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FIXED POINT RESULTS FOR α_* ψ-MULTIVALUED MAPPINGS

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Abstract. The aim of this paper is to establish fixed point results for semi α_*-admissible multivalued mappings satisfying generalized locally α_*-ψ-contractive conditions in left (right) K-sequentially complete dislocated quasi metric space. Applications have been given. An example has been constructed to demonstrate the novelty of our results.

1. Introduction

Let $H : M \longrightarrow M$ be a mapping. A point $x \in M$ is said to be a fixed point of M, if $x = Hx$. Fixed point results are a tool to approximate the unique solution of non linear functional equations. Many results appeared in literature related to the fixed point of mappings which are contractive on the whole domain. It may happens that $H : M \longrightarrow M$ is not a contraction but is a contraction on a subset of M. Recently, Beg et al. [8] proved a result concerning the existence of fixed points of a mapping satisfying locally contractive conditions on a closed ball (see also [3, 4, 5, 13, 22, 23, 24]). It is also possible that mapping satisfying locally contractive conditions on a sequence contained in a closed ball in M. One can obtain fixed point results for such mapping by using the suitable conditions.

The notion of dislocated topologies have useful applications in the context of logic programming semantics (see [10]). Dislocated metric space (metric-like space) (see [15, 19]) is a generalization of partial metric space (see [17]). Furthermore, dislocated quasi metric space (quasi-metric-like space) (see [8, 21, 26, 27]) generalized the idea of dislocated metric space and quasi-partial metric space (see [16, 22]).

Nadler [18], introduced a study of fixed point theorems involving multivalued mappings (see also [7]). Asl et al. [6] introduced the concepts of α_*-ψ contractive multifunctions, α_*-admissible mapping and obtained some fixed point results for these multifunctions (see also [1, 11]). In this paper, we discuss some new fixed point results for α_*-ψ-contractive type multivalued mappings in a closed ball in left (right) K-sequentially complete dislocated quasi metric space. Our results unify, extend and generalize several comparable results in the existing literature. We give the following definitions and results which will be needed in the sequel.

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Definition 1.1 [26] Let X be a nonempty set and let $d_q : X \times X \to [0,\infty)$ be a function, called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x, y, z \in X$:

(i) If $d_q(x,y) = d_q(y,x) = 0$, then $x = y$;

(ii) $d_q(x,y) \leq d_q(x,z) + d_q(z,y)$.

The pair (X, d_q) is called a dislocated quasi metric space.

It is clear that if $d_q(x,y) = d_q(y,x) = 0$, then from (i), $x = y$. But if $x = y$, $d_q(x,y)$ may not be 0. It is observed that if $d_q(x,y) = d_q(y,x)$ for all $x, y \in X$, then (X, d_q) becomes a dislocated metric space (metric-like space) (X, d_l). For $x \in X$ and $\varepsilon > 0$, $B_{d_q}(x,\varepsilon) = \{y \in X : d_q(x,y) < \varepsilon$ and $d_q(y,x) < \varepsilon\}$ and $\overline{B_{d_q}(x,\varepsilon)} = \{y \in X : d_q(x,y) \leq \varepsilon$ and $d_q(y,x) \leq \varepsilon\}$ are open ball and closed ball in (X, d_q) respectively. Also $B_{d_l}(x,\varepsilon) = \{y \in X : d_q(x,y) \leq \varepsilon\}$ is a closed ball in (X, d_l).

Example 1.2 [8] Let $X = R^+ \cup \{0\}$ and $d_q(x,y) = x + \max\{x,y\}$ for any $x, y \in X$. Now,

(i) If $d_q(x,y) = d_q(y,x) = 0$, then $x + \max\{x,y\} = y + \max\{y,x\} = 0$, which implies that $x = y = 0$.

(ii) Case 1: If $x \geq y$, then $d_q(x,y) = x + \max\{x,y\} = 2x$. Let $z \in X$. If $z \leq x$, then,

$$\begin{aligned} d_q(x,z) + d_q(z,y) &= x + \max\{x,z\} + z + \max\{z,y\} \\ &= x + x + z + \max\{z,y\} \geq 2x = d_q(x,y) \end{aligned}$$

If $z > x$, then, $d_q(x,z) + d_q(z,y) = x + z + z + z \geq 2x = d_q(x,y)$.

Case 2: If $x < y$, then $d_q(x,y) = x + y$. If $z \geq y$, then, $d_q(x,z) + d_q(z,y) = x + z + z + z \geq x + y = d_q(x,y)$. If $z < y$, then, $d_q(x,z) + d_q(z,y) = x + \max\{x,z\} + z + y \geq x + y = d_q(x,y)$.

Hence both the conditions of Definition 1.1 hold, so $d_q(x,y) = x + \max\{x,y\}$ defines a dislocated quasi metric on X.

Definition 1.3 [8] Let (X, d_q) be a dislocated quasi metric space.

(a) A sequence $\{x_n\}$ in (X, d_q) is called left (right) K-Cauchy if $\forall\, \varepsilon > 0$, $\exists\, n_0 \in N$ such that $\forall\, n > m \geq n_0$ (respectively $\forall\, m > n \geq n_0$), $d_q(x_m, x_n) < \varepsilon$.

(b) A sequence $\{x_n\}$ dislocated quasi-converges (for short d_q -converges) to x if $\lim\limits_{n\to\infty} d_q(x_n, x) = \lim\limits_{n\to\infty} d_q(x, x_n) = 0$ or for any $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $n > n_0$, $d_q(x, x_n) < \varepsilon$ and $d_q(x_n, x) < \varepsilon$. In this case x is called a d_q-limit of $\{x_n\}$.

(c) (X, d_q) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in X converges to a point $x \in X$ such that $d_q(x,x) = 0$.

Definition 1.4 Let (X, d_q) be a dislocated quasi metric space. Let K be a nonempty subset of X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$\begin{aligned} d_q(x,K) &= d_q(x,y_0),\ \text{where } d_q(x,K) = \inf_{y\in K} d_q(x,y) \\ \text{and } d_q(K,x) &= d_q(y_0,x),\ \text{where } d_q(K,x) = \inf_{y\in K} d_q(y,x). \end{aligned}$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

We denote $P(X)$ be the set of all proximinal subsets of X.

Definition 1.5 Let (X, d_q) be a dislocated quasi metric space, $S : X \to P(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, +\infty)$. Let $A \subseteq X$, we say that S is semi α_*-admissible on A, whenever $\alpha(x, y) \geq 1$ implies that $\alpha_*(Sx, Sy) \geq 1$, for all $x, y \in A$, where $\alpha_*(Sx, Sy) = \inf\{\alpha(a, b) : a \in Sx,\ b \in Sy\}$. If $A = X$, then we say that S is α_*-admissible on X.

Definition 1.6 The function $H_{d_q} : P(X) \times P(X) \to X$, defined by

$$H_{d_q}(A, B) = \max\{\sup_{a \in A} d_q(a, B),\ \sup_{b \in B} d_q(A, b)\}$$

is called dislocated quasi Hausdorff metric on $P(X)$. Also $(P(X), H_{d_q})$ is known as dislocated quasi Hausdorff metric space.

Lemma 1.7 Let (X, d_q) be a dislocated quasi metric space. Let $(P(X), H_{d_q})$ be a dislocated quasi Hausdorff metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$, such that $H_{d_q}(A, B) \geq d_q(a, b_a)$ and $H_{d_q}(B, A) \geq d_q(b_a, a)$.

Proof. We know that $H_{d_q}(A, B) = \max\{\sup\limits_{a \in A} d_q(a, B), \sup\limits_{b \in B} d_q(A, b)\}$. If $H_{d_q}(A, B) = \sup\limits_{a \in A} d_q(a, B)$, then $H_{d_q}(A, B) \geq d_q(a, B)$ for each $a \in A$. As B is a proximinal set, so for each $a \in X$, there exist at least one best approximation $b_a \in B$ satisfies $d_q(a, B) = d_q(a, b_a)$ and $d_q(B, a) = d_q(b_a, a)$. Now, we have $H_{d_q}(A, B) \geq d_q(a, b_a)$. Also, if $H_{d_q}(A, B) = \sup\limits_{b \in B} d_q(A, b) \geq \sup\limits_{a \in A} d_q(a, B)$. Hence, for each $a \in A$ there exists $b_a \in B$, such that $H_{d_q}(A, B) \geq d_q(a, b_a)$. Similarly, we can prove that $H_{d_q}(B, A) \geq d_q(b_a, a)$.

Definition 1.8 Let (X, d_q) be a dislocated quasi metric space. Let $U \subseteq X$, then

(a) U is called an open set, if for each $x \in U$, there exist an open ball $B_{d_q}(x, \varepsilon)$, such that $B_{d_q}(x, \varepsilon) \subseteq U$.

(b) a point $x \in X$ is called a limit point of U, if for each open ball $B_{d_q}(x, \varepsilon)$, $B_{d_q}(x, \varepsilon) \cap U - \{x\} \neq \phi$.

(c) the set of all limit points of U is denoted by $D(U)$.

(d) U is said to be a closed set if $D(U) \subseteq U$.

(e) the set $\overline{U}$ is the intersection of all closed super sets of U.

(f) the set U^c is the compliment of U.

Lemma 1.9 If U^c is an open set, then U is a closed set.

Proof. Let $x \in D(U)$. Suppose $x \notin U$, then $x \in U^c$. So, there exist an open ball $B_{d_q}(x, \varepsilon)$ such that $B_{d_q}(x, \varepsilon) \subseteq U^c$. As $U^c \cap U = \phi$, then $B_{d_q}(x, \varepsilon) \cap U = \phi$, which implies that $B_{d_q}(x, \varepsilon) \cap U - \{x\} = \phi$. Therefore, $x \notin D(U)$. A contradiction, so $x \in U$. Hence U is closed.

Lemma 1.10 Every closed ball in a dislocated quasi metric space is a closed set.

Proof. Let $\overline{B_{d_q}(x_0, r)}$ be a closed ball in a dislocated quasi metric space X. Let $x \in \overline{B_{d_q}(x_0, r)}^c$. It implies that $d_q(x_0, x) > r$. Let $\varepsilon = d_q(x_0, x) - r$ and $y \in B_{d_q}(x, \varepsilon)$. Then $d_q(x_0, x) \leq d_q(x_0, y) + d_q(y, x) < d_q(x_0, y) + \varepsilon \Rightarrow d_q(x_0, y) > r$. It follows that $y \in \overline{B_{d_q}(x_0, r)}^c$. It implies that $B_{d_q}(x, \varepsilon) \subseteq \overline{B_{d_q}(x_0, r)}^c$. Now by Lemma 1.9, $\overline{B_{d_q}(x_0, r)}$ is a closed set.

Lemma 1.11 Every closed ball Y in a left (right) K-sequentially complete dislocated quasi metric space X is left (right) K-sequentially complete.

Proof. Let $\{x_n\}$ be a left (right) K-Cauchy sequence in Y. As $Y \subseteq X$, then $\{x_n\}$ be a left (right) K-Cauchy sequence in X. As X is left (right) K-sequentially complete, so $x_n \to x \in X$. Therefore, for any $\varepsilon > 0$, there exists $n_0 \in N$, such that for all $n > n_0$, $d_q(x, x_n) < \varepsilon$ and $d_q(x_n, x) < \varepsilon$. It implies that each open ball $B_{d_q}(x, \varepsilon)$ contains all the points of $\{x_n\}$, except a finite number of points. As $\{x_n\}$ is a sequence in Y, so $B_{d_q}(x, \varepsilon)$ contains at least one point of Y, different from x. It implies that $x \in D(Y)$ and therefore $x \in \overline{Y}$. As Y is closed, so $\overline{Y} = Y$. This shows that $x \in Y$. Hence Y is left (right) K-sequentially complete.

2. Main Result

Let (X, d_q) be a dislocated quasi metric space, $x_0 \in X$ and $S : X \to P(X)$ be a multivalued mapping on X. As Sx_0 is a proximinal set, then there exists $x_1 \in Sx_0$ such that $d_q(x_0, Sx_0) = d_q(x_0, x_1)$ and $d_q(Sx_0, x_0) = d_q(x_1, x_0)$. Now, for $x_1 \in X$, there exist $x_2 \in Sx_1$ be such that $d_q(x_1, Sx_1) = d_q(x_1, x_2)$ and $d_q(Sx_1, x_1) = d_q(x_2, x_1)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Sx_n$, $d_q(x_n, Sx_n) = d_q(x_n, x_{n+1})$ and $d_q(Sx_n, x_n) = d_q(x_{n+1}, x_n)$. We denote this iterative sequence $\{XS(x_n)\}$ and say that $\{XS(x_n)\}$ is a sequence in X generated by x_0.

Theorem 2.1 Let (X, d_q) be a left (right) K-sequentially complete dislocated quasi metric space, $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$, $\alpha : X \times X \to [0, +\infty)$, $S : X \to P(X)$ be a semi α_*-admissible multifunction on $\overline{B_{d_q}(x_0, r)}$ and $\{XS(x_n)\}$ be a sequence in X generated by x_0 with $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_0) \geq 1$. Assume that for some $\psi \in \Psi$, the following hold:

$$\alpha_*(Sx, Sy) H_{d_q}(Sx, Sy) \leq \psi(d_q(x, y)), \text{ for all } x, y \in \overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\} \tag{2.1}$$

$$\text{and } \max\{\sum_{i=0}^{n} \psi^i(d_q(x_1, x_0)), \sum_{i=0}^{n} \psi^i(d_q(x_0, x_1))\} \leq r, \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$, $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(x_{n+1}, x_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0, r)}$. Also, if $\alpha(x_n, x^*) \geq 1$, $\alpha(x^*, x_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (2.1) holds for all $x, y \in (\overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_q}(x_0, r)}$.

Proof. Let $\{XS(x_n)\}$ be a sequence in X generated by x_0. Then, we have $x_{n+1} \in Sx_n$, and $d_q(x_n, Sx_n) = d_q(x_n, x_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$. By Lemma 1.7, we have

$$d_q(x_n, x_{n+1}) \leq H_{d_q}(Sx_{n-1}, Sx_n), \tag{2.3}$$

$$d_q(x_{n+1}, x_n) \leq H_{d_q}(Sx_n, Sx_{n-1}), \tag{2.4}$$

for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for any $n \in \mathbb{N} \cup \{0\}$, then x_n is a fixed point in $\overline{B_{d_q}(x_0, r)}$ of S. Let $x_n \neq x_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. From (2.2), we have

$$\max\{d_q(x_1, x_0), d_q(x_0, x_1))\} \leq \max\{\sum_{i=0}^{n} \psi^i(d_q(x_1, x_0)), \sum_{i=0}^{n} \psi^i(d_q(x_0, x_1))\} \leq r.$$

It follows that, $d_q(x_1, x_0) \leq r$ and $d_q(x_0, x_1) \leq r$. Hence, we have

$$x_1 \in \overline{B_{d_q}(x_0, r)}.$$

As $\alpha(x_0,x_1) \geq 1$ and S is semi α_*-admissible multifunction on $\overline{B_{d_q}(x_0,r)}$, so $\alpha_*(Sx_0,Sx_1) \geq 1$. As $\alpha_*(Sx_0,Sx_1) \geq 1$, $x_1 \in Sx_0$ and $x_2 \in Sx_1$, so $\alpha(x_1,x_2) \geq 1$. Let $x_2,\cdots,x_j \in \overline{B_{d_q}(x_0,r)}$ for some $j \in \mathbb{N}$. As S is semi α_*-admissible multifunction on $\overline{B_{d_q}(x_0,r)}$, thus, we have $\alpha_*(Sx_1,Sx_2) \geq 1$. As $\alpha_*(Sx_1,Sx_2) \geq 1$, we have $\alpha(x_2,x_3) \geq 1$, which further implies $\alpha_*(Sx_2,Sx_3) \geq 1$. Continuing this process, we have $\alpha_*(Sx_{j-1},Sx_j) \geq 1$. Now, by using (2.3), we have

$$\begin{aligned} d_q(x_j,x_{j+1}) \leq & \quad H_{d_q}(Sx_{j-1},Sx_j) \\ \leq & \quad \alpha_*(Sx_{j-1},Sx_j)H_{d_q}(Sx_{j-1},Sx_j) \\ \leq & \quad \psi(d_q(x_{j-1},x_j)) \leq ... \leq \psi^j(d_q(x_0,x_1)). \end{aligned} \tag{2.5}$$

Also, we have

$$\begin{aligned} d_q(x_0,x_{j+1}) \leq & \quad d_q(x_0,x_1)+...+d_q(x_j,x_{j+1}) \\ \leq & \quad d_q(x_0,x_1)+...+\psi^j(d_q(x_0,x_1)), \quad \text{by (2.5)} \\ d_q(x_0,x_{j+1}) \leq & \quad \sum_{i=0}^{j}\psi^i(d_q(x_0,x_1)) \\ \leq & \quad \max\{\sum_{i=0}^{n}\psi^i(d_q(x_1,x_0)),\sum_{i=0}^{n}\psi^i(d_q(x_0,x_1))\} \leq r. \end{aligned} \tag{2.6}$$

Also, by using $\alpha(x_1,x_0) \geq 1$, we have $\alpha_*(Sx_j,Sx_{j-1}) \geq 1$. Now, by using (2.4), we have

$$\begin{aligned} d_q(x_{j+1},x_j) \leq & \quad H_{d_q}(Sx_j,Sx_{j-1}) \leq \alpha_*(Sx_j,Sx_{j-1})H_{d_q}(Sx_j,Sx_{j-1}) \\ \leq & \quad \psi(d_q(x_j,x_{j-1})) \leq ... \leq \psi^j(d_q(x_1,x_0)). \end{aligned} \tag{2.7}$$

Now,

$$\begin{aligned} d_q(x_{j+1},x_0) \leq & \quad d_q(x_{j+1},x_j)+...+d_q(x_1,x_0) \\ \leq & \quad \psi^j(d_q(x_1,x_0))+...+d_q(x_1,x_0), \quad \text{by (2.7)} \\ d_q(x_{j+1},x_0) \leq & \quad \max\{\sum_{i=0}^{n}\psi^i(d_q(x_1,x_0)),\sum_{i=0}^{n}\psi^i(d_q(x_0,x_1))\} \leq r. \end{aligned} \tag{2.8}$$

By (2.6) and (2.8), we have $x_{j+1} \in \overline{B_{d_q}(x_0,r)}$. Hence, by mathematical induction, $x_n \in \overline{B_{d_q}(x_0,r)}$. As $\alpha_*(Sx_{j-1},Sx_j) \geq 1$ and $\alpha_*(Sx_j,Sx_{j-1}) \geq 1$, then we have $\alpha(x_j,x_{j+1}) \geq 1$ and $\alpha(x_{j+1},x_j) \geq 1$. Also S is semi α_*-admissible multifunction on $\overline{B_{d_q}(x_0,r)}$, therefore $\alpha_*(Sx_j,Sx_{j+1}) \geq 1$ and $\alpha_*(Sx_{j+1},Sx_j) \geq 1$. This further implies that $\alpha(x_{j+1},x_{j+2}) \geq 1$ and $\alpha(x_{j+2},x_{j+1}) \geq 1$. Continuing this process, we have $\alpha(x_n,x_{n+1}) \geq 1$ and $\alpha(x_{n+1},x_n) \geq 1$ for all $n \in \mathbb{N}$. Now, inequalities (2.5) and(2.7) can be written as

$$d_q(x_n,x_{n+1}) \leq \quad \psi^n(d_q(x_0,x_1)), \text{ for all } n \in \mathbb{N}. \tag{2.9}$$

$$d_q(x_{n+1},x_n) \leq \quad \psi^n(d_q(x_1,x_0)), \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$, such that $\sum\limits_{k \geq k_1(\varepsilon)} \psi^k(d_q(x_0, x_1)) < \varepsilon$. Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$. Now, we have

$$\begin{aligned} d_q(x_n, x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k(d_q(x_0, x_1)), \quad \text{by (2.9)} \\ d_q(x_n, x_m) &\leq \sum_{k \geq k_1(\varepsilon)} \psi^k(d_q(x_0, x_1)) < \varepsilon. \end{aligned}$$

Thus, $\{XS(x_n)\}$ is a left K-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$. Now, let $k_2(\varepsilon) \in \mathbb{N}$, such that $\sum\limits_{k \geq k_2(\varepsilon)} \psi^k(d_q(x_1, x_0)) < \varepsilon$. Let $s, t \in \mathbb{N}$ with $s > t > k_2(\varepsilon)$. Now, we have

$$\begin{aligned} d_q(x_s, x_t) &\leq \sum_{k=s-1}^{t} \psi^k(d_q(x_1, x_0)), \quad \text{by (2.10)} \\ d_q(x_s, x_t) &\leq \sum_{k \geq k_2(\varepsilon)} \psi^k(d_q(x_1, x_0)) < \varepsilon. \end{aligned}$$

Thus, $\{XS(x_n)\}$ is a right K-Cauchy sequence in $(\overline{B_{d_q}(x_0, r)}, d_q)$. As every closed ball in a left (right) K-sequentially complete dislocated quasi metric space is left (right) K-sequentially complete, so there exists $x^* \in \overline{B_{d_q}(x_0, r)}$ such that $\{XS(x_n)\} \to x^*$, and

$$\lim_{n \to \infty} d_q(x_n, x^*) = \lim_{n \to \infty} d_q(x^*, x_n) = 0. \tag{2.11}$$

By assumption, we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\alpha_*(Sx_n, Sx^*) \geq 1$. Now, we have

$$\begin{aligned} d_q(x^*, Sx^*) &\leq d_q(x^*, x_{n+1}) + d_q(x_{n+1}, Sx^*) \\ &\leq d_q(x^*, x_{n+1}) + H_{d_q}(Sx_n, Sx^*) \\ &\leq d_q(x^*, x_{n+1}) + \alpha_*(Sx_n, Sx^*) H_{d_q}(Sx_n, Sx^*) \\ &\leq d_q(x^*, x_{n+1}) + \psi(d_q(x_n, x^*)). \end{aligned}$$

Letting $n \to \infty$ and by using inequality (2.11), we obtain $d_q(x^*, Sx^*) = 0$. Similarly, as $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then, $\alpha_*(Sx^*, Sx_n) \geq 1$. Now, we have

$$d_q(Sx^*, x^*) \leq \psi(d_q(x^*, x_n)) + d_q(x_{n+1}, x^*).$$

We obtain $d_q(Sx^*, x^*) = 0$. Hence, $x^* \in Sx^*$. So S has a fixed point in $\overline{B_{d_q}(x_0, r)}$.

Corollary 2.2 Let (X, d_l) be a complete dislocated metric space, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $\alpha : X \times X \to [0, +\infty)$, $S : X \to P(X)$ be a semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$ and let for a sequence $\{XS(x_n)\}$ in X generated by x_0, $\alpha(x_0, x_1) \geq 1$. Assume that for some $\psi \in \Psi$, the following hold:

$$\alpha_*(Sx, Sy) H_{d_l}(Sx, Sy) \leq \psi(d_l(x, y)) \text{ for all } x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\} \tag{2.12}$$

$$\text{and } \sum_{i=0}^{n} \psi^i(d_l(x_0, Sx_0)) \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$, $\alpha(x_n,x_{n+1}) \geq 1$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0,r)}$. Also, if $\alpha(x_n,x^*) \geq 1$ or $\alpha(x^*,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (2.12) holds for all $x,y \in (\overline{B_{d_l}(x_0,r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_q}(x_0,r)}$. Then, S has a fixed point x^* in $\overline{B_{d_l}(x_0,r)}$.

Let X be a nonempty set. Then $(X,\preceq,d_q)$ is called a preordered dislocated quasi metric space if d_q is a dislocated quasi metric on X and $\preceq$ is a preorder on X. Let $(X,\preceq,d_q)$ be a preordered dislocated quasi metric space and $A,B \subseteq X$. We say that $A \preceq B$ whenever for each $a \in A$ there exists $b \in B$ such that $a \preceq b$. Also,we say that $A \preceq_r B$ whenever for each $a \in A$ and $b \in B$, we have $a \preceq b$.

Corollary 2.3 Let $(X,\preceq,d_q)$ be a preordered left (right) K-sequentially complete dislocated quasi metric space, $r > 0$, $x_0 \in \overline{B_{d_q}(x_0,r)}$, $S : X \to P(X)$ and for a sequence $\{XS(x_n)\}$ in X generated by x_0 with $x_0 \preceq x_1$ and $x_1 \preceq x_0$. Assume that for some $\psi \in \Psi$, the following hold:

$$H_{d_q}(Sx,Sy) \leq \psi(d_q(x,y)), \text{ for all } x,y \in \overline{B_{d_q}(x_0,r)} \cap \{XS(x_n)\} \text{ with } x \preceq y \quad (2.13)$$

$$\text{and } \max\{\sum_{i=0}^{n} \psi^i(d_q(x_1,x_0)), \sum_{i=0}^{n} \psi^i(d_q(x_0,x_1))\} \leq r, \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $x,y \in \overline{B_{d_q}(x_0,r)}$, such that $x \preceq y$ implies $Sx \preceq_r Sy$. Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$, $x_n \preceq x_{n+1}$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0,r)}$. Also, if $x^* \preceq x_n$ and $x_n \preceq x^*$ for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (2.13) holds for all $x,y \in (\overline{B_{d_q}(x_0,r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_q}(x_0,r)}$.

Corollary 2.4 Let $(X,\preceq,d)$ be a preordered complete metric space, $r > 0$, $S : X \to P(X)$ and for a sequence $\{XS(x_n)\}$ in X generated by x_0 with $x_0 \preceq x_1$. Assume that there exists $k \in [0,1)$ with

$$H(Sx,Sy) \leq kd(x,y), \text{ for all } x,y \in \overline{B(x_0,r)} \cap \{XS(x_n)\} \text{ with } x \preceq y \quad (2.14)$$

$$\text{and } \sum_{i=0}^{n} k^i(d(x_0,x_1)) \leq r \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

If $x,y \in \overline{B(x_0,r)}$, such that $x \preceq y$ implies $Sx \preceq_r Sy$. Then, $\{XS(x_n)\}$ is a sequence in $\overline{B(x_0,r)}$, $x_n \preceq x_{n+1}$ and $\{XS(x_n)\} \to x^* \in \overline{B(x_0,r)}$. Also, if $x^* \preceq x_n$ or $x_n \preceq x^*$ for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (2.14) holds for all $x,y \in (\overline{B(x_0,r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then S has a fixed point x^* in $\overline{B(x_0,r)}$.

Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to [0,\infty)$ be a mapping, then the mapping f is called semi α-admissible if, $A \subseteq X$, $x,y \in A$, $\alpha(x,y) \geq 1 \Rightarrow \alpha(fx,fy) \geq 1$. If $A = X$, then the mapping f is called α-admissible.

Corollary 2.5 Let (X,d) be a complete metric space, $S : X \to X$, $r > 0$, $\{x_n\}$ be a Picard sequence in X with initial guess x_0. Let $\alpha : X \times X \to [0,+\infty)$ be a semi α-admissible mapping on $\overline{B(x_0,r)}$ with $\alpha(x_0,x_1) \geq 1$. For some $\psi \in \Psi$, assume that:

$$\alpha(x,y)d(Sx,Sy) \leq \psi(d(x,y)), \text{ for all } x,y \in \overline{B(x_0,r)} \cap \{x_n\} \quad (2.15)$$

and

$$\sum_{i=0}^{j} \psi^i(d(x_0,x_1)) \leq r \text{ for all } j \in \mathbb{N} \cup \{0\}.$$

Then, $\{x_n\}$ is a sequence in $\overline{B(x_0,r)}$, $\alpha(x_n,x_{n+1})\geq 1$ and $\{x_n\}\to x^*\in\overline{B(x_0,r)}$. Also, if $\alpha(x_n,x^*)\geq 1$ or $\alpha(x^*,x_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$ and the inequality (2.15) holds for all $x,y\in(\overline{B(x_0,r)}\cap\{x_n\})\cup\{x^*\}$. Then, S has a fixed point x^* in $\overline{B(x_0,r)}$.

Recall that if $(X,\preceq)$ is a preordered set and $T:X\to X$ is such that for $x,y\in X$, with $x\preceq y$ implies $Tx\preceq Ty$, then the mapping T is said to be non-decreasing.
Corollary 2.6 Let (X,d) be a complete metric space, $S:X\to X$ be nondecreasing mapping, $r>0$, $\{x_n\}$ be a Picard sequence in X with initial guess x_0 with $x_0\preceq x_1$. For some $k\in[0,1)$, assume that:

$$d(Sx,Sy)\leq kd(x,y),\ \text{for all } x,y\in\overline{B(x_0,r)}\cap\{x_n\} \text{ with } x\preceq y \tag{2.16}$$

and

$$\sum_{i=0}^{j}k^i(d(x_0,x_1))\leq r \text{ for all } j\in\mathbb{N}\cup\{0\}.$$

Then, $\{x_n\}$ is a sequence in $\overline{B(x_0,r)}$, $x_n\preceq x_{n+1}$ and $\{x_n\}\to x^*\in\overline{B(x_0,r)}$. Also, if $x^*\preceq x_n$ or $x_n\preceq x^*$ for all $n\in\mathbb{N}\cup\{0\}$ and the inequality (2.16) holds for all $x,y\in(\overline{B(x_0,r)}\cap\{x_n\})\cup\{x^*\}$. Then S has a fixed point x^* in $\overline{B(x_0,r)}$.
Example 2.7 Let $X=Q^+\cup\{0\}$ and let $d_q:X\times X\to X$ be the complete dislocated quasi metric on X defined by

$$d_q(x,y)=x+y \text{ for all } x,y\in X.$$

Define the multivalued mapping, $S:X\to P(X)$ by

$$Sx=\begin{cases}[\frac{2x}{3},\frac{3x}{4}] \text{ if } x\in[0,1]\cap X\\ [x,x+1] \text{ if } x\in(1,\infty)\cap X\end{cases}.$$

Considering, $x_0=1$, $r=8$, then $\overline{B_{d_q}(x_0,r)}=[0,7]\cap X$. Now, $d_q(x_0,Sx_0)=d_q(1,S1)=d_q(1,\frac{2}{3})=\frac{5}{3}$. So we obtain a sequence $\{XS(x_n)\}=\{1,\frac{2}{3},\frac{4}{9},\frac{8}{27},...\}$ in X generated by x_0. Let $\psi(t)=\frac{3t}{4}$ and

$$\alpha(x,y)=\begin{cases}1 \text{ if } x,y\in[0,1]\cap X\\ \frac{3}{2} \quad \text{otherwise.}\end{cases}$$

Now,

$$\alpha_*(S2,S3)H_{d_q}(S2,S3)=(\frac{3}{2})6>\psi(d_q(x,y))=\frac{15}{4}.$$

So the contractive condition does not hold on X and $\overline{B_{d_q}(x_0,r)}$. Now, for all $x,y \in \overline{B_{d_q}(x_0,r)} \cap \{XS(x_n)\}$, we have

$$\begin{aligned}
\alpha_*(Sx,Sy)H_{d_q}(Sx,Sy) &= 1[\max\{\sup_{a\in Sx} d_q(a,Sy), \sup_{b\in Sy} d_q(Sx,b)\}] \\
&= \max\{\sup_{a\in Sx} d_q(a,[\frac{2y}{3},\frac{3y}{4}]), \sup_{b\in Sy} d_q([\frac{2x}{3},\frac{3x}{4}],b)\} \\
&= \max\{d_q(\frac{3x}{4},[\frac{2y}{3},\frac{3y}{4}]), d_q([\frac{2x}{3},\frac{3x}{4}],\frac{3y}{4})\} \\
&= \max\{d_q(\frac{3x}{4},\frac{2y}{3}), d_q(\frac{2x}{3},\frac{3y}{4})\} \\
&= \max\{\frac{3x}{4}+\frac{2y}{3}, \frac{2x}{3}+\frac{3y}{4}\} \\
&\le \frac{3x}{4}+\frac{3y}{4} = \psi(x+y) = \psi(d_q(x,y)).
\end{aligned}$$

So the contractive condition holds on $\overline{B_{d_q}(x_0,r)} \cap \{XS(x_n)\}$. Also,

$$\sum_{i=0}^{n} \psi^i(d_q(x_0,x_1)) = \frac{5}{3}\sum_{i=0}^{n}(\frac{3}{4})^i < 8 = r.$$

Hence, all the conditions of Theorem 2.1 are satisfied. Now, we have $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0,r)}$, $\alpha(x_n,x_{n+1}) \ge 1$, $\alpha(x_{n+1},x_n) \ge 1$ and $\{XS(x_n)\} \to 0 \in \overline{B_{d_q}(x_0,r)}$. Also, $\alpha(x_n,0) \ge 1$ or $\alpha(0,x_n) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$. Moreover, S has a fixed point 0.

Theorem 2.8 Let (X,d_l) be a complete dislocated metric space, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0,r)}$, $\alpha : X \times X \to [0,+\infty)$, $S : X \to P(X)$ be a semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0,r)}$ and $\{XS(x_n)\}$ in X generated by x_0 with $\alpha(x_0,x_1) \ge 1$. Assume that for $t \in [0,\frac{1}{2})$, such that

$$\alpha_*(Sx,Sy)H_{d_l}(Sx,Sy) \le t(d_l(x,Sx)+d_l(y,Sy)) \text{ for all } x,y \in \overline{B_{d_l}(x_0,r)}\cap\{XS(x_n)\} \tag{2.17}$$

$$\text{and } d_l(x_0,x_1)) \le (1-\theta)r, \tag{2.18}$$

where $\theta = \frac{t}{1-t}$. Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0,r)}$, $\alpha(x_n,x_{n+1}) \ge 1$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0,r)}$. Also, if $\alpha(x_n,x^*) \ge 1$ or $\alpha(x^*,x_n) \ge 1$ for all $n \in \mathbb{N}\cup\{0\}$ and the inequality (2.17) holds for all $x,y \in (\overline{B_{d_l}(x_0,r)}\cap\{XS(x_n)\})\cup\{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_l}(x_0,r)}$.

Proof. Let $\{XS(x_n)\}$ be a sequence in X generated by x_0. Then, we have $x_{n+1} \in Sx_n$, and $d_q(x_n,Sx_n) = d_q(x_n,x_{n+1})$, for all $n \in \mathbb{N}\cup\{0\}$. By Lemma 1.7, we have $d_l(x_n,x_{n+1}) \le H_{d_l}(Sx_{n-1},Sx_n)$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for any $n \in \mathbb{N}\cup\{0\}$, then x_n is a fixed point in $\overline{B_{d_q}(x_0,r)}$ of S. Let $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}\cup\{0\}$. From (2.18), we get

$$d_l(x_0,x_1) \le (1-\theta)r < r.$$

It follows that,

$$x_1 \in \overline{B_{d_l}(x_0,r)}.$$

Let $x_2, \cdots, x_j \in \overline{B_{d_l}(x_0, r)}$, for some $j \in \mathbb{N}$. By using the similar technique given in Theorem 2.1, we have $\alpha_*(Sx_{j-1}, Sx_j) \geq 1$. Now, we have

$$\begin{aligned} d_l(x_j, x_{j+1}) &\leq H_{d_l}(Sx_{j-1}, Sx_j) \leq \alpha_*(Sx_{j-1}, Sx_j) H_{d_l}(Sx_{j-1}, Sx_j) \\ &\leq t(d_l(x_{j-1}, Sx_{j-1}) + d_l(x_j, Sx_j)) \quad \text{by (2.17)} \\ &\leq \theta(d_l(x_{j-1}, x_j)) \leq \ldots \leq \theta^j(d_l(x_0, x_1)) \end{aligned}$$

Now, we have

$$\begin{aligned} d_l(x_0, x_{j+1}) &\leq d_l(x_0, x_1) + d_l(x_1, x_2) + \ldots + d_l(x_j, x_{j+1}) \\ &\leq d_l(x_0, x_1) + \theta d_l(x_0, x_1) + \ldots + \theta^j(d_l(x_0, x_1)) \\ &\leq (1-\theta) r \frac{(1-\theta^{j+1})}{(1-\theta)} \leq r. \end{aligned}$$

Thus, $x_{j+1} \in \overline{B_{d_l}(x_0, r)}$. Hence, by induction, $x_n \in \overline{B_{d_l}(x_0, r)}$. As $\alpha_*(Sx_{j-1}, Sx_j) \geq 1$, $x_j \in Sx_{j-1}$, $x_{j+1} \in Sx_j$, we have $\alpha(x_j, x_{j+1}) \geq 1$. Also S is semi α_*-admissible multifunction on $\overline{B_{d_l}(x_0, r)}$, therefore $\alpha_*(Sx_j, Sx_{j+1}) \geq 1$. This further implies that $\alpha(x_{j+1}, x_{j+2}) \geq 1$. Continuing this process, we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Now, we have

$$d_l(x_n, x_{n+1}) \leq \theta^n(d_l(x_0, x_1)), \text{ for all } n \in \mathbb{N}.$$

Now, have

$$\begin{aligned} d_l(x_n, x_{n+i}) &\leq d_l(x_n, x_{n+1}) + \ldots + d_l(x_{n+i-1}, x_{n+i}) \\ &\leq \frac{\theta^n(1-\theta^i)}{1-\theta} d_l(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\{XS(x_n)\}$ is a Cauchy sequence in $(\overline{B_{d_l}(x_0, r)}, d_l)$. As every closed ball in a complete dislocated metric space is complete, so there exists $x^* \in \overline{B_{d_l}(x_0, r)}$ such that $\{XS(x_n)\} \to x^*$, and

$$\lim_{n \to \infty} d_l(x_n, x^*) = 0.$$

By assumption, we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\alpha_*(Sx_n, Sx^*) \geq 1$. Now,

$$\begin{aligned} d_l(x^*, Sx^*) &\leq d_l(x^*, x_{n+1}) + d_l(x_{n+1}, Sx^*) \\ &\leq d_l(x^*, x_{n+1}) + H_{d_l}(Sx_n, Sx^*) \\ &\leq d_l(x^*, x_{n+1}) + \alpha_*(Sx_n, Sx^*) H_{d_l}(Sx_n, Sx^*) \\ &\leq d_l(x^*, x_{n+1}) + t(d_l(x_n, x_{n+1}) + d_l(x^*, Sx^*)). \end{aligned}$$

Letting $n \to \infty$, we obtain $d_l(x^*, Sx^*) = 0$. Similarly, if $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain $d_l(Sx^*, x^*) = 0$. Hence, $x^* \in Sx^*$. So S has a fixed point in $\overline{B_{d_l}(x_0, r)}$.

Corollary 2.9 Let $(X, \preceq, d_l)$ be an ordered complete dislocated metric space, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $S : X \to P(X)$ and let for a sequence $\{XS(x_n)\}$ in X generated by x_0, $x_0 \preceq x_1$. Assume that there exists $t \in [0, \frac{1}{2})$ with

$$H_{d_l}(Sx, Sy) \leq t(d_l(x, Sx) + d_l(y, Sy)), \text{ for all } x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\} \text{ with } x \preceq y$$
$$\text{and } d_l(x_0, Sx_0) \leq (1-\theta) r,$$

where $\theta = \frac{t}{1-t}$. If $x, y \in \overline{B_{d_l}(x_0, r)}$, such that $x \preceq y$ implies $Sx \preceq_r Sy$. Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$, $x_n \preceq x_{n+1}$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_l}(x_0, r)}$. Also, if $x^* \preceq x_n$ or $x_n \preceq x^*$ for all $n \in \mathbb{N} \cup \{0\}$, then S has a fixed point x^* in

$\overline{B_{d_l}(x_0, r)}$.

3. Fixed point results for graphic contractions

Consistent with Jachymski [14], let (X, d) be a metric space and Δ denotes the diagonal of the cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph (see [14]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N $(N \in \mathbb{N})$ is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$. A graph G is connected if there is a path between any two vertices (see for details [9, 12, 14, 25]).

Definition 3.1 [25] Let X be a nonempty set and $G = (V(G), E(G))$ be a graph such that $V(G) = X$, and let $T : X \to CB(X)$. T is said to be graph preserving if it satisfies the following:

if $(x, y) \in E(G)$, then $(u, v) \in E(G)$ for all $u \in Tx$ and $v \in Ty$.

Theorem 3.2 Let (X, d_q) be a left (right) K-sequentially complete dislocated quasi metric space endowed with a graph G, $r > 0$, $x_0 \in \overline{B_{d_q}(x_0, r)}$, $S : X \longrightarrow P(X)$ and let for a sequence $\{XS(x_n)\}$ in X generated by x_0, with $(x_0, x_1), (x_1, x_0) \in E(G)$. Suppose the following assertions hold:

(i) S is graph preserving for all x , $y \in \overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}$;

(ii) there exists $\psi \in \Psi$, such that

$$H_{d_q}(Sx, Sy) \leq \psi(d_q(x, y)),$$

for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}$;

(iii) $\max\{\sum_{i=0}^n \psi^i(d_q(x_1, x_0)), \sum_{i=0}^n \psi^i(d_q(x_0, x_1))\} \leq r$, for all $n \in \mathbb{N} \cup \{0\}$.

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$, $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ and $\{XS(x_n)\} \to x^*$. Also, if $(x_n, x^*), (x^*, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (ii) holds for all $x, y \in (\overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_q}(x_0, r)}$.

Proof. Define, $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G) \\ 0, & \text{otherwise} \end{cases}.$$

As $(x_0, x_1), (x_1, x_0) \in E(G)$, then $\alpha(x_0, x_1) = \alpha(x_1, x_0) = 1$. Let, $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. From (i), we have $(u, v) \in E(G)$ for all $u \in Sx$ and $v \in Sy$. This implies that $\alpha(u, v) = 1$ for all $u \in Sx$ and $v \in Sy$. This further implies that $\inf\{\alpha(u, v) : u \in Sx, v \in Sy\} = 1$. Thus S is a semi α_*-admissible multifunction on $\overline{B_{d_q}(x_0, r)}$. Also, if $(x, y) \in E(G)$, we have $\alpha(x, y) = 1$ and hence, $\alpha_*(Sx, Sy) = 1$. Now, condition (ii) can be written as

$$\alpha_*(Sx, Sy)H_{d_q}(Sx, Sy) = H_{d_q}(Sx, Sy) \leq \psi(d_q(x, y)),$$

for all $x, y \in \overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}$. By including condition (iii), we obtain all the conditions of Theorem 2.1. Now, by Theorem 2.1, we have $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_q}(x_0, r)}$, $\alpha(x_n, x_{n+1}) \geq 1$, $\alpha(x_{n+1}, x_n) \geq 1$ that is $(x_n, x_{n+1}), (x_{n+1}, x_n) \in E(G)$ and $\{XS(x_n)\} \to x^* \in \overline{B_{d_q}(x_0, r)}$. Also, if $(x_n, x^*), (x^*, x_n) \in E(G)$ for all

$n \in \mathbb{N} \cup \{0\}$. Then, we have $\alpha(x_n, x^*) \geq 1$ and $\alpha(x^*, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Again, by Theorem 2.1, S has a fixed point x^* in $\overline{B_{d_q}(x_0, r)}$.

Theorem 3.3 Let (X, d_l) be a complete dislocated metric space endowed with a graph G, $r > 0$, $x_0 \in \overline{B_{d_l}(x_0, r)}$, $S : X \longrightarrow P(X)$ and let for a sequence $\{XS(x_n)\}$ in X generated by x_0 with $(x_0, x_1) \in E(G)$. Suppose the following assertions hold:
(i) S is graph preserving for all $x\ , y \in \overline{B_{d_l}(x_0, r)}$;
(ii) there exists $t \in \left[0, \frac{1}{2}\right)$, such that

$$H_{d_l}(Sx, Sy) \leq t\left[d_l(x, Sx) + d_l(y, Sy)\right],$$

for all $x, y \in \overline{B_{d_l}(x_0, r)} \cap \{XS(x_n)\}$ and $(x, y) \in E(G)$;
(iii) $d_l(x_0, Sx_0) \leq (1 - \lambda)\, r$, where $\lambda = \dfrac{t}{1-t}$.

Then, $\{XS(x_n)\}$ is a sequence in $\overline{B_{d_l}(x_0, r)}$, $(x_n, x_{n+1}) \in E(G)$ and $\{XS(x_n)\} \to x^*$. Also, if $(x_n, x^*) \in E(G)$ or $(x^*, x_n) \in E(G)$ for all $n \in \mathbb{N} \cup \{0\}$ and the inequality (ii) holds for all $x, y \in (\overline{B_{d_q}(x_0, r)} \cap \{XS(x_n)\}) \cup \{x^*\}$. Then, S has a fixed point x^* in $\overline{B_{d_l}(x_0, r)}$.

Competing interests

The author declares that he has no competing interests.

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References

[1] MU. Ali, T. Kamran, E. Karapınar, Further discussion on modified multivalued $\alpha_* - \psi$-contractive type mapping, Filomat, 29(8), 2015, 1893–1900.
[2] R.P. Agarwal, M.A. El-Gebeily, D. O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Anal. 87 (2008), 109–116.
[3] M. Arshad, A. Shoaib, and P. Vetro, Common Fixed Points of a Pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Dislocated Metric Spaces, Journal of Function Spaces, 2013 (2013), Article ID 63818.
[4] M. Arshad , A. Shoaib, M. Abbas and A. Azam, Fixed Points of a pair of Kannan Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, Miskolc Mathematical Notes, 14(3), 2013, 769–784.
[5] M. Arshad, A. Azam, M. Abbas and A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity U.P.B. Sci. Bull., Series A, 76(2), 2014.
[6] J. H. Asl, S. Rezapour and N. Shahzad, On fixed points of $\alpha - \psi$ contractive multifunctions, Fixed Point Theory Appl., 2012 2012:212.
[7] A. Azam, M. Arshad, Fixed points of a sequence of locally contractive multi-valued maps, Comp. Math. Appl., 57 (2009), 96-100.
[8] I. Beg, M. Arshad , A. Shoaib,Fixed Point on a Closed Ball in ordered dislocated Metric Space, Fixed Point Theory, 16(2), 2015.
[9] F.Bojor, Fixed point theorems for Reich type contraction on metric spaces with a graph, Nonlinear Anal, 75 (2012) 3895-3901.
[10] P. Hitzler and A. K. Seda, Dislocated Topologies, Journal of Electrical Engineering, 51(12/s)(2000), 3-7.
[11] N. Hussain, J. Ahmad and A. Azam, Generalized fixed point theorems for multi-valued α-ψ-contractive mappings, J. Inequal. Appl., 2014, 2014:348.

[12] N. Hussain, S. Al-Mezel and P. Salimi, Fixed points for ψ-graphic contractions with application to integral equations, Abstract and Applied Analysis, Volume 2013, Article ID 575869.

[13] N. Hussain, M. Arshad, A. Shoaib and Fahimuddin, Common Fixed Point results for α-ψ-contractions on a metric space endowed with graph, J. Inequalities and Appl., 2014, 2014:136.

[14] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 1 (136), (2008), 1359-1373.

[15] E. Karapınar, H. Piri, and H. H. Alsulami, Fixed points of modified F-contractive mappings in complete metric-like spaces, J. Funct. Spaces, Volume 2015, Article ID 270971, 9 pages.

[16] E. Karapınar, İ.M. Erhan and A. Öztürk, Fixed point theorems on quasi-partial metric spaces, Math. Comp. Model., (2012), doi:10.1016/j.mcm.2012.06.036.

[17] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., 728 (1994), 183-197.

[18] Jr. Nadler, Multi-valued contraction mappings. Pacific J. Math. 30(1969), 475-478.

[19] Y. Ren, J. Li, and Y. Yu, Common Fixed Point Theorems for Nonlinear Contractive Mappings in Dislocated Metric Spaces, Abstr. Appl. Anal., Vol. 2013 (2013), Article ID 483059, 5 pages.

[20] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. 75(2012), 2154-2165.

[21] M. Sarwar, Mujeeb Ur Rahman and G. Ali, Some fixed point results in dislocated quasi metric (dq-metric) spaces, J. Inequalities and Appl., 2014, 2014:278.

[22] A. Shoaib, M. Arshad and J. Ahmad, Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces, The Scientific World Journal, 2013 (2013), Article ID 194897, 8 pages.

[23] A. Shoaib, M. Arshad and M. A. Kutbi, Common fixed points of a pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, J. Comput. Anal. Appl., 17(2014), 255-264.

[24] A. Shoaib, $\alpha - \eta$ Dominated Mappings and Related Common Fixed Point Results in Closed Ball, Journal of Concrete and Applicable Mathematics, 13(1-2), 2015, 152-170.

[25] J. Tiammee, S. Suantai, Coincidence point theorems for graph-preserving multi-valued mappings, Fixed Point Theory Appl., 2014, 2014:70.

[26] F. M. Zeyada, G. H. Hassan, and M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, Arabian J. Science and Engineering A, 31(1),(2006), 111–114.

[27] L. Zhu, C. Zhu, C. Chen and Z. Stojanović, Multidimensional fixed points for generalized Ψ-quasi-contractions in quasi-metric-like spaces, J. Inequalities and Appl., 2014, 2014:27.

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