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RIESZ TYPE INTEGRATED AND DIFFERENTIATED SEQUENCE SPACES (COMMUNICATED BY FEYZI BASAR)

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Abstract. Let $\int bv$ and $d(bv)$ denote the spaces of integrated and differentiated sequence spaces, which introduced by [4] and $\int bv(R)$ and $d(bv(R))$ also be matrix domain of the Riesz mean in the sequence spaces $\int bv$ and $d(bv)$. In this paper, some important properties of these spaces are studied and dual spaces of the spaces $\int bv(R)$ and $d(bv(R))$ are determined. Finally, the classes $(\int bv(R) : Y)$, $(d(bv(R)) : Y)$, $(Y : \int bv(R))$ and $(Y : (d(bv(R)))$ of infinite matrices are characterized, where Y is any given sequence space.

1. Introduction

The theory of sequence spaces is the fundamental of summability. Summability is wide field of mathematics, mainly in analysis and functional analysis, and has many applications, for instance in numerical analysis to speed up the rate of convergence, in operator theory, the theory of orthogonal series and approximation theory.

The classical summability theory deals with the generalization of the convergence of sequences or series of real or complex numbers. The idea is to assign a limit of some sort to divergent sequences or series by considering a transform of a sequence or series rather than original sequence or series.

One can ask why we employ the special transformations represented by infinite matrices instead of general linear operators? The answer to this question is, in many cases, the most general linear operators between two sequence spaces is given by an infinite matrix. So the theory of matrix transformations has always been of great inretest in the study of sequence spaces. The study of the general theory of matrix transformations was motivated by special results in summability theory.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors.

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In [7], it can be seen the qualified studies related to the matrix domains. Although in most cases the new sequence space X_A generated by the summability matrix A from a sequence space X is the expansion or the contraction of the original space X, it may be observed in some cases that those spaces are overlap.

Now, we introduce the necessary information and definitions which will be used throughout the paper.

The set of all sequences denotes with $\omega := \mathbb{C}^{\mathbb{N}} := \{x = (x_k) : x : \mathbb{N} \to \mathbb{C}, k \to x_k := x(k)\}$ where $\mathbb{C}$ denotes the complex field and $\mathbb{N}$ is the set of positive integers. Each linear subspace of ω (with the induced addition and scalar multiplication) is called a *sequence space*. The following subsets of ω are obviously sequence spaces:

$$\ell_\infty = \{x = (x_k) \in \omega : \sup_k |x_k| < \infty\} \qquad c = \{x = (x_k) \in \omega : \lim_k x_k \ \text{ exists }\}$$

$$c_0 = \{x = (x_k) \in \omega : \lim_k x_k = 0\} \qquad bs = \left\{x = (x_k) \in \omega : \sup_n \left|\sum_{k=1}^{n} x_k\right| < \infty\right\}$$

$$cs = \left\{x = (x_k) \in \omega : \left(\sum_{k=1}^{n} x_k\right) \in c\right\} \qquad \ell_p = \left\{x = (x_k) \in \omega : \sum_k |x_k|^p < \infty, \quad 1 \le p < \infty\right\}.$$

These sequence spaces are Banach space with the norms; $\|x\|_{\ell_\infty} = \sup_k |x_k|$, $\|x\|_{bs} = \|x\|_{cs} = \sup_n |\sum_{k=1}^{n} x_k|$ and $\|x\|_{\ell_p} = (\sum_k |x_k|^p)^{1/p}$ as usual, respectively. Let X is one of the above mentioned sequence spaces. The concept of integrated and differentiated sequence spaces was employed as

$$\int X = \{x = (x_k) \in \omega : (kx_k) \in X\} \qquad \text{and} \qquad d(X) = \left\{x = (x_k) \in \omega : (k^{-1}x_k) \in X\right\},$$

in [4].

By $\mathcal{F}$, we will denote the collection of all finite subsets on $\mathbb{N}$. For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞. Also we use the convention that any term with negative subscript is equal to zero.

A coordinate space (or $K-$*space*) is a vector space of numerical sequences, where addition and scalar multiplication are defined pointwise. That is, a sequence space X with a linear topology is called a K-space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A $K-$space is called an $FK-$*space* provided X is a complete linear metric space. An $FK-$*space* whose topology is normable is called a $BK-$ *space*.

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is unique sequence of scalars (α_n) such that

$$\lim_{n\to\infty} \|x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n)\| = 0$$

then (b_n) is called *Schauder basis* for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. An $FK-$*space* X is said to have AK property, if $\phi \subset X$ and $\{e^k\}$ is a basis for X, where e^k is a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$

and $\phi = span\{e^k\}$, the set of all finitely non-zero sequences.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} and $x = (x_k) \in \omega$, where $k, n \in \mathbb{N}$. Then the sequence Ax is called as the $A-$transform of x defined by the usual matrix product. Hence, we transform the sequence x into the sequence $Ax = \{(Ax)_n\}$ where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

for each $n \in \mathbb{N}$, provided the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$.

Let X and Y be two sequence spaces. If Ax exists and is in Y for every sequence $x = (x_k) \in X$, then we say that A defines a matrix mapping from X into Y, and we denote it by writing $A : X \to Y$ if and only if the series on the right hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y$ for all $x \in X$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x.

Let X be a sequence space and A be an infinite matrix. The sequence space

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{1.2}$$

is called the domain of A in X which is a sequence space.

Let (q_k) be a sequence of positive numbers and $Q_n = \sum_{k=0}^{n} q_k$ for all $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean [1] is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n} & , \quad (0 \le k \le n) \\ 0 & , \quad (k > n) \end{cases}$$

It is known that the Riesz mean is regular if and only if $Q_n \to \infty$ as $n \to \infty$. Also, for every Riesz mean $\sum_{k=1}^{\infty} r^q_{nk} = \sum_{k=1}^{\infty} |r^q_{nk}| = 1$. This means that every Riesz mean is a limitation method, [8, p.10].

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Altay and Baar [1] defined the Riesz sequence spaces as

$$r^q(p) = \left\{ x = (x_k) \in w : \sum_k \left| \frac{1}{Q_k} \sum_{j=0}^{k} q_j x_j \right|^{p_k} < \infty \right\}, \quad (0 < p_k \le H < \infty)$$

In [3], Başar and Altay have studied the sequence space bv_p which consists of all sequences whose Δ-transforms are in ℓ_p; i.e.,

$$bv_p = \left\{ x = (x_k) \in w : \sum_k |x_k - x_{k-1}|^p < \infty \right\}, \quad (1 \le p < \infty).$$

where Δ denotes the matrix $\Delta = (\delta_{nk})$

$$\delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad (n-1 \le k \le n) \\ 0 & , \quad (0 \le k < n-1 \text{ or } k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$.

If we state the matrix domain of the space bv_p as the notation (1.2), then we can write $bv_p = (\ell_p)_\Delta$.

We define the matrices $C = (c_{nk})$ and $D = (d_{nk})$ by

$$c_{nk} = \begin{cases} k(q_k - q_{k+1})/Q_n & , \quad (k < n) \\ nq_n/Q_n & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \tag{1.3}$$

$$d_{nk} = \begin{cases} (q_k - q_{k+1})/kQ_n & , \quad (k < n) \\ q_n/(nQ_n) & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \tag{1.4}$$

for all $k, n \in \mathbb{N}$.

Now, we can give the matrices $C^{-1} = (e_{nk})$ and $D^{-1} = (f_{nk})$ which are inverse of above matrices, by

$$e_{nk} := \begin{cases} \frac{1}{n}Q_k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) & , \quad (k < n) \\ Q_n/(nq_n) & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \quad \text{and} \quad f_{nk} := \begin{cases} nQ_k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) & , \quad (k < n) \\ nQ_n/q_n & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Now, we give the following lemmas which are needed in the text. Especially, in [2], Başar and Altay developed very useful tools for duals and matrix transformations of sequence spaces as Lemma 1.2 and Lemma 1.3.

Lemma 1.1. *Matrix transformations between $BK-$spaces are continuous.*

Lemma 1.2. [3, Lemma 5.3] *Let X, Y be any two sequence spaces, A be an infinite matrix and U a triangle matrix matrix.Then, $A \in (X : Y_U)$ if and only if $UA \in (X : Y)$.*

Lemma 1.3. [2, Theorem 3.1] *$B^U = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} a_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then,

$$\lambda_U^\beta = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}$$

and

$$\lambda_U^\gamma = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.$$

Let $\int bv$ and $d(\ell_1)$ denote the integrated and differentiated spaces of bv and ℓ_1, respectively. These spaces are given in [4]. The main purpose of this paper is to define the new integrated and differentiated sequence spaces using the Riesz mean and is to study their some properties. In section 3, we compute the alpha-, beta- and gamma duals of these spaces. Afterward, we characterize the classes of matrix transformations from these spaces to the well-known sequence spaces such as ℓ_∞,

c, c_0, bs, cs and c_0s.

2. Riesz Type New Sequence Spaces

In this section, we will give new spaces defined by a weighted mean.

Goes and Goes [4] firstly mentioned the integrated and differentiated sequence spaces. In Section 2 of [4], it was given some definitions which also including the integarted and differentiated sequence spaces. In Section 3 of the same paper, it was defined the Hahn sequence space by $h = \{x = (x_k) \in w : \sum_k k|x_k - x_{k+1}| < \infty$ and $\lim_{k\to\infty} x_k = 0\}$. Hahn [5] was proved that $h \subset \ell_1 \cap \int c_0$, where $\int c_0$ denotes the integrated sequence space. In this section, the functional analytic properties of the space $h = \ell_1 \cap \int bv$ and $dh = bv_0 \cap d\ell_1$ are investigated. In Theorem 3.2, Goes and Goes proved that the Hahn space is in the intersection of the spaces ℓ_1 and $\int bv$. Also, Goes and Goes defined the differentiated spaces dh depending in Theorem 3.2 by $dh = bv_0 \cap d\ell_1$. Therefore, in [4], it was shown that the integrated and differentiated sequence spaces are associated with each other.

In [6], new integrated and differentiated sequence spaces and matrices related to these spaces are constructed and some properties of the integrated and differentiated sequence spaces which are both new spaces and mentioned in [4], were discussed. The space $\int bv$ was defined in [4]. The new spaces $\int \ell_1$, $d(\ell_1)$ and $d(bv)$ were defined which is mentioned paper. In Section 2 of [6], the properties Banach spaces, $BK-$spaces, monotone norms, Schauder base, separability and, $AK-$property, $AB-$property and, isomorphism between new spaces and original space, were investigated. Besides this, dual spaces are computed and matrix classes are characterized by Kirişci[6].

Following Hahn [5], Goes and Goes [4], Altay and Başar [1] and Kirişci [6], we will define the new integrated and differentiated sequence spaces using the Riesz mean.

The Riesz type integrated spaces defined by

$$\int bv(R) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{n} (q_k \Delta(kx_k))/Q_n < \infty \right\}$$

and the Riesz type differentiated spaces defined by

$$d(bv(R)) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{n} (q_k \Delta(k^{-1}x_k))/Q_n < \infty \right\}$$

where $\Delta(kx_k) = kx_k - (k-1)x_{k-1}$ and $\Delta(k^{-1}x_k) = k^{-1}x_k - (k-1)^{-1}x_{k-1}$.

Consider the notation (1.2) and the matrices (1.3), (1.4). From here, we can re-define the spaces $\int bv(R)$ and $d(bv(R))$ by

$$(\ell_1)_C = \int bv(R) \tag{2.1}$$

and

$$(\ell_1)_D = d(bv(R)). \tag{2.2}$$

Let $x = (x_k) \in \int bv(R)$. The $C-$transform of a sequence $x = (x_k)$ is defined by

$$y_n = \sum_{k=1}^{n-1} [k(q_k - q_{k+1})/Q_n]\, x_k + [(nq_n)/Q_n]x_n \tag{2.3}$$

where C is defined by (1.3). Let $x = (x_k) \in d(bv(R))$. The $D-$transform of a sequence $x = (x_k)$ is defined by

$$y_n = \sum_{k=1}^{n-1} \left[k^{-1}(q_k - q_{k+1})/Q_n\right] x_k + [q_n/(nQ_n)]/x_n \tag{2.4}$$

where D is defined by (1.4).

Theorem 2.1. *The following statements hold:*

(i) *The space $\int bv(R)$ is a $BK-$space with the norm $\|x\|_{\int bv(R)} = \|Cx\|_{\ell_1}$.*
(ii) *The space $d(bv(R))$ is a $BK-$space with the norm $\|x\|_{d(bv(R))} = \|Dx\|_{\ell_1}$.*

Proof. Since $\int bv(R) = [\ell_1]_C$ and $d(bv(R)) = [\ell_1]_D$ holds, ℓ_1 is a $BK-$space with the norm $\|.\|_{\ell_1}$ and C and D are triangle matrices, then Theorem 4.3.2 of Wilansky[9] gives the fact that the spaces $\int bv(R)$ and $d(bv(R))$ are $BK-$spaces. □

Theorem 2.2. *The spaces $\int bv(R)$ and $d(bv(R))$ are norm isomorphic to ℓ_1.*

Proof. We consider the spaces $\int bv(R)$ and ℓ_1. To prove the theorem, we should show the existence of a linear bijection between these spaces.

Now, with the notation (2.3), we define the transformation T from $\int bv(R)$ to ℓ_1 by $x \mapsto y = Tx$. It is clear that T is linear and also $x = \theta$ whenever $Tx = \theta$. Therefore, T is injective.

Let us take $y = (y_k) \in \ell_1$ and consider the sequence $x = (x_k)$ using the inverse C^{-1}, defined by

$$x_k = \sum_{j=1}^{k-1} \frac{1}{k} Q_j \left(\frac{1}{q_j} - \frac{1}{q_{j+1}}\right) y_j + \frac{Q_k y_k}{k.q_k}.$$

for all $k \in \mathbb{N}$. Then, we have

$$\|x\|_{\int bv(R)} = \sum_k \left| \sum_{j=1}^{k-1} k \frac{1}{Q_k}(q_j - q_{j+1})x_j + \frac{1}{Q_k} k q_k x_k \right| = \sum_k |y_k| = \|y\|_{\ell_1} < \infty.$$

for all $k \in \mathbb{N}$, which leads us to the fact that $x \in \int bv(R)$. Consequently, we see from here that T is surjective and norm preserving. Hence T is a linear bijection, which therefore says that the spaces $\int bv(R)$ and ℓ_1 are norm isomorphic.

Similarly, using the relation (2.4), we can define the transformation S from $d(bv(R))$ to ℓ_1 by $x \mapsto y = Sx$. Therefore, by using the inverse D^{-1} we obtain the sequence $x = (x_k)$ as follows

$$x_k = \sum_{j=1}^{k-1} kQ_j \left(\frac{1}{q_j} - \frac{1}{q_{j+1}} \right) y_j + k\frac{Q_k y_k}{q_k}$$

while $y \in \ell_1$, then we obtain the space $d(bv(R))$ is norm isomorphic to ℓ_1 with the norm $\|x\|_{d(bv(R))}$. $\square$

Theorem 2.3. *The space $d(bv(R))$ has $AK-$property.*

Proof. Let $x = (x_k) \in d(bv(R))$ and $x^{[n]} = \{x_1, x_2, \cdots, x_n, 0, 0, \cdots\}$. Hence,

$x - x^{[n]} = \{0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots\} \Rightarrow \|x - x^{[n]}\|_{d(bv(R))} = \|(0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots)\|$

and since $x \in d(bv(R))$,

$$\|x - x^{[n]}\|_{d(bv(R))} = \sum_{k \geq n+1} \left| \frac{1}{k} \frac{1}{Q_n}(q_k - q_{k+1})x_k + \frac{1}{n}\frac{q_n}{Q_n} x_n \right| \to 0 \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} \|x - x^{[n]}\|_{d(bv(R))} = 0 \Rightarrow x^{[n]} \to x \quad \text{as} \quad n \to \infty \quad \text{in} \quad d(bv(R)).$$

Then the space $d(bv(R))$ has $AK-$property. $\square$

Theorem 2.4. *Define a sequence $s^{(k)(q)} = \{s_n^{(k)}(q)\}_{n\in\mathbb{N}}$ of elements of the space $\int bv(R)$ for every fixed $k \in \mathbb{N}$ by*

$$s_n^{(k)}(q) = \begin{cases} \frac{1}{n} Q_k \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) & , \quad (1 < k < n) \\ \frac{Q_n}{nq_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{s^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $\int bv(R)$ and any $x \in \int bv(R)$ has a unique representation of the form

$$x = \sum_k (Cx)_k(q)s^{(k)}(q). \tag{2.5}$$

Proof. Let $e^{(k)}$ be a sequence whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}$. We know that

$$Cs^{(k)}(q) = e^{(k)} \in \ell_1 \tag{2.6}$$

for all $k \in \mathbb{N}$. Then, we have $\{s^{(k)}(q)\} \subset \int bv(R)$.

We take $x \in \int bv(R)$. Then, we put,

$$x^{[m]} = \sum_{k=1}^{m} (Cx)_k(q)s^{(k)}(q), \tag{2.7}$$

for every positive integer m. Then, we have

$$Cx^{[m]} = \sum_{k=1}^{m} (Cx)_k(q)Cs^{(k)}(q) = \sum_{k=1}^{m} (Cx)_k(q)e^{(k)}$$

and

$$\left(C(x - x^{[m]}) \right)_i = \begin{cases} 0 & , \quad (1 \leq i < m) \\ (Cx)_i & , \quad (i > m) \end{cases}$$

by applying C to (2.7) with (2.6), for $i, m \in \mathbb{N}$. For $\varepsilon > 0$, there exists an integer m_0 such that

$$\left[\sum_{i=m}^{\infty} |(Cx)_i|\right] < \varepsilon/2$$

for all $m \geq m_0$. Hence,

$$\|x - x^{[m]}\|_{\int bv(R)} =$$

for all $m \geq m_0$. Therefore, $x \in \int bv(R)$ is represented as in (2.5), as we desired. □

Theorem 2.5. *Define a sequence $t^{(k)}(q) = \{t_n^{(k)}(q)\}_{n\in\mathbb{N}}$ of elements of the space $d(bv(R))$ for every fixed $k \in \mathbb{N}$ by*

$$t_n^{(k)}(q) = \begin{cases} nQ_k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) & , \quad (1 < k < n) \\ n\frac{Q_n}{q_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases}$$

Therefore, the sequence $\{t^{(k)}(q)\}_{k\in\mathbb{N}}$ is a basis for the space $d(bv(R))$ and any $x \in d(bv(R))$ has a unique representation of the form

$$x = \sum_k (Dx)_k(q)t^{(k)}(q).$$

Remark. *It is well known that every Banach space X with a Schauder basis is separable*

From Theorem 2.4, Theorem 2.5 and Remark 2, we can give following corollary:

Corollary 2.6. *The spaces $\int bv(R)$ and $d(bv(R))$ are separable.*

3. Dual Spaces of the Spaces $\int bv(R)$ and $d(bv(R))$

In this section, we state and prove the theorems determining the α-, β- and γ-duals of the sequence spaces $\int bv(R)$ and $d(bv(R))$.

The set $S(\lambda, \mu)$ defined by

$$S(\lambda, \mu) = \{z = (z_k) \in \omega : xz = (x_k z_k) \in \mu \quad \text{for all} \quad x = (x_k) \in \lambda\} \tag{3.1}$$

is called the *multiplier space* of the sequence spaces λ and μ. One can easily observe for a sequence space υ with $\lambda \supset \upsilon \supset \mu$ that the inclusions

$$S(\lambda, \mu) \subset S(\upsilon, \mu) \quad \text{and} \quad S(\lambda, \mu) \subset S(\lambda, \upsilon)$$

hold. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda, \ell_1), \quad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as *Köthe-Toeplitz dual*, *generalized Köthe-Toeplitz dual* and *Garling dual* of a sequence space, respectively.

To give the alpha-, beta- and gamma- duals of the spaces $\int bv(R)$ and $d(bv(R))$, we need the following lemmas:

Lemma 3.1. *Let $A = (a_{nk})$ be an infinite matrix. $A \in (\ell_1 : \ell_\infty)$ if and only if*

$$\sup_{k,n\in\mathbb{N}} |a_{nk}| < \infty. \tag{3.2}$$

Lemma 3.2. *Let $A = (a_{nk})$ be an infinite matrix. $A \in (\ell_1 : c)$ if and only if (3.2) holds, and there is $\alpha_k \in \mathbb{C}$ such that*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \textit{for each} \quad k \in \mathbb{N}. \tag{3.3}$$

Lemma 3.3. *Let $A = (a_{nk})$ be an infinite matrix. $A \in (\ell_1 : \ell_1)$ if and only if*

$$\sup_{k\in\mathbb{N}} \sum_n |a_{nk}| < \infty. \tag{3.4}$$

Theorem 3.4. *We define the matrix $M = (m_{nk})$ as*

$$m_{nk} = \begin{cases} \frac{1}{n} Q_k \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) a_n & , \quad (1 \le k < n) \\ \frac{Q_n a_n}{n q_n} & , \quad (n = k) \\ 0 & , \quad (k > n) \end{cases} \tag{3.5}$$

for all $k, n \in \mathbb{N}$, where $a = (a_k) \in \omega$. The $\alpha-$dual of the space $\int bv(R)$ is the set

$$d_1 = \left\{ a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} m_{nk} \right| < \infty \right\}$$

Proof. Let $a = (a_k) \in \omega$. We can easily derive that with the notation (2.3) that

$$a_n x_n = \sum_{k=1}^{n-1} \frac{Q_k}{n} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) a_n y_k + \frac{Q_n a_n}{n q_n} y_n = \sum_{k=1}^{n} m_{nk} y_k = (My)_n \tag{3.6}$$

for all $k, n \in \mathbb{N}$, where $M = (m_{nk})$ is defined by (3.5). It follows from (3.6) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \int bv(R)$ if and only if $My \in \ell_1$ whenever $y \in \ell_1$. We obtain that $a \in [\int bv(R)]^\alpha$ whenever $x \in \int bv(R)$ if and only if $M \in (\ell_1 : \ell_1)$. Therefore, we get by Lemma 3.3 with M instead of A that $a \in \left[\int bv(R)\right]^\alpha$ if and only if $\sup_{k\in\mathbb{N}} \sum_n |m_{nk}| < \infty$. This gives us the result that $\left[\int bv(R)\right]^\alpha = d_1$. □

Theorem 3.5. *The $\alpha-$dual of the space $d(bv(R))$ is the set*

$$d_2 = \left\{ a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sum_k \left| \sum_{n\in N} p_{nk} \right| < \infty \right\}.$$

Theorem 3.6. *The $\beta-$dual of the space $\int bv(R)$ is $d_3 \cap cs$, where*

$$d_3 = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{n} \left| \frac{1}{k} \frac{Q_k a_k}{q_k} + Q_k \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \sum_{j=k+1}^{n} \frac{1}{j} a_j \right| < \infty \right\}$$

Proof. Consider the equality

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n} a_k \left[\sum_{j=1}^{k-1} \frac{1}{k} \left(\frac{1}{q_j} - \frac{1}{q_{j+1}}\right) Q_j y_j + \frac{Q_k y_k}{k q_k} \right] \tag{3.7}$$

$$= \sum_{k=1}^{n} \left| \frac{1}{k} \frac{Q_k a_k y_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k y_k \sum_{j=k+1}^{n} \frac{1}{j} a_j \right| = (Sy)_n$$

for all $n \in \mathbb{N}$, where the matrix $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k \sum_{j=k+1}^{n} \frac{1}{j} a_j & , \quad (k > n) \\ \frac{1}{n} \frac{Q_n a_n}{q_n} & , \quad (n = k) \\ 0 & , \quad (k < n) \end{cases} \tag{3.8}$$

for all $k, n \in \mathbb{N}$. Therefore, we deduce from Lemma 3.2 with (3.7) that $ax = (a_n x_n) \in cs$ whenever $x \in \int bv(R)$ if and only if $Sy \in c$ whenever $y \in \ell_1$. From (3.2) and (3.3), we have

$$\lim_n s_{nk} = \alpha_k \quad \text{and} \quad \sup_k \sum_n |s_{nk}| < \infty$$

which shows that $\left[\int bv(R) \right]^{\beta} = d_3 \cap cs$. □

Theorem 3.7. $\left[\int bv(R) \right]^{\gamma} = d_3$.

Proof. We obtain from Lemma 3.1 with (3.7) that $ax = (a_n x_n) \in bs$ whenever $x \in \int bv(R)$ if and only if $Sy \in \ell_\infty$ whenever $y \in \ell_1$. Then, we see from (3.2) that $\left[\int bv(R) \right]^{\gamma} = d_3$.

□

Theorem 3.8. *The $\beta-$dual of the space $d(bv(R))$ is $d_4 \cap cs$, where*

$$d_4 = \left\{ a = (a_k) \in \omega : \sum_{k=1}^{n} \left| \frac{k Q_k a_k}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k \sum_{j=k+1}^{n} j a_j \right| < \infty \right\}$$

Theorem 3.9. $[d(bv(R))]^{\gamma} = d_4$.

4. Matrix transformations

In this section, we characterize the matrix transformations from new spaces into any given sequence space X.

We shall write for brevity that

$$\overline{a}_{nk} = \sum_{k=1}^{n} \left| \frac{1}{k} \frac{Q_k a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k \sum_{j=k+1}^{n} \frac{1}{j} a_{nj} \right|,$$

$$\widetilde{a}_{nk} = \sum_{k=1}^{n} \left| \frac{k Q_k a_{nk}}{q_k} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k \sum_{j=k+1}^{n} j a_{nj} \right|,$$

$$\overline{b}_{nk} = \sum_{j=1}^{n-1} j \frac{1}{Q_n} (q_j - q_{j+1}) a_{jk} + n \frac{q_n}{Q_n} a_{nk},$$

$$\widetilde{b}_{nk} = \sum_{j=1}^{n-1} \frac{1}{j} \frac{1}{Q_n} (q_j - q_{j+1}) a_{jk} + \frac{1}{n} \frac{q_n}{Q_n} a_{nk}$$

for all $k, n \in \mathbb{N}$.

Theorem 4.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$a_{nk} = \sum_{j=k}^{\infty} \frac{1}{Q_j} j(q_k - q_{k+1}) b_{nj} \quad or \quad b_{nk} = \overline{a}_{nk} \tag{4.1}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (\int bv(R) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^{\beta}$ for all $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$.

Proof. Let Y be any given sequence. Suppose that (4.1) holds between the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$, and take into account that the spaces $\int bv(R)$ and ℓ_1 are linearly isomorphic.

Let $A \in (\int bv(R) : Y)$ and take any $y = (y_k) \in \ell_1$. Then BC exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^{\beta}$ which yields that (4.1) is necessary and $\{b_{nk}\}_{k\in\mathbb{N}} \in \ell_1^{\beta}$ for each $n \in \mathbb{N}$. Hence, By exists for each $y \in \ell_1$ and thus by letting $m \to \infty$ in the equality

$$\sum_{k=1}^{m} a_{nk} x_k = \sum_{k=1}^{m} \left| \frac{1}{k} \frac{Q_k y_k}{q_k} a_{nk} + \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) Q_k y_k \sum_{j=k+1}^{m} \frac{1}{j} a_{nj} \right|$$

for all $m, n \in \mathbb{N}$. Therefore, we obtain that $Ax = By$ which leads us to the consequence $B \in (\ell_1 : Y)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^{\beta}$ for each $n \in \mathbb{N}$ and $B \in (\ell_1 : Y)$, and take any $x = (x_k) \in \int bv(R)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=1}^{m} b_{nk} y_k = \sum_{k=1}^{m} a_{nk} x_k$$

for all $m, n \in \mathbb{N}$, as $m \to \infty$ the result that $By = Ax$ and this shows that $A \in (\int bv(R) : Y)$. This completes the proof. □

Theorem 4.2. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $G = (g_{nk})$ are connected with the relation $g_{nk} = \overline{b}_{nk}$ for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : \int bv(R))$ if and only if $G \in (Y : \ell_1)$.*

Proof. Let $z = (z_k) \in Y$ and consider the following equality:

$$\sum_{k=1}^{m} g_{nk} z_k = \frac{1}{Q_n} \sum_{j=1}^{m} j(q_j - q_{j+1}) a_{jk} \left(\sum_{k=1}^{m} a_{jk} z_k \right) \tag{4.2}$$

for all $m, n \in \mathbb{N}$. Equation (4.2) yields as $m \to \infty$ the result that $(Gz)_n = \{C(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in \int bv(R)$ whenever $z \in Y$ if and only if $Gz \in \ell_1$ whenever $z \in Y$. □

Theorem 4.3. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $H = (h_{nk})$ are connected with the relation*

$$a_{nk} = \sum_{j=k}^{\infty} \frac{1}{j} \frac{1}{Q_j} (q_k - q_{k+1}) h_{nj} \quad or \quad h_{nk} = \widetilde{a}_{nk}$$

for all $k,n \in \mathbb{N}$ and Y be any given sequence space. Then $A \in (d(bv(R)) : Y)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ for all $n \in \mathbb{N}$ and $H \in (\ell_1 : Y)$.

Theorem 4.4. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $J = (j_{nk})$ are connected with the relation $j_{nk} = \widetilde{b}_{nk}$ for all $k,n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y : d(bv(R)))$ if and only if $J \in (Y : \ell_1)$.*

Lemma 4.5. (i) *$A \in (\ell_\infty : \ell_1) = (c : \ell_1) = (c_0 : \ell_1)$ if and only if*

$$\sup_{N,K\in\mathcal{F}} \left|\sum_{n\in N}\sum_{k\in K} a_{nk}\right| < \infty \tag{4.3}$$

(ii) *$A \in (bs : \ell_1)$ if and only if*

$$\lim_k a_{nk} = 0 \quad \textit{for each} \quad n \in \mathbb{N}. \tag{4.4}$$

$$\sup_{N,K\in\mathcal{F}} \left|\sum_{n\in N}\sum_{k\in K} (a_{nk} - a_{n,k+1})\right| < \infty \tag{4.5}$$

(iii) *$A \in (cs : \ell_1)$ if and only if*

$$\sup_{N,K\in\mathcal{F}} \left|\sum_{n\in N}\sum_{k\in K} (a_{nk} - a_{n,k-1})\right| < \infty \tag{4.6}$$

(iv) *$A \in (c_0s : \ell_1)$ if and only if (4.5) holds.*

Lemma 4.6. (i) *$A \in (\ell_1 : bs)$ if and only if*

$$\sup_{k,m\in\mathbb{N}} \left|\sum_{n=0}^{m} a_{nk}\right| < \infty. \tag{4.7}$$

(ii) *$A \in (\ell_1 : cs)$ if and only if (4.7) holds, and*

$$\sum_n a_{nk} \quad \textit{convergent for each} \quad k \in \mathbb{N}. \tag{4.8}$$

(iii) *$A \in (\ell_1 : c_0s)$ if and only if (4.7) holds, and*

$$\sum_n a_{nk} = 0 \quad \textit{for each} \quad k \in \mathbb{N}. \tag{4.9}$$

Now, we can give the following results:

Corollary 4.7. *The following statements hold:*

(i) *$A = (a_{nk}) \in (\int bv(R) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\overline{a}_{nk}$ instead of a_{nk}.*

(ii) *$A = (a_{nk}) \in (\int bv(R) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (3.3) hold with $\overline{a}_{nk}$ instead of a_{nk}.*

(iii) *$A \in (\int bv(R) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (3.3) hold with $\alpha_k = 0$ as $\overline{a}_{nk}$ instead of a_{nk}.*

(iv) *$A = (a_{nk}) \in (\int bv(R) : bs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and (4.7) holds with $\overline{a}_{nk}$ instead of a_{nk}.*

(v) *$A = (a_{nk}) \in (\int bv(R) : cs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ for all $n \in \mathbb{N}$ and (4.7), (4.8) hold with $\overline{a}_{nk}$ instead of a_{nk}.*

(vi) $A = (a_{nk}) \in (\int bv(R) : c_0s)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.7), (4.9) hold with* $\overline{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.8. *The following statements hold:*

(i) $A = (a_{nk}) \in (d(bv(R)) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) holds with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
(ii) $A = (a_{nk}) \in (d(bv(R)) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) and (3.3) hold with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
(iii) $A \in (d(bv(R)) : c_0)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ *for all* $n \in \mathbb{N}$ *and (3.2) and (3.3) hold with* $\alpha_k = 0$ *as* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
(iv) $A = (a_{nk}) \in (d(bv(R)) : bs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.7) holds with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
(v) $A = (a_{nk}) \in (d(bv(R)) : cs)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{d(bv(R))\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.7), (4.8) hold with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.
(vi) $A = (a_{nk}) \in (d(bv(R)) : c_0s)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{\int bv(R)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.7), (4.9) hold with* $\widetilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 4.9. *We have:*

(i) $A = (a_{nk}) \in (\ell_\infty : \int bv(R)) = (c : \int bv(R)) = (c_0 : \int bv(R))$ *if and only if (4.3) hold with* $\overline{b}_{nk}$ *instead of* a_{nk}.
(ii) $A = (a_{nk}) \in (bs : \int bv(R))$ *if and only if (4.4) and (4.5) hold with* $\overline{b}_{nk}$ *instead of* a_{nk}.
(iii) $A = (a_{nk}) \in (cs : \int bv(R))$ *if and only if (4.6) holds with* $\overline{b}_{nk}$ *instead of* a_{nk}.
(iv) $A = (a_{nk}) \in (c_0s : \int bv(R))$ *if and only if (4.5) holds with* $\overline{b}_{nk}$ *instead of* a_{nk}.

Corollary 4.10. *We have:*

(i) $A = (a_{nk}) \in (\ell_\infty : d(bv(R))) = (c : d(bv(R))) = (c_0 : d(bv(R)))$ *if and only if (4.3) hold with* $\widetilde{b}_{nk}$ *instead of* a_{nk}.
(ii) $A = (a_{nk}) \in (bs : d(bv(R)))$ *if and only if (4.4) and (4.5) hold with* $\widetilde{b}_{nk}$ *instead of* a_{nk}.
(iii) $A = (a_{nk}) \in (cs : d(bv(R)))$ *if and only if (4.6) holds with* $\widetilde{b}_{nk}$ *instead of* a_{nk}.
(iv) $A = (a_{nk}) \in (c_0s : d(bv(R)))$ *if and only if (4.5) holds with* $\widetilde{b}_{nk}$ *instead of* a_{nk}.

5. Conclusion

Goes and Goes [4] firstly mentioned to the integrated and differentiated sequence spaces. In Section 2 of [4], it was given some definitions which also including the integarted and differentiated sequence spaces. In Section 3 of the same paper, it was defined the Hahn sequence spaces by $h = \{x = (x_k) \in w : \sum_k k|x_k - x_{k+1}| < \infty \text{ and } \lim_{k\to\infty} x_k = 0\}$. Hahn [5] was proved that $h \subset \ell_1 \cap \int c_0$, where ℓ_1 and $\int c_0$ are denote the spaces of absolutely summable and the integrated sequences, respectively. In this section, the functional analytic properties of the spaces $h = \ell_1 \cap \int bv$ and $dh = bv_0 \cap d\ell_1$ are investigated. In Theorem 3.2, Goes and Goes proved that the Hahn space is in the intersection of the spaces ℓ_1 and $\int bv$. Also, Goes and Goes defined the differentiated spaces dh depending on Theorem 3.2 as $dh = bv_0 \cap d\ell_1$. Therefore, in [4], it was shown that the integrated and differentiated sequence spaces are associated with each other.

In [6], new integrated and differentiated sequence spaces and matrices related to these spaces are constructed and some properties of the integrated and differentiated sequence spaces which are both new spaces and mentioned in [4], were discussed. The space $\int bv$ was defined in [4]. The new spaces $\int \ell_1$, $d(\ell_1)$ and $d(bv)$ were defined which is mentioned paper. In Section 2 of [6], the properties Banach spaces, $BK-$spaces, monotone norms, Schauder base, separability and, $AK-$property, $AB-$property and, isomorphism between new spaces and original space, were investigated. Besides this, dual spaces are computed and matrix classes are characterized by Kirişci[6].

Let $\int bv$ and $d(\ell_1)$ denote the integrated and differentiated spaces, respectively. The main purpose of this paper is to define the new integrated and differentiated sequence spaces using the Riesz mean and to study their some properties. In section 3, we compute the alpha-, beta- and gamma duals of these spaces. Afterward, we characterize matrix classes $(\int b(R) : Y), (d(bv(R)) : Y)$ and $(Y : \int b(R)), (Y : d(bv(R)))$, where Y is one of the well-known sequence spaces such as ℓ_∞, c, c_0, bs, cs and c_0s.

As a natural continuation of this paper, one can study the domain of different matrices instead of R^q. Additionally, sequence spaces in this paper can be defined by a index p for $1 \leq p < \infty$ and a bounded sequence of strictly positive real numbers (p_k) for $0 < p_k \leq 1$ and $1 < p_k < \infty$ and the concept almost convergence. And also it may be characterized several classes of matrix transformations between new sequence spaces in this work and sequence spaces which obtained with the domain of different matrices.

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