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FRACTIONAL POWER THEORY FOR EIGENFUNCTIONS OF HANKEL TRANSFORMS

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Abstract. The eigenvalues of Hankel transform with corresponding eigenfunctions are considered. Translation and convolution for eigenfunctions of Hankel transforms are defined. These operators are used to define eigenfunction Hankel wavelet transform. Certain boundedness, continuity results and inversion formulae for the continuous eigenfunction of Hankel wavelet transforms are obtained.

1. Introduction

For suitable functions ϕ, Namias defined the Hankel transform of ϕ of Bessel order ν by $H\phi$ where

$$(H\phi)(x) = \int_0^\infty yJ_\nu(xy)\,\phi(y)\,dy. \tag{1.1}$$

The eigenvalues of H are $\left\{e^{inx}\right\}_{n=0}^{\infty}$ with corresponding eigenfunctions in [8]

$$\psi_n^{(\nu)}(x) = x^\nu \exp\left(-x^2/2\right) L_n^{(\nu)}\left(x^2\right), \tag{1.2}$$

where $L_n^{(\nu)}$ are the generalized Laguerre polynomials defined by

$$L_n^{(\nu)}(x) = \frac{1}{n!}e^x x^{-\nu} D^n\left[e^{-x}x^{n+\nu}\right],\ n \in \mathbb{N}_0 \equiv \mathbb{N}\cup\{0\},\ x > 0. \tag{1.3}$$

The Hankel eigenfunction transform of $f \in L_2(R^+)$ and $\nu \in R$, with $\nu > -1$ is defined by

$$\hat{f}(n) = \left(f, \psi_n^{(\nu)}\right) = \int_\alpha^\beta f(x)\ \overline{\psi_n^{(\nu)}(x)}\,dx;\ \psi_n^{(\nu)} \in L_2\left(R^+\right). \tag{1.4}$$

A classical result [6, Zemanian], states that $\{\psi_n\}$ is complete orthonormal if and only if, for every $f \in L_2(I)$, the coefficients (f, ψ_n) satisfy Parseval's equation:

$$\sum_{n=0}^{\infty} |(f,\psi_n)|^2 = \int_\alpha^\beta |f(x)|^2\,dx = \|f\|_2^2\,. \tag{1.5}$$

For various properties of the eigenfunctions transform [5, 6, 7] may be referred.

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Theorem 1.1 Let $\{\psi_n\}_{n=0}^{\infty}$ be a complete orthonormal system as specified above, and let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |c_n|^2$ converges. Then, there exists a unique $f \in L_2(I)$ such that $c_n = (f, \psi_n)$. Consequently,

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x) \tag{1.6}$$

in the sense of convergence in $L_2(I)$.
A new formula for generating the self-fractional Hankel functions was proposed in [1]. In [3], the authors developed a distributional theory of fractional transformations. A constructive approach, based on the eigenfunction expansion method pioneered by Zemanian, was used to produce an appropriate space of test functions and corresponding space of generalized functions.

2. Translation and convolution for Hankel eigenfunction transform

2.1. **Preliminary results.** The Hankel eigenfunctions transform $\psi_n^{(\nu)}(x)$ the basic function $u^{(\nu)}(x, y, z)$ is defined by

$$\psi_n^{(\nu)}(x)\, \psi_n^{(\nu)}(y) = \int_\alpha^\beta \psi_n^{(\nu)}(z)\; u^{(\nu)}(x, y, z)\, dz, \tag{2.1}$$

where

$$u^{(\nu)}(x, y, z) = \sum_{n=0}^{\infty} \overline{\psi_n^{(\nu)}(x)\, \psi_n^{(\nu)}(y)}\, \psi_n^{(\nu)}(z)\,. \tag{2.2}$$

Assuming $\psi_0^{(\nu)}(x) = x^\nu \exp\left(-x^2/2\right)$, equation (2.1) becomes

$$\int_\alpha^\beta z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x, y, z)\; dz = (xy)^{2\nu} \exp\left(-\left(x^2 + y^2\right)/2\right).$$

Assuming $\left|(xy)^{2\nu} \exp\left(-\left(x^2 + y^2\right)/2\right)\right| \le C$, then

$$\int_\alpha^\beta \left|z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x, y, z)\right|\; dz \le C; \int_\alpha^\beta \left|x^\nu \exp\left(-x^2/2\right) u^{(\nu)}(x, y, z)\right|\; dx \le C \tag{2.3}$$

for some constant $C > 0$. In most of the special cases $u^{(\nu)}(x, y, z)$ is a nonnegative and symmetric in x, y and z, then equation (2.3) follows from equation (2.2), cf. [4].
Definition 2.1: Translation associated with Hankel eigenfunctions transform of a function $f \in L_2(R^+)$ is defined by

$$\left(\tau_x^{(\nu)} f\right)(y) = f(x, y) = \int_\alpha^\beta f(z) \left\{z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x, y, z)\right\} dz. \tag{2.4}$$

Lemma 2.1 Let $f \in L_2(R^+)$, then

$$\left\|\tau_y^{(\nu)} f\right\|_2 \le C \|f\|_2\,, \quad C > 0 \tag{2.5}$$

and the map $f \to \tau_y^{(\nu)} f$ is linear and continuous in $f \in L_2(R^+)$.
Proof Equation (2.4), gives

$$\left(\tau_y^{(\nu)} f\right)(x) = \int_\alpha^\beta f(z) \left\{z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x, y, z)\right\}\; dz;$$

so

$$\left|\left(\tau_y^{(\nu)} f\right)(x)\right| \leq \int_\alpha^\beta |f(z)| \left|z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z)\right| \, dz. \tag{2.6}$$

Using Schwarz's inequality, equation (2.6) becomes

$$\left|\left(\tau_y^{(\nu)} f\right)(x)\right| \leq \left(\int_\alpha^\beta |f(z)|^2 \left|z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z)\right| \, dz\right)^{1/2}$$

$$\times \left(\int_\alpha^\beta \left|z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z)\right| \, dz\right)^{1/2}.$$

Using equation (2.3), it gives

$$\int_\alpha^\beta \left|\left(\tau_y^{(\nu)} f\right)(x)\right|^2 \, dx \leq \int_\alpha^\beta |f(z)|^2 \, dz \left(\int_\alpha^\beta \left|z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z)\right| \, dz\right). \tag{2.7}$$

Thus

$$\left\|\tau_y^{(\nu)} f\right\|_2 \leq C \|f\|_2 .$$

The continuity of the map $f \to \tau_y^{(\nu)} f$ follows from the above inequality.

Definition 2.2 The convolution of f_1 and f_2 is defined by

$$(f_1 * f_2)(x) = \int_\alpha^\beta \left(\tau_x^{(\nu)} f_1\right)(y) \, f_2(y) \, dy \tag{2.8}$$

$$(f_1 * f_2)(x) = \int_\alpha^\beta f_1(z) \, f_2(y) \, z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z) \, dydz. \tag{2.9}$$

From (2.3) and (2.9) it follows that

$$\|f_1 * f_2\|_1 \leq C \|f_1\|_1 \|f_2\|_1 .$$

Lemma 2.2 Let $f_1 \in L_2(R^+)$ and $f_2 \in L_2(R^+)$. Then the convolution $f_1 * f_2$ defined by equation (2.8) satisfies the following:

$$1. \, \|f_1 * f_2\|_\infty \leq C \|f_1\|_2 \|f_2\|_2 . \tag{2.10}$$

$$2. \, (f_1 * f_2)^\wedge (n) = f_1^\wedge (n) \, f_2^\wedge (n) . \tag{2.11}$$

Proof

1. Applying Schwartz's inequality, from equation (2.8) and Lemma 2.1, it follows

$$|(f_1 * f_2)(x)| = \left|\int_\alpha^\beta f_1(x,y) \, f_2(y) \, dy\right|$$
$$\leq \left(\int_\alpha^\beta |f_1(x,y)|^2 \, dy\right)^{1/2} \left(\int_\alpha^\beta |f_2(y)|^2 \, dy\right)^{1/2}.$$
$$\|f_1 * f_2\|_\infty \leq C \|f_1\|_2 \|f_2\|_2 .$$

2. Multiplying $\overline{\psi_n^{(\nu)}(z)}$ to equation (2.9) and integrating with respect to z and using representation (2.2)

$$\int_\alpha^\beta (f_1 * f_2)(z) \, \overline{\psi_n^{(\nu)}(z)} \, dz$$

$$= \int_\alpha^\beta \int_\alpha^\beta \int_\alpha^\beta f_1(x) \, f_2(y) \, \overline{\psi_n^{(\nu)}(z)} \left\{z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(x,y,z)\right\} dxdydz.$$

$$\int_{\alpha}^{\beta} (f_1 * f_2)(z)\, \overline{\psi_n^{(\nu)}(z)}\, dz$$
$$= \sum_{n=0}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} f_1(x)\, f_2(y)\, \overline{\psi_n^{(\nu)}(z)\, \psi_m^{(\nu)}(x)\, \psi_m^{(\nu)}(y)}\, \psi_m^{(\nu)}(z)\, dxdydz.$$

Using orthonormality conditions, follows

$$(f_1 * f_2)^{\wedge}(n) = f_1^{\wedge}(n)\,(f_2)^{\wedge}(n).$$

3. Hankel eigenfunction wavelet transform (HEWT)

For a function $\psi^{(\nu)} \in L_2(R^+)$, define the dilation D_a by

$$D_a\, \psi^{(\nu)}(t) = \psi^{(\nu)}(at),\ \ a > 0. \tag{3.1}$$

Using eigenfunction translation (2.4) and the dilation in (3.1), Hankel eigenfunction wavelet $\psi_{b,a}^{(\nu)}(t)$ is defined as follows:

$$\psi_{b,a}^{(\nu)}(t) = \tau_b^{(\nu)} D_a\, \psi^{(\nu)}(t) = \tau_b^{(\nu)} \psi^{(\nu)}(at). \tag{3.2}$$

$$\psi_{b,a}^{(\nu)}(t) = \int_{\alpha}^{\beta} \psi^{(\nu)}(az)\, z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(b,t,z)\ \ dz. \tag{3.3}$$

where $a > 0$ and $b \geq 0$. Using $\psi_{b,a}^{(\nu)}(t) \in L_2(R^+)$ by equation (2.5) the HEWT of $f \in L_2(R^+)$ by

$$(H_\psi f)(b,a) = \int_{\alpha}^{\beta} f(t)\ \overline{\psi_{b,a}^{(\nu)}(t)}\ dt \tag{3.4}$$

$$(H_\psi f)(b,a) = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} f(t)\ \overline{\psi^{(\nu)}(az)\, z^{2\nu} \exp(-z^2)\, u^{(\nu)}(b,t,z)}\ \ dtdz. \tag{3.5}$$

provided the right-hand side of (3.5) is convergent. Applying Schwartz's inequality to (3.4) and invoking equation (2.5),

$$\left\| (H_\psi f)(b,a) \right\|_{\infty} \leq C \left\| f \right\|_2 \left\| \psi^{(\nu)}(a) \right\|_2. \tag{3.6}$$

4. Reconstruction formula

In order to derive the reconstruction formula, the following Lemma is stated:
Lemma 4.1 Let $f \in L_2(R^+)$ and let $\psi \in L_2(R^+)$ is a basic wavelet which defines HEWT (3.4). Then

$$(H_\psi f)^{\wedge}(n,a) = \overline{\left(\overline{f}\right)^{\wedge}(n)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,n), \tag{4.1}$$

where

$$\psi^{(\nu)\wedge}(a,n) = \int_{\alpha}^{\beta} \psi^{(\nu)}(az)\, \overline{\psi_n^{(\nu)}(z)}\, dz. \tag{4.2}$$

Proof From equations (3.5) and (2.2), it follows

$$(H_\psi f)(b,a) = \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} f(t)\ \overline{\psi^{(\nu)}(az)\left(\sum_{m=0}^{\infty} \overline{\psi_m^{(\nu)}(b)\, \psi_m^{(\nu)}(t)}\, \psi_m^{(\nu)}(z)\right)} dtdz$$
$$= \sum_{m=0}^{\infty} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} f(t)\ \overline{\psi^{(\nu)}(az)}\ \psi_m^{(\nu)}(b)\, \psi_m^{(\nu)}(t)\ \overline{\psi_m^{(\nu)}(z)}\ dtdz$$

$$= \sum_{m=0}^{\infty} \psi_m^{(\nu)}(b) \int_\alpha^\beta f(t)\, \psi_m^{(\nu)}(t)\, dt \int_\alpha^\beta \overline{\psi^{(\nu)}(az)\, \psi_m^{(\nu)}(z)}\, dz.$$

$$(H_\psi f)(b,a) = \sum_{m=0}^{\infty} \psi_m^{(\nu)}(b)\, \overline{(\overline{f})^\wedge(m)} \left(\overline{\psi^{(\nu)}}\right)^\wedge(a,m). \tag{4.3}$$

Multiplying both sides of equation (4.3) by $\overline{\psi_n^{(\nu)}(b)}$ and integrating with respect to b, then

$$\int_\alpha^\beta (H_\psi f)(b,a)\, \overline{\psi_n^{(\nu)}(b)}\, db = \sum_{m=0}^{\infty} \overline{(\overline{f})^\wedge(m)} \left(\overline{\psi^{(\nu)}}\right)^\wedge(a,m) \int_\alpha^\beta \psi_m^{(\nu)}(b)\, \overline{\psi_n^{(\nu)}(b)}\, db. \tag{4.4}$$

As

$$\int_\alpha^\beta (H_\psi f)(b,a)\, \overline{\psi_n^{(\nu)}(b)}\, db = (H_\psi f)^\wedge(n,a). \tag{4.5}$$

and

$$\int_\alpha^\beta \psi_m^{(\nu)}(b)\, \overline{\psi_n^{(\nu)}(b)}\, db = 1. \tag{4.6}$$

Using orthonormality condition, it gives

$$(H_\psi f)^\wedge(n,a) = \overline{(\overline{f})^\wedge(n)} \left(\overline{\psi^{(\nu)}}\right)^\wedge(a,n).$$

Thus proved.

Lemma 4.2 Let $\psi \in L_2(R^+)$ be a basic wavelet and $\psi^{(\nu)^\wedge}(a,n)$ be defined by equation (4.2), then

$$\left(\overline{\psi_{b,a}^{(\nu)}}\right)^\wedge(m) = \psi_m^{(\nu)}(b) \left(\overline{\psi^{(\nu)}}\right)^\wedge(a,m). \tag{4.7}$$

Proof From equation (3.3), it follows

$$\psi_{b,a}^{(\nu)}(t) = \int_\alpha^\beta \psi^{(\nu)}(az)\, z^{2\nu} \exp\left(-z^2\right) u^{(\nu)}(b,t,z)\, dz.$$

Multiplying both sides $\psi_m^{(\nu)}(t)$ and integrating with respect to t,

$$\int_\alpha^\beta \psi_{b,a}^{(\nu)}(t)\, \psi_m^{(\nu)}(t)\, dt$$

$$= \sum_{n=0}^{\infty} \overline{\psi_n^{(\nu)}(b)} \int_\alpha^\beta z^{2\nu} \exp\left(-z^2\right) \psi_n^{(\nu)}(az)\, \psi_n^{(\nu)}(z)\, dz \int_\alpha^\beta \overline{\psi_n^{(\nu)}(t)}\, \psi_m^{(\nu)}(t)\, dt$$

Then orthonormality of $\left\{\psi_n^{(\nu)}\right\}_{n=0}^{\infty}$ yields

$$\overline{\left(\psi_{b,a}^{(\nu)}\right)^\wedge(m)} = \overline{\psi_m^{(\nu)}(b)}\, \overline{\left(\overline{\psi^{(\nu)}}\right)^\wedge(a,m)}.$$

Theorem 4.3 Let $f \in L_2(R^+)$ and ψ be a basic wavelet which defines HEWT by equation (3.5). Let $q(a) > 0$ be a weight function as in [2, equation (6)],

$$Q(m) = \int_\alpha^\beta q(a) \left|\left(\overline{\psi^{(\nu)}}\right)^\wedge(a,m)\right|^2 da. \tag{4.8}$$

Let

$$\widehat{\psi^{(\nu)}}^{b,a}(m) = \frac{\left(\overline{\psi_{b,a}^{(\nu)}}\right)^{\wedge}(m)}{Q(m)}. \tag{4.9}$$

Then

$$f(t) = \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\psi^{(\nu)b,a}(t)}\,dadb. \tag{4.10}$$

Proof From Lemma 4.1 and equation (4.5), it is obtained as

$$\int_{\alpha}^{\beta}(H_{\psi}f)(b,a)\,\overline{\psi_m^{(\nu)}(b)}\,db = \overline{\left(\overline{f}\right)^{\wedge}(m)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m). \tag{4.11}$$

Multiplying both sides by $\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}$ and weight function $q(a)$ and integrating with respect to a, the result obtained is

$$\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\psi_m^{(\nu)}(b)}\;\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}\,dbda$$

$$= \int_{\alpha}^{\beta} q(a)\,\overline{\left(\overline{f}\right)^{\wedge}(m)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)\,\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}\,da. \tag{4.12}$$

Since

$$\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)\,\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)} = \left|\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)\right|^{2}. \tag{4.13}$$

Using equation (4.13) in (4.12), becomes

$$\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\psi_m^{(\nu)}(b)}\;\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}\,dbda = \int_{\alpha}^{\beta} q(a)\,\overline{\left(\overline{f}\right)^{\wedge}(m)}\left|\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)\right|^{2}da. \tag{4.14}$$

From (4.8), $Q(m) = \int_{\alpha}^{\beta} q(a)\left|\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)\right|^{2}da$ is substituted in (4.14), then

$$\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\psi_m^{(\nu)}(b)}\;\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}\,dbda = Q(m)\,\overline{\left(\overline{f}\right)^{\wedge}(m)}. \tag{4.15}$$

Hence

$$\overline{\left(\overline{f}\right)^{\wedge}(m)} = \frac{1}{Q(m)}\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\psi_m^{(\nu)}(b)}\;\overline{\left(\overline{\psi^{(\nu)}}\right)^{\wedge}(a,m)}\,dbda. \tag{4.16}$$

Using (4.7) in (4.16);

$$\overline{\left(\overline{f}\right)^{\wedge}(m)} = \frac{1}{Q(m)}\int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\left(\overline{\psi_{b,a}^{(\nu)}}\right)^{\wedge}(m)}\,dbda. \tag{4.17}$$

From equation (4.9) in (4.17),

$$\overline{\left(\overline{f}\right)^{\wedge}(m)} = \int_{\alpha}^{\beta}\int_{\alpha}^{\beta} q(a)(H_{\psi}f)(b,a)\,\overline{\widehat{\psi^{(\nu)}}^{b,a}(m)}\,dbda. \tag{4.18}$$

Since in [8], page 116,

$$f(x) = \sum_{n=0}^{\infty} f\wedge(n)\;\psi_n^{(\nu)}(x) \tag{4.19}$$

Then

$$\left(\overline{f}\right)(x) = \sum_{n=0}^{\infty}\left(\overline{f}\right)\wedge(n)\;\psi_n^{(\nu)}(x) \tag{4.20}$$

From equation (4.18) becomes

$$\left(\overline{f}\right)(t)=\sum_{m=0}^{\infty}\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}q\left(a\right)\overline{\left(H_{\psi}f\right)\left(b,a\right)}\,\widehat{\psi^{(\nu)}}^{\,b,a}\left(m\right)\psi_{m}^{(\nu)}\left(t\right)dbda\,.$$

$$=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}q\left(a\right)\overline{\left(H_{\psi}f\right)\left(b,a\right)}\left(\sum_{m=0}^{\infty}\widehat{\psi^{(\nu)}}^{\,b,a}\left(m\right)\psi_{m}^{(\nu)}\left(t\right)\right)dbda.$$

That implies

$$\left(\overline{f}\right)(t)=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}q\left(a\right)\overline{\left(H_{\psi}f\right)\left(b,a\right)}\,\psi^{(\nu)b,a}\left(t\right)dbda. \tag{4.21}$$

Then

$$f\left(t\right)=\int_{\alpha}^{\beta}\int_{\alpha}^{\beta}q\left(a\right)\left(H_{\psi}f\right)\left(b,a\right)\overline{\psi^{(\nu)^{b},a}\left(t\right)}\,dbda.$$

This completes the proof.

4.1. **Characterization Theorem. Theorem 4.4** Assume that there exist positive constants A and B such that

$$0<A\leq Q\left(m\right)\leq B<\infty. \tag{4.22}$$

Let

$$\left(\psi^{(\nu)}\right)^{a}\left(t\right)=\sum_{m=0}^{\infty}\frac{1}{Q\left(m\right)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}\left(a,m\right)\psi_{m}^{(\nu)}\left(t\right). \tag{4.23}$$

Then

(1)

$$\left(\psi^{(\nu)}\right)^{b,a}\left(t\right)=\tau_{b}^{(\nu)}\left(\psi^{(\nu)}\right)^{a}\left(t\right). \tag{4.24}$$

(2)

$$\left\|\left(\psi^{(\nu)}\right)^{b,a}\right\|_{2}\leq A^{-1}\left\|\left(\psi^{(\nu)}\right)_{b,\,a}\right\|_{2}. \tag{4.25}$$

Proof Using (1.6), (4.7) and (4.8)

$$\left(\psi^{(\nu)}\right)^{b,a}\left(t\right)=\sum_{m=0}^{\infty}\left(\widehat{\psi^{(\nu)}}\right)^{b,a}\left(m\right)\psi_{m}^{(\nu)}\left(t\right)$$

$$=\sum_{m=0}^{\infty}\frac{1}{Q\left(m\right)}\left(\overline{\psi^{(\nu)}}\right)_{b,a}^{\wedge}\left(m\right)\psi_{m}^{(\nu)}\left(t\right)$$

$$=\sum_{m=0}^{\infty}\frac{1}{Q\left(m\right)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}\left(a,m\right)\psi_{m}^{(\nu)}\left(b\right)\psi_{m}^{(\nu)}\left(t\right)$$

$$=\sum_{m=0}^{\infty}\frac{1}{Q\left(m\right)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}\left(a,m\right)\int_{\alpha}^{\beta}\psi_{m}^{(\nu)}\left(z\right)\,z^{2\nu}\exp\left(-z^{2}\right)u^{(\nu)}\left(b,t,z\right)dz$$

$$=\int_{\alpha}^{\beta}z^{2\nu}\exp\left(-z^{2}\right)u^{(\nu)}\left(b,t,z\right)\left(\sum_{m=0}^{\infty}\frac{1}{Q\left(m\right)}\left(\overline{\psi^{(\nu)}}\right)^{\wedge}\left(a,m\right)\psi_{m}^{(\nu)}\left(z\right)\right)dz$$

From (4.23)

$$\left(\psi^{(\nu)}\right)^{b,a}\left(t\right)=\int_{\alpha}^{\beta}z^{2\nu}\exp\left(-z^{2}\right)u^{(\nu)}\left(b,t,z\right)\left(\psi^{(\nu)}\right)^{a}\left(z\right)\,dz$$

$$=\tau_{b}^{(\nu)}\left(\psi^{(\nu)}\right)^{a}\left(t\right).$$

(ii) From equation (4.9),

$$\left|\widehat{\psi^{(\nu)}}^{b,a}(m)\right| \le \frac{1}{A}\left|\left(\overline{\psi^{(\nu)}}\right)^{\wedge}_{b,a}(m)\right|.$$

So that

$$\sum_{m=0}^{\infty}\left|\widehat{\psi^{(\nu)}}^{b,a}(m)\right|^2 \le \frac{1}{A^2}\sum_{m=0}^{\infty}\left|\left(\overline{\psi^{(\nu)}}\right)^{\wedge}_{b,a}(m)\right|^2.$$

Using Parseval relation (1.5),

$$\left\|\psi^{(\nu)^{b,a}}\right\|_2^2 \le \frac{1}{A^2}\left\|\left(\overline{\psi^{(\nu)}_{b,a}}\right)\right\|_2^2,$$

i.e.

$$\left\|\psi^{(\nu)^{b,a}}\right\|_2 \le \frac{1}{A}\left\|\psi^{(\nu)}_{b,a}\right\|_2.$$

5. Applications

In this section, Hankel eigen function wavelet transforms can be constructed for given f and ψ in the explicit form. From (1.2) and (1.3), it follows that

$$L_n^{\nu}\left(x^2\right) = \sum_{m=0}^{n}\begin{pmatrix} n+\lambda \\ n-m \end{pmatrix}\frac{\left(-x^2\right)^m}{m!} \tag{5.1}$$

Choose $f(x) = e^{-x^2/2} \in L_2\left(R^+\right)$.
Thus
$\hat{f}(n) = \left(f, \psi_n^{(\nu)}\right) = \int_0^{\infty} e^{-x^2/2}\ \overline{\psi_n^{(\nu)}(x)}\ dx$

$$= \sum_{m=0}^{n}\begin{pmatrix} n+\lambda \\ n-m \end{pmatrix}\frac{\left(-x^2\right)^m}{m!}\int_0^{\infty} x^{(\nu+2m)}e^{-x^2}dx$$

$$= \sum_{m=0}^{n}\begin{pmatrix} n+\lambda \\ n-m \end{pmatrix}\frac{\left(-x^2\right)^m}{m!}\frac{1}{2}\Gamma\left(m+\nu/2+1/2\right)$$

Therefore

$$\hat{f}(n) = \sum_{m=0}^{n}\begin{pmatrix} n+\lambda \\ n-m \end{pmatrix}\frac{\left(-x^2\right)^m}{m!}\frac{1}{2}\Gamma\left(m+\nu/2+1/2\right). \tag{5.2}$$

Let the mother wavelet be

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \le t < 1/2, \\ -1 & \text{if } 1/2 \le t < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

This wavelet is called the Haar wavelet. It is piecewise continuous. Using this wavelet the expression for $\psi(at)$ is as follows

$$\psi(at) = \begin{cases} 1 & \text{if } 0 \le t < 1/2a, \\ -1 & \text{if } 1/2\text{a} \le t < 1/a, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Now

$$\psi^{(\nu)} \wedge (a,n) = \overline{\psi^{(\nu)}} \wedge \overline{(a,n)}$$
$$= \int_0^\infty \overline{\psi\,(at)}\psi_n^{(\nu)}\,(t)dt$$
$$= \int_0^\infty \psi\,(at)\,\psi_n^{(\nu)}\,(t)\,dt,$$

since ψ and $\psi_n^{(\nu)}$ are both real valued. Using equation (5.4), follows

$$\psi^{(\nu)} \wedge (a,n) = \int_0^{1/2a} \psi_n^{(\nu)}\,(t)\,dt - \int_{1/2a}^{1/a} \psi_n^{(\nu)}\,(t)\,dt$$
$$= \int_0^{1/2a} t^\nu e^{-t^2/2} L_n^{(\nu)}\left(t^2\right) dt - \int_{1/2a}^{1/a} t^\nu e^{-t^2/2} L_n^{(\nu)}\left(t^2\right) dt$$

for n =0,1,2,3,....

$$= \sum_{m=0}^{n} \begin{pmatrix} n+\lambda \\ n-m \end{pmatrix} \frac{(-1)^m}{m!} \left[\begin{array}{l} 2^{m+\frac{\nu}{2}+\frac{1}{2}} \left\{\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{8a^2}\right)\right\} \\ -16^{m+\frac{\nu}{2}} \left\{ \begin{array}{l} \left(\frac{1}{2}\right)^{3m+\frac{3\nu}{2}-\frac{1}{2}} \left\{\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{8a^2}\right)\right\} \\ -\left(\frac{1}{2}\right)^{3m+\frac{3\nu}{2}-\frac{1}{2}} \left\{\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{2a^2}\right)\right\} \end{array} \right\} \end{array} \right].$$

$$= \sum_{m=0}^{n} \begin{pmatrix} n+\lambda \\ n-m \end{pmatrix} \frac{(-1)^m}{m!} \left[2^{m+\frac{\nu}{2}+\frac{1}{2}} \left\{\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{2a^2}\right)\right\}\right]$$

Thus by choosing $2\,Re(m) + Re(\nu) > -1$; it follows

$$\psi^{(\nu)} \wedge (a,n) = \sum_{m=0}^{n} \begin{pmatrix} n+\lambda \\ n-m \end{pmatrix} \frac{(-1)^m}{m!} \left[2^{m+\frac{\nu}{2}+\frac{1}{2}} \left\{\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{2a^2}\right)\right\}\right]. \tag{5.5}$$

From equation (5.2) and (5.5), substituting in (4.3),

$$(H_\psi f)\,(b,a) = \sum_{q=0}^{\infty} \psi_q^{(\nu)}\,(b)\,\overline{\left(\overline{f}\right)^{\wedge}(m)} \left(\overline{\psi^{(\nu)}}\right) \wedge (a,p)$$

$$= \sum_{q=0}^{\infty} \sum_{m=0}^{q} \psi_q(\nu)\,(b) \begin{pmatrix} q+\lambda \\ q-m \end{pmatrix} \frac{(-1)^m}{m!} \left[2^{m+\frac{\nu}{2}+\frac{1}{2}} \left\{\gamma\left(q+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(q+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{2a^2}\right)\right\}\right]$$
$$\times \sum_{p=0}^{q} \begin{pmatrix} q+\lambda \\ q-p \end{pmatrix} \frac{\left(-x^2\right)^p}{p!} \frac{1}{2} \Gamma\left(q+\nu/2+1/2\right).$$

Thus

$$(H_\psi f)\,(b,a)$$
$$= \sum_{q=0}^{\infty} \sum_{m=0}^{q} \sum_{p=0}^{q} \begin{pmatrix} q+\lambda \\ q-m \end{pmatrix} \frac{(-1)^m}{m!} \begin{pmatrix} q+\lambda \\ q-m \end{pmatrix} \frac{\left(-x^2\right)^p}{p!} \frac{1}{2} \Gamma\left(q+\nu/2+1/2\right)$$
$$\times \left[2^{m+\frac{\nu}{2}+\frac{1}{2}} \left\{\Gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{1}{2a^2}\right)\right\}\right] \tag{5.6}$$

Let the mother wavelet for Kekre wavelet from [9] be

$$K\,(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ -N + (t-1) & \text{if } 1 \le t < 2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

It is piecewise continuous. Here let $N = 2$. Using Kekre wavelet the expression for $K(at)$ is as follows

$$K(at) = \begin{cases} 1 & \text{if } 0 \le t < a, \\ -N + (t-1) & \text{if } a \le t < a+1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Now using equation (5.8), follows

$$\psi^{(\nu)} \wedge (a, n) = \int_0^a \psi_n^{(\nu)}(t)\, dt - \int_a^{2a} \psi_n^{(\nu)}(t)\, dt,$$

$= \int_{0^a} t^\nu e^{-t^2/2} L_n^{(\nu)}\left(t^2\right) dt - \int_a^{a+1} (-3+t)\; t^\nu e^{-t^2/2} L_n^{(\nu)}\left(t^2\right) dt$, for n =0,1,2,3,....

$$= \sum_{m=0}^{n} \begin{pmatrix} n+\lambda \\ n-m \end{pmatrix} \frac{(-1)^m}{m!} \left[\begin{array}{l} 2^{m+\frac{\nu}{2}-\frac{1}{2}} \left\{ \gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{a^2}{2}\right) \right\} \\ -2^{m+\frac{\nu}{2}} \left\{ -i\gamma\left(m+\frac{\nu}{2}+1, \frac{(a+1)^2}{2}\right) + i\gamma\left(m+\frac{\nu}{2}+1, \frac{a^2}{2}\right) \right\} \\ +3^{m+\frac{\nu}{2}} \left\{ -i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{(a+1)^2}{2}\right) + i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{a^2}{2}\right) \right\} \end{array} \right] \tag{5.9}$$

From equation (5.2) and (5.9), substituting in (4.3), it follows as

$$(H_\psi f)(b, a) = \sum_{q=0}^{\infty} \psi_q^{(\nu)}(b)\, \overline{\left(\overline{f}\right)^{\wedge}(m)} \left(\overline{\psi^{(\nu)}}\right) \wedge (a, p)$$

$$= \sum_{q=0}^{\infty} \sum_{m=0}^{q} \sum_{p=0}^{q} \begin{pmatrix} q+\lambda \\ q-m \end{pmatrix} \frac{(-1)^m}{m!} \begin{pmatrix} q+\lambda \\ q-m \end{pmatrix} \frac{(-1)^p}{p!} \frac{1}{2} \Gamma\left(q+\nu/2+1/2\right) \frac{(-1)^m}{m!}$$

$$\times \left[\begin{array}{l} 2^{m+\frac{\nu}{2}-\frac{1}{2}} \left\{ \gamma\left(m+\frac{\nu}{2}+\frac{1}{2}\right) - i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{a^2}{2}\right) \right\} \\ -2^{m+\frac{\nu}{2}} \left\{ -i\gamma\left(m+\frac{\nu}{2}+1, \frac{(a+1)^2}{2}\right) + i\gamma\left(m+\frac{\nu}{2}+1, \frac{a^2}{2}\right) \right\} \\ +3^{m+\frac{\nu}{2}} \left\{ -i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{(a+1)^2}{2}\right) + i\gamma\left(m+\frac{\nu}{2}+\frac{1}{2}, \frac{a^2}{2}\right) \right\} \end{array} \right]$$

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