ON THE HYPERGEOMETRIC MATRIX k-FUNCTIONS

G. RAHMAN, A. GHAFFAR, S. D. PUROHIT, S. MUBEEN, M. ARSHAD

Abstract. The hypergeometric functions with the matrix arguments are considered to be important as they have many applications described by a number of researchers. In this present paper, we deal with the study of the hypergeometric k-functions i.e., $F_k(P,Q;R;z)$ with the matrix arguments P, Q and R and these matrix arguments satisfy the matrix differential equation in terms of the new parameter $k > 0$ which is the improved version of generalization of classical hypergeometric matrix functions. Further, we obtain an integral representation of $F_k(P,Q;R;z)$ for the case where Q, R and $R-Q$ are positive stable matrices with the property that $QR = RQ$ by using the definitions of gamma and beta matrix k-functions recently defined by the researchers.

1. Introduction

Most of the special functions encountered in physics, engineering, analytic functions and probability theory are special cases of hypergeometric functions ([12, 19],[26]-[29]). A function of matrix argument is a real or complex valued function of the elements of a matrix. Special matrix functions appear in the literature related to Statistics [1], Lie groups theory [11], and more recently in connection with matrix analogues of Laguerre matrix polynomial and system of second order differential equations for matrix arguments, orthogonal matrix polynomial and second order differential equations, Hermite and Legendre differential equations and the corresponding polynomial families (see [13]-[16]). Also, many researchers ([8]-[10],[17, 18]) have defined matrix computation, Bessel function of matrix arguments, ordinary differential equation of matrix arguments, properties of gamma and beta matrices and hypergeometric matrix arguments. Apart from the close relationship with the well-known beta and gamma matrix k-functions, the emerging theory of orthogonal matrix polynomials ([4]-[6]) and its operational calculus suggest the study of hypergeometric matrix k-functions.
The paper is organized as follows: In section 2, we deal with the study of new properties of the beta and gamma matrix k-functions. We are mainly concerned

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with the matrix analog of the formula

$$B_k(P,Q) = \frac{\Gamma_k(P)\Gamma_k(Q)}{\Gamma_k(P+Q)} \tag{1.1}$$

and may be regarded as a continuation of [25]. In Section 3, the Gauss hypergeometric matrix k-functions $F_k(P,Q;R;z)$ is introduced as a matrix power series. Conditions for the convergence on the boundary of the unit disc are treated. In section 4, we define some results of an integral representations of hypergeometric matrix k-functions. Recently the researchers have worked on special k-functions (see [20]-[22],[24]). Mubeen *et al.* [23] defined the solution of hypergeometric k-differential equations. In this paper, we also prove that if matrices Q and R commutes and are positive stables (where positive stable means if every eigenvalue of the matrix has positive real part), then $F_k(P,Q;R;z)$ is a solution of the differential equation

$$kz(1-kz)\omega'' - kzP\omega' + (R - (Q+kI)kz)\omega' - PQ\omega = 0.$$

If P is an arbitrary matrix in $\mathbb{C}^{r \times r}$ and R is an invertible matrix whose eigenvalues are not negative integers then we prove that equation

$$kz(1-kz)\omega'' - kzP\omega' + (R + (n-k)kIz)\omega' - nP\omega = 0$$

has matrix polynomial solutions of degree n for all integer $n \geq 1$.

Finally in Section 4, we define an integral representation of the hypergeometric matrix k-functions. Throughout in this paper, for a matrix P in $\mathbb{C}^{r \times r}$ and its spectrum $\sigma(P)$ denotes the set of all the eigenvalues of P. The 2-norm of P will be denoted by $\parallel P \parallel$ and it is defined by

$$\parallel P \parallel = \sup_{x \neq 0} \frac{\parallel Px \parallel_2}{\parallel x \parallel_2} \tag{1.2}$$

where for a y in $C^{r \times r}$, $\parallel Y \parallel_2 = (yTy)^{\frac{1}{2}}$ is the Euclidean norm of y. Let us denote $\sigma(P)$ and $\beta(P)$ the real numbers

$$\alpha(P) = \max\{Re(z) : z \in \sigma(P)\}, \qquad \beta(Q) = \min\{Re(z) : z \in \sigma(P)\}. \tag{1.3}$$

Let $f(z)$ and $g(z)$ be two holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and P is a matrix in $\mathbb{C}^{r \times r}$ with $\sigma(P) \subset \Omega$, then from the properties of matrix functional calculus ([3],p.558), it follows that

$$f(P)g(Q) = g(Q)f(P). \tag{1.4}$$

The reciprocal of gamma k-function denoted by $\Gamma_k^{-1} = \frac{1}{\Gamma_k}$ is an entire function of the complex variable. Like wise the image of the inverse gamma matrix k-function acting on the matrix P, denoted by $\Gamma_k^{-1}(P)$ is a well defined matrix for $k > 0$. Now, if

$P + nkI$ is invertible matrix for every integer $n \geq 0$ and $k > 0$,

then Γ_k is invertible and its inverse coincides with $\Gamma_k^{-1}(P)$, and recently Mubeen *et al.* [25] defined

$$P(P+kI)(P+2kI)\cdots(P+(n-1)kI)\Gamma_k^{-1}(P+nkI) = \Gamma_k^{-1}(P), n \geq 1, k > 0. \tag{1.5}$$

In the same paper, they introduced by using the condition that $P+nkI$ is invertible matrix, then equation (1.5) can be written as

$$P(P + kI)(P + 2kI) \cdots (P + (n-1)kI) = \Gamma_k(P + nkI)\Gamma_k^{-1}(P), n \geq 1, k > 0, \quad (1.6)$$

and like the Pochhammer k-symbol for any matrix P in $\mathbb{C}^{r \times r}$ by application of the matrix functional calculus, they defined

$$(P)_{n,k} = P(P + kI)(P + 2kI) \cdots (P + (n-1)kI), \quad n > 0, \quad (P)_0 = I. \quad (1.7)$$

The Schur deposition of a matrix P is given by (see [8], pp. 192-193)

$$\|e^{tP}\| \leq e^{t\alpha(P)} \sum_{i=0}^{r-1} \frac{(\|P\| r^{\frac{1}{2}} t)}{i!}, \quad t \geq 0 \quad (1.8)$$

2.   On Gamma, Beta Matrix k-Functions

In this section, we used the property of commutativity of matrices and extend the matrix framework of gamma and beta k-functions which will be used in section 4 to obtain an integral representation of the hypergeometric matrix k-function. Further more for the sake of clarity in the presentation, we recall the following results recently defined in [25].

Definition 1. Gamma Matrix k-Function [25].   If P is a positive stable matrix in $\mathbb{C}^{r \times r}$, $n \geq 1$ is an integer and $k > 0$, then

$$\Gamma_k(P) = \lim_{n \to \infty} n! k^n (P)_{n,k}^{-1} (nk)^{\frac{P}{k} - I}. \quad (2.1)$$

Definition 2. Beta Matrix k-Function [25].   If P and Q are positive stable matrices in $\mathbb{C}^{r \times r}$, then beta k-function is defined by

$$\beta_k(P, Q) = \frac{1}{k} \int_0^1 t^{\frac{P}{k} - I} (1 - t)^{\frac{Q}{k} - I} dt. \quad (2.2)$$

Hence the authors defined that if P and Q are commuting positive stable matrices then $\beta_k(P, Q) = \beta_k(Q, P)$, and commutativity is a necessary condition for the symmetry of beta k-function, see [25].

Lemma 2.1. *Let P and Q be positive matrices in $\mathbb{C}^{r \times r}$ such that $PQ = QP$ and satisfy the condition $P + Q + mkI$ is invertible for all integer $m \geq 0$ and $k > 0$.*
If $n \geq 0$ is an integer, then the following identities hold:
(i)

$$\beta_k(P, Q + nkI) \quad = \quad (P + Q)_{n,k}^{-1} (Q)_{n,k} \beta_k(P, Q),$$

(ii)

$$\beta_k(P + nkI, Q + nkI) \quad = \quad (P + Q)_{2n,k}^{-1} (Q)_{n,k} \beta_k(P, Q).$$

Proof. (i) For $n = 0$ the proof is obvious. Let us assume that $0 < m \geq n$ and using the fact that $PQ = QP$, it follows that

$$
\beta_k(P, Q + mkI) = \frac{1}{k} \int_0^1 t^{\frac{P}{k} - I} (1 - t)^{\frac{Q}{k} + (m-1)I} dt
$$

$$
= \frac{1}{k} \lim_{\delta \to 0} \int_\delta^{1-\delta} t^{\frac{P}{k} - I} (1 - t)^{\frac{Q}{k} + (m-1)I} dt
$$

$$
= \frac{1}{k} \lim_{\delta \to 0} \int_\delta^{1-\delta} t^{\frac{P+Q}{k} + (m-2)I} (1 - t)^{\frac{Q}{k} + (m-1)I} t^{-(\frac{Q}{k} + (m-1)I)} dt
$$

$$
= \frac{1}{k} \lim_{\delta \to 0} \int_\delta^{1-\delta} u(t) v(t) dt, \tag{2.3}
$$

where

$$
u(t) = (1 - t)^{\frac{Q}{k} + (m-1)I} t^{-(\frac{Q}{k} + (m-1)I)} t^{\frac{P}{k}}, \quad v(t) = t^{\frac{P+Q}{k} + (m-2)I}.
$$

Integrating equation (2.3) by parts, we get

$$
\beta_k(P, Q + mkI) = \lim_{\delta \to 0} [k(P + Q + (m-1)kI)^{-1} (1 - t)^{\frac{Q}{k} + (m-1)I} t^{\frac{P}{k}}]_{t=\delta}^{t=1-\delta}
$$

$$
+ \lim_{\delta \to 0} k(P + Q + (m-1)kI)^{-1}
$$

$$
\times \int_\delta^{1-\delta} \{ \frac{1}{k} (Q + (m-1)kI)(1 - t)^{\frac{Q}{k} + (m-2)I} t^{\frac{P}{k}}
$$

$$
+ \frac{1}{k} (Q + (m-1)kI)(1 - t)^{\frac{Q}{k} + (m-1)} t^{\frac{P}{k}} \} dt
$$

$$
= k(P + Q + (m-1)kI)^{-1} (Q + (m-1)kI)
$$

$$
\times \frac{1}{k} \int_0^1 (1 - t)^{\frac{Q}{k} + (m-1)} t^{\frac{P}{k}} dt
$$

$$
= (P + Q + (m-1)kI)^{-1} (Q + (m-1)kI)
$$

$$
\times \beta_k(P, Q + (m-1)kI).
$$

Hence by using an induction, we obtain

$$
\beta_k(P, Q + nkI) = (P + Q)_{n,k}^{-1} (Q)_{n,k} \beta_k(P, Q).
$$

(ii). To prove (ii), we apply (i) by taking $\hat{P} = P + nkI$ where $n \geq 1$. Then by (i) it follows that

$$
\beta_k(\hat{P}, Q + nkI) = (\hat{P} + Q)_{n,k}^{-1} (Q)_{n,k} \beta_k(\hat{P}, Q). \tag{2.4}
$$

Since $PQ = QP$, therefore we have $\hat{P}Q = Q\hat{P}$ and $\beta_k(\hat{P}, Q) = \beta_k(Q, \hat{P})$. By (2.4) it follows that

$$
\beta_k(\hat{P}, Q + nkI) = (\hat{P} + Q)_{n,k}^{-1} (Q)_{n,k} \beta_k(Q, \hat{P}). \tag{2.5}
$$

Also by (i), we have

$$\beta_k(Q, P + nkI) = (Q + P)^{-1}_{n,k}(P)_{n,k}\beta_k(Q, P) = (Q + P)^{-1}_{n,k}(P)_{n,k}\beta_k(P, Q). \quad (2.6)$$

By equations (2.4) and (2.5), we get

$$\beta_k(P + nkI, Q + nkI) = \beta_k(\hat{P}, Q + nkI) \quad = \quad (P + Q + nkI)^{-1}_{n,k}(Q)_{n,k}(P)_{n,k}(2.7)$$
$$\times \quad (Q + P)^{-1}_{n,k}\beta_k(P, Q). \quad (2.8)$$

Now by definition, we have $(P + Q + nkI)_{n,k}(Q + P)_{n,k} = (P + Q)_{2n,k}$. Hence by substituting in equation (2.7), we get the required result as

$$\beta_k(P + nkI, Q + nkI) = (P + Q)^{-1}_{2n,k}(P)_{n,k}(Q)_{n,k}\beta_k(P, Q).$$

$$\square$$

Lemma 2.2. *Let P and Q be commuting matrices in $\mathbb{C}^{r \times r}$ such that P, Q and $P + Q$ are positive stable matrices, then*

$$\beta_k(P, Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P + Q).$$

Proof. Since the matrices P and Q are stable and also $PQ = QP$, we can write it as

$$\Gamma_k(P)\Gamma_k(Q) = (\int_0^\infty u^{P-I}e^{-\frac{u^k}{k}}\,du)(\int_0^\infty v^{Q-I}e^{-\frac{v^k}{k}}\,dv). \quad (2.9)$$

By changing of variables $x = \frac{u^k}{u^k + v^k}$ and $y = u^k + v^k$, then equation (2.9) becomes

$$\Gamma_k(P)\Gamma_k(Q) \quad = \quad \int_0^\infty \int_0^1 (xy)^{\frac{1}{k}(P-I)}e^{-\frac{1}{k}(xy)}\frac{1}{k}x^{\frac{1}{k}-I}y^{\frac{1}{k}}(y(1-x))^{\frac{1}{k}(Q-I)}e^{-\frac{1}{k}(y(1-x))}$$
$$\times \quad \frac{1}{k}y^{\frac{1}{k}-I}(1-x)^{\frac{1}{k}}\,dxdy$$

$$= \quad (\frac{1}{k}\int_0^\infty (y)^{\frac{1}{k}(P+Q)-I}e^{-\frac{y}{k}}\,dy)(\frac{1}{k}\int_0^1 x^{\frac{P}{k}-I}(1-x)^{\frac{Q}{k}-I}\,dx). \quad (2.10)$$

Now by replacing $y = t^k$ in the first integral of (2.10), we get

$$\Gamma_k(P)\Gamma_k(Q) \quad = \quad (\int_0^\infty t^{P+Q-I}e^{-\frac{t^k}{k}}\,dt)(\frac{1}{k}\int_0^1 x^{\frac{P}{k}-I}(1-x)^{\frac{Q}{k}-I}\,dx)$$
$$= \quad \Gamma_k(P + Q)\beta_k(P, Q).$$

$$\square$$

Definition 3. Let us consider P and Q be two commuting matrices in $\mathbb{C}^{r \times r}$ such that for all integer $n \geq 0$ and satisfy the condition

$$P + nkI, \qquad Q + nkI, \qquad P + Q + nkI \quad \forall \quad k > 0, \quad (2.11)$$

are invertible matrices.
Let $\alpha(P, Q) = \min\{\alpha(P), \alpha(Q), \alpha(P + Q)\}$ and let $n_0 = n_0(P, Q) = [|\alpha(P, Q)|] + 1$,

where $[|\alpha(P,Q)|]$ denotes the entire part function. Then beta k-function $\beta_k(P,Q)$ is defined by

$$\beta_k(P,Q) = (P)_{n_0,k}^{-1}(Q)_{n_0,k}^{-1}(P+Q)_{2n_0,k}\beta_k(P+n_0kI, Q+n_0kI). \qquad (2.12)$$

Theorem 2.3. *Let P and Q be two commuting matrices in $\mathbb{C}^{r \times r}$ satisfying the condition (2.11) for all integer $n \geq 0$. then*

$$\beta_k(P,Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P+Q).$$

Proof. Suppose that $n_0 = n_0(P,Q)$ be defined in definition 3, then we can write

$$\beta_k(P,Q) = (P)_{n_0,k}^{-1}(Q)_{n_0,k}^{-1}(P+Q)_{2n_0,k}\beta_k(P+n_0kI, Q+n_0kI),$$

where $P + nkI$ and $Q + nkI$ are positive stable matrices. By (2.4) we can write

$$\begin{aligned}
\Gamma_k(P) &= \Gamma_k(P+n_0kI)(P+(n_0-1)kI)^{-1}\cdots(P+kI)^{-1}P^{-1} \\
&= \Gamma_k(P+n_0kI)(P)_{n_0,k}^{-1},
\end{aligned}$$

$$\Gamma_k(Q) = \Gamma_k(Q+n_0kI)(Q)_{n_0,k}^{-1}$$

and

$$\Gamma_k(P+Q) = \Gamma_k(P+Q+2n_0kI)(P+Q)_{2n_0,k}^{-1}.$$

Since $PQ = QP$, we can write

$$\Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P+Q)$$

$$= \Gamma_k(P+n_0kI)\Gamma_k(Q+n_0kI)\Gamma_k^{-1}(P+Q+2n_0kI)(P)_{n_0,k}^{-1}(Q)_{n_0,k}^{-1}(P+Q)_{2n_0,k}. \qquad (2.13)$$

Since we know that the matrices $P+n_0kI$, $Q+n_0kI$ and $P+Q+2n_0kI$ are positive stable, so by Lemma 2 we get

$$\Gamma_k(P+n_0kI)\Gamma_k(Q+n_0kI)\Gamma_k^{-1}(P+Q+2n_0kI) = \beta_k(P+n_0kI, Q+n_0kI), \qquad (2.14)$$

and by Lemma 2.1 (ii), we have

$$\beta_k(P+n_0kI, Q+n_0kI) = (P)_{n_0,k}(Q)_{n_0,k}(P+Q)_{2n_0,k}^{-1}\beta_k(P,Q). \qquad (2.15)$$

Hence by (2.13)-(2.15), it follows that

$$\beta_k(P,Q) = \Gamma_k(P)\Gamma_k(Q)\Gamma_k^{-1}(P+Q).$$

$\square$

3. On the Hypergeometric Matrix k-Functions

In this section, we define the hypergeometric matrix k-function which is denoted by $F_k(P,Q;R;z)$ where $k > 0$ and defined as

$$F_k(P,Q;R;z) = \sum_{n=0}^{\infty} \frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}}{n!}z^n, \qquad (3.1)$$

where the matrices P, Q and R are in $\mathbb{C}^{r \times r}$ such that $R + nkI$ is invertible matrix for all $n \geq 0$. Now we prove that the hypergeometric matrix k-function converges for $|z| = 1$ and $k > 0$.

Theorem 3.1. *Let P, Q and R be positive stable matrices in $\mathbb{C}^{r \times r}$ such that*

$$\beta(R) > \alpha(P) + \alpha(Q). \tag{3.2}$$

Then the series (3.1) is absolutely convergent for $|z| = 1$.

Proof. Assume that there exist a positive number δ, then by hypothesis (3.2) we have

$$\beta(R) - \alpha(P) - \alpha(Q) = 2\delta. \tag{3.3}$$

Now let us write

$$(nk)^{1+\frac{\delta}{k}} [\frac{1}{n!} (P)_{n,k} (Q)_{n,k} (R)_{n,k}^{-1}]$$

$$= \frac{(nk)^{1+\frac{\delta}{k}}}{n!} \frac{(n-1)! k^{n-1} (nk)^{\frac{P}{k}} (nk)^{-\frac{P}{k}} (P)_{n,k}}{(n-1)! k^{n-1}}$$

$$\times \frac{(n-1)! k^{n-1} (nk)^{\frac{Q}{k}} (nk)^{-\frac{Q}{k}} (Q)_{n,k}}{(n-1)! k^{n-1}} (R)_{n,k}^{-1} (nk)^{\frac{R}{k}} (nk)^{-\frac{R}{k}}$$

$$= \frac{(nk)^{1+\frac{\delta}{k}}}{n} (\frac{(nk)^{-\frac{P}{k}} (P)_{n,k}}{(n-1)! k^{n-1}}) (nk)^{\frac{P}{k}} k^{n-1} (\frac{(nk)^{-\frac{Q}{k}} (Q)_{n,k}}{(n-1)! k^{n-1}}) (nk)^{\frac{Q}{k}}$$

$$\times k^{n-1} (n-1)! (R)_{n,k}^{-1} (nk)^{\frac{R}{k}} (nk)^{-\frac{R}{k}}$$

or

$$(nk)^{1+\frac{\delta}{k}} [\frac{1}{n!} (P)_{n,k} (Q)_{n,k} (R)_{n,k}^{-1}]$$

$$= k^n (nk)^{\frac{\delta}{k}} (\frac{(nk)^{-\frac{P}{k}} (P)_{n,k}}{(n-1)! k^{n-1}}) (nk)^{\frac{P}{k}} (\frac{(nk)^{-\frac{Q}{k}} (Q)_{n,k}}{(n-1)! k^{n-1}})$$

$$\times (nk)^{\frac{Q}{k}} ((n-1)! k^{n-1} (R)_{n,k}^{-1} (nk)^{\frac{R}{k}}) (nk)^{-\frac{R}{k}}. \tag{3.4}$$

By (1.8), we are taking into account that $\alpha(-R) = -\beta(R)$ thus we can write

$$\|(nk)^{\frac{P}{k}}\| \quad \|(nk)^{\frac{Q}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\| \leq (nk)^{\frac{1}{k}(\alpha(P)+\alpha(Q)-\beta(R))} \{\sum_{j=0}^{r-1} \frac{(\|P\| r^{\frac{1}{2}} \ln n)^j}{kj!}\}$$

$$\times \{\sum_{j=0}^{r-1} \frac{(\|Q\| r^{\frac{1}{2}} \ln n)^j}{kj!}\} \{\sum_{j=0}^{r-1} \frac{(\|R\| r^{\frac{1}{2}} \ln n)^j}{kj!}\}.$$

By (3.3), we obtain

$$\|(nk)^{\frac{P}{k}}\| \quad \|(nk)^{\frac{Q}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\| \leq (nk)^{-\frac{2\delta}{k}} \{\sum_{j=0}^{r-1} \frac{[\max\{\|P\|, \|Q\|, \|R\|\} r^{\frac{1}{2}}]^j}{kj!} (\ln n)^j\}^3. \tag{3.5}$$

Thus with the aid of (3.3)-(3.5) and for $|z| = 1$, we get

$$\lim_{n \to \infty} (nk)^{1+\frac{\delta}{k}} \|\frac{(P)_{n,k} (Q)_{n,k} (R)_{n,k}^{-1} z^n}{n!}\|$$

$$\leq \quad \lim_{n\to\infty} k^n (nk)^{-\frac{\delta}{k}} \|\frac{(nk)^{-\frac{P}{k}}(P)_{n,k}}{(n-1)!k^{n-1}}\| \quad \|(nk)^{\frac{P}{k}}\|$$

$$\times \quad \|\frac{(nk)^{-\frac{Q}{k}}(Q)_{n,k}}{(n-1)!k^{n-1}}\| \quad \|(nk)^{\frac{Q}{k}}\|$$

$$\times \quad \|(n-1)!k^{n-1}(R)_{n,k}^{-1}(nk)^{\frac{R}{k}}\| \quad \|(nk)^{-\frac{R}{k}}\|$$

$$\leq \quad \|\Gamma_k^{-1}(P)\| \quad \|\Gamma_k^{-1}(Q)\| \quad \|\Gamma_k(R)\|$$

$$\times \quad \lim_{n\to\infty} k^{2n-2}(nk)^{-\frac{\delta}{k}}\{\sum_{j=0}^{r-1} \frac{[\max\{\|P\|,\|Q\|,\|R\|\}r^{\frac{1}{2}}]^j}{kj!}$$

$$\times \quad (\ln n)^j\}^3$$

$$= \quad 0,$$

because

$$\lim_{n\to\infty} n^{-\frac{\delta}{k}}(\ln n)^j = 0, \quad \forall \quad j \geq 0, \quad k > 0.$$

Thus

$$\lim_{n\to\infty} (nk)^{1+\frac{\delta}{k}}\|\frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}z^n}{n!}\| = 0; \quad |z| = 1,$$

therefore the series (3.1) is absolutely convergent for $|z| = 1$. Now we show that under certain condition the hypergeometric matrix k-function $F_k(P, Q; R; z)$ is a solution of matrix differential equation of bilateral type. $\qquad\square$

Theorem 3.2. *Let R is matrix in $\mathbb{C}^{r\times r}$ satisfying $R+nkI$ is invertible matrix and $QR = RQ$. Then $F_k(P, Q; R; z)$ is the solution of*

$$kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW = 0, \quad 0 \leq |z| < 1 \quad (3.6)$$

satisfying $F_k(P, Q; R; 0) = I$.

Proof. By the given hypothesis $QR = RQ$, so we can write

$$F_{n,k} = \frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}}{n!} = \frac{(P)_{n,k}(R)_{n,k}^{-1}(Q)_{n,k}}{n!}.$$

Let us denote

$$W(z) = F_k(P, Q; R; z) = \sum_{n=0}^{\infty} F_{n,k}z^n, \quad |z| < 1. \quad (3.7)$$

Since $W(z)$ is a power series convergent for $|z| < 1$, so it is termwise differentiable in the given domain and

$$W'(z) = \sum_{n=1}^{\infty} nF_{n,k}z^{n-1}, \quad W''(z) = \sum_{n=2}^{\infty} n(n-1)F_{n,k}z^{n-2}, \quad |z| < 1.$$

Hence

$$kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW$$

$$= \sum_{n=2}^{\infty} nk(n-1)F_{n,k}z^{n-1} - \sum_{n=2}^{\infty} nk^2(n-1)F_{n,k}z^n - P\sum_{n=1}^{\infty} nkF_{n,k}z^n$$

$$+ \sum_{n=1}^{\infty} nF_{n,k}Rz^{n-1} - \sum_{n=1}^{\infty} nkF_{n,k}(Q+kI)z^n - \sum_{n=0}^{\infty} PF_{n,k}Qz^n,$$

replacing $n = n + 1$ in the first and fourth summation, we obtain

$$kz(1-kz)W'' - kzPW' + W'(R - kz(Q+kI)) - PQW$$

$$= \sum_{n=1}^{\infty} nk(n+1)F_{n+1,k}z^{n-1} - \sum_{n=2}^{\infty} nk^2(n-1)F_{n,k}z^n - P\sum_{n=1}^{\infty} nkF_{n,k}z^n$$

$$+ \sum_{n=1}^{\infty} nF_{n,k}Rz^{n-1} - \sum_{n=1}^{\infty} nkF_{n,k}(Q+kI)z^n - \sum_{n=0}^{\infty} PF_{n,k}Qz^n$$

$$= \sum_{n=1}^{\infty} \{nk(n+1)F_{n,k} - nk^2(n-1)F_{n,k} - nkPF_{n,k} + (n+1)F_{n+1,k}R$$

$$- nkF_{n,k}(Q+kI) - PF_{n,k}Q\}z^n + 2kF_{2,k}z - kPF_{1,k}z + F_{1,k}R + 2F_{2,k}Rz$$

$$- F_{1,k}(Q+kI)kz - PF_{0,k}Q - PF_{1,k}Qz = 0.$$

By equating the coefficients of each power z^n and noting that $F_{0,k} = I$, we get

$$z^0 \quad : \quad F_{1,k}R - PIQ = 0,$$

$$z^1 \quad : \quad 2kF_{2,k} - kPF_{1,k} + 2F_{2,k}R - F_{1,k}(Q+kI)kz - PF_{1,k}Q$$

$$= \quad 2F_{2,k}(kI+R) - PF_{1,k}(kI+Q) - F_{1,k}(Q+kI)k = 0$$

$$\vdots \quad \vdots$$

$$\Rightarrow \quad F_{n+1,k} = \frac{(P+nkI)F_{n,k}(Q+nkI)(R+nkI)^{-1}}{n+1}.$$

Hence $W(z) = F_k(P, Q; R; z)$ is the solution of (3.6) satisfying $W(0) = I$. $\qquad \square$

Corollary 3.3. *Let R be a matrix in $\mathbb{C}^{r \times r}$ satisfying that $R + nkI$ is invertible matrix for $n \geq 0$ and let P be an arbitrary matrix in $\mathbb{C}^{r \times r}$ and n be a positive integer. Then equation*

$$kz(1-kz)W'' - kzPW' + W'(R + z(n-k)kI) + nPW = 0 \qquad (3.8)$$

has matrix polynomial solutions of degree n.

Proof. Let $Q = -nI$, then by theorem 3.2 the function $W(z) = F_k(P, -nI; R; z)$ satisfies equation (3.6) for $Q = -nI$. Hence

$$W(z) = F_k(P, Q; R; z) = \sum_{l=0}^{n} \frac{(P)_{l,k}(-nI)_{l,k}(R)_{l,k}}{l!} z^l$$

is a matrix polynomial of degree n of equation (3.8). $\qquad \square$

4. An Integral Representation of Hypergeometric matrix k-function

In this section, we define the integral representation of hypergeometric matrix k-function. If y and b are complex numbers with $|y| < 1$, then the Taylor series expansion of $(1 - ky)^{-\frac{b}{k}}$ about $y = 0$ is given by [2]

$$(1 - ky)^{-\frac{b}{k}} = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} y^n, \quad |y| < 1, \quad a \in \mathbb{C}. \tag{4.1}$$

Let $f_{n,k}(a)$ be a function defined by

$$f_{n,k}(a) = \frac{(a)_{n,k}}{n!} y^n = \frac{a(a+k)(a+2k)\cdots(a+(n-1)k)}{n!} y^n, \quad a \in \mathbb{C} \quad k > 0, \tag{4.2}$$

for a fixed complex number y with $|y| < 1$. Clearly the function $f_{n,k}$ is an holomorphic function of variable a defined in the complex plane for $k > 0$. For a given closed disc $D_\alpha = \{a \in \mathbb{C} : |a| \le \alpha\}$, we have

$$|f_{n,k}(a)| \le \frac{(|a|)_{n,k}|y|^n}{n!} \le \frac{(\alpha)_{n,k}|y|^n}{n!}, \quad n \ge 0, \quad |a| \le \alpha, \quad k > 0.$$

Since

$$\sum_{n=0}^{\infty} \frac{(\alpha)_{n,k}|y|^n}{n!} \le +\infty,$$

so by the Weierstrass theorem for the convergence of holomorphic functions [27, 2] it follows that

$$g(a) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} y^n = (1 - ky)^{-\frac{a}{k}}$$

is holomorphic in R for $k > 0$. Thus by the application of the holomorphic functional calculus [3], for any matrix P in $\mathbb{C}^{r \times r}$, the image of g by this functional calculus acting on P yields

$$(1 - ky)^{-\frac{P}{k}} = g(P) = \sum_{n=0}^{\infty} \frac{(P)_{n,k}}{n!} y^n, \quad |y| < 1, \tag{4.3}$$

where

$$(P)_{n,k} = P(P + kI)\cdots(P + (n-1)kI), \quad k > 0.$$

Suppose that Q and R are matrices in $\mathbb{C}^{r \times r}$ such that $QR = RQ$ and Q, R and $R - Q$ are positive stable matrices. Thus by (1.5), (1.7) and with the aid of the condition that Q, R and $R - Q$ are positive stable matrices, we obtain

$$(Q)_{n,k}(R)_{n,k}^{-1} = \Gamma_k^{-1}(Q)\Gamma_k(Q + nkI)\Gamma_k(R)\Gamma_k^{-1}(R + nkI),$$

$$= \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R - Q)\Gamma_k(R - Q)\Gamma_k(Q + nkI)\Gamma_k^{-1}(R + nkI)\Gamma_k(R). \tag{4.4}$$

By positive stability condition of the matrices and by Lemma 2.2 it follows that

$$\frac{1}{k}\int_0^1 t^{\frac{Q}{k} + (n-1)I}(1 - t)^{\frac{R-Q}{k} - I} dt$$

$$= \beta_k(Q + nkI, R - Q) = \Gamma_k(R - Q)\Gamma_k(Q + nkI)\Gamma_k^{-1}(R + nkI), \tag{4.5}$$

by (4.4) and (4.5), we get

$$(Q)_{n,k}(R)_{n,k}^{-1} = \Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)[\frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}dt]\Gamma_k(R). \quad (4.6)$$

Hence, for $|z| < 1$ we can write

$$
\begin{aligned}
F_k(P,Q;R;z) &= \sum_{n=0}^{\infty} \frac{(P)_{n,k}(Q)_{n,k}(R)_{n,k}^{-1}}{n!}z^n \\
&= \sum_{n=0}^{\infty} \frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)z^n}{\Gamma_k(R)n!} \\
&\quad \times [\frac{1}{k}\int_0^1 t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}dt]
\end{aligned}
$$

$$= \sum_{n=0}^{\infty}[\frac{1}{k}\int_0^1 \frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}\Gamma_k(R)z^n}{n!}dt]. \quad (4.7)$$

Now let us consider

$$S_{n,k}(t) = \frac{(P)_{n,k}\Gamma_k^{-1}(Q)\Gamma_k^{-1}(R-Q)t^{\frac{Q}{k}+(n-1)I}(1-t)^{\frac{R-Q}{k}-I}\Gamma_k(R)z^n}{n!}, \quad 0 \le t \le 1,$$

and note that for $0 < t < 1$ and $n \ge 0$, we have

$$\|S_{n,k}(t)\|$$

$$\le \frac{(\|P\|)_{n,k}\|\Gamma_k^{-1}(Q)\|\|\Gamma_k^{-1}(R-Q)\|\|\Gamma_k(R)\|\|t^{\frac{Q}{k}-I}\|(1-t)^{\frac{R-Q}{k}-I}\||z|^n}{n!}, k > 0. (4.8)$$

By (1.8) it follows that

$$
\begin{aligned}
\|t^{\frac{Q}{k}-I} \quad \|(1-t)^{\frac{R-Q}{k}-I}\| &\le t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}[\sum_{j=0}^{r-1}\frac{(\|Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}] \\
&\quad \times [\sum_{j=0}^{r-1}\frac{(\|R-Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}]
\end{aligned}
$$

and noting that for $0 < t < 1$, we have $\ln t < t < 1$ and $\ln(1-t) < 1-t < 1$, hence from above expression we get

$$\|t^{\frac{Q}{k}-I} \quad \|(1-t)^{\frac{R-Q}{k}-I}\| \le \Lambda t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}, \quad 0 < t < 1, \quad k > 0 \quad (4.9)$$

where

$$\Lambda = [\sum_{j=0}^{r-1}\frac{(\|Q-kI\|r^{\frac{1}{2}})^j}{kj!}][\sum_{j=0}^{r-1}\frac{(\|R-Q-kI\|r^{\frac{1}{2}}\ln t)^j}{kj!}]. \quad (4.10)$$

Now let S be the sum of the convergent series

$$S = \sum_{n=0}^{r-1}\frac{(\|P\|)_{n,k}|z|^n}{n!}, \quad |z| < 1 \quad k > 0. \quad (4.11)$$

By (4.8)-(4.10), we obtain

$$\sum_{n=0}^{\infty} \|S_{n,k}(t)\| \le \phi(t) = \frac{1}{k}[L\Lambda S t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}], 0 < t < 1, k > 0, \quad (4.12)$$

where

$$L = \|\Gamma_k^{-1}(Q)\| \quad \|\Gamma_k^{-1}(R-Q)\| \quad \|\Gamma_k(R)\|.$$

Since $\alpha(Q) > 0$, $\alpha(R-Q) > 0$ and $k > 0$, then the function
$\phi(t) = \frac{1}{k}[L\Lambda t^{\frac{\alpha(Q)}{k}-1}(1-t)^{\frac{\alpha(R-Q)}{k}-1}]$ is integrable and

$$\int_0^1 \phi(t)dt = L\Lambda S B_k(\alpha(Q), \alpha(R-Q)).$$

Thus by dominated convergence theorem ([7], p.83), the series and the integral can be computed in (4.7) and using $QR = RQ$, we can write

$$F_k(P, Q; R; z)$$

$$= \frac{1}{k}\int_0^1 \{\sum_{n=0}^{\infty}(\frac{(P)_{n,k}(tz)^n}{n!})t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}\}dt\Gamma_k^{-1}(Q)\Gamma_k(R-Q)\Gamma_k(R). \quad (4.13)$$

Now by (4.3), we obtain

$$\sum_{n=0}^{\infty}\frac{(P)_{n,k}(tz)^n}{n!} = (1-ktz)^{-\frac{P}{k}}, \quad |z| < 1, \quad 0 < t < 1, \quad (4.14)$$

and (4.13) becomes

$$F_k(P, Q; R; z)$$

$$= \Gamma_k^{-1}(Q)\Gamma_k(R-Q)\Gamma_k(R)\frac{1}{k}\int_0^1 t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}(1-ktz)^{-\frac{P}{k}}dt. \quad (4.15)$$

Summarizing, the following result has been established:

Theorem 4.1. *Let P, Q and R be matrices in $\mathbb{C}^{r \times r}$ such that $QR = RQ$ and Q, R, $R - Q$ are positive stable matrices. Then for $|z| < 1$ it follows that;*

$$F_k(P, Q; R; z) = \Gamma_k^{-1}(Q)\Gamma_k(R-Q)\Gamma_k(R)[\frac{1}{k}\int_0^1 t^{\frac{Q}{k}-I}(1-t)^{\frac{R-Q}{k}-I}(1-ktz)^{-\frac{P}{k}}dt].$$

$$(4.16)$$

Corollary 4.2. *Let P, Q and R be matrices in $\mathbb{C}^{r \times r}$ and let $\hat{\alpha}(Q, R) = \min\{\alpha(Q), \alpha(R), \alpha(R-Q)\}$ and $n_1 = n_1(Q, R) = [|\hat{\alpha}(Q, R)|] + 1$, where $[|\hat{\alpha}(Q, R)|]$ denotes the entire part function. Suppose that $QR = RQ$, and*

$$\begin{aligned}
\sigma(Q) &\subset R \sim \{-n; n \ge n_1, n \in Z\} \\
\sigma(R-Q) &\subset R \sim \{-n; n \ge n_1, n \in Z\} \\
\sigma(R) &\subset R \sim \{-2n; n \ge n_1, n \in Z\}.
\end{aligned}$$

Then for $|z| < 1$, we have

$$
\begin{aligned}
F_k(P, Q + n_1 kI; R + 2n_1 kI; z) &= \Gamma_k^{-1}(Q + n_1 kI)\Gamma_k(R - Q + n_1 kI) \\
&\times \Gamma_k(R + 2n_1 kI) \\
&\times \frac{1}{k}\int_0^1 t^{\frac{Q}{k} + (n-1)I}(1 - t)^{\frac{R-Q}{k} + (n-1)I} \\
&\times (1 - ktz)^{-\frac{P}{k}} dt.
\end{aligned}
$$

Proof. Consider the matrices P, $\hat{Q} = Q + nkI$, $\hat{R} = R + nkI$ and $\hat{R}$, $\hat{Q}$, $\hat{R} - \hat{Q} = R - Q + n_1 kI$ are positive stable matrices. The result is now a consequence of theorem 4.1.

Conclusion. In this paper, the authors conclude that for each positive value of k there exist gamma, beta and hypergeometric matrix arguments which are the improved generalized version of the said classical functions with matrix arguments. Obviously the substitution $k = 1$ will lead to the results of the said classical functions with matrix arguments.

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Conflict of Interests

The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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GAUHAR RAHMAN

DEPARTMENT OF MATHEMATICS, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD, PAKISTAN.
E-mail address: gauhar55uom@gmail.com

ABDUL GHAFFAR

DEPARTMENT OF MATHEMATICS, BALOCHISTAN UNIVERSITY OF INFORMATION TECHNOLOGY, ENGINEERING AND MANAGEMENT SCIENCES
E-mail address: abdulghaffar.jaffar@gmail.com

SUNIL DUTT PUROHIT

DEPARTMENT OF HEAS (MATHEMATICS), RAJASTHAN TECHNICAL UNIVERSITY, KOTA-234010, INDIA.
E-mail address: sunil_a_purohit@yahoo.com

SHAHID MUBEEN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SARGODHA, SARGODHA 40100, PAKISTAN.
E-mail address: smjhanda@gmail.com

MUHAMMAD ARSHAD

DEPARTMENT OF MATHEMATICS, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD, PAKISTAN.
E-mail address: marshad_zia@gmail.com