anal. **58**(2004), pp.909–931.

[21] L. Kong, Q. Kong, *Multi-point boundary value problems of second-order differential equations (II)*, Comm. Appl. Nonlinear. Anal ,**14**(2007), pp.93–111.

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