Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 8 Issue 1(2016), Pages 22-26.

ON SUMMABILITY METHODS $|A_f|_k$ |AND $|C,0|_s$

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Abstract. In this paper we give necessary and sufficient conditions for $|C,0| \Rightarrow |A_f|_s$ and vise versa, and $|A_f| \Rightarrow |C,0|_s$ and vise versa for the case $1 \leq s < \infty$, where $|A_f|_k$ is absolute factorable summability. So we also complete some open problems in the paper of Sarıgöl [10].

1. Introduction

Let Σx_v be a given infinite series with partial sums (s_n). By σ_n^α we denote n-th Cesàro mean of order α, $\alpha > -1$, of the sequence (s_n). The series Σx_v is said to be absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k$, $k \geq 1$, if (*see* [5])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k < \infty. \tag{1.1}$$

By $\sigma_n^0 = s_n$, the summability $|C,0|_k$ is equivalent to the condition

$$\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty. \tag{1.2}$$

Let $A_f = (a_{nv})$ be a factorable matrix which is the lower triangular with entries

$$a_{nv} = \begin{cases} \widehat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v > n, \end{cases} \tag{1.3}$$

where $(\widehat{a}_n)$ and (a_n) are any sequences of real numbers. Then the series Σx_v is said to be summable $|A_f|_k$, $k \geq 1$, if (*see* [10])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\widehat{a}_n \sum_{v=1}^{n} a_v x_v\right|^k < \infty. \tag{1.4}$$

Note that if one takes $\widehat{a}_n = p_n/P_nP_{n-1}$, $a_v = P_{v-1}$ and $\widehat{a}_n = 1/n(n+1)$, $a_v = v$, then $|A_f|_k$ are reduced to the well known summabilities $|R,p_n|_k$ and $|C,1|_k$,

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2000 *Mathematics Subject Classification.* 40C05,40D25,40F05, 46A45.

Key words and phrases. Absolute Riesz summability, high indices theorem, matrix transformation.

Submitted February 2, 2016. Published March 1, 2016.

respectively, where (p_n) be a sequence of positive real constants with $P_n = p_0 + p_1 + ... + p_n \to \infty$ as $n \to \infty$, [15].

If A and B are methods of summability, B is said to include A (written $A \Rightarrow B$) if every series summable by the method A is also summable by the method B. A and B said to be equivalent (written $A \Leftrightarrow B$) if each methods includes the other.

Problems on inclusion dealing absolute Cesàro and absolute weighted mean summabilities have been examined by many authors (see, [1-5], [7-17]). In this direction, Bor [1] proved sufficient conditions for equivalence of the summabilities $|R, p_n|_k$ and $|C, 0|_k$. The more general result including Bor's result has given by Sarıgöl [12] under necessary and sufficient conditions. Quite recently, the main resuls of [12] have been extended by a factorable matrix in [10] as follows.

Theorem 1.1. Let $1 < k \le s < \infty$ and A be a factorable matrix given by (1.3) such that $\widehat{a}_n, a_n \neq 0$ for all n. Then, $|A_f|_k \ |\Rightarrow |C, 0|_s$ if and only if

$$\left(\sum_{v=m-1}^{m} \frac{1}{v\,|\widehat{a}_v|^{k^*}}\right)^{1/k^*} \left(\sum_{n=m}^{m+1} \frac{n^{s-1}}{|a_n|^s}\right)^{1/s} = O(1), \tag{1.5}$$

where k^* denotes the conjugate index of k, i.e., $\frac{1}{k} + \frac{1}{k^*} = 1$

Theorem 1.2. Let $1 < k \le s < \infty$ and A be a factorable matrix given by (1.3). Then, $|C, 0|_k \Rightarrow |A_f|_s$ if and only if

$$\left(\sum_{v=1}^{m} \frac{1}{v} |a_v|^{k*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} n^{s-1} |\widehat{a}_n|^s\right)^{1/s} = O(1), \tag{1.6}$$

where k^*denotes the conjugate index of k.

Corollary 1.3. Let $1 < k < \infty$ and A be a factorable matrix given by (1.3) such that $\widehat{a}_n, a_n \neq 0$ for all n. Then, $|C, 0|_k \Leftrightarrow |A_f|_k$ if and only if conditions (1.5) and (1.6) satisfied.

2. Main Results

Note that Theorem 1.1 and Theorem 1.2 do not include results $|C, 0| \Rightarrow |A_f|_s$ vise versa, and $|A_f| \ |\Rightarrow |C, 0|_s$ vise versa for the case $1 \le s < \infty$. So, motivated by these theorems, a natural problem is that, what are the necessary and sufficient conditions in order that these results should be satisfied. The aim of this paper is to answer this open problem proving the following theorems.

Theorem 2.1. Let $1 < s < \infty$ and A be a factorable matrix given by (1.3) such that $\widehat{a}_v$, $a_v \neq 0$ for all v. Then, $|A_f|_s \ |\Rightarrow |C, 0|$ if and only if

$$\sum_{v=1}^{\infty} \frac{1}{v} \left\{ \frac{1}{|\widehat{a}_v|} \left(\frac{1}{|a_v|} + \frac{1}{|a_{v+1}|} \right) \right\}^{s^*} < \infty, \tag{2.1}$$

where s^* denotes the conjugate index of s, i.e., $\frac{1}{s} + \frac{1}{s^*} = 1$.

Theorem 2.2. Let $1 < s < \infty$ and A be a factorable matrix given by (1.3). Then, $|C, 0|_s \ |\Rightarrow |A_f|$ if and only if

$$\sum_{v=1}^{\infty} \frac{1}{v} \left(|a_v| \sum_{n=v}^{\infty} |\widehat{a}_n| \right)^{s^*} < \infty, \tag{2.2}$$

where s^* denotes the conjugate index of s.

Theorem 2.3 Let $1 \leq s < \infty$ and A be a factorable matrix given by (1.3). Then, $|C,0| \Rightarrow |A_f|_s$ if and only if

$$\sum_{n=\nu}^{\infty} n^{s-1} \left|\widehat{a}_n a_\nu\right|^s = O\left(1\right) \text{ as } v \to \infty. \tag{2.3}$$

Theorem 2.4. Let $1 \leq s < \infty$ and A be a factorable matrix given by (1.3) such that $\widehat{a}_v$, $a_v \neq 0$ for all v.Then, $|A_f| \Rightarrow |C,0|_s$ if and only if

$$\frac{v^{s-1}}{\left|\widehat{a}_\nu\right|^s}\left(\frac{1}{\left|a_\nu\right|^s} + \frac{1}{\left|a_{\nu+1}\right|^s}\right) = O\left(1\right) \text{ as } v \to \infty. \tag{2.4}$$

If one takes $\widehat{a}_n = p_n \backslash P_n P_{n-1}$ and $a_\nu = P_{\nu-1}$ in Theorem 2.1 and Theorem 2.2, then the conditions (2.1) and (2.2) are reduced to

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{P_{v-1}}{p_\nu} + \frac{P_v}{p_\nu}\right)^{s^*} < \infty \text{ and } \sum_{v=1}^{\infty} \frac{1}{v} < \infty$$

respectively, which is impossible. So we get the following results.

Corollary 2.5. If $s > 1$, then $|R,p_n|_s \nRightarrow |C,0|$ and also $|C,0|_s \nRightarrow |R,p_n|$.

Also, by taking $\widehat{a}_n = p_n \backslash P_n P_{n-1}$ and $a_\nu = P_{\nu-1}$ in Theorem 2.3 and Theorem 2.4, we get the following results concerning the summability methods $|C,0|$, $|R,p_n|_s$, $|R,p_n|$ and $|C,0|_s$.

Corollary 2.6. Let $s \geq 1$. Then, $|C,0| \Rightarrow |R,p_n|_s$ if and only if

$$\sum_{n=\nu}^{\infty} n^{s-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^s = O\left(\frac{1}{P_{\nu-1}^s}\right).$$

Corollary 2.7. Let $s \geq 1$. Then, $|R,p_n| \Rightarrow |C,0|_s$ if and only if

$$v^{1/s^*} P_v = O\left(p_v\right) \text{ as } v \to \infty.$$

3. **Needed Lemmas**

In this subtitle we give the following lemmas which are needed in proving our Theorems.

Lemma 3.1 ([11]). Let $1 < s < \infty$. Then, $A : \ell_s \to \ell$ if and only if

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty} \left|a_{nv}\right|\right)^{s^*} < \infty. \tag{3.1}$$

Lemma 3.2 ([7]). Let $1 \leq s < \infty$. Then, $A : \ell \to \ell_s$ if and only if

$$\sum_{n=0}^{\infty} \left|a_{nv}\right|^s = O(1) \text{ as } v \to \infty. \tag{3.2}$$

4. Proof of Theorems

Since the proofs of Theorem 2.2 and Theorem 2.4 are similar to those of Theorem 2.1 and Theorem 2.3, respectively, we only give proofs of Theorem 2.1 and Theorem 2.3.

Proof of Theorem 2.1. Let $A_n^*(x) = n^{1/s^*} A_n(x)$ for $n \geq 1$, where

$$A_n(x) = \widehat{a}_n \sum_{\nu=1}^{n} a_\nu x_\nu. \tag{4.1}$$

Then Σx_n is summable $|A_f|_s$ and $|C,0|$ iff $A^*(x) \in l_s$ and $x \in l$, respectively. On the other hand, it can be written from (4.1) that

$$x_n = \frac{1}{a_n}\left(\frac{A_n^*(x)}{n^{1/s^*}\widehat{a}_n} - \frac{A_{n-1}^*(x)}{(n-1)^{1/s^*}\widehat{a}_{n-1}}\right)$$

which gives us

$$x_n = \sum_{\nu=1}^{\infty} b_{n\nu} A_\nu^*(x),$$

where

$$b_{n\nu} = \begin{cases} \dfrac{1}{a_n}\left(-\frac{1}{(n-1)^{1/s^*}\widehat{a}_{n-1}}\right), & v = n-1 \\ \dfrac{1}{a_n}\left(\frac{1}{n^{1/s^*}\widehat{a}_{n-1}}\right), & v = n \\ 0, & v \neq n-1, n \end{cases} \tag{4.2}$$

Then $|A_f|_s \Rightarrow |C,0|$ if and only if

$$\sum_{n=1}^{\infty} |A_n^*(x)|^s < \infty \Longrightarrow \sum_{n=1}^{\infty} |x_n| < \infty, \ \textit{i.e.}, \ B : l_s \longrightarrow l,$$

where B is the matrix whose entries are defined by (4.2). Therefore applying (3.1) to the matrix B, by Lemma 2.1, we have that $|A_f|_s \Rightarrow |C,0|$ iff the condition (2.1) holds, which completes the proof.

Proof of Theorem 2.3. Let, for $n \geq 1$,

$$A_n^*(x) = n^{1/s^*}\widehat{a}_n \sum_{v=1}^{n} a_\nu x_\nu.$$

Then Σx_n is summable $|A_f|_s$ whenever Σx_n is summable$|C,0|$ if and only if $A(x) \in l_s$ whenever $x \in l$. Also, it follows that

$$A_n^*(x) = n^{1/s^*}\widehat{a}_n \sum_{v=1}^{n} a_\nu x_\nu = \sum_{v=1}^{\infty} h_{n\nu} x_\nu$$

where

$$h_{n\nu} = \begin{cases} n^{1/s^*}\widehat{a}_n a_v, 1 \leq \nu \leq n \\ 0, \quad \nu > n. \end{cases}$$

Hence $|C,0|_s \Rightarrow |A_f|$if and only if $H : l \to l_s$. So, applying the matrix H to (3.2) gives

$$\sum_{v=1}^{\infty}\frac{1}{v}\left(|a_v|\sum_{n=v}^{\infty}|\widehat{a}_n|\right)^{s^*}<\infty,$$

which completes the proof by Lemma 3.2.

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