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# MODIFIED GENERALIZED WEAKLY CONTRACTIVE AND $F$-CONTRACTION MAPPINGS WITH FIXED POINT RESULTS

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Abstract. The purpose of this article is to present the existence and uniqueness results of a fixed point for cyclic generalized weakly contractive mappings as well as for cyclic $F$-contraction mappings in metric spaces. In this way, we extend and improve the conclusions of Xue [Zhiqun Xue, Fixed point theorems for generalized weakly contractive mappings, Bull. Aust. Math. Soc., (2015) 1-9] and Wardowski [Dariusz Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl., (2012) 1-6]. Examples are given to useability of our conclusions.

## 1. Introduction and Preliminaries

In 2003, an interesting generalization of the Banach contraction principle was presented by Kirk, Srinivasan and Veeramani as follows.

**Theorem 1.1.** *([9]). Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $X = (X, d)$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping i.e. $T(A) \subseteq B$, $T(B) \subseteq A$, such that*

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

*for some $\alpha \in ]0, 1[$ and for all $x \in A, y \in B$. Then $A \cap B \neq \emptyset$ and $T$ has a unique fixed point in $A \cap B$.*

It is worth noticing that the cyclic mapping $T$ considered in Theorem 1.1 is not continuous on it's domain necessary. We refer to [1, 2] for more information about the existence of fixed points for various classes of cyclic mappings.

Very recently, Xue ([13]) established another interesting extension of the Banach fixed point theorem as below (see also [14] for similar results).

**Theorem 1.2.** *Let $(X, d)$ be a complete metric space and $T : X \to X$ be a mapping such that for all $x, y \in X$,*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

*where $M(x, y) := \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$ and $\psi, \varphi$ satisfy the following conditions:*

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*(i)* $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ *are two functions with* $\psi(t) = \varphi(t) = 0$ *iff* $t = 0$,
*(ii)* $\liminf_{\tau \to t} \psi(\tau) > \limsup_{\tau \to t} \psi(\tau) - \liminf_{\tau \to t} \varphi(\tau)$ *for all* $t > 0$.
*Then* $T$ *has a unique fixed point.*

Theorem 1.2 extends and improves some recent fixed point theorems appeared in [4, 5]. We also refer to [3, 6, 7] for some recent relevant results.

Another interesting fixed point theorem was established by Wardowski in [11]. In what follows $\mathbb{R}^+$ denotes the set of all positive real numbers.

**Definition 1.3.** *Let* $F : \mathbb{R}^+ \to \mathbb{R}$ *be a function satisfying:*
$(F_1)$ *F is strictly increasing;*
$(F_2)$ *For each sequence* $\{\alpha_n\}$ *of positive numbers* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(F_3)$ *There exists* $r \in (0, 1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^r F(\alpha) = 0$.
*Suppose* $(X, d)$ *is a metric space. Then a self-mapping* $T : X \to X$ *is called* $F$*-contraction if there exists* $\tau > 0$ *so that*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)),$$

*for any* $x, y \in X$.

We refer to Examples 2.1, 2.2 and 2.3 of [11] to illustrate this contractivity condition on self-mappings.

Next existence and uniqueness theorem is a main result of [11] (Theorem 2.1 of [11]).

**Theorem 1.4.** *(Wardowski's fixed point theorem) Let* $(X, d)$ *be a complete metric space and let* $T : X \to X$ *be an* $F$*-contraction mapping. Then* $T$ *has a unique fixed point* $x^* \in X$ *and for any* $x_0 \in X$ *a sequence* $\{T^n x_0\}$ *is convergent to* $x^*$.

In this article, we extend and improve Theorems 1.2 and 1.4 to cyclic mappings under modified conditions. We also present examples to illustrate our main conclusions.

## 2. Cyclic generalized weakly contractive mappings

We begin the main results of this paper with the following theorem.

**Theorem 2.1.** *Let* $A$ *and* $B$ *be two nonempty and closed subsets of a complete metric space* $(X, d)$ *and* $T : A \cup B \to A \cup B$ *be a cyclic mapping such that for all* $(x, y) \in A \times B$,

$$\psi(d(Tx, Ty)) \le \psi(M(x, y)) - \varphi(M(x, y)),$$

*where* $M$ *is defined as in Theorem 1.2 and* $\psi, \varphi : cl(\mathrm{rand}) \to [0, \infty)$ *are functions so that*
*(i)* $\psi(t) = \varphi(t) = 0$ *iff* $t = 0$,
*(ii)* $\liminf_{\tau \to t} \psi(\tau) > \limsup_{\tau \to t} \psi(\tau) - \liminf_{\tau \to t} \varphi(\tau)$ *for all* $t > 0$,
*where* $cl(\mathrm{rand})$ *denotes the closure of the value of the metric* $d$ *defined on* $X \times X$.
*Then* $A \cap B$ *is nonempty and* $T$ *has a unique fixed point in* $A \cap B$.

*Proof.* Choose $x_0 \in A$ and consider the Picard iteration $x_{n+1} = Tx_n$, where $n \in \mathbb{N} \cup \{0\}$. Since $T$ is cyclic on $A \cup B$, $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are sequences in $A$ and $B$ respectively. We show that $d(x_n, x_{n+1}) \to 0$. If $d(x_n, x_{n+1}) = 0$ for some $n \ge 0$, then we are finished. So we assume that $d(x_n, x_{n+1}) > 0$ for all $n$. It follows from the condition $(ii)$ that

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) &= \psi(d(Tx_{2n}, Tx_{2n-1})) & (2.1)\\ &\le \psi(M(x_{2n}, x_{2n-1})) - \varphi(M(x_{2n}, x_{2n-1})) & (2.2)\\ &\le \psi(M(x_{2n}, x_{2n-1})), & (2.3) \end{aligned}$$

where $M(x_{2n}, x_{2n-1}) = \max\{d(x_{2n}, x_{2n-1}), d(x_{2n+1}, x_{2n})\}$. Now if $d(x_{2n+1}, x_{2n}) > d(x_{2n}, x_{2n-1})$, then from (2.3), we obtain

$$\psi(d(x_{2n+1}, x_{2n})) \le \psi(d(x_{2n+1}, x_{2n})) - \varphi(d(x_{2n+1}, x_{2n})),$$

which is a contradiction. Thus we must have $d(x_{2n+1}, x_{2n}) \le d(x_{2n}, x_{2n-1})$. Again, by using (2.3) we conclude that

$$\begin{aligned} \psi(d(x_{2n+1}, x_{2n})) &\le \psi(d(x_{2n}, x_{2n-1})) & (2.4)\\ &\le \psi(M(x_{2n-1}, x_{2n-2})) - \varphi(M(x_{2n-1}, x_{2n-2})) & (2.5)\\ &\le \psi(M(x_{2n-1}, x_{2n-2})), & (2.6) \end{aligned}$$

for which $M(x_{2n-1}, x_{2n-2}) = \max\{d(x_{2n-1}, x_{2n-2}), d(x_{2n}, x_{2n-1})\}$. If $d(x_{2n}, x_{2n-1}) > d(x_{2n-1}, x_{2n-2})$, then we get a contradiction by (2.6). Thus

$$d(x_{2n+1}, x_{2n}) \le d(x_{2n}, x_{2n-1}) \le d(x_{2n-1}, x_{2n-2}), \tag{2.7}$$

which concludes that $M(x_{2n-1}, x_{2n-2}) = d(x_{2n-1}, x_{2n-2})$ and so $\psi(d(x_{2n+1}, x_{2n})) \le \psi(d(x_{2n-1}, x_{2n-2}))$. Thereby, the sequences $\{\psi(d(x_{2n+1}, x_{2n}))\}$ and $\{d(x_{2n+1}, x_{2n})\}$ are decreasing. Suppose

$$\lim_{n\to\infty} d(x_{2n+1}, x_{2n}) = r \quad \text{and} \quad \lim_{n\to\infty} \psi(d(x_{2n+1}, x_{2n})) = R.$$

Note that if $r > 0$, then from the relation (2.6)

$$\begin{aligned} R &= \lim_{n\to\infty} \psi(d(x_{2n+1}, x_{2n}))\\ &\le \lim_{n\to\infty} \psi(d(x_{2n-1}, x_{2n-2})) - \liminf_{n\to\infty} \varphi(d(x_{2n-1}, x_{2n-2}))\\ &= R - \liminf_{n\to\infty} \varphi(d(x_{2n-1}, x_{2n-2})), \end{aligned}$$

which is a contradiction with $(ii)$. Therefore, $r = 0$. Moreover, $d(x_{2n}, x_{2n-1}) \to 0$ from (2.7). We now prove that the sequence $\{x_{2n}\}$ is a cauchy sequence. Suppose the contrary. Then there exist $\varepsilon > 0$ and the subsequences $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$ and

$$d(x_{2m_k}, x_{2n_k}) \ge \varepsilon \quad \text{and} \quad d(x_{2m_k}, x_{2n_k-1}) < \varepsilon.$$

So,

$$\begin{aligned} \varepsilon &\le d(x_{2m_k}, x_{2n_k})\\ &\le d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k})\\ &< \varepsilon + d(x_{2n_k-1}, x_{2n_k}) \to \varepsilon \quad (k \to \infty). \end{aligned}$$

That is, $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \varepsilon$. We now have

$$\psi(d(x_{2m_k}, x_{2n_k+1})) \le \psi(M(x_{2m_k-1}, x_{2n_k})) - \varphi(M(x_{2m_k-1}, x_{2n_k})), \tag{2.8}$$

for all $k \in \mathbb{N}$. Also,

$$\begin{aligned} M(x_{2m_k-1}, x_{2n_k}) = \max\{&d(x_{2m_k-1}, x_{2n_k}), d(x_{2m_k-1}, x_{2m_k}), d(x_{2n_k}, x_{2_k+1}),\\ &\frac{1}{2}[d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2m_k}, x_{2n_k})]\} \end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{1}{2}[d(x_{2m_k-1}, x_{2n_k+1}) + d(x_{2m_k}, x_{2n_k})]\\ \leq &\frac{1}{2}[d(x_{2m_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}) + d(x_{2m_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2n_k})]\\ &\leq \frac{1}{2}[2d(x_{2m_k-1}, x_{2n_k}) + d(x_{2n_k}, x_{2n_k+1}) + d(x_{2m_k}, x_{2m_k-1})].\end{aligned}$$

Hence,

$$d(x_{2m_k-1}, x_{2n_k}) \leq M(x_{2m_k-1}, x_{2n_k}) \leq d(x_{2m_k-1}, x_{2n_k})+d(x_{2n_k}, x_{2n_k+1})+d(x_{2m_k}, x_{2m_k-1}).$$

Letting $k \to \infty$, we obtain $\lim_{k\to\infty} M(x_{2m_k-1}, x_{2n_k}) = \lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k})$. Also,

$$|d(x_{2m_k-1}, x_{2n_k}) - d(x_{2n_k}, x_{2m_k})| \leq d(x_{2m_k-1}, x_{2m_k}) \to 0.$$

Therefore, $|\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) - \varepsilon| = 0$, that is,

$$\lim_{k\to\infty} M(x_{2m_k-1}, x_{2n_k}) = \lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \varepsilon.$$

Besides,

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) \leq \lim_{k\to\infty}[d(x_{2m_k}, x_{2n_k}) + d(x_{2n_k}, x_{2n_{k+1}})] = \varepsilon.$$

It follows from (2.8) that

$$\begin{aligned}\inf_{j\geq k} \psi(d(x_{2m_j}, x_{2n_j+1})) + \inf_{j\geq k} \varphi(M(x_{2m_j-1}, x_{2n_j}))&\\ \leq \inf_{j\geq k} \psi(M(x_{2m_j-1}, x_{2n_j}))&\\ \leq \sup_{i\geq k} \psi(M(x_{2m_i-1}, x_{2n_i})),&\end{aligned}$$

which implies that

$$\liminf_{t\to\varepsilon} \psi(t) + \liminf_{t\to\varepsilon} \varphi(t) \leq \limsup_{t\to\varepsilon} \psi(t),$$

which is a contradiction with the condition (ii). Thereby, $\{x_{2n}\}$ is a Cauchy sequence in $A$. Since $A$ is closed, $\{x_{2n}\}$ converges to a point of $A$, namely $p$. We have

$$d(p, x_{2n+1}) \leq d(p, x_{2n}) + d(x_{2n} + x_{2n+1}).$$

Taking $n \to \infty$ in above relation, we obtain $d(p, x_{2n+1}) \to 0$ and so, $x_{2n+1} \to p$. Hence, $p \in A \cap B$ and the sequence $\{x_n\}$ converges to the point $p \in A \cap B$. It now follows from the similar argument of the proof of Theorem 1.2 that $p$ is a unique fixed point for the mapping $T$ and this completes the proof of theorem.

□

Next result is straightforward consequence of Theorem 2.1.

**Corollary 2.2.** *(Extension of Theorem 2.2 of* [5]*, Theorem 4 of* [8]*) Let $A$ and $B$ be two nonempty and closed subsets of a complete metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping such that for all $(x, y) \in A \times B$,*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

*where $M$ is defined as in Theorem 1.2 and $\varphi, \psi$ satisfy the following conditions:*
*(i) $\psi : [0, \infty) \to [0, \infty)$ is a monotone and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,*
*(ii) $\varphi : [0, \infty) \to [0, \infty)$ is a function with $\varphi(t) = 0$ if and only if $t = 0$ and*

*$\liminf_{n\to\infty}\varphi(a_n) > 0$ if $\lim_{n\to\infty} a_n = a > 0$.*
*(iii) $\varphi(a) > \psi(a) - \psi(a-)$ for any $a > 0$, where $\psi(a-)$ is the left limit of $\psi$ at $a$.*
*Then $A \cap B$ is nonempty and $T$ has a unique fixed point in $A \cap B$.*

Let us illustrate Theorem 2.1 with the following example.

**Example 2.1.** Let $X = [-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and define a metric $d$ on $X$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{|x|, |y|\} & \text{if } x \neq y. \end{cases}$$

Obviously, $cl(\text{rand}) = [0, \frac{\sqrt{2}}{2}]$. Let $A = [-\frac{1}{2}, 0]$ and $B = [0, \frac{1}{2}]$ and $T : A\cup B \to A\cup B$ the cyclic mapping given by

$$Tx = \begin{cases} x^2 & \text{if } x \in A, \\ -x^2 & \text{if } x \in B. \end{cases}$$

Define $\psi, \varphi : cl(\text{rand}) \to [0, \infty)$ with

$$\psi(t) = \sqrt{t}, \quad \& \quad \varphi(t) = \sqrt{t} - t.$$

It is easy to verify that the condition

$$\liminf_{\tau\to t} \psi(\tau) > \limsup_{\tau\to t} \psi(\tau) - \liminf_{\tau\to t} \varphi(\tau),$$

holds for all $t \in cl(\text{rand})$ with $t > 0$. On the other hand, by the definition of $M$ and the metric $d$

$$d(Tx, Ty) = \max\{x^2, y^2\},$$

$$M(x, y) = \max\{\max\{-x, y\}, \max\{-x, x^2\}, \max\{y, y^2\}, \frac{1}{2}[\max\{-x, y^2\}+\max\{y, x^2\}]\}$$

$$= \max\{-x, y\}.$$

for all $(x, y) \in A \times B$. We now have the following two cases.
**Case 1.** If $x = y = 0$, then

$$\psi(d(Tx, Ty)) = 0 = \psi(M(x, y)) - \varphi(M(x, y)).$$

**Case 2.** If $x \neq y$, then

$$\psi(d(Tx, Ty)) = \psi(\max\{x^2, y^2\}) = \sqrt{\max\{x^2, y^2\}} = \max\{-x, y\}$$

$$= M(x, y) = \sqrt{M(x, y)} - (\sqrt{M(x, y)} - M(x, y)) = \psi(M(x, y)) - \varphi(M(x, y)).$$

Therefore, all of the conditions of Theorem 2.1 hold and so $T$ has a unique fixed point in $A \cap B$ and it is clear that this point is $p = 0$. It is interesting to note that our result cannot be obtained from Theorem 1.2 due to Xue because of the fact that the considered function $\varphi$ does not map $[0, \infty)$ to $[0, \infty)$ (indeed, $\varphi(t) < 0$ for any $t > 1$).

## 3. $F$-cyclic contractions

Motivated by Theorem 2.1, we extend and improve the Wardowski's fixed point theorem as follows.

**Theorem 3.1.** *Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$ and $T : A\cup B \to A\cup B$ be a cyclic mapping such that*

$$d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(M(x,y)), \quad \forall (x,y)\in A\times B \tag{3.1}$$

*where $M$ is defined as in Theorem 1.2, $\tau > 0$ and $F : cl(\mathrm{rand})^+ \to \mathbb{R}$ is a function satisfying:*
*$(F_1)$ $F$ is strictly increasing;*
*$(F_2)$ For each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n\to\infty}\alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
*$(F_3)$ There exists $r\in(0,1)$ such that $\lim_{\alpha\to 0^+}\alpha^r F(\alpha) = 0$.*
*If $A$ is complete and $T|_A$ is continuous, then $A\cap B$ is nonempty and $T$ has a unique fixed point in $A\cap B$.*

*Proof.* Let $x_0\in A$ and define $x_{n+1} = Tx_n$. Since $T$ is cyclic on $A\cup B$, $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are sequences in $A$ and $B$ respectively. Put $\delta_n = d(x_n,x_{n+1})$. If we have $x_{n_0+1} = x_{n_0}$ for some $n_0\in\mathbb{N}$, then we are finished. So assume that $x_n \neq x_{n+1}$ for all $n\in\mathbb{N}$. We now have

$$\begin{aligned}
&F(\delta_{2n}) = F(d(x_{2n},x_{2n+1})) = F(d(Tx_{2n-1},Tx_{2n}))\\
&\leq F(\max\{d(x_{2n-1},x_{2n}), d(x_{2n-1},x_{2n}), d(x_{2n},x_{2n+1}), \frac{d(x_{2n-1},x_{2n+1})}{2}\}) - \tau\\
&\leq F(\max\{d(x_{2n-1},x_{2n}), d(x_{2n},x_{2n+1}), \frac{d(x_{2n-1},x_{2n}) + d(x_{2n},x_{2n+1})}{2}\}) - \tau\\
&= F(\max\{d(x_{2n-1},x_{2n}), d(x_{2n},x_{2n+1})\}) - \tau.
\end{aligned}$$

Notice that if $\max\{d(x_{2n-1},x_{2n}), d(x_{2n},x_{2n+1})\} = d(x_{2n},x_{2n+1})$ for some $n\in\mathbb{N}$, then we obtain $F(\delta_{2n}) \leq F(\delta_{2n}) - \tau$, which is a contradiction. So we must have $d(x_{2n},x_{2n+1}) < d(x_{2n-1},x_{2n})$ for all $n\in\mathbb{N}$, which implies that $F(\delta_{2n}) \leq F(\delta_{2n-1}) - \tau$ for all $n\in\mathbb{N}$. Continuing this process and by induction, we conclude that

$$F(\delta_{2n}) \leq F(\delta_0) - 2n\tau.$$

Thus $\lim_{n\to\infty} F(\delta_{2n}) = -\infty$ which concludes that $\delta_{2n}\to 0$. Similarly, we can see that $F(\delta_{2n-1}) \leq F(\delta_1) - 2(n-1)\tau$ and so $\delta_{2n-1}\to 0$. Therefore, $\delta_n\to 0$. It now follows from the condition of $(F_3)$ that there exist $r_1, r_2\in(0,1)$ so that

$$\lim_{n\to\infty}\delta_{2n}^{r_1}F(\delta_{2n}) = 0, \quad \lim_{n\to\infty}\delta_{2n-1}^{r_2}F(\delta_{2n-1}) = 0.$$

Thereby, for all $n\in\mathbb{N}$ we have

$$\delta_{2n}^{r_1}F(\delta_{2n}) - \delta_{2n}^{r_1}F(r_0) \leq \delta_{2n}^{r_1}(F(r_0) - 2n\tau) - \delta_{2n}^{r_1}F(r_0) = -2n\delta_{2n}^{r_1}\tau,$$

which deduces that $\lim_{n\to\infty} n\delta_{2n}^{r_1} = 0$. Equivalently,

$$\begin{aligned}
&\delta_{2n-1}^{r_2}F(\delta_{2n-1}) - \delta_{2n-1}^{r_2}F(r_1)\\
&\leq \delta_{2n-1}^{r_2}(F(r_1) - 2(n-1)\tau) - \delta_{2n-1}^{r_2}F(r_1)\\
&= -2(n-1)\delta_{2n-1}^{r_2}\tau,
\end{aligned}$$

and so, $\lim_{n\to\infty} n\delta_{2n-1}^{r_2} = 0$. Suppose there exists $N \in \mathbb{N}$ such that $n\delta_{2n}^{r_1} \leq 1$ and $n\delta_{2n-1}^{r_2} \leq 1$ for all $n \geq N$. Hence,

$$\delta_{2n} \leq \frac{1}{n^{\frac{1}{r_1}}}, \quad \delta_{2n-1} \leq \frac{1}{n^{\frac{1}{r_2}}}, \quad \forall n \geq N.$$

Now, for each $m > n \geq N$ we have

$$\begin{aligned} d(x_{2m}, x_{2n}) &\leq \sum_{j=n}^{m+1} [d(x_{2j}, x_{2j+1}) + d(x_{2j+1}, x_{2j+2})] \\ &= \sum_{j=n}^{m} (\delta_{2j} + \delta_{2j+1}) \leq \sum_{j=n}^{\infty} (\delta_{2j} + \delta_{2j+1}) \\ &\leq \sum_{j=1}^{\infty} (\frac{1}{j^{\frac{1}{r_1}}} + \frac{1}{(j+1)^{\frac{1}{r_2}}}) < \infty. \end{aligned}$$

This concludes that $\{x_{2n}\}$ is a Cauchy sequence in $A$ and by the fact that $A$ is complete, $x_{2n} \to p$ for some $p \in A$. Continuity of $T$ on $A$ implies that $x_{2n+1} = Tx_{2n} \to Tp$. So

$$d(p, Tp) = \lim_{n\to\infty} \delta_{2n} = 0.$$

The uniqueness of the fixed point for the mapping $T$ can be obtained from a similar way of the proof of Theorem 2.4 of [12].

□

**Remark.** *It is worth noticing that we can replace the condition of continuity of the mapping $T$ on $A$ with the condition of continuity of the function $F$ (see Theorem 2.4 of* [12] *for more details).*

We mention that the condition (3.1) of Theorem 3.1 is a sufficient but not necessary condition. Let us illustrate this fact with the following example.

**Example 3.1.** Consider $X = [0,1] \cup \mathbb{Z}$ with the following metric

$$d(x,y) = \begin{cases} 0 \quad \text{if } x = y, \\ \max\{x,y\} \quad \text{if } x \neq y\ , x,y \in [0,1], \\ \max\{\frac{1}{|x|}, \frac{1}{|y|}\} \quad \text{if } x \neq y,\ x,y \in \mathbb{Z} - \{0,1\}, \\ \max\{x, \frac{1}{|y|}\}, \quad \text{if } x \in [0,1],\ y \in \mathbb{Z} - \{0,1\}. \end{cases}$$

Then it is easy to see that $(X,d)$ is a complete metric space and that $cl(\text{rand}) = [0,1]$. It is worth noticing that any convergent sequence in $X$ converges to 0. Let $A = [0,1]$ and $B = \mathbb{Z} - \mathbb{N}$ and define the cyclic mapping $T : A \cup B \to A \cup B$ as

$$Tx = \begin{cases} 0 \quad \text{if } x \in A, \\ \frac{1}{|x|+1} \quad \text{if } x \in B - \{0\}. \end{cases}$$

We also define the function $F : cl(\text{rand})^+ \to \mathbb{R}$ as $F(\alpha) = -\cot\sqrt{\alpha}$. We have the following facts about the function $F$:

$(F_1)$ $F$ is strictly increasing. To see this, it is sufficient to note that

$$F'(\alpha) = \frac{1}{2\sqrt{\alpha}}(1 + \cot^2\sqrt{\alpha}) > 0, \quad \forall \alpha \in (0,1].$$

$(F_2)$ It is clear that for a sequence $\{\alpha_n\}$ of positive real numbers, $\alpha_n \to 0$ if and only if $\lim_{n\to\infty} -\cot\sqrt{\alpha_n} = -\infty$.
$(F_3)$ For any $r \in (0, \frac{1}{2})$ we have

$$\lim_{\alpha\to 0^+} \alpha^r \cot\sqrt{\alpha} = \lim_{\alpha\to 0^+} \frac{\alpha^r}{\tan\sqrt{\alpha}} =^H 2r \lim_{\alpha\to 0^+} \frac{\alpha^{\frac{1}{2}-r}}{1+\tan^2\sqrt{\alpha}} = 0.$$

Now, assume that $(x, y) \in A \times (B - \{0\})$. Then we have

$$d(Tx, Ty) = \max\{0, \frac{1}{|y|+1}\}, \quad d(x, y) = \max\{x, \frac{1}{|y|}\}.$$

Since $F$ is strictly increasing, we conclude that

$$F(d(Tx, Ty)) = -\cot\sqrt{\frac{1}{|y|+1}}$$
$$< F(d(x, y)) = -\cot\sqrt{\max\{x, \frac{1}{|y|}\}}.$$

Put

$$\tau := \inf_{n\in\mathbb{N}}(\cot\sqrt{\frac{1}{n+1}} - \cot\sqrt{\frac{1}{n}}).$$

Since

$$\lim_{n\to\infty}(\cot\sqrt{\frac{1}{n+1}} - \cot\sqrt{\frac{1}{n}}) = 0,$$

we must have $\tau = 0$. Thereby,

$$F(d(Tx, Ty)) < F(d(x, y)), \quad \forall (x, y) \in A \times (B - \{0\}),$$

that is, the condition (3.1) does not hold where as $T$ has a unique fixed point in $A \cap B$.

**Remark.** *It is interesting to note that Example 3.1 cannot be concluded from Theorem 1.4 due to Wardowski because the function $F$ is not defined on $\mathbb{R}^+$.*

**Remark.** *The results of the current paper can be extended to generating space of a b-quasi-metric family which was introduced by P. Kumari and D. Panthi in* [10] *in order to find coincidence and common fixed points of two cyclic mappings.*

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