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COMPLEX VALUED RECTANGULAR METRIC SPACES AND COMMON FIXED POINT THEOREMS

(COMMUNICATED BY VLADIMIR MULLER)

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Abstract. Acknowledging the concept of complex valued metric spaces introduced by Azam et al.[3] many authors proved several fixed point results for mappings satisfying certain contraction conditions. In this note, some common fixed point theorems for two pairs of weakly compatible mapping satisfying a contractive condition having rational type terms in complex valued rectangular (generalized) metric spaces are proved. Further application of property (E.A.) and common limit range (CLR) property are employed. Moreover same results are also obtained in complex valued metric spaces. We suggest some examples distinguishing these two spaces. On the other hand illustrative example is also furnished to support our results. Our results generalize the results of Azam et al.[3] and Rouzkard et al.[9].

1. Introduction and Preliminaries

Fixed point theory is very interesting in study of nonlinear analysis and has a broad set of applications in Mathematics and Engineering. In this theory Banach contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Large number of generalizations has been made on this principle. In 2000, Branciari[6] introduced the concept of a rectangular (generalized) metric spaces where the triangle inequality of a metric space was replaced by another inequality which involves four (or more) points instead of three and improved Banach contraction principle. In 2011, Azam et al.[3] introduced the notion of complex valued metric spaces. After the establishment of complex valued metric spaces many researchers have contributed with their works in this space. Rouzkard et al.[9], Bhatt et al.[4], Nashine et al.[8] proved common fixed point theorems under rational contractions in complex valued metric spaces. For more detailed development one can see in ([5],[7],[10],[12]).
Recently, M. Abbas et al.[2] introduced a notion of complex valued generalized metric spaces and obtained common fixed point result for mappings in such spaces.

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Consistent with Azam et al.[3], the following definitions and results will be needed in the sequel.

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows :

$$z_1 \precsim z_2 \ \textit{if and only if} \ Re(z_1) \leq Re(z_2) \ \textit{and} \ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions are satisfied :
(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.
In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C2)$, $(C3)$ and $(C4)$ is satisfied and we write $z_1 \prec z_2$ if only $(C4)$ is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3.$$

Definition 1.1. [3] *Let X be a nonempty set such that the mapping $d : X \times X \to C$ satisfies the following conditions:*
(CM1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.1. *Let $X = C$ be a set of complex number. Define $d : C \times C \to C$ by*

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|.$$

Where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C, d) is a complex valued metric space.

very recently, M.Abbas et al.[2] defined the notion of complex valued rectangular(generalized) metric spaces as follows:

Definition 1.2. [2] *Let X be a non empty set. If a mapping $d : X \times X \to C$ satisfies:*
(a) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(c) $d(x, y) \precsim d(x, u) + d(u, v) + d(v, y)$ for all $x, y \in X$ and all distinct $u, v \in X$, each one is different from x and y.
Then d is called a complex valued rectangular (generalized) metric on X and (X, d) is called a complex valued rectangular (generalized) metric space.

To illustrate this we suggested two examples as follows:

Example 1.2. *Let $X = \{1 + i, -1 + i, -1 - i, 1 - i\}$. Define $d : X \times X \to C$ as follows:*
$d(1 + i, -1 + i) = d(-1 + i, 1 + i) = 3e^{i\theta}$;
$d(-1 + i, -1 - i) = d(-1 - i, -1 + i) = d(1 + i, -1 - i) = d(-1 - i, 1 + i) = e^{i\theta}$;
$d(1 + i, 1 - i) = d(1 - i, 1 + i) = d(-1 + i, 1 - i) = d(1 - i, 1 + i) = d(-1 - i, 1 - i) = d(1 - i, -1 - i) = 4e^{i\theta}$;
$d(1 + i, 1 + i) = d(-1 + i, -1 + i) = d(-1 - i, -1 - i) = d(1 - i, 1 - i) = 0$.
Obviously (X, d) is a complex valued rectangular(generalized) metric space, when $\theta \in [0, \frac{\pi}{2}]$. On the other hand (X, d) is not a complex valued metric space.

Example 1.3. *Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\}$ and $B = [1, 3]$. Define the generalized complex valued metric d on X as follows :*

$$d\Big(\frac{1}{2}, \frac{2}{3}\Big) = d\Big(\frac{3}{4}, \frac{4}{5}\Big) = d\Big(\frac{2}{3}, \frac{1}{2}\Big) = d\Big(\frac{4}{5}, \frac{3}{4}\Big) = 0.2i;$$
$$d\Big(\frac{1}{2}, \frac{4}{5}\Big) = d\Big(\frac{2}{3}, \frac{3}{4}\Big) = d\Big(\frac{4}{5}, \frac{1}{2}\Big) = d\Big(\frac{3}{4}, \frac{2}{3}\Big) = 0.3i;$$
$$d\Big(\frac{1}{2}, \frac{3}{4}\Big) = d\Big(\frac{2}{3}, \frac{4}{5}\Big) = d\Big(\frac{3}{4}, \frac{1}{2}\Big) = d\Big(\frac{4}{5}, \frac{2}{3}\Big) = 0.6i;$$
$$d\Big(\frac{1}{2}, \frac{1}{2}\Big) = d\Big(\frac{2}{3}, \frac{2}{3}\Big) = d\Big(\frac{3}{4}, \frac{3}{4}\Big) = d\Big(\frac{4}{5}, \frac{4}{5}\Big) = 0;$$

and $d(x, y) = i|x - y|$, if $x, y \in B$ or $x \in A, y \in B$ or $x \in B, y \in A$.
It is easy to show that d is a complex valued rectangular(generalized) metric but d is not a complex valued metric. Indeed,

$$0.6i = d\Big(\frac{1}{2}, \frac{3}{4}\Big) \succsim d\Big(\frac{1}{2}, \frac{3}{4}\Big) + d\Big(\frac{2}{3}, \frac{4}{4}\Big) = 0.2i + 0.3i = 0.5i.$$

Shows d is not a complex valued metric.

Definition 1.3. [2] *Let (X, d) be a complex valued rectangular (generalized) metric space and $\{x_n\}$ be a sequence in X.*

(1) *If for every $c \in C$ with $0 \prec c$, there exist $n_0 \in N$ such that $d(x_n, x) \prec c$ for all $n > n_0$, then $\{x_n\}$ is said to be convergent to $x \in X$, and we denote this by $x_n \to x$ as $n \to \infty$ or $lim_{n\to\infty} x_n = x$.*
(2) *If for every $c \in C$ with $0 \prec c$, there exist $n_0 \in N$ such that for all $n, m > n_0$, $d(x_n, x_m) \prec c$, then $\{x_n\}$ is called Cauchy sequence in X.*
(3) *If every Cauchy sequence in X, then (X, d) is called a complete complex valued rectangular(generalized) metric space.*

Lemma 1.1. [2] *Let (X, d) be a complex valued rectangular (generalized) metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2. [2] *Let (X, d) be a complex valued rectangular (generalized) metric spaces and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_m)| \to 0$ as $n, m \to \infty$.*

Definition 1.4. *Let A and S be self mappings on a set X, if $w = Ax = Sx$ for some x in X, then x is called coincidence point of A and S and w is called a point of coincidence of A and S.*

Definition 1.5. *A pair of self mappings $A, S : X \to X$, is called weakly compatible if they commute at their coincidence points.*

Example 1.4. *Let $X = C$ and $d : X \times X \to C$ be any complex valued metric on X. Define $A, S : X \to X$ by $A(z) = z$ and $S(z) = 2z - i, \quad \forall z \in X$. We see that $z = i$ is the only coincidence point and (A, S) are weakly compatible mappings, since they commute at their coincidence point $z = i$. i.e. $\quad AS(z) = SA(z)$ for $z = i \in X$.*

Definition 1.6. *Let $A, S : X \to X$, be two self mappings of a complex valued rectangular metric s pace (X, d). The pair (A, S) is said to satisfy property (E.A.), if there exist a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t, \text{ for some } t \in X.$$

Example 1.5. *Let $X = C$ and $d : X \times X \to C$ be any complex valued metric on X. Define $A, S : X \to X$ by $A(z) = 4z$ and $S(z) = 4i|z|$ for all $z \in X$. Consider a sequence $\{z_n\} = \{i + \frac{1}{n+1}\}_{n \geq 1}$ in X, then*

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} A\Big\{i + \frac{1}{n+1}\Big\} = \lim_{n\to\infty} 4\Big\{i + \frac{1}{n+1}\Big\} = 4i$$

and

$$\lim_{n\to\infty} S(z_n) = \lim_{n\to\infty} S\Big\{i + \frac{1}{n+1}\Big\} = \lim_{n\to\infty} 4i\sqrt{1^2 + \frac{1}{(n+1)^2}} = 4i$$

hence the pair (A, S) satisfy property (E.A.) for the sequence $\{z_n\}$ in X with $t = 4i \in X$.

Definition 1.7. [11] *Two self mappings A and S from X to X are said to satisfy the common limit in the range of S property (CLR_s property) if*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = St, \text{ for some } t \in X.$$

Example 1.6. *Let $X = C$ and $d : X \times X \to C$ be any complex valued metric on X. Define $A, S : X \to X$ by $A(z) = z + 3i$ and $S(z) = 4z$ for all $z \in X$. Consider a sequence $\{z_n\} = \{i + \frac{1}{n}\}_{n \geq 1}$ in X, then*

$$\lim_{n\to\infty} A(z_n) = \lim_{n\to\infty} A\{z_n + 3i\} = \lim_{n\to\infty} 4\Big\{i + \frac{1}{n} + 3i\Big\} = 4i$$

and

$$\lim_{n\to\infty} S(z_n) = \lim_{n\to\infty} S\Big\{i + \frac{1}{n}\Big\} = \lim_{n\to\infty} 4\Big\{i + \frac{1}{n}\Big\} = 4i$$

hence the pair (A, S) satisfy property (CLR_S) in X with $z = 0 + i \in X$.

2. Main result

In this section, we prove some common fixed point results with rational type contraction conditions. Our main result is the following.

Theorem 2.1. *Let A, B, S and T be four mappings of a complete complex valued rectangular (generalized) metric space (X, d) satisfying:*

(1) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*
(2)

$$d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Sx, Ax)d(By, Ty)}{1 + d(Sx, Ty)} + \gamma \frac{d(Sx, By)d(Ax, Ty)}{1 + d(Sx, Ty)} + \eta \frac{d(Sx, Ax)d(Sx, Ty)}{1 + d(Sx, Ty)} + \xi \frac{d(Bx, Sy)d(By, Ty)}{1 + d(Bx, Sy)}, \qquad \forall x, y \in X; \tag{2.1}$$

Where $\alpha, \beta, \gamma, \eta$ and ξ are non negative reals such that $\alpha + \beta + \gamma + \eta + \xi < 1$,

(3) *Pairs $(A, S), (B, T)$ are weakly compatible.*

If $B(X)$ is a closed subset of X. Then A, B, S and T have a unique common fixed point.

Proof. Consider a sequence $\{y_n\}$ in X, such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \;\; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}.$$

Where $\{x_n\}$ is a another sequence in X.
First of all we show that $\{y_n\}$ is a Cauchy sequences of X, for this, consider

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\
&\precsim \alpha d(Sx_{2n}, Tx_{2n+1}) + \beta \frac{d(Sx_{2n}, Ax_{2n})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&+ \gamma \frac{d(Sx_{2n}, Bx_{2n+1})d(Ax_{2n}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} + \eta \frac{d(Sx_{2n}, Ax_{2n})d(Sx_{2n}, Tx_{2n+1})}{1 + d(Sx_{2n}, Tx_{2n+1})} \\
&+ \xi \frac{d(Bx_{2n}, Sx_{2n+1})d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Bx_{2n}, Sx_{2n+1})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \gamma \frac{d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \\
&+ \eta \frac{d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \xi \frac{d(y_{2n}, y_{2n})d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n})} \\
&\precsim \alpha d(y_{2n-1}, y_{2n}) + \beta \frac{d(y_{2n-1}, y_{2n})d(y_{2n+1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} + \eta \frac{d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})}.
\end{aligned}$$

Or

$$\begin{aligned}
|d(y_{2n}, y_{2n+1})| \leq &\alpha |d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})| \left| \frac{d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \right| \\
&+ \eta |d(y_{2n-1}, y_{2n})| \left| \frac{d(y_{2n-1}, y_{2n})}{1 + d(y_{2n-1}, y_{2n})} \right|
\end{aligned}$$

$\Rightarrow |d(y_{2n}, y_{2n+1})| \leq (\alpha + \eta)|d(y_{2n-1}, y_{2n})| + \beta |d(y_{2n+1}, y_{2n})|$.
Or consequently
$d(y_{2n}, y_{2n+1}) \precsim (\alpha + \eta)d(y_{2n-1}, y_{2n}) + \beta d(y_{2n+1}, y_{2n})$
or
$d(y_{2n}, y_{2n+1}) \precsim \frac{\alpha+\eta}{1-\beta} d(y_{2n-1}, y_{2n})$ since $\frac{\alpha+\eta}{1-\beta} = k < 1$,
then $d(y_{2n}, y_{2n+1}) \precsim kd(y_{2n-1}, y_{2n})$.
Proceeding in similar way, we have

$$d(y_{2n}, y_{2n+1}) \precsim kd(y_{2n-1}, y_{2n}) \precsim k^2 d(y_{2n-2}, y_{2n-1}) \precsim \dots \precsim k^{2n} d(y_0, y_1).$$

Finally, we can conclude that $d(y_n, y_{n+1}) \precsim k^n d(y_0, y_1)$.
Now for all $m > n$, $m, n \in N$

$$\begin{aligned}
d(y_n, y_m) &\precsim d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m). \\
&\precsim k^n d(y_0, y_1) + k^{n+1} d(y_0, y_1) + \cdots + k^{m-1} d(y_0, y_1). \\
&\precsim \frac{k^n}{1-k} d(y_0, y_1). \\
Or \quad |d(y_n, y_m)| &\leq \frac{k^n}{1-k} |d(y_0, y_1)|.
\end{aligned}$$

Taking limit $n \to \infty$, we have $|d(y_n, y_m)| \to 0$, since $k < 1$.
This implies that $\{y_n\}$ is a Cauchy sequence in X.
Since X is a complete therefore there exist a point $z \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Now since $B(X)$ is a closed subset of X and so $z \in B(X)$.
Now $B(X) \subseteq S(X)$, then there exists a point $u \in X$,such that $z = Su$.

Now we show that $Au = Su = z$, by the inequality (2.1), we have

$$\begin{aligned} d(Au, z) \precsim & d(Au, Bx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, z) \\ \precsim & \alpha d(Su, Tx_{2n+1}) + \beta \frac{d(Su, Au)d(Bx_{2n+1}, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \\ & + \gamma \frac{d(Su, Bx_{2n+1})d(Au, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} + \eta \frac{d(Su, Au)d(Su, Tx_{2n+1})}{1 + d(Su, Tx_{2n+1})} \\ & + \xi \frac{d(Bu, Sx_{2n+1})d(Bx_{2n-1}, Tx_{2n+1})}{1 + d(Bu, Sx_{2n+1})} \\ & + d(Bx_{2n+1}, Tx_{2n+1}) + d(Tx_{2n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(Au, z) \precsim & \alpha d(z, z) + \beta \frac{d(z, Au)d(z, z)}{1 + d(z, z)} + \gamma \frac{d(z, z)d(Au, z)}{1 + d(z, z)} + \eta \frac{d(z, Au)d(z, z)}{1 + d(z, z)} + \\ & \xi \frac{d(Bu, z)d(z, z)}{1 + d(Bu, z)} + d(z, z) + d(z, z). \\ \Rightarrow \quad d(Au, z) = & 0 \ \ or \ \ Au = z. \end{aligned}$$

Thus $Au = Su = z$.
This implies that u is a coincidence point of (A, S).
Since $A(X) \subseteq T(X)$ and now $z \in A(X)$, then there exist a point $v \in X$ such that $z = Tv$. Now we show that $Bv = z$. By inequality (2.1) and by $Au = Su = Tv = z$, we have

$$\begin{aligned} d(z, Bv) = & d(Au, Bv) \\ \precsim & \alpha d(Su, Tv) + \beta \frac{d(Su, Au)d(Bv, Tv)}{1 + d(Su, Tv)} + \gamma \frac{d(Su, Bv)d(Au, Tv)}{1 + d(Su, Tv)} \\ & + \eta \frac{d(Su, Au)d(Su, Tv)}{1 + d(Su, Tv)} + \xi \frac{d(Bu, Sv)d(Bv, Tv)}{1 + d(Bu, Sv)} \\ \precsim & \alpha d(z, z) + \beta \frac{d(z, z)d(Bv, z)}{1 + d(z, z)} + \gamma \frac{d(z, Bv)d(z, z)}{1 + d(z, z)} \\ & + \eta \frac{d(z, z)d(z, z)}{1 + d(z, z)} + \xi \frac{d(Bu, Sv)d(Bv, z)}{1 + d(Bu, Sv)}. \end{aligned}$$

$Or \quad |d(z, Bv)| \leq \xi \left| \frac{d(Bu,Sv)}{1+d(Bu,Sv)} \right| |d(z, Bv)|$.
$\Rightarrow |d(z, Bv)| = 0$, since $\xi < 1$ and $\left| \frac{d(Bu,Sv)}{1+d(Bu,Sv)} \right| < 1$.
$\Rightarrow Bv = z \Rightarrow Bv = Tv = z$.
$\Rightarrow v$ is a coincidence point of (B, T).
Now, we have $Au = Su = Bv = Tv = z$.
Since A and S are weakly compatible mapping then $ASu = SAu \Rightarrow Az = Sz$.
Now we show that z is a fixed point of A. If $Az \neq z$ then by using inequality (2.1),

we have

$$\begin{aligned}
d(Az,z) =& d(Az,Bv)\\
&\precsim \alpha d(Sz,Tv)+\beta\frac{d(Sz,Az)d(Bv,Tv)}{1+d(Sz,Tv)}+\gamma\frac{d(Sz,Bv)d(Az,Tv)}{1+d(Sz,Tv)}\\
&\quad+\eta\frac{d(Sz,Az)d(Sz,Tv)}{1+d(Sz,Tv)}+\xi\frac{d(Bz,Sv)d(Bv,Tv)}{1+d(Bz,Sv)}\\
&\precsim \alpha d(Az,z)+\beta\frac{d(Az,Az)d(z,z)}{1+d(Az,z)}+\gamma\frac{d(Az,z)d(Az,z)}{1+d(Az,z)}\\
&\quad+\eta\frac{d(Az,Az)d(Az,z)}{1+d(Az,z)}+\xi\frac{d(Bz,Sv)d(z,z)}{1+d(Bz,Sv)}.\\
&\qquad\text{(Since } Az=Sz \text{ and } Su=Au=Tv=Bv=z.)
\end{aligned}$$

This follows

$|d(Az,z)|\le\alpha|d(Az,z)|+\gamma\left|\frac{d(Az,z)}{1+d(Az,z)}\right||d(Az,z)|.$

So that $|d(Az,z)|\le(\alpha+\gamma)|d(Az,z)|$, *since* $\alpha+\gamma<1$,
implies $d(Az,z)=0\Rightarrow Az=z$. Hence we have $Az=Sz=z$.
Now (B,T) is weakly compatible pair, so $BTv=TBv\Rightarrow Tz=Bz$.
Next we show that z is a fixed point of B. Suppose $Bz\neq z$, then inequality (2.1) yields

$$\begin{aligned}
d(z,Bz) =& d(Az,Bz)\\
&\precsim \alpha d(Sz,Tz)+\beta\frac{d(Sz,Az)d(Bz,Tz)}{1+d(Sz,Tz)}+\gamma\frac{d(Sz,Bz)d(Az,Tz)}{1+d(Sz,Tz)}\\
&\quad+\eta\frac{d(Sz,Az)d(Sz,Tz)}{1+d(Sz,Tz)}+\xi\frac{d(Bz,Sz)d(Bz,Tz)}{1+d(Bz,Sz)}.\\
&\precsim \alpha d(z,Bz)+\beta\frac{d(z,z)d(Bz,Bz)}{1+d(z,Bz)}+\gamma\frac{d(z,Bz)d(z,Bz)}{1+d(z,Bz)}\\
&\quad+\eta\frac{d(z,z)d(z,Bz)}{1+d(z,Bz)}+\xi\frac{d(Bz,z)d(Bz,Bz)}{1+d(Bz,z)}.
\end{aligned}$$

Or $\quad |d(z,Bz)|\le\alpha|d(z,Bz)|+\gamma\left|\frac{d(z,Bz)}{1+d(z,Bz)}\right||d(z,Bz)|.$
$\Rightarrow \quad |d(z,Bz)|\le(\alpha+\gamma)|d(z,Bz)|.$
This implies that $d(z,Bz)=0\Rightarrow Bz=z$. Hence $Az=Bz=Sz=Tz=z$.
Therefore z is a common fixed point of A,B,S and T.
Now we show that this fixed point is unique.
Let w be another fixed point of A,B,S and T, such that $z\neq w$. Then by inequality (2.1)

$$\begin{aligned}
d(z,w) =& d(Az,Tw)\\
&\precsim \alpha d(Sz,Tw)+\beta\frac{d(Sz,Az)d(Bw,Tw)}{1+d(Sz,Tw)}+\gamma\frac{d(Sz,Bw)d(Az,Tw)}{1+d(Sz,Tw)}\\
&\quad+\eta\frac{d(Sz,Az)d(Sz,Tw)}{1+d(Sz,Tw)}+\xi\frac{d(Bw,Sz)d(Bw,Tw)}{1+d(Bz,Sw)}.
\end{aligned}$$

Or $\quad |d(z,w)|\le\alpha|d(z,w)|+\gamma\left|\frac{d(z,w)}{1+d(z,w)}\right||d(z,w)|.$
$\Rightarrow \quad |d(z,w)|\le(\alpha+\gamma)|d(z,w)|.$
We immediately obtain that $d(z,w)=0$ and therefore $z=w$.

Thus z is a unique fixed point of A, B, S and T.
This complete the proof. $\square$

Now we furnish an illustrative example to highlight the utility of Theorem (2.1).

Example 2.1. *Let* $X = P \cup Q$, *where* $P = \{1, -i\}, Q = \{i, -1\}$. *Define* $d : X \times X \to C$ *as follows*
$d(1, -1) = d(-1, 1) = 3e^{i\theta}$;
$d(-1, i) = d(i, -1) = d(1, i) = d(i, 1) = e^{i\theta}$;
$d(1, -i) = d(-i, 1) = d(-1, -i) = d(-i, -1) = d(i, -i) = d(-i, i) = 5e^{i\theta}$;
$d(1, 1) = d(-1, -1) = d(i, i) = d(-i, -i) = 0$.
Then (X, d) *is a complex valued generalized metric space, where* $\theta \in \left[0, \frac{\pi}{4}\right)$. *Set* $A = B$ *and* $S = T$. *Define* $A, S : X \to X$ *as follows*

$$A(x) = \begin{cases} -1, & if\ x \in P, \\ i, & if\ x \in Q \end{cases} \qquad and\ S(x) = \begin{cases} -i, & if\ x \in P, \\ -1, & if\ x = -1, \\ i, & if\ x = i. \end{cases}$$

Clearly the pair (A, S) *is weakly compatible and* $A(X) \subseteq S(X)$.
Before discussing different cases, one needs to notice that

$$0 \precsim d(Ax, Ay), d(Sx, Sy), \frac{d(Sx, Ax)d(Ay, Sy)}{1 + d(Sx, Sy)}, \frac{d(Sx, Ay)d(Ax, Sy)}{1 + d(Sx, Sy)}, \frac{d(Sx, Ax)d(Sx, Sy)}{1 + d(Sx, Sy)}, \frac{d(Ax, Sy)d(Ay, Sy)}{1 + d(Ax, Sy)}$$

for all $x, y \in X$ *and for* $\theta \in \left[0, \frac{\pi}{4}\right)$.
It is sufficient to show that $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$, *with* $\alpha, \beta, \gamma, \eta, \xi \geq 0$ *and* $\alpha + \beta + \gamma + \eta + \xi < 1$.
Following cases for $x, y \in X$ *are discussed with* $\alpha = \frac{1}{3}, \beta = \frac{1}{4}, \gamma = \frac{1}{5}, \eta = \frac{1}{10}$ *and* $\xi = \frac{1}{15}$. *Notice that* $\alpha + \beta + \gamma + \eta + \xi < 1$.
Case I: *When* $x \in P$ *and* $y \in P$, *we have*
$d(Ax, Ay) = d(-1, -1) = 0$ *and* $d(Sx, Sy) = d(-i, -i) = 0$
then we find that $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $0 \precsim 0$, *where* $\theta \in [0, \pi/4)$.
Case II: *When* $x \in P$ *and* $y = -1$, *we get*
$d(Ax, Ay) = d(-1, i) = e^{i\theta}$ *and* $d(Sx, Sy) = d(-i, -1) = 5e^{i\theta}$
then we find that $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $e^{i\theta} \precsim \frac{5}{3}e^{i\theta}$, *where* $\theta \in [0, \pi/4)$.
Case III: *When* $x \in P$ *and* $y = -i$, *we find that*
$d(Ax, Ay) = d(-1, i) = e^{i\theta}$ *and* $d(Sx, Sy) = d(-i, i) = 5e^{i\theta}$
$\Rightarrow d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $e^{i\theta} \precsim \frac{5}{3}e^{i\theta}$, *where* $\theta \in [0, \pi/4)$.
Case IV: *When* $x = i$ *and* $y \in P$, *then*
$d(Ax, Ay) = d(i, -1) = e^{i\theta}$ *and* $d(Sx, Sy) = d(i, -i) = 5e^{i\theta}$
we get $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $e^{i\theta} \precsim \frac{5}{3}e^{i\theta}$, *where* $\theta \in [0, \pi/4)$.
Case V: *When* $x = i$ *and* $y = -1$, *we get*
$d(Ax, Ay) = d(i, i) = 0$ *and* $d(Sx, Sy) = d(i, -1) = e^{i\theta}$
we find that $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $0 \precsim \frac{1}{3}e^{i\theta}$, *where* $\theta \in [0, \pi/4)$.
Case VI: *When* $x = i$ *and* $y = i$, *we get*
$d(Ax, Ay) = d(i, i) = 0$ *and* $d(Sx, Sy) = d(i, i) = 0$
then $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $0 \precsim 0$, *where* $\theta \in [0, \pi/4)$.
Case VII: *When* $x = -1$ *and* $y \in P$, *we get*
$d(Ax, Ay) = d(i, -1) = e^{i\theta}$ *and* $d(Sx, Sy) = d(-1, -i) = 5e^{i\theta}$
we find that $d(Ax, Ay) \precsim \alpha d(Sx, Sy)$ *as* $e^{i\theta} \precsim \frac{5}{3}e^{i\theta}$, *where* $\theta \in [0, \pi/4)$.

Case VIII: *When $x=-1$ and $y=-1$, we get*
$d(Ax,Ay)=d(i,i)=0$ *and* $d(Sx,Sy)=d(-1,-1)=0$
then $d(Ax,Ay)\precsim \alpha d(Sx,Sy)$ *as* $0\precsim 0$, *where* $\theta\in[0,\pi/4)$.
Case IX: *When $x=-1$ and $y=i$, we get*
$d(Ax,Ay)=d(i,-1)=e^{i\theta}$ *and* $d(Sx,Sy)=d(i,-i)=5e^{i\theta}$
then we find that $d(Ax,Ay)\precsim \alpha d(Sx,Sy)$ *as* $e^{i\theta}\precsim \frac{5}{3}e^{i\theta}$ *where* $\theta\in[0,\pi/4)$.
Thus mapping A and S satisfy inequality(2.1) of Theorem(2.1).
Therefore all the conditions of Theorem(2.1) are satisfied. Here, $i\in X$ is a unique common fixed point of pair (A,S).

If we set $S=T=I$ in Theorem(2.1), then we get the following corollary.

Corollary 2.2. *Let A,B be two self mappings of a complex valued rectangular (generalized) metric space (X,d) satisfying :*

$$d(Ax,By)\precsim \alpha d(x,y)+\beta\frac{d(x,Ax)d(y,By)}{1+d(x,y)}+\gamma\frac{d(x,By)d(y,Ay)}{1+d(x,y)}+\eta\frac{d(x,Ax)d(x,y)}{1+d(x,y)}+\xi\frac{d(y,Bx)d(y,By)}{1+d(y,Bx)},\qquad \forall x,y\in X.$$

Where α,β,γ,η and ξ are non negative reals such that $\alpha+\beta+\gamma+\eta+\xi<1$. Then A and B have a unique common fixed point.

Again if we set $S=T$ in Theorem(2.1), then following corollary is obtained.

Corollary 2.3. *Let A,B and S be three self mappings of a complex valued rectangular (generalized) metric space (X,d) satisfying :*
(i) $A(X)\subseteq S(X)$ *and* $B(X)\subseteq S(X)$;
(ii)

$$d(Ax,By)\precsim \alpha d(Sx,Sy)+\beta\frac{d(Sx,Ax)d(By,Sy)}{1+d(Sx,Sy)}+\gamma\frac{d(Sx,By)d(Ax,Sy)}{1+d(Sx,Sy)}+\eta\frac{d(Sx,Ax)d(Sx,Sy)}{1+d(Sx,Sy)}+\xi\frac{d(Bx,Sy)d(By,By)}{1+d(Bx,Sy)},\qquad \forall x,y\in X.$$

Where α,β,γ,η and ξ are non negative reals such that $\alpha+\beta+\gamma+\eta+\xi<1$. If pairs (A,S) and (B,T) are weakly compatible and $B(X)$ is a closed subset of X. Then A and B have a unique common fixed point.

3. Application of property (E.A.) in complex valued rectangular (generalized) metric space:

In this section, application of property (E.A.) is invoked to obtain fixed point theorem.

Theorem 3.1. *Let A,B,S and T be four self mappings of a complex valued rectangular (generalized) metric space (X,d) satisfying the conditions (1),(2),(3) of Theorem(2.1) and also one of the pair (A,S) and (B,T) satisfies property (E.A.). If one of the mappings $S(X)$ and $T(X)$ is closed subset of X. Then the mappings A,B,S and T have a unique common fixed point in X.*

Proof. Firstly, suppose the pair (B,T) satisfies E.A. property then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}Bx_n=\lim_{n\to\infty}Tx_n=t,\text{ for some } t\in X.$$

Since $B(X) \subseteq S(X)$ then we find a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$.
Hence $\lim_{n\to\infty} Sy_n = t$.
We claim that $\lim_{n\to\infty} Ay_n = t$. Utilizing inequality(2.1) with $x = y_n$ and $y = x_n$, we have

$$\begin{aligned} d(Ay_n, Bx_n) \precsim& \alpha d(Sy_n, Tx_n) + \beta \frac{d(Sy_n, Ay_n)d(Bx_n, Tx_n)}{1 + d(Sy_n, Tx_n)} \\ &+ \gamma \frac{d(Sy_n, Bx_n)d(Ay_n, Tx_n)}{1 + d(Sy_n, Tx_n)} + \eta \frac{d(Sy_n, Ay_n)d(Sy_n, Tx_n)}{1 + d(Sy_n, Tx_n)} \\ &+ \xi \frac{d(By_n, Sx_n)d(Bx_n, Tx_n)}{1 + d(By_n, Sx_n)}. \end{aligned}$$

Which on letting $n \to \infty$, reduces to
$\lim_{n\to\infty} |d(Ay_n, Bx_n)| = 0 \Rightarrow \lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = t$.
Now suppose $S(X)$ is closed subset of X, then for some $u \in X$, we have $Su = t$.
Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = t = Su. \tag{3.1}$$

Now we show that $Au = Su$.
Putting $x = u$ and $y = x_n$ in inequality(2.1), we get

$$\begin{aligned} d(Au, Bx_n) \precsim& \alpha d(Su, Tx_n) + \beta \frac{d(Su, Au)d(Bx_n, Tx_n)}{1 + d(Su, Tx_n)} + \gamma \frac{d(Su, Bx_n)d(Au, Tx_n)}{1 + d(Su, Tx_n)} + \\ &\eta \frac{d(Su, Au)d(Su, Tx_n)}{1 + d(Su, Tx_n)} + \xi \frac{d(Bu, Sx_n)d(Bx_n, Tx_n)}{1 + d(Bu, Sx_n)}. \end{aligned}$$

Letting $n \to \infty$ and using equation(3.1), we have
$|d(Au, t)| = 0$ implies $Au = t = Su$.
Which yield that u is a coincidence point of (A, S).
Since (A, S) is weakly compatible pair so we have $ASu = SAu \Rightarrow At = St$.
Since $A(X) \subseteq T(X)$ then there exist $v \in X$, such that $Au = Tv$.
Thus

$$Su = Au = Tv = t. \tag{3.2}$$

Now we show that $Bv = t$, for this, putting $x = u$ and $y = v$ in inequality (2.1), we have

$$\begin{aligned} d(Au, Bv) \precsim& \alpha d(Su, Tv) + \beta \frac{d(Su, Au)d(Bv, Tv)}{1 + d(Su, Tv)} + \gamma \frac{d(Su, Bv)d(Au, Tv)}{1 + d(Su, Tv)} + \\ &\eta \frac{d(Su, Au)d(Su, Tv)}{1 + d(Su, Tv)} + \xi \frac{d(Bu, Sv)d(Bv, Tv)}{1 + d(Bu, Sv)}. \end{aligned}$$

Now using equation (3.2), we have $|d(t, Bv)| \leq \xi \left| \frac{d(Bu,Sv)}{1+d(Bu,Sv)} \right| |d(Bv, t)|$,
since $\xi < 1$ and $\left| \frac{d(Bu,Sv)}{1+d(Bu,Sv)} \right| < 1$. Then we must have $|d(t, Bv)| = 0$.
or $Bv = t$. Hence $Bv = Tv = t$.
This shows that v is a coincidence point of (B, T).
By weak compatibility of pair (B, T), we have $BTv = TBv \Rightarrow Bt = Tt$.
Therefore t is a coincidence point of B and T.

Now we show that t is a common fixed point of mappings B and T, for this putting $x = y$, and $y = t$ in inequality (2.1), we have

$$\begin{aligned} d(t, Bt) =& d(Au, Bt) \\ &\precsim \alpha d(Su, Tt) + \beta \frac{d(Su, Au)d(Bt, Tt)}{1 + d(Su, Tt)} + \gamma \frac{d(Su, Bt)d(Au, Tt)}{1 + d(Su, Tt)} \\ &+ \eta \frac{d(Su, Au)d(Su, Tt)}{1 + d(Su, Tt)} + \xi \frac{d(Bu, St)d(Bt, Tt)}{1 + d(Bu, St)} \end{aligned}$$

or $\quad |d(t, Bt)| \leq \alpha |d(t, Bt)| + \gamma \left| \frac{d(y,Bt)}{1+d(t,Bt)} \right| |d(t, Bt)|$.
$\Rightarrow \quad |d(t, Bt)| \leq (\alpha + \gamma)|d(t, Bt)|$, since $\alpha + \gamma < 1$.
So we have $d(t, Bt) = 0 \Rightarrow Bt = t$ and so $Bt = t = Tt$.
Now to show that t is also a common fixed point of mappings A and S.
From inequality (2.1), we have

$$\begin{aligned} d(At, t) =& d(At, Bt) \\ &\precsim \alpha d(St, Tt) + \beta \frac{d(St, At)d(Bt, Tt)}{1 + d(St, Tt)} + \gamma \frac{d(St, Bt)d(At, Tt)}{1 + d(St, Tt)} + \\ &\eta \frac{d(St, At)d(St, Tt)}{1 + d(St, Tt)} + \xi \frac{d(Bt, St)d(Bt, Tt)}{1 + d(Bt, St)}. \end{aligned}$$

Or $\quad |d(At, t)| \leq \alpha |d(At, t)| + \gamma \left| \frac{d(At,t)}{1+d(At,t)} \right| |d(At, t)|$.
$\Rightarrow \quad |d(At, t)| \leq (\alpha + \gamma)|d(At, t)|$, since $\alpha + \gamma < 1$.
So we have $d(At, t) = 0 \Rightarrow At = t$ and so $At = t = Tt$.
Hence $At = Bt = St = Tt = t$, *i.e.* t is a common fixed point of A, B, S, and T.
We arrive at same conclusion if we assume that $T(X)$ is a closed subset of X.
Similar result can be obtained when we take E.A. property for another pair (A, S).
Arguing the same as in Theorem(2.1), uniqueness of fixed point follows immediately □

4. CLR property and its application in complex valued rectangular (generalized) metric space:

In this section, we establish the fixed point theorem using CLR property in complex valued rectangular metric spaces.

Theorem 4.1. *Let A, B, S and T be four self mappings of a complex valued rectangular (generalized) metric space (X, d) satisfying the conditions (1),(2),(3) of Theorem(2.1). If the pair (A, S) satisfies CLR_A property or the pair (B, T) satisfies CLR_B property. Then the mappings A, B, S and T have a unique common fixed point in X.*

Proof. First we suppose that pair (B, T) satisfies the (CLR_B) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Bx, for \ \ some \quad x \in X. \tag{4.1}$$

Since $B(X) \subseteq S(X)$, then we have $Bx = Su,$ for some $u \in X$.
Now we show that $Au = Su$, so putting $x = u, y = x_n$ in inequality (2.1)

$$d(Au, Bx_n) \precsim \alpha d(Su, Tx_n) + \beta \frac{d(Su, Au)d(Bx_n, Tx_n)}{1 + d(Su, Tx_n)} + \gamma \frac{d(Su, Bx_n)d(Au, Tx_n)}{1 + d(Su, Tx_n)} + \eta \frac{d(Su, Au)d(Su, Tx_n)}{1 + d(Su, Tx_n)} + \xi \frac{d(Bu, Sx_n)d(Bx_n, Tx_n)}{1 + d(Bu, Sx_n)}.$$

Letting $n \to \infty$, and using equation (4.1), we have
$|d(Au, Bx)| = 0 \quad \Rightarrow Au = Bx$.
Thus $Au = Su = Bx = t$. (say) Or $Au = Su = t$.
Which amounts to say that u is a coincidence point of A and S.

Now remaining part of the theorem can be obtained on similar lines as in previous Theorem(3.1). □

We observe that all the theorems and corollaries discussed in this note can also be proved on the similar lines in the **setting of complex valued metric spaces** due to Azam et al.[3].
In the context of complex valued metric space, our Corollary (2.2), generalizes the results of Azam et al.[3] and of Rouzkard et al. [9].

Remark. *If we put $\gamma = \eta = \xi = 0$ in Corollary(2.2) in the context of complex valued metric spaces, we obtain Theorem 4 of Azam et al.*[3] *as follows:*
Let A and B be two self mappings of a complex valued metric space (X, d) satisfying

$$d(Ax, By) \precsim \alpha d(x, y) + \beta \frac{d(x, Ax)d(y, By)}{1 + d(x, y)}, \qquad \forall x, y \in X.$$

Where α, β are non negative reals such that $\alpha + \beta < 1$.
Then A and B have a unique common fixed point.

Remark. *If we put $\eta = \xi = 0$ in Corollary (2.2) in the setting of complex valued metric spaces, we get Theorem (2.1) of Rouzkard et al.* [9] *stated as:*
Let A and B be two self mappings of a complex valued metric space (X, d) satisfying

$$d(Ax, By) \precsim \alpha d(x, y) + \beta \frac{d(x, Ax)d(y, By)}{1 + d(x, y)} + \gamma \frac{d(x, By)d(y, Ax)}{1 + d(x, y)}, \qquad \forall x, y \in X.$$

Where α, β, γ are non negative reals such that $\alpha + \beta + \gamma < 1$.
Then A and B have a unique common fixed point.

Conclusion: In this paper, in Section (II) we used classical method to prove the common fixed point of four mappings through Cauchy sequence and completeness of space with rational type of contraction condition in complex valued rectangular metric space. In Section (III) we applied E.A. property for the existence of common fixed point which relaxed completeness of space but it requires closedness of sub space. In Section (IV) we used CLR property to claim the existence of common fixed point, this property never requires any condition of closedness of sub spaces, continuity of one or more mappings.

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