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THE KUNZE-STEIN PHENOMENON ASSOCIATED WITH JACOBI-DUNKL CONVOLUTION

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Abstract. The main purpose of this paper is to establish the endpoint estimate for the Kunze-Stein phenomenon in Lorentz spaces associated with Jacobi-Dunkl convolution.

1. Introduction

Let μ denote the Haar measure on locally compact group G. The convolution of two compactly supported continuous functions f and g is the function $f * g$ on G defined by setting

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\mu(y), \quad \forall x \in G\,.$$

In 1960, R.A. kunze and E.M. Stein proved in [8] that, if G is $SL(2, \mathbb{R})$, then the continuous inclusion

$$L^p(G) * L^2(G) \subseteq L^2(G) \tag{1}$$

holds whenever $1 \leq p < 2$. A group satisfying (1) for any $1 \leq p < 2$ is called the Kunze-Stein phenomenon group and for such a group (1) is called the Kunze-Stein phenomenon for G. In 1978 M. Cowling proved in [2] that every connected real semisimple Lie group with finite center is a Kunze-Stein phenomenon group. After twenty years Cowling, S. Meda and A.G. Setti proved that if G is the group of isometries of a homogeneous tree $\mathbf{H}$ (see [4]) or a semisimple Lie group of real rank 1(see [3]), then G satisfies a more accurate version of the Kunze-Stein phenomenon. By using the Lorentz spaces $L^{p,\,q}$, they proved that for $1 < p < 2$, the continuous inclusion

$$L^{p,\,q_1}(G) * L^{p,\,q_2}(G) \subseteq L^{p,q}(G) \tag{2}$$

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holds if and only if $\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2} - 1$. In 2000, A.D. Ionescu [6] studied the validity of (2) when $p \to 2$. He proved that

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G), \tag{3}$$

when G is a semisimple Lie group of rank one. In this paper we establish the endpoint estimate for the Kunze-Stein phenomenon associated with Jacobi-Dunkl convolution similar to (3) and (2).

2. **Preliminaries**

In this section we recapitulate some results about harmonic analysis associated to Jacobi-Dunkl operators and the Lorentz spaces. For details the reader is referred to ([1], [5]) .

Let $\alpha \geq \beta \geq -\frac{1}{2}$, we consider the Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ defined on $\mathcal{E}(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}$ by:

$$\Lambda_{\alpha,\beta}(f)(x) = f^{'}(x) + ((2\alpha+1)\coth(x) + (2\beta+1)\tanh(x))\Big(\frac{f(x) - f(-x)}{2}\Big).$$

For $\lambda \in \mathbb{C}$, the initial problem

$$\Lambda_{\alpha,\beta}(f)(x) = i\lambda f(x), \quad f(0) = 1, \quad x \in \mathbb{R},$$

has a unique solution $\Psi_\lambda^{\alpha,\beta}$ (called the *Jacobi-Dunkl kernel*) given by

$$\Psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + \frac{i\lambda}{2(\alpha+1)}\sinh(x)\cosh(x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

where

$$\lambda^2 = \mu^2 + \rho^2 \quad \text{with} \quad \rho = \alpha + \beta + 1$$

and

$$\varphi_\mu^{\alpha,\beta}(x) = {}_2F_1(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}; \alpha+1; -\sinh^2 x),$$

here ${}_2F_1$ denotes the Gaussian hypergeometric function.

Notation. For all $x, y, z \in \mathbb{R}$ and $\chi \in [0, \pi]$, we put:

- $\sigma_{x,y,z}^\chi = \begin{cases} -\frac{\cosh z \cos\chi - \cosh x \cosh y}{\sinh x \sinh y}, & \text{if } xy \neq 0 \\ 0, & \text{otherwise} \end{cases}$
- $\varrho^\chi(x,y,z) = 1 - \sigma_{x,y,z}^\chi + \sigma_{z,y,x}^\chi + \sigma_{z,x,y}^\chi$.
- $I_{x,y} = \Big[- |x| - |y|, -\Big||x| - |y|\Big|\Big] \cup \Big[\Big||x| - |y|\Big|, |x| + |y|\Big]$.
- $K_{\alpha,\beta}(x,\ y,\ z) = M_{\alpha,\beta}(\sinh|x| \sinh|y| \sinh|z|)^{-2\alpha} 1_{I_{x,y}}(z)$
$$\times \int_0^\pi \varrho^\chi(x,\ y,\ z)(g(x,\ y,\ z,\ \chi))_+^{\alpha-\beta-1} \sin^{2\beta}\chi d\chi,$$

where $g(x,\ y,\ z,\ \chi) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + \cosh x \cosh y \cosh z \cos\chi$. Here

$$t_+ = \begin{cases} t, & \text{if } t > 0 \\ 0, & \text{otherwise} \end{cases}$$

and

$$M_{\alpha,\beta} = \frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}.$$

Proposition 2.1. *For $z \in I_{x,y}$, there exist positive constants C such that*

$$|K_{\alpha,\beta}(x,y,z)| \leq Ce^{-\rho(|x|+|y|+|z|)}\Big(\frac{(1+|x|)(1+|y|)(1+|z|)}{|xyz|}\Big)^{2\alpha}\times$$

$$\Big(\frac{(|x|+|y|+|z|)(|x|+|y|-|z|)(|z|-|x|+|y|)(|z|+|x|-|y|)}{(1+|x|+|y|+|z|)(1+|x|+|y|-|z|)(1+|z|-|x|+|y|)(1+|z|+|x|-|y|)}\Big)^{\alpha-\frac{1}{2}}$$

Proof. For $x,\ y \in \mathbb{R}\setminus\{0\}$, we have

$$\begin{aligned}|K_{\alpha,\beta}(x,y,z)| &\leq M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}1_{I_{x,y}}(z)\\ &\times \int_{\Delta_{x,y}}|\varrho^{\chi}(x,\ y,\ z)|(g(x,\ y,\ z,\ \chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi d\chi,\end{aligned}$$

where $\Delta_{x,y}=\{\chi\in[0,\ \pi]:\ g(x,\ y,\ z,\ \chi)>0\}$, we have

$$\chi\in\Delta_{x,y}\Leftrightarrow \cosh z\cos\chi > \frac{\cosh^2 x\cosh^2 y\sinh^2 z}{2\cosh x\cosh y},$$

then

$$|\sigma^{\chi}_{x,y,z}|\leq 1.$$

Using the fact that

$$z\in I_{x,y}\Leftrightarrow x\in I_{z,y}\Leftrightarrow y\in I_{z,x},$$

we get $|\sigma^{\chi}_{x,z,y}|\leq 1$ and $|\sigma^{\chi}_{y,z,x}|\leq 1$. Then we deduce

$$\begin{aligned}|K_{\alpha,\beta}(x,y,z)| &\leq M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}1_{I_{x,y}}(z)\\ &\times \int_{\Delta_{x,y}}(g(x,\ y,\ z,\ \chi))_+^{\alpha-\beta-1}\sin^{2\beta}\chi d\chi.\end{aligned}$$

From [7], we get

$$|K_{\alpha,\beta}(x,y,z)|\leq 2\,W_{\alpha,\beta}(|x|,|y|,|z|),\quad |z|\in[\,\Big||x|-|y|\Big|,\ |x|+|y|\,],$$

where $W_{\alpha,\beta}$ is the Jacobi kernel given by

$$\begin{aligned}W_{\alpha,\beta}(|x|,|y|,|z|) &= \frac{2^{-2\rho}\Gamma(\alpha+1)(\cosh x\cosh y\cosh z)^{\alpha-\beta-1}}{\sqrt{\pi}\Gamma(\alpha+1/2)(\sinh|x|\sinh|y|\sinh|z|)^{2\alpha}}\\ &\times\ (1-B^2)^{\alpha-\frac{1}{2}}F_1(\alpha+\beta,\ \alpha-\beta;\ \alpha+\frac{1}{2};\ \frac{1-B}{2}),\end{aligned}$$

and

$$B=\frac{(\cosh x)^2+(\cosh y)^2+(\cosh z)^2-1}{2\cosh x\cosh y\cosh z}.$$

We have

$$1-B^2=\frac{\sinh(|x|+|y|+|z|)\sinh(|x|+|y|-|z|)\sinh(|z|+|y|-|x|)\sinh(|x|+|x|-|y|)}{4(\cosh x\cosh y\cosh z)^2}.$$

Using the fact that for $t\geq 0$, $\sinh t\sim\frac{t}{1+t}e^t$ and $\cosh t\sim e^t$ we obtained the result. □

The Jacobi-Dunkl translation operator $\tau^x_{\alpha,\beta}, x \in \mathbb{R}$ is defined for a continuous function f on $\mathbb{R}$, by

$$\tau^x_{\alpha,\beta} f(y) = \int_{\mathbb{R}} f(z) d\nu^{\alpha,\beta}_{x,y}(z), \quad y \in \mathbb{R}, \tag{4}$$

where $\nu^{\alpha,\beta}_{x,y}$ are a signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z) d\mu_{\alpha,\beta}(z), & \text{if } x, y \in \mathbb{R}\backslash\{0\} \\ d\delta_x(z), & \text{if } y = 0 \\ d\delta_y(z), & \text{if } x = 0. \end{cases}$$

Let $d\mu_{\alpha,\beta}(x) := A_{\alpha,\beta}(x)dx$, where $A_{\alpha,\beta}(x) = 2^{2\rho}(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$. We denote by $L^p(\mu_{\alpha,\beta})$, $p \in [1,\infty]$, the Lebesque space on $\mathbb{R}$ with respect to the measure $\mu_{\alpha,\beta}$. In the following we use the shorter notation $\|f\|_{p,A}$ instead of $\|f\|_{L^p(\mu_{\alpha,\beta})}$.

For all $x \in \mathbb{R}$ and $f \in L^q(\mu_{\alpha,\beta})$, $q \in [1,\infty]$, we have $\|\tau^x_{\alpha,\beta} f\|_{q,A} \leq 4\,\|f\|_{q,A}$.

Let $p, q, r \in [1,\infty]$ such that $1/p + 1/q = 1/r + 1$. The convolution product of $f \in L^p(\mu_{\alpha,\beta})$ and $g \in L^q(\mu_{\alpha,\beta})$ is defined by

$$f *_{\alpha,\beta} g(x) = \int_{\mathbb{R}} \tau^x_{\alpha,\beta}(f)(-y) g(y) d\mu_{\alpha,\beta}(y), \quad a.e.\ x \tag{5}$$

and we have

$$\|f *_{\alpha,\beta} g\|_{r,A} \leq 4\|f\|_{p,A}\|g\|_{q,A}. \tag{6}$$

Now we recall first basic definitions about the Lorentz spaces.
Let f be a measurable function defined on $(\mathbb{R},\, \mu_{\alpha,\beta})$. we assume the function f be finite almost everywhere and for $y > 0$, $\mu_{\alpha,\beta}(E_y) < \infty$, where

$$E_y = \{x \in \mathbb{R} \,:\, \mid f(x) \mid > y\}.$$

The distribution function of f is defined by $\lambda_f(y) = \mu_{\alpha,\beta}(E_y)$, $y > 0$ and the (nonnegative) rearrangement of f is defined by

$$f^*(t) = \inf\{y > 0 \,:\, \lambda_f(y) \leq t\} = \sup\{y > 0 \,:\, \lambda_f(y) > t\},\ t > 0\,.$$

The Lorentz space $L^{(p,\,q)}(\mathbb{R},\, \mu_{\alpha,\beta})$ (shortly $L^{(p,\,q)}(\mu_{\alpha,\beta})$) is defined to be vector space of all (equivalence classes) of measurable functions such that $\|f\|^*_{(p,\,q),A} < \infty$ where

$$\|f\|^*_{(p,\,q),\,A} = \Big(\frac{q}{p}\int_0^\infty \Big[t^{\frac{1}{p}} f^*(t)\Big]^q \frac{dt}{t}\Big)^{\frac{1}{q}},\ 0 < p,\, q < \infty,$$

$$\|f\|^*_{(p,\,\infty),A} = \sup_{t>0} t^{\frac{1}{p}} f^*(t), \qquad 0 < p \leq \infty\,.$$

It is known that $\|f\|^*_{(p,\,p),A} = \|f\|_{p,\,A}$ and so $L^{(p,\,p)}(\mu_{\alpha,\beta}) = L^p(\mu_{\alpha,\beta})$ and if $0 < q_1 \leq q_2 \leq \infty$, $0 < p < \infty$ then $\|f\|^*_{(p,\,q_2),\,A} \leq \|f\|^*_{(p,\,q_1),\,A}$ holds and hence $L^{(p,\,q_1)}(\mu_{\alpha,\beta}) \subseteq L^{(p,\,q_2)}(\mu_{\alpha,\beta})$. Notice that if χ_E is the characteristic function of a measurable set $E \subset \mathbb{R}$, we have

$$\|\chi_E\|^*_{(2,\,q),A} = \mu_{\alpha,\beta}(E)^{\frac{1}{2}} = \|\chi_E\|_{2,A}\,. \tag{7}$$

The following lemma is shown in [6]:

Lemma 2.2. *If $\delta \neq 0$ and $d\mu_1(t) = e^{\delta t}dt$, $d\mu_2(t) = e^{2\delta t}dt$ ar tow measures on $\mathbb{R}$, then*

$$\|f\|_{L^1(\mathbb{R},\,\mu_1)} \leq C_\delta \|f\|_{L^{(2,\,1)}(\mathbb{R},\,\mu_2)}.$$

3. The Kunze-Stein phenomenon

The main purpose of this paper is to prove the following endpoint estimate.

Theorem 3.1. *The convolution operator defined by (5) satisfies*

$$L^{(2,1)}(\mu_{\alpha,\beta}) *_{\alpha,\beta} L^{(2,1)}(\mu_{\alpha,\beta}) \subseteq L^{(2,\infty)}(\mu_{\alpha,\beta})\,.$$

Proof. In view of the general theory of Lorentz spaces, it suffices to prove that

$$\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq C\|f\|^*_{(2,1),A}\|g\|^*_{(2,1),A}\|h\|^*_{(2,1),A} \tag{8}$$

whenever f, g and h are characteristic functions of open sets with finite measure.

First suppose that one function of f, g or h is supported in $[-1, 1]$, say f, we have

$$\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq \|f *_{\alpha,\beta} g\|_{2,A}\|h\|_{2,A}\,.$$

Using (6) and (7), we get

$$\begin{aligned}\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) &\leq \|f\|_{1,A}\|g\|_{2,A}\|h\|_{2,A} \\ &\leq C\|f\|_{2,A}\|g\|_{2,A}\|h\|_{2,A} \\ &\leq C\|f\|^*_{(2,1),A}\|g\|^*_{(2,1),A}\|h\|^*_{(2,1),A}\,.\end{aligned}$$

Assume now f, g and h are characteristic functions of open sets with finite measure, we can writ $f = f_0 + f_1$, $g = g_0 + g_1$ and $h = h_0 + h_1$, where f_0, g_0 and h_0 are supported in $[-1, 1]$ and f_1, g_1 and h_1 are supported in $(-\infty, -1]\bigcup[1, \infty)$. By the first step, it suffices to prove (8) for $f = f_1$, $g = g_1$ and $h = h_1$.

If $\alpha \geq \frac{1}{2}$, from (5) and (4) we have

$$\begin{aligned}\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)|\tau^z_{\alpha,\beta} g(-x)| h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(z) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)\left|\int_{\mathbb{R}} g(y) d\nu_{z,-x}(y)\right| h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(z)\end{aligned}$$

Using the fact $K_{\alpha,\beta}(z, -x, y) = K_{\alpha,\beta}(-x, z, y) = K_{\alpha,\beta}(x, y, z)$, we obtain

$$\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) \leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|K_{\alpha,\beta}(x, y, z)\Big| f(x) g(y) h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(y) d\mu_{\alpha,\beta}(z)$$

For $|x|$, $|y|$, $|z| \geq 1$, By Proposition 2.1, there exists a constant C such that

$$|K_{\alpha,\beta}(x, y, z)| \leq Ce^{-\rho(|x|+|y|+|z|)}.$$

Using the fact that for $t \geq 0$, $A_{\alpha,\beta}(t) = 2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1} \leq Ce^{2\rho t}$, we get

$$\begin{aligned}\int_{\mathbb{R}} |(f *_{\alpha,\beta} g)(z)| h(z) d\mu_{\alpha,\beta}(z) &\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|K_{\alpha,\beta}(x, y, z)\Big| f(x) g(y) h(z) d\mu_{\alpha,\beta}(x) d\mu_{\alpha,\beta}(y) d\mu_{\alpha,\beta}(z) \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|K_{\alpha,\beta}(x, y, z)\Big| f(x) g(y) h(z) A_{\alpha,\beta}(x) A_{\alpha,\beta}(y) A_{\alpha,\beta}(z) dxdydz \\ &\leq C \int_{|x|\geq 1}\int_{|y|\geq 1}\int_{|z|\geq 1} f(x) g(y) h(z) e^{\rho(|x|+|y|+|z|)} dxdydz.\end{aligned}$$

Using Lemma 2.2 we get (8).

If $-\frac{1}{2} < \alpha < \frac{1}{2}$, we only need to prove (8) in which the integral is taken over the domain $D = \{(x,\, y,\, z) \,:\, |x| \le |y| \le |z| \le |x| + |y|\,,\; |x|,\, |y|,\, |z| \ge 1\}$.
By proposition 2.1, for all $(x,\, y,\, z) \in D$, we have

$$|K_{\alpha,\beta}(x,y,z)|A_{\alpha,\beta}(x)A_{\alpha,\beta}(y)A_{\alpha,\beta}(z) \le Ce^{\rho(|x|+|y|+|z|)}\Big(\frac{|x|+|y|-|z|}{1+|x|+|y|-|z|}\Big)^{\alpha-\frac{1}{2}}$$

If $|x|+|y|-|z| \ge 1$, by proceeding as in the analysis of the case $\alpha \ge \frac{1}{2}$, we get (8). If $|x|+|y|-|z| \le 1$, it suffices to prove (8) for $(x,\, y,\, z) \in D_1 = \{(x,\, y,\, z) \in D,\; y \ge 1\}$, for this, make the change of variable $y = u - |x|$, we get

$$\int_{(x,\, y,\, z)\in D_1} f(x)g(y)h(z)e^{\rho(|x|+y+|z|)}(|x|+y-|z|)^{\alpha-\frac{1}{2}}dxdydz$$

$$\le \int_{|z|\ge 1}\Big(\int_1^\infty\Big[\int_{1\le|x|\le u-|x|\le|z|} f(x)e^{\rho|x|}g(u-|x|)e^{\rho(u-|x|)}dx\Big](u-|z|)^{\alpha-\frac{1}{2}}du\Big)h(z)e^{\rho|z|}dz\,.$$

By Holder's inequality, we get

$$\int_{1\le|x|\le u-|x|\le z} f(x)g(u-|x|)e^{\rho|x|}e^{\rho(u-|x|)}dx$$

is bounded by $\|f\|_{2,\,A}\|g\|_{2,\,A}$ uniformly in u. Then

$$\begin{aligned}&\int_{(x,\, y,\, z)\in D_1} f(x)g(y)h(z)e^{\rho(|x|+y+|z|)}(|x|+y-|z|)^{\alpha-\frac{1}{2}}dxdydz\\ &\quad\le C\|f\|_{2,\,A}\|g\|_{2,\,A}\int_1^\infty\Big(\int_{|z|}^{|z|+1}(u-|z|)^{\alpha-\frac{1}{2}}du\Big)h(z)e^{|z|}dz\,,\end{aligned}$$

since $\alpha - \frac{1}{2} > -1$, then

$$\int_{(x,\, y,\, z)\in D_1} f(x)g(y)h(z)e^{\rho(|x|+y+|z|)}(|x|+y-|z|)^{\alpha-\frac{1}{2}}dxdydz \le C\|f\|^*_{(2,\,1),\,A}\|g\|^*_{(2,\,1),\,A}\|h\|^*_{(2,\,1),\,A}\,.$$

This completes the proof of the theorem. □

From the last Theorem and the bilinear interpolation theorem (see ([3], Theorem 1.2), we deduce the following result:

Theorem 3.2. *Let $1 < p < 2$ and $(q,\, q_1,\, q_2) \in [1,\, \infty]^3$ such that $1 + \frac{1}{q} \le \frac{1}{q_1} + \frac{1}{q_2}$, then*

$$L^{(p,\, q_1)}(\mu_{\alpha,\beta}) *_{\alpha,\beta} L^{(p,\, q_2)}(\mu_{\alpha,\beta}) \subseteq L^{(p,\, q)}(\mu_{\alpha,\beta})\,.$$

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