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COMMON FIXED POINT RESULTS USING (φ_1-φ_2-φ_3)-CONTRACTIVE CONDITIONS IN ORBITALLY 0-COMPLETE PARTIAL METRIC SPACES

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S.K. PADHAN†, G.V.V. JAGANNADHA RAO†, HEMANT K. NASHINE ‡ AND Z. KADELBURG§

Abstract. In this paper, several results are presented on common fixed points using (φ_1-φ_2-φ_3)-contraction condition including rational expressions in orbitally 0-complete partial metric spaces. A counterexample illustrates that the results are distinct from some known ones. It is also observed that the work of Kadelburg and Radenović [Kadelburg, Z., Radenović, S.: Fixed points under ψ-α-β conditions in ordered partial metric spaces, Intern. J. Anal. Appl. 5, 91–101 (2014)] is a particular case of the present investigation. As applications of our results, we prove two theorems for the existence of solutions of certain system of Volterra type integral equations, as well as for a nonlinear fractional differential equation.

1. Introduction

Matthews [14, 16] introduced the concept of partial metric spaces. He proved that the Banach contraction mapping theorem can be generalized to the partial metric structure for applications in program verifications. Since then, many researchers have extended this principle by considering contractive mappings on partial metric spaces. Throughout this paper, we abbreviate 'partial metric space' and 'partial metric spaces' to PMS and PMS's, respectively.

Completeness of the underlying space in fixed point results has been replaced and relaxed by the so-called orbital completeness in the papers by Browder and Petryshin [6] and Ćirić [8]. Romaguera noticed in [19] that in the case of partial metric, it is natural to use 0-completeness instead. Fixed point theorems for generalized contraction mappings in metric spaces using orbital completeness, f-orbital completeness or 0-completeness of the given space have been proved by many researchers.

The aim of this paper is to prove common fixed point results in orbitally 0-complete PMS's using a (φ_1-φ_2-φ_3)-contractive condition that involves rational

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Correspondence author: H. K. Nashine (drhknashine@gmail.com).

expressions. Moreover, several results are constructed and a counterexample is presented in support of the work. By considering rational type weakly contraction condition for two mappings, it is shown that our results are generalizations of the ones due to Kadelburg and Radenović [11]. Two applications are provided in order to prove existence and uniqueness of solution for a system of Volterra-type integral equations and a nonlinear fractional differential equation.

Remark 1.1. It was shown in [9] that in some cases fixed point results in PMS's can be directly reduced to their standard metric counterparts. We note that the results of the present paper do not fall into this category.

2. Prerequisites

In this section, we recollect some definitions and notions which are used in our results; for more details on PMS's one can see [7, 15, 14, 21, 16].

Definition 2.1. [16] Let $\mathcal{X}$ be a nonempty set. A function $p\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ is a partial metric if the following conditions are satisfied, for all $u, v, t \in \mathcal{X}$:

a_1) $u = v$ if and only if $p(u,u) = p(u,v) = p(v,v)$,
a_2) $p(u,u) \le p(u,v)$,
a_3) $p(u,v) = p(v,u)$,
a_4) $p(u,v) \le p(u,t) + p(t,v) - p(t,t)$.

The pair $(\mathcal{X}, p)$ is called a PMS.

A trivial example for the above definition is $(\mathbb{R}^+, p)$, where $p(u,v) = \max\{u,v\}$ for all $u, v \in \mathbb{R}^+$. Obviously, in this, as well as in many other examples, it may happen that $p(u,u) > 0$ for some $u \in \mathcal{X}$. Other examples of PMS's may be found in [7, 14].

A sequence $\{u_n\}$ in $(\mathcal{X}, p)$ converges to a point $u \in \mathcal{X}$ (more precisely "p-converges") if $p(u,u) = \lim_{n\to\infty} p(u,u_n)$. It can be denoted as $u_n \to u$ as $n \to \infty$.

Remark 2.1. [1, 16]

(1) Clearly, in a PMS the limit of a sequence need not be unique. Furthermore, the function $p(\cdot,\cdot)$ need not be continuous, that is $u_n \to u$, $v_n \to v \Rightarrow p(u_n, v_n) \to p(u,v)$ might not hold.
(2) A sequence $\{u_n\}$ in $(\mathcal{X}, p)$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(u_n, u_m)$ exists (and is finite). If every Cauchy sequence $\{u_n\}$ in $\mathcal{X}$ converges in $\mathcal{X}$ then $(\mathcal{X}, p)$ is called complete.
(3) If $\lim_{n,m\to\infty} p(u_n, u_m) = 0$ then a sequence $\{u_n\}$ in $(\mathcal{X}, p)$ is called a 0-Cauchy sequence.
(4) The space $(\mathcal{X}, p)$ is said to be 0-complete if every 0-Cauchy sequence in $\mathcal{X}$ converges (in p) to a point $u \in \mathcal{X}$ such that $p(u,u) = 0$.

Lemma 2.1. [16, 12, 19] *Let $(\mathcal{X}, p)$ be a PMS.*

(i) *If $p(u_n, t) \to p(t,t) = 0$ as $n \to \infty$, then $p(u_n, v) \to p(t, v)$, for all $v \in \mathcal{X}$.*
(ii) *If $(\mathcal{X}, p)$ is complete, then it is 0-complete.*

The following simple example shows that the converse assertion of (ii) does not hold.

Example 2.1. [16, 19] Let $\mathcal{X} = [0, +\infty) \cap Q$ be a space endowed with partial metric $p(u, v) = \max\{u, v\}$ for all $u, v \in \mathcal{X}$. It is 0-complete, but not complete. Furthermore, $\{u_n\}$, with $u_n = 1$ for all $n \in \mathbb{N}$, is a Cauchy sequence in $(\mathcal{X}, p)$ but it is not a 0-Cauchy sequence.

Every closed (in the topology generated by p) subset of a 0-complete PMS is 0-complete.

Recall that the set $\mathcal{O}(u_0; \mathcal{R}) = \{\mathcal{R}^n u_0 : n = 0, 1, 2, \cdots\}$ is called the orbit of a self-map $\mathcal{R}\colon \mathcal{X} \to \mathcal{X}$ at a point $u_0 \in \mathcal{X}$.

Definition 2.2. [16] Let $(\mathcal{X}, p)$ be a PMS and let $\mathcal{R}\colon \mathcal{X} \to \mathcal{X}$ be a mapping. If every 0-Cauchy sequence contained in $\mathcal{O}(u; \mathcal{R})$ (for some $u \in \mathcal{X}$) converges in $\mathcal{X}$ to a point t, with $p(t, t) = 0$, then $(\mathcal{X}, p)$ is called $\mathcal{R}$-orbitally 0-complete.

Observe that every 0-complete PMS is $\mathcal{R}$-orbitally 0-complete for each $\mathcal{R}$, but the converse does not hold.

Consider now the following sets of functions:
$\Phi_1 = \{\varphi_1\colon [0, \infty) \to [0, \infty) \mid \varphi_1$ is lower semicontinuous and nondecreasing$\}$,
$\Phi_2 = \{\varphi_2\colon [0, \infty) \to [0, \infty) \mid \varphi_2$ is upper semicontinuous $\}$,
$\Phi_3 = \{\varphi_3\colon [0, \infty) \to [0, \infty) \mid \varphi_3$ is lower semicontinuous $\}$.
Recently, Karapinar and Salimi [13] proved the following

Theorem 2.2. [11] Let $(\mathcal{X}, \preceq, d)$ be a complete ordered metric space and let $\mathcal{R}\colon \mathcal{X} \to \mathcal{X}$ be a nondecreasing selfmap. Let us assume that there exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ and $\varphi_3 \in \Phi_3$ such that

$$\varphi_1(t) - \varphi_2(s) + \varphi_3(s) > 0, \text{ for all } t > 0, \text{ and either } s = t \text{ or } s = 0 \tag{1}$$

and

$$\varphi_1(d(\mathcal{R}u, \mathcal{R}v)) \leq \varphi_2(d(u, v)) - \varphi_3(d(u, v)). \tag{2}$$

for all comparable $u, v \in \mathcal{X}$. Suppose that $\mathcal{X}$ is regular, or $\mathcal{R}$ is continuous. If $u_0 \in \mathcal{X}$ exists such that $u_0 \preceq \mathcal{R}u_0$, then $\mathcal{R}$ has a fixed point.

Kadelburg and Radenović [11] used this concept to prove fixed point theorems in PMS's. Here, we extend these results for two mappings, using a more involved (rational type) contraction condition.

3. Pairs of mappings and common fixed point results

In this section, Theorem 2.2 is extended to mappings in orbitally 0-complete PMS's with a rational contraction (φ_1-φ_2-φ_3)-condition, more general than (2). A numerical example is given in support of this extension.

Sastry *et al.* [16, 20] extended the concepts of orbit and orbital completeness for two and three maps and utilised them to obtain common fixed point results. We evoke such definitions for two mappings in PMS's.

Definition 3.1. [15] Let $\mathcal{P}, \mathcal{R}$ be two self-maps defined on a PMS $(\mathcal{X}, p)$.

(1) If a sequence $\{u_n\}$ and a point u_0 are in $\mathcal{X}$, such that $u_{2n+1} = \mathcal{P}u_{2n}$, $u_{2n+2} = \mathcal{R}u_{2n+1}$, $n = 0, 1, 2, \cdots$, then $\mathcal{O}(u_0; \mathcal{P}, \mathcal{R}) = \{u_n \colon n = 1, 2, \cdots\}$ is said to be the orbit of $(\mathcal{P}, \mathcal{R})$ at u_0.
(2) The space $(\mathcal{X}, p)$ is said to be $(\mathcal{P}, \mathcal{R})$-orbitally 0-complete at u_0 if every 0-Cauchy sequence in $\mathcal{O}(u_0; \mathcal{P}, \mathcal{R})$ converges to a point $t \in \mathcal{X}$ such that $p(t, t) = 0$.

Theorem 3.1. Let $(\mathcal{X}, p)$ be a PMS and $\mathcal{R}, \mathcal{P} : \mathcal{X} \to \mathcal{X}$ be a pair of selfmaps. Assume that there exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ and $\varphi_3 \in \Phi_3$ such that for all $r > 0$,

$$\varphi_1(r) - \varphi_2(r) + \varphi_3(r) > 0, \tag{3}$$

and for all $u, v \in \overline{\mathcal{O}(u_0; \mathcal{R}, \mathcal{P})}$ (for some u_0),

$$\varphi_1(p(\mathcal{R}u, \mathcal{P}v)) \leq \varphi_2(\Theta(u, v)) - \varphi_3(\Theta(u, v)), \tag{4}$$

where

$$\Theta(u, v) = \max \left\{ \begin{array}{c} p(u, v), p(\mathcal{R}u, u), p(\mathcal{P}v, v), \frac{1}{2}[p(\mathcal{R}u, v) + p(\mathcal{P}v, u)], \\ \dfrac{p(\mathcal{R}u, u)p(\mathcal{P}v, v)}{1 + p(u, v)}, \dfrac{p(\mathcal{R}u, u)p(\mathcal{P}v, v)}{1 + p(\mathcal{R}u, \mathcal{P}v)} \end{array} \right\}.$$

Suppose that $(\mathcal{X}, p)$ is $(\mathcal{P}, \mathcal{R})$-orbitally 0-complete at u_0. Then $\mathcal{P}, \mathcal{R}$ have a common fixed point $t \in \mathcal{X}$ such that $p(t, t) = 0$. Moreover, if each common fixed point t of $\mathcal{P}$, $\mathcal{R}$ in $\overline{\mathcal{O}(u_0; \mathcal{R}, \mathcal{P})}$ satisfies $p(t, t) = 0$, then the common fixed point of the mappings $\mathcal{P}$, $\mathcal{R}$ in $\overline{\mathcal{O}(u_0; \mathcal{R}, \mathcal{P})}$ is unique.

Proof. Our first priority is to show that, if one of the mappings $\mathcal{P}$ or $\mathcal{R}$ has a fixed point with the zero self-distance, then $\mathcal{P}$ and $\mathcal{R}$ have a common fixed point. Indeed, let the fixed point of $\mathcal{P}$ is t such that $p(t, t) = 0$ and suppose that $p(t, \mathcal{R}t) > 0$. If we use $u = v = t$ in the equation (4), we get

$$\varphi_1(p(\mathcal{R}t, t)) = \varphi_1(p(\mathcal{R}t, \mathcal{P}t)) \leq \varphi_2(p(t, \mathcal{R}t)) - \varphi_3(p(t, \mathcal{R}t)),$$

which contradicts the condition (3), Therefore $p(t, \mathcal{R}t) = 0$, and so the pair $\mathcal{P}$ and $\mathcal{R}$ has a common fixed point t.

Starting with the given point u_0, let the sequence $\{u_n\}$ be defined in $\mathcal{X}$ in the following manner:

$$u_{2n+1} = \mathcal{P}u_{2n}, \qquad u_{2n+2} = \mathcal{R}u_{2n+1}, \qquad \text{for all } n \in \{0, 1, 2, \cdots\}.$$

If $p(u_{n_0}, \mathcal{P}u_{n_0}) = 0$ or $p(u_{n_0}, \mathcal{R}u_{n_0}) = 0$ for some $n_0 \in \{0, 1, 2, \cdots\}$, then our proof is completed. So let us assume that the consecutive terms of $\{u_n\}$ are distinct.

Furthermore, we show that $\{u_n\}$ is a 0-Cauchy sequence in $\mathcal{O}(u_0, \mathcal{P}, \mathcal{R})$. For this, we first show that the sequence $\{p(u_{n+1}, u_n)\}$ is non-increasing. Indeed, putting $u = u_1$, $v = u_0$ in (4) we have

$$\varphi_1(p(u_2, u_1)) = \varphi_1(p(\mathcal{R}u_1, \mathcal{P}u_0)) \leq \varphi_2(\Theta(u_1, u_0)) - \varphi_3(\Theta(u_1, u_0)), \tag{5}$$

where

$$\Theta(u_1, u_0) = \max \left\{ \begin{array}{c} p(u_2, u_1), p(u_1, u_0), p(u_0, u_1), \frac{1}{2}[p(u_2, u_0) + p(u_1, u_1)], \\ \dfrac{p(u_2, u_1)p(u_1, u_0)}{1 + p(u_0, u_1)}, \dfrac{p(u_2, u_1)p(u_1, u_0)}{1 + p(u_2, u_1)} \end{array} \right\}.$$

Using property (p_4),

$$\Theta(u_1, u_0) = \max\{p(u_0, u_1)p(u_1, u_2)\}.$$

Suppose that $p(u_1, u_2) > p(u_0, u_1)$. Then (5) implies that

$$\varphi_1(p(u_1, u_2)) \leq \varphi_2(p(u_1, u_2)) - \varphi_3(p(u_1, u_2)),$$

which is a contradiction to the condition (3). It follows that $p(u_1, u_2) = 0$ and, hence, $u_2 = u_1$ which is already excluded. Hence, $p(u_2, u_1) \leq p(u_1, u_0)$ and $\Theta(u_1, u_0) = p(u_1, u_0)$. It follows also that

$$\varphi_1(p(u_2, u_1)) \leq \varphi_2(p(u_1, u_0)) - \varphi_3(p(u_1, u_0)).$$

In a similar manner, starting from $u = u_1$ and $v = u_2$, we get that $p(u_3, u_2) \leq p(u_2, u_1)$. Repeating similar procedure, we obtain by induction that $\{p(u_{n+1}, u_n)\}$ is non-increasing and also

$$\varphi_1(p(u_{n+2}, u_{n+1})) \leq \varphi_2(p(u_{n+1}, u_n)) - \varphi_3(p(u_{n+1}, u_n)) \tag{6}$$

holds for each $n \in \mathbb{N} \cup \{0\}$.

Since the sequence $\{p(u_{n+1}, u_n)\}$ is bounded from below, $\delta \geq 0$ exists such that $\lim_{n\to\infty} p(u_n, u_{n+1}) = \delta$. Using the properties of functions φ_1, φ_2 and φ_3, and (6) we get that

$$\begin{aligned}
\varphi_1(\delta) &\leq \liminf \varphi_1(p(u_{n+2}, u_{n+1}))) \leq \limsup \varphi_1(p(u_{n+2}, u_{n+1})) \\
&\leq \limsup[\varphi_2(p(u_{n+1}, u_n)) - \varphi_3(p(u_{n+1}, u_n))] \\
&\leq \limsup \varphi_2(p(u_{n+1}, u_n)) - \liminf \varphi_3(p(u_{n+1}, u_n)) \\
&\leq \varphi_2(\delta) - \varphi_3(\delta).
\end{aligned}$$

Thus, we obtain $\varphi_1(\delta) \leq \varphi_2(\delta) - \varphi_3(\delta)$, which is not possible by the condition (3) if $\delta > 0$. Hence, $\delta = \lim_{n\to\infty} p(u_n, u_{n+1}) = 0$.

Now, we have to show that $\{u_n\}$ is a 0-Cauchy sequence in $\overline{\mathcal{O}(u_0, \mathcal{R}, \mathcal{P})}$. Suppose the contrary; then, using [11, Lemma 2.6], we obtain that there exist $\epsilon > 0$ and subsequences $\{m_r\}$ and $\{n_r\}$ of positive integers such that $n_r > m_r > r$ and that sequences $\{p(u_{m_r}, u_{n_r})\}$, $\{p(u_{m_r}, u_{n_r+1})\}$, $\{p(u_{m_r-1}, u_{n_r})\}$, $\{p(u_{m_r-1}, u_{n_r+1})\}$ tend to ϵ^+ as $r \to \infty$. Repeating the arguments in the proof of [11, Theorem 3.1], we obtain a contradiction.

Thus, $\{u_n\}$ is a 0-Cauchy sequence in $\overline{\mathcal{O}(u_0, \mathcal{R}, \mathcal{P})}$. Since at u_0, $\mathcal{X}$ is $(\mathcal{R}, \mathcal{P})$-orbitally complete, therefore some $t \in \mathcal{X}$ exists with $\lim_{n\to\infty} u_n = t$ and $p(t, t) = 0$. Using the contraction condition (4), we get

$$\varphi_1(p(u_{2n}, \mathcal{P}t)) = \varphi_1(p(\mathcal{R}u_{2n-1}, \mathcal{P}t)) \leq \varphi_2(\Theta(u_{2n-1}, t)) - \varphi_3(\Theta(u_{2n-1}, t)),$$

where

$$\Theta(u_{2n-1}, t) = \max \left\{ \begin{array}{c} p(u_{2n}, u_{2n-1}), p(\mathcal{P}t, t), p(t, u_{2n-1}), \frac{1}{2}[p(u_{2n}, t) + p(\mathcal{P}t, u_{2n-1})], \\ \frac{p(u_{2n}, u_{2n-1})p(\mathcal{P}t, t)}{1+p(u_{2n-1}, t)}, \frac{p(u_{2n}, u_{2n-1})p(\mathcal{P}t, t)}{1+p(u_{2n}, \mathcal{P}t)} \end{array} \right\}.$$

By Lemma 2.1, $p(u_{2n-1}, \mathcal{P}t) \to p(t, \mathcal{P}t)$ and $p(u_{2n}, \mathcal{P}t) \to p(t, \mathcal{P}t)$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} \Theta(u_{2n-1}, t) = \max \left\{ 0, p(\mathcal{P}t, t), 0, \frac{1}{2}[0 + p(\mathcal{P}t, t)], 0, 0 \right\} = p(\mathcal{P}t, t).$$

Thus, using the properties of functions φ_1, φ_2, φ_3, we get

$$\begin{aligned}
\varphi_1(p(z, Sz)) &\leq \liminf \varphi_1(p(u_{2n}, \mathcal{P}t)) \leq \limsup \varphi_1(p(u_{2n}, \mathcal{P}t)) \\
&\leq \limsup[\varphi_2(\Theta(u_{2n-1}, \mathcal{P}t)) - \varphi_3(\Theta(u_{2n-1}, \mathcal{P}t))] \\
&\leq \limsup \varphi_2(\Theta(u_{2n-1}, \mathcal{P}t)) - \liminf \varphi_3(\Theta(u_{2n-1}, \mathcal{P}t)) \\
&\leq \varphi_2(p(t, \mathcal{P}t)) - \varphi_3(p(t, \mathcal{P}t)).
\end{aligned}$$

By the condition (3), this is only possible if $p(\mathcal{P}t, t) = 0$. Hence, t is the fixed point of $\mathcal{P}$ such that $p(t, t) = 0$. Therefore, t is a common fixed point of $\mathcal{R}, \mathcal{P}$, such that $p(t, t) = 0$.

We have to prove that the common fixed point of $\mathcal{R}$ and $\mathcal{P}$ in $\overline{\mathcal{O}(u_0; \mathcal{R}, \mathcal{P})}$ is unique. Suppose to the contrary $p(u, \mathcal{P}u') = p(u', \mathcal{R}u') = 0$ and $p(v', \mathcal{P}v') =$

$p(v',\mathcal{R}v')=0$ but $u'\neq v'$. By the assumption, we can replace u by u' and v by v' in the condition (4). Since

$$\varphi_1(p(u',v'))=\varphi_1(p(\mathcal{R}u',\mathcal{P}v'))\leq\varphi_2(\Theta(u',v'))-\varphi_3(\Theta(u',v')),$$

where

$$\Theta(u',v')=\max\left\{\begin{array}{c}p(\mathcal{R}u',u'),p(\mathcal{P}v',v'),p(u',v'),\frac{1}{2}[p(\mathcal{R}u',v')+p(\mathcal{P}v',u')],\\ \dfrac{p(\mathcal{R}u',u')p(\mathcal{P}v',v')}{1+p(u',v')},\dfrac{p(\mathcal{R}u',u')p(\mathcal{P}v',v')}{1+p(\mathcal{R}u',\mathcal{P}v')}\end{array}\right\}=p(u',v'),$$

we get

$$\varphi_1(p(u',v'))\leq\varphi_2(p(u',v'))-\varphi_3(p(u',v')),$$

a contradiction to condition (3), Therefore, $u'=v'$. □

We state the following consequence of Theorem 3.1.

Corollary 3.1. Let $(\mathcal{X},p)$ be a PMS and $\mathcal{X}$ be $\mathcal{R}$-orbitally 0-complete at a point u_0 of $\mathcal{X}$. Suppose that $\varphi_1\in\Phi_1$, $\varphi_2\in\Phi_2$ and $\varphi_3\in\Phi_3$ exist such that

$$\varphi_1(r)-\varphi_2(r)+\varphi_3(r)>0,\ \text{for all } r>0, \tag{7}$$

and the given mapping $\mathcal{R}:\mathcal{X}\to\mathcal{X}$ satisfies

$$\varphi_1(p(\mathcal{R}u,\mathcal{R}v))\leq\varphi_2(\Theta(u,v))-\varphi_3(\Theta(u,v)), \tag{8}$$

for all $u,v\in\overline{\mathcal{O}(u_0;\mathcal{R})}$, where

$$\Theta(u,v)=\max\left\{\begin{array}{c}p(\mathcal{R}u,u),p(\mathcal{R}v,v),p(v,u),\frac{1}{2}[p(\mathcal{R}u,v)+p(\mathcal{R}v,u)],\\ \dfrac{p(\mathcal{R}u,u)p(\mathcal{R}v,v)}{1+p(u,v)},\dfrac{p(\mathcal{R}u,u)p(\mathcal{R}v,v)}{1+p(\mathcal{R}u,\mathcal{R}v)}\end{array}\right\}.$$

Then $\mathcal{R}$ has a fixed point. If, furthermore, each fixed point t of $\mathcal{R}$ in $\overline{\mathcal{O}(u_0;\mathcal{R})}$ satisfies that $p(t,t)=0$ then the fixed point of $\mathcal{R}$ in $\overline{\mathcal{O}(u_0;\mathcal{R})}$ is unique.

Theorem 3.2. Suppose all the conditions of Theorem 3.1 are satisfied, apart from the condition (4) which is replaced by

$$\varphi_1(p(\mathcal{R}u,\mathcal{P}v))\leq\varphi_2(\Theta_1(u,v))-\varphi_3(\Theta_1(u,v)),$$

for all $u,v\in\overline{\mathcal{O}(u_0;\mathcal{R},\mathcal{P})}$ (for some u_0), where

$$\Theta_1(u,v)=\max\left\{\begin{array}{c}\frac{1}{2}[p(u,\mathcal{R}u)+p(v,\mathcal{P}v)],\frac{1}{2}[p(\mathcal{R}u,v)+p(\mathcal{P}v,u)],p(u,v)\\ \dfrac{1}{2}\left[\dfrac{p(\mathcal{R}u,u)p(\mathcal{P}v,v)}{1+p(u,v)}+\dfrac{p(\mathcal{R}u,u)p(\mathcal{P}v,v)}{1+p(\mathcal{R}u,\mathcal{P}v)}\right]\end{array}\right\}.$$

Then all the conclusions of Theorem 3.1 hold.

Corollary 3.2. Let all the conditions of Corollary 3.1 are satisfied, apart from the condition (8) which is replaced by

$$\varphi_1(p(\mathcal{R}u,\mathcal{R}v))\leq\varphi_2(\Theta_2(u,v))-\varphi_3(\Theta_2(u,v)),$$

for all $u,v\in\overline{\mathcal{O}(u_0;\mathcal{R})}$, where

$$\Theta_2(u,v)=\max\left\{\begin{array}{c}p(u,v),\frac{1}{2}[p(\mathcal{R}u,v)+p(u,\mathcal{R}v)],\\ \dfrac{1}{2}\left[\dfrac{p(\mathcal{R}u,u)p(\mathcal{R}v,v)}{1+p(u,v)}+\dfrac{p(\mathcal{R}u,u)p(\mathcal{R}v,v)}{1+p(\mathcal{R}u,\mathcal{R}v)}\right]\end{array}\right\}.$$

Then all the conclusions of Corollary 3.1 hold.

The following example (inspired by [11]) demonstrates a possible usage of Theorem 3.1.

Example 3.1. Let $p(u,v) = \max\{u,v\}$ be a partial metric defined on the set $\mathcal{X} = [0,+\infty)$. Consider the self-mappings $\mathcal{P}$ and $\mathcal{R}$ on $\mathcal{X}$ given as

$$\mathcal{R}(u) = \begin{cases} \frac{u}{5}, & u \in [0,1] \\ 1, & \text{otherwise} \end{cases} \quad \text{and } \mathcal{P}(u) = \begin{cases} \frac{u}{7}, & u \in [0,1] \\ 3, & \text{otherwise.} \end{cases}$$

Take $u_0 = 1$. Now it is very simple to prove that
$\mathcal{O}(u_0;\mathcal{P},\mathcal{R}) \subset \left\{ \frac{1}{5^p 7^q} : p,q \in \mathbb{N}\cup\{0\} \right\}$ and $\overline{\mathcal{O}(u_0;\mathcal{P},\mathcal{R})} = \mathcal{O}(u_0;\mathcal{P},\mathcal{R}) \cup \{0\}$,
and that $(\mathcal{X},p)$ is $(\mathcal{P},\mathcal{R})$-orbitally 0-complete at u_0. Define the following three functions:

$$\varphi_1(r) = \begin{cases} r + \frac{3}{2}, & r \in (0,1] \\ 1, & r = 0, \end{cases} \quad \varphi_2(r) = r + \frac{5}{2} \text{ and } \varphi_3(r) = \frac{r}{2} + 1 \text{ for all } r \geq 0.$$

Clearly, φ_1 is lower semicontinuous, also $\varphi_1(r)-\varphi_2(r)+\varphi_3(r) = r+\frac{3}{2}-r-\frac{5}{2}+\frac{r}{2}+1 > 0$ for all $r > 0$. Now, we shall prove that all the conditions of Theorem 3.1 are satisfied.

Take $u,v \in \overline{\mathcal{O}(u_0;\mathcal{P},\mathcal{R})}$, and so $0 \leq u,v \leq 1$. Consider two cases (if one of u,v is 0, the contraction condition is fulfilled):

Case I: $0 < v \leq u \leq 1$. Then

$$\varphi_1(p(\mathcal{R}u,\mathcal{P}v)) = \varphi_1\left(p\left(\frac{u}{5},\frac{v}{7}\right)\right) = \varphi_1\left(\frac{u}{5}\right) = \frac{u}{5} + \frac{3}{2}$$

and

$$\Theta(u,v) = \left\{ u, v, u, \frac{1}{2}[\max\{\frac{u}{5}, v\} + u], \frac{uv}{1+u}, \frac{uv}{1+\frac{u}{5}} \right\} = u.$$

Therefore the condition (4) reduces to

$$\frac{u}{5} + \frac{3}{2} \leq u + \frac{5}{2} - \frac{u}{2} - 1 = \frac{u}{2} + \frac{3}{2}$$

and obviously holds true.

Case II: $0 < u \leq v \leq 1$. Then

$$\varphi_1(p(\mathcal{R}u,\mathcal{P}v)) = \varphi_1\left(p\left(\frac{u}{5},\frac{v}{7}\right)\right) = \begin{cases} \frac{v}{7} + \frac{3}{2}, & \frac{u}{5} < \frac{v}{7} \\ \frac{u}{5} + \frac{3}{2}, & \frac{u}{5} \geq \frac{v}{7} \end{cases}$$

and

$$\Theta(u,v) = \left\{ u, v, v, \frac{1}{2}[v + \max\{\frac{v}{7}, u\}], \frac{uv}{1+v}, \frac{uv}{1+\max\{\frac{u}{5},\frac{v}{7}\}} \right\} = v.$$

In the case when $\frac{u}{5} < \frac{v}{7}$, the condition (4) reduces to $\frac{v}{7} + \frac{3}{2} \leq \frac{v}{2} + \frac{3}{2}$, and in the case when $\frac{u}{5} \geq \frac{v}{7}$, it reduces to $\frac{u}{5} + \frac{3}{2} \leq \frac{v}{2} + \frac{3}{2}$ and it is fulfilled in both cases.

Therefore, all the conditions of Theorem 3.1 are fulfilled and $t = 0$ is the common fixed point of $\mathcal{P}$, $\mathcal{R}$ in $\overline{\mathcal{O}(u_0;\mathcal{P},\mathcal{R})}$.

Observe that $\mathcal{P}$ and $\mathcal{R}$ do not satisfy the contraction condition for arbitrary $u, v \in \mathcal{X}$. Indeed, e.g., for $u = v = 2$, $\Theta(u, v) = 3$ and

$$\varphi_1(p(\mathcal{R}u, \mathcal{P}v)) = 3 + \frac{3}{2} > 3 + \frac{5}{2} - \frac{3}{2} - 1 = \varphi_2(\Theta(u, v)) - \varphi_3(\Theta(u, v)).$$

4. An application to integral equations

In this section, an existence theorem for a solution of the following system of Volterra-type integral equations is discussed by using Theorem 3.1.

$$\begin{cases} u(t) = g(t) + \int_0^t h_1(t, u(t))\, dt, \text{ for all } t \in [0, S], \\ u(t) = g(t) + \int_0^t h_2(t, u(t))\, dt, \text{ for all } t \in [0, S], \end{cases} \tag{9}$$

where $S > 0$. Let $\mathcal{X} = C([0, S], \mathbb{R})$ be the space of all real continuous functions defined on $[0, S]$ and endowed with the metric

$$d(u, v) = \|u - v\|_\infty = \max_{t\in[0,S]} |u(t) - v(t)|.$$

It is well known that it is a complete metric space.

Theorem 4.1. *Let $\mathcal{R}, \mathcal{P}$ be two operators from $\mathcal{X}$ into itself defined by*

$$\begin{cases} \mathcal{R}u(t) = g(t) + \int_0^t h_1(t, u(t))\, dt, \text{ for all } t \in [0, S] \\ \mathcal{P}u(t) = g(t) + \int_0^t h_2(t, u(t))\, dt, \text{ for all } t \in [0, S]. \end{cases}$$

Suppose the following assertions hold:

(i) $h_1, h_2 \in C([0, S] \times \mathbb{R}, \mathbb{R})$ *and* $g \in \mathcal{X}$*;*
(ii) *there exists* $\tau \in [1, +\infty)$ *such that*
$$|h_1(t, u(t)) - h_2(t, v(t))| \leq \frac{3\tau}{4}\Theta(u, v)(t), \text{ for all } t \in [0, S] \text{ and } u, v \in \mathcal{X},$$

where

$$\Theta(u, v)(t) = \max \left\{ \begin{array}{c} |u(t) - v(t)|, |\mathcal{R}u(t) - u(t)|, |\mathcal{P}v(t) - v(t)|, \\ \frac{1}{2}[|\mathcal{R}u(t) - v(t)| + |\mathcal{P}v(t) - u(t)|], \\ \dfrac{|\mathcal{R}u(t) - u(t)|\,|\mathcal{P}v(t) - v(t)|}{e^{\tau t}[1 + \max_{t\in[0,S]} |u(t) - v(t)|e^{-\tau t}]}, \\ \dfrac{|\mathcal{R}u(t) - u(t)|\,|\mathcal{P}v(t) - v(t)|}{e^{\tau t}[1 + \max_{t\in[0,S]} |\mathcal{R}u(t) - \mathcal{P}v(t)|e^{-\tau t}]}. \end{array} \right\}.$$

Then the system (9) has at least one solution $u^* \in \mathcal{X}$.

Proof. For $u \in \mathcal{X}$, define $\|u\|_\tau = \max_{t\in[0,S]}\{|u(t)|e^{-\tau t}\}$, where $\tau \geq 1$ is given in (ii). Observe that $\|.\|_\tau$ is a norm equivalent to the maximum norm and $(\mathcal{X}, \|\cdot\|_\tau)$ is a Banach space, see [3, 5, 18]. The respective metric is given as

$$d_\tau(u, v) = \max_{t\in[0,S]}\{|u(t) - v(t)|e^{-\tau t}\}, \text{ for all } u, v \in \mathcal{X}. \tag{10}$$

Now consider $\mathcal{X}$ endowed with the partial metric

$$p_\tau(u, v) = \begin{cases} d_\tau(u, v), & \text{if } \|u\|_\tau, \|v\|_\tau \leq 1, \\ d_\tau(u, v) + \tau, & \text{otherwise.} \end{cases}$$

It is known that $(\mathcal{X}, p_\tau)$ is not complete but it is 0-complete. In fact, the associated metric $p^s_\tau(u,v) = 2p_\tau(u,v) - p_\tau(u,u) - p_\tau(v,v)$ (see [14]) is given by

$$p^s_\tau(u,v) = \begin{cases} 2d_\tau(u,v), & \text{if } (\|u\|_\tau, \|v\|_\tau \le 1) \text{ or } (\|u\|_\tau, \|v\|_\tau \ge 1), \\ 2d_\tau(u,v) + \tau, & \text{otherwise,} \end{cases}$$

and consequently $(\mathcal{X}, p^s_\tau)$ is not complete (see [17]). It is evident that $u^* \in \mathcal{X}$ is a solution of (9) iff u^* is a fixed point of $\mathcal{R}$. Now, we have to prove that the condition (4) holds true. Notice that this condition need not be checked for $u = v \in \mathcal{X}$. From assertion (ii), we have

$$\begin{aligned}
&|\mathcal{R}u(t) - \mathcal{P}v(t)| \\
&= \left| \int_0^t h_1(t,u(t)) - h_2(t,v(t))\, dt \right| \le \int_0^t |h_1(t,u(t)) - h_2(t,v(t))|\, dt \\
&\le \frac{3\tau}{4} \int_0^t \Theta(u,v)(t)\, dt = \frac{3\tau}{4} \int_0^t e^{\tau t} \Theta(u,v)(t) \frac{e^{-2\tau t}}{e^{-\tau t}}\, dt \\
&\le \frac{3\tau}{4} \left(\int_0^t e^{\tau t}\, dt \right) \left(\max \left\{ \begin{array}{c} \|u-v\|_\tau, \|\mathcal{R}u-u\|_\tau, \|\mathcal{P}v-v\|_\tau, \frac{1}{2}[\|\mathcal{R}u-v\|_\tau + \|\mathcal{P}v-u\|_\tau], \\ \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|u-v\|_\tau}, \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|\mathcal{R}u-\mathcal{P}v\|_\tau} \end{array} \right\} \right) \\
&\le \frac{3\tau}{4} \frac{e^{\tau t}}{\tau} \left(\max \left\{ \begin{array}{c} \|u-v\|_\tau, \|\mathcal{R}u-u\|_\tau, \|\mathcal{P}v-v\|_\tau, \frac{1}{2}[\|\mathcal{R}u-v\|_\tau + \|\mathcal{P}v-u\|_\tau], \\ \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|u-v\|_\tau}, \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|\mathcal{R}u-\mathcal{P}v\|_\tau} \end{array} \right\} \right)
\end{aligned}$$

which implies that, if $\|u\|_\tau, \|v\|_\tau \le 1$, then

$$\begin{aligned}
&|\mathcal{R}u(t) - \mathcal{P}v(t)| e^{-\tau t} \\
&\qquad \le \frac{3}{4} \left(\max \left\{ \begin{array}{c} \|u-v\|_\tau, \|\mathcal{R}u-u\|_\tau, \|\mathcal{P}v-v\|_\tau, \frac{1}{2}[\|\mathcal{R}u-v\|_\tau + \|\mathcal{P}v-u\|_\tau], \\ \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|u-v\|_\tau}, \frac{\|\mathcal{R}u-u\|_\tau \|\mathcal{P}v-v\|_\tau}{1+\|\mathcal{R}u-\mathcal{P}v\|_\tau} \end{array} \right\} \right).
\end{aligned}$$

Now, by considering the control functions $\varphi_1, \varphi_2, \varphi_3 : [0,+\infty) \to [0,+\infty)$ described by:

$$\varphi_1(t) = t + \frac{3}{2}, \quad \varphi_2(t) = t + \frac{5}{2} \quad \text{and} \quad \varphi_3(t) = \frac{t}{4} + 1, \text{ for } t \ge 0,$$

we get

$$\begin{aligned}
\varphi_1(p_\tau(\mathcal{R}u,\mathcal{P}v)) \le \varphi_2 &\left(\max \left\{ \begin{array}{c} p_\tau(u,v), p_\tau(\mathcal{R}u,u), p_\tau(\mathcal{P}v,v), \frac{1}{2}[p_\tau(\mathcal{R}u,v) + p_\tau(\mathcal{P}v,u)] \\ \frac{p_\tau(\mathcal{R}u,u) p_\tau(\mathcal{P}v,v)}{1+p_\tau(u,v)}, \frac{p_\tau(\mathcal{R}u,u) p_\tau(\mathcal{R}v,v)}{1+p_\tau(\mathcal{R}u,\mathcal{P}v)} \end{array} \right\} \right) \\
- \varphi_3 &\left(\max \left\{ \begin{array}{c} p_\tau(u,v), p_\tau(\mathcal{R}u,u), p_\tau(\mathcal{P}v,v), \frac{1}{2}[p_\tau(\mathcal{R}u,v) + p_\tau(\mathcal{P}v,u)] \\ \frac{p_\tau(\mathcal{R}u,u) p_\tau(\mathcal{P}v,v)}{1+p_\tau(u,v)}, \frac{p_\tau(\mathcal{R}u,u) p_\tau(\mathcal{R}v,v)}{1+p_\tau(\mathcal{R}u,\mathcal{P}v)} \end{array} \right\} \right).
\end{aligned}$$

Therefore by Theorem 3.1 we conclude that there is $u^* \in \mathcal{X}$ which is a common fixed point for the mappings $\mathcal{R}, \mathcal{P}$ and, hence, also a solution to the system of integral equations (9). □

5. Application to fractional differential equations

This section is devoted to the existence of solutions for a nonlinear fractional differential equation as an application of Corollary 3.1. It is inspired by the paper [4].

The Caputo derivative of fractional order β is defined as

$$C_{D^\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\, ds \quad (n-1<\beta<n,\;\; n=[\beta]+1),$$

where $g:[0,\infty)\to\mathbb{R}$ is a continuous function, $[\beta]$ denotes the integer part of a positive real number β and Γ is the gamma function.

We consider the nonlinear fractional differential equation of the form

$$C_{D^\beta}(x(t)) = f(t,x(t)) \quad (0<t<1,\;\; 1<\beta\le 2) \tag{11}$$

with the integral boundary conditions

$$x(0)=0, \quad x(1)=\int_0^\eta x(s)\, ds \quad (0<\eta<1),$$

where $x\in C([0,1],\mathbb{R})$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. We denote $\mathcal{X}=C([0,1],\mathbb{R})$ endowed with the metric

$$d(u,v)=\|u-v\|_\infty=\max_{t\in[0,1]}|u(t)-v(t)|,$$

making it a complete metric space.

Theorem 5.1. *Let $\mathcal{R}\colon\mathcal{X}\to\mathcal{X}$ be the operator defined by*

$$\begin{aligned}\mathcal{R}u(t) = {} & \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{(\beta-1)} f(s,u(s))\, ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^t (1-s)^{(\beta-1)} f(s,u(s))\, ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \left(\int_0^s (s-m)^{(\beta-1)} f(m,u(m))\, dm\right) ds, \end{aligned} \tag{12}$$

where $t\in[0,1]$. Suppose that there exists $\tau\in[1,+\infty)$ such that

$$|f(t,u(t))-f(t,v(t))| \le \frac{5}{2e^\tau}\Gamma(\beta+1)\Theta(u,v)(t), \text{ for all } t\in[0,1] \text{ and } u,v\in\mathcal{X},$$

where

$$\Theta(u,v)(t)=\max\left\{\begin{array}{c} |u(t)-v(t)|, |\mathcal{R}u(t)-u(t)|, |\mathcal{R}v(t)-v(t)|, \\ \frac{1}{2}[|\mathcal{R}u(t)-v(t)|+|\mathcal{R}v(t)-u(t)|], \\ \dfrac{|\mathcal{R}u(t)-u(t)|\,|\mathcal{R}v(t)-v(t)|}{e^{\tau t}[1+\max_{t\in[0,1]}|u(t)-v(t)|e^{-\tau t}]}, \\ \dfrac{|\mathcal{R}u(t)-u(t)|\,|\mathcal{R}v(t)-v(t)|}{e^{\tau t}[1+\max_{t\in[0,1]}|\mathcal{R}u(t)-\mathcal{R}v(t)|e^{-\tau t}]} \end{array}\right\}.$$

Then the equation (11) has at least one solution $u^*\in\mathcal{X}$.

Proof. We can define metric and partial metric on $\mathcal{X}$ as in Section 4. Here we have only to check contraction condition (7) for $u, v \in \mathcal{X}$. From (12), we have

$$
\begin{aligned}
|\mathcal{R}u(t) - \mathcal{R}v(t)| = \Bigg| & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} f(s,u(s))\,ds \\
& - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} f(s,u(s))\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{(\beta-1)} f(m,u(m))\,dm \right) ds \Bigg| \\
- \Bigg| & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{(\beta-1)} f(s,v(s))\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} f(s,v(s))\,ds \\
& - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{(\beta-1)} f(m,v(m))\,dm \right) ds \Bigg| \\
\le & \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{(\beta-1)} |f(s,u(s)) - f(s,v(s))|\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} |f(s,u(s)) - f(s,v(s))|\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \int_0^s (s-m)^{(\beta-1)} \big| f(m,u(m)) - f(m,v(m))\,dm \big|\,ds \\
\le & \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(s)\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(s)\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s |s-m|^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(m)\,dm \right) ds, \\
= & \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(s) e^{\tau s} e^{-\tau s}\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(s) e^{\tau s} e^{-\tau s}\,ds \\
& + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s |s-m|^{(\beta-1)} \frac{\Gamma(\beta+1)}{5} \Theta(u,v)(m) e^{\tau m} e^{-\tau m}\,dm \right) ds, \\
\le & \frac{5}{2e^\tau} \Gamma(\beta+1) \Theta(u,v) \times \sup_{t\in(0,1)} \begin{pmatrix} \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{(\beta-1)} e^{\tau s}\,ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{(\beta-1)} e^{\tau s}\,ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \int_0^s |s-m|^{(\beta-1)} e^{\tau m}\,dm\,ds \end{pmatrix} \\
\le & \frac{1}{2} \Theta(u,v)
\end{aligned}
$$

which implies that

$$|\mathcal{R}u(t) - \mathcal{R}v(t)| e^{-\tau t} \le \frac{1}{2} \Theta(u,v) e^{-\tau t} \le \frac{1}{2} \Theta(u,v),$$

where

$$\Theta(u,v)=\max\left\{\begin{array}{c}\|u-v\|_\tau,\|\mathcal{R}u-u\|_\tau,\|\mathcal{R}v-v\|_\tau,\\ \frac{1}{2}[\|\mathcal{R}u-v\|_\tau+\|\mathcal{R}v-u\|_\tau],\\ \dfrac{\|\mathcal{R}u-u\|_\tau\|\mathcal{R}v-v\|_\tau}{1+\|u-v\|_\tau},\dfrac{\|\mathcal{R}u-u\|_\tau\,\|\mathcal{R}v-v\|_\tau}{1+\|\mathcal{R}u-\mathcal{R}v\|_\tau}\end{array}\right\}.$$

Now, by considering the control functions $\varphi_1,\varphi_2,\varphi_3:[0,+\infty)\to[0,+\infty)$ defined by:

$$\varphi_1(t)=t+\frac{3}{2},\quad \varphi_2(t)=t+\frac{5}{2}\quad\text{and}\quad \varphi_3(t)=\frac{t}{2}+1,\ \text{for } t\geq 0,$$

we get

$$\begin{aligned}\varphi_1(p_\tau(\mathcal{R}u,\mathcal{R}v)\leq\varphi_2&\left(\max\left\{\begin{array}{c}p_\tau(u,v),p_\tau(\mathcal{R}u,u),p_\tau(\mathcal{R}v,v),\dfrac{p_\tau(\mathcal{R}u,u)p_\tau(\mathcal{R}v,v)}{1+p_\tau(u,v)},\\ \dfrac{1}{2}[p_\tau(\mathcal{R}u,v)+p_\tau(\mathcal{R}v,u)],\dfrac{p_\tau(\mathcal{R}u,u)p_\tau(\mathcal{R}v,v)}{1+p_\tau(\mathcal{R}u,\mathcal{R}v)}\end{array}\right\}\right)\\ &-\varphi_3\left(\max\left\{\begin{array}{c}p_\tau(u,v),p_\tau(\mathcal{R}u,u),p_\tau(\mathcal{R}v,v),\dfrac{p_\tau(\mathcal{R}u,u)p_\tau(\mathcal{R}v,v)}{1+p_\tau(u,v)},\\ \dfrac{1}{2}[p_\tau(\mathcal{R}u,v)+p_\tau(\mathcal{R}v,u)],\dfrac{p_\tau(\mathcal{R}u,u)p_\tau(\mathcal{R}v,v)}{1+p_\tau(\mathcal{R}u,\mathcal{R}v)}\end{array}\right\}\right).\end{aligned}$$

Therefore, by Corollary 3.1 we conclude that there is a fixed point $u^*\in\mathcal{X}$ of the operator $\mathcal{R}$ and u^* is also a solution to the integral equation (12) and the fractional differential equation (11). □

6. Conclusion

This paper delivers some generalizations of the results of Kadelburg and Radenović [11] in the sense of rational type weakly contraction condition for two mappings. An immediate applications are given to prove existence and uniqueness of solution for a system of Volterra-type integral equations and existence of solutions for a nonlinear fractional differential equation.

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† Department of Mathematics, Veer Surendra Sai University of Technology, Burla - 768018, INDIA

E-mail address: `skpadhan_math@vssut.ac.in & gvvjagan1@gmail.com`

‡ Department of Mathematics, Texas A & M University - Kingsville - 78363-8202, Texas, USA, & Department of Mathematics, Amity University, Chhattisgarh, Raipur - 493225, INDIA

E-mail address: `drhknashine@gmail.com`

§ Department of Mathematics, University of Belgrade, Studentski trg 16, 11000, SERBIA

E-mail address: `kadelbur@matf.bg.ac.rs`