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TAUBERIAN THEOREMS FOR THE GENERALIZED DE LA VALLÉE-POUSSIN MEAN-CONVERGENT SEQUENCES OF FUZZY NUMBERS

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Abstract. In this paper, we will show Tauberian theorems for the generalized de la Vallée-Poussin mean-convergent sequences of fuzzy numbers.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [18] and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming. Matloka [11] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Nuray and Savas [13], Kwon [9], Altin et al. [1], Gokhan et al. [8], Et et al. [5] and many others. The notion of statistical convergence was introduced by Fast [6] and Steinhaus [15], independently.

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $K \in \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the natural density of K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ if the limit exists, where the vertical bars indicate the number of elements in the enclosed set. The sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero (cf. [6, 15]), i.e. for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $L = st - \lim x$. Note that every convergent sequence is statistically convergent but not conversely.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized de la Vallée-Poussin mean is

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defined by

$$T_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j,$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_j)$ is said to be (V, λ)-summable to a number L (see [10]) if $T_n(x) \to L$ as $n \to \infty$. In this case L is called the λ-limit of x, and we say that $x = (x_n)$ is λ-statistical convergent to L, if

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

for every given $\epsilon > 0$, and will write $st_\lambda - \lim_n x_n = L$.

A sequence $x = (x_n)$ is said to be statistically $\lambda-$convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{n} |\{k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.1}$$

holds. In this case we write that $st - \lim_n T_n = L$.

In what follows we will define the following type of the statistical convergence. A sequence $x = (x_n)$ is said to be $(V, \lambda)-$statistically convergent to L if for every $\epsilon > 0$ the following relation

$$\lim_n \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |T_k(x) - L| \geq \epsilon\}| = 0, \tag{1.2}$$

holds. In this case we write that $st_\lambda - \lim_n T_n = L$.

In paper [4], was given conditions under which for every bounded sequence (x_k) the implication

$$st_\lambda - \lim_k x_k = L \quad \text{implies} \quad st_\lambda - \lim_k T_k = L$$

holds. The converse of the above fact is known as Tauberian Theorem. Also this fact, for the above summability method, is given in [4]. The theory of Tauberian theorems are intensively investigated by several authors, see [12], [10], [4],[16]. In this paper we will prove Tauberian theorems for fuzzy sequence spaces. Denote by

$$L(\mathbb{R}^n) = \{u : \mathbb{R}^n \to [0,1] : u \quad \text{satisfies} \quad (1) - (4) \quad \text{bellow}\}$$

where

(1) u is normal, there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)]$,

(3) u is upper semicontinuous,

(4) the closure of $\{x \in \mathbb{R}^n : u(x) > 0\}$, denoted by $[u]^0$, is compact.

If $u \in L(\mathbb{R}^n)$, then u is called fuzzy number, and $L(\mathbb{R}^n)$ is said to be fuzzy number space. For $0 < \alpha \leq 1$, the $\alpha-$ level set $[u]^\alpha$ of u is defined by $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. Then from (1)-(4), it follows that the $\alpha-$level sets $[u]^\alpha$ give information about $C(\mathbb{R}^n)$.

We will denote by E the set of all fuzzy numbers on $\mathbb{R}$. The set of real numbers can be embedded in E, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r} = \begin{cases} 1; & \text{if} \quad x = r, \\ 0; & \text{if} \quad x \neq r. \end{cases}$$

Let $u, v, w \in E$ and $k \in \mathbb{R}$. Then the operations addition and scalar multiplications are defined in E as follows:

$$u + v = w \Leftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha \quad \text{for all} \quad \alpha \in [0, 1],$$

$$\Leftrightarrow w_\alpha^- = u_\alpha^- + v_\alpha^- \quad \text{and} \quad w_\alpha^+ = u_\alpha^+ + v_\alpha^+ \quad \text{for all} \quad \alpha \in [0, 1],$$

$$[ku]_\alpha = k[u]_\alpha \quad \text{for all} \quad \alpha \in [0, 1].$$

Further details related to the structural properties of the fuzzy numbers, are given in [3]. Let us denote by W the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A = [\underline{A}, \overline{A}]$. Define the relation d on $W$by

$$d(A, B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be easily observed that d is a metric on W and (W, d) is a complete metric space, ([**?**]). Now, we may define the metric D on E by means of the Hausdorff metric d as follows

$$D(u, v) = \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha) = \sup_{\alpha \in [0;1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\},$$

and

$$D(u, 0) = \sup \alpha \in [0; 1] \max\{|u^-(\alpha)|, |u^+(\alpha)|\} = \max\{|u^-(\alpha)|, |u^+(\alpha)|\}.$$

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, into the set E. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the $k-$th term of the sequence. By $w(F)$, we denote the set of all sequences of fuzzy numbers. A sequence $(u_n) \in w(F)$ is said to be convergent to $u \in E$, if for every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$D(u_n, u) < \epsilon \quad \text{for all} \quad n > n_0.$$

Definition 1.1. [14] *Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to converge statistically to a fuzzy number X_0, if for every $\epsilon > 0$*

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : D(X_k, X_0) \geq \epsilon\}| = 0.$$

The above type of convergence we will denote by

$$st_{F\lambda} - \lim_n X_n = X_0.$$

Definition 1.2. *Let $X = (X_k)$ be a sequence of fuzzy numbers. The sequence X is said to be statistically generalized de la Vallée-Poussin summable to a fuzzy number X_0 if the sequence*

$$T_n(X) = \frac{1}{\lambda_n} \sum_{j \in I_n} X_j$$

is statistically convergent to X_0, where the sum in $T_n(X)$ is usual addition of fuzzy real numbers through $\alpha-$ level sets. That is (X_k) is statistically generalized de la Vallée-Poussin summable to the fuzzy number X_0, if for every $\epsilon > 0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in I_n : D(T_k,X_0)\geq\epsilon\}\right|=0.$$

The above type of convergence we will denote by

$$st_{F\lambda}-\lim_n T_n = X.$$

Theorem 1.3. *Let us suppose that (X_k) is a bounded sequence of fuzzy numbers such that exists $st_{F\lambda}-\lim_k X_k = L$, then it follows that $st_{F\lambda}-\lim_k T_k = L$, but not conversely.*

Proof. Let us suppose that $st_{F\lambda}-\lim_k X_k = L$. Let $\epsilon > 0$ be any given number. Then

$$D(T_k,L)=D\left(\frac{1}{\lambda_k}\sum_{j\in I_k}X_j, L\right)=D\left(\frac{1}{\lambda_k}\sum_{j\in I_k}X_j, \frac{\lambda_n}{\lambda_n}L\right)=\left|\frac{1}{\lambda_n}\right|D\left(\sum_{j\in I_k}X_j,\lambda_k L\right)$$

$$\leq\frac{1}{\lambda_n}D\left(\sum_{j\in I_k}X_j, L\right)\leq\frac{1}{\lambda_n}\sup_{j\in I_k}\{D\left(X_j,L\right)\}\to 0,\quad \text{as}\quad n\to\infty,$$

because

$$\sup_{k\in I_n}\{D(X_k,L)\}\subset\sup\{k\in I_n : D(X_k,L)\leq\epsilon\}\cup\sup\{k\in I_n : D(X_k,L)\geq\epsilon\}.$$

And from last relation we have:

$$\lim_n\frac{|\sup_{k\in I_n}\{D(X_k,L)\}|}{\lambda_n}\leq\lim_n\frac{|\{k\in I_n : D(X_k,L)\leq\epsilon\}|}{\lambda_n}+\lim_n\frac{|\{k\in I_n : D(X_k,L)\geq\epsilon\}|}{\lambda_n}.$$

To prove that converse is not true, we construct this example

Example 1.4. *Let us define the following sequence of fuzzy numbers*

$$X_k(t)=\begin{cases}\left.\begin{array}{cc} t-1, & \text{for } 1\leq t\leq 2\\ -t+3, & \text{for } 2\leq t\leq 3\\ 0, & \text{otherwise}\end{array}\right\} := L_1, & \text{if } k \text{ is odd}\\ \left.\begin{array}{cc} t-5, & \text{for } 5\leq t\leq 6\\ -t+7, & \text{for } 6\leq t\leq 7\\ 0, & \text{otherwise}\end{array}\right\} := L_2, & \text{if } k \text{ is even}\end{cases}$$

Then, we calculate the $\alpha-$level sets of sequences (X_k) as follows

$$[X_k]^\alpha=\begin{cases}[\alpha+1,3-\alpha]:=L_1^\alpha, & \text{if } k \text{ is odd}\\ [\alpha+5,7-\alpha]:=L_2^\alpha, & \text{if } k \text{ is even}\end{cases}$$

Of course this sequence is not $st_\lambda-$convergent.

On the other hand, consider the sequence

$$X_k(t)=\begin{cases}\left.\begin{array}{cc}\frac{k}{2}t+1, & \text{for } -\frac{2}{k}\leq t\leq 0\\ -\frac{k}{2}t+1, & \text{for } 0\leq t\leq\frac{2}{k}\\ 0, & \text{otherwise}\end{array}\right\}, & \text{if } k=n^3\\ \bar{0}, & \text{otherwise}\end{cases}$$

Then, the $\alpha-$level sets of sequences (X_k) is

$$[X_k]^\alpha = \begin{cases} \left[\frac{2(\alpha-1)}{k}, \frac{2(1-\alpha)}{k}\right], & \text{if } k = n^3 \\ [0,0], & \text{otherwise} \end{cases}$$

After some arithmetic operations, we conclude that (X_k) *is* $(V,\lambda)-$*summable to* $\bar{0}$ *and hence* $(V,\lambda)-$*statistically convergent to* $\bar{0}$.

□

In this paper our aim is to find conditions (so-called Tauberian) under which the converse implication holds, for defined convergence. Exactly, we will prove under which conditions $\lambda-$statistical convergence follows from (V,λ)-statistically convergence.

2. MAIN RESULTS

Theorem 2.1. *Let* (λ_n) *be a sequence of real numbers defined as above and*

$$st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} > 1, \quad t > 1 \tag{2.1}$$

where t_n, *denotes the integral parts of the* $[tn]$ *for every* $n \in \mathbb{N}$, *and let* (X_k) *be a sequence of fuzzy real numbers such that* $st_{F\lambda} - \lim_k T_k = L$. *Then* (X_k) *is* $st_{F\lambda}-$ *convergent to the same fuzzy number* L *if and only if the following to conditions holds:*

$$\inf_{t>1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : D\left(\frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (X_j - X_k), X_k \right) \geq \epsilon \right\} \right| = 0 \tag{2.2}$$

and

$$\inf_{0<t<1} \limsup_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : D\left(\frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} (X_k - X_j), X_k \right) \geq \epsilon \right\} \right| = 0. \tag{2.3}$$

Remark 2.2. *Let us suppose that* $st_{F\lambda} - \lim_k X_k = L$; $st_{F\lambda} - \lim_k T_k = L$ *and relation (2.1) satisfies, then for every* $t > 1$, *is valid the following relation:*

$$st_{F\lambda} - \lim_n \frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} X_j = 0 \tag{2.4}$$

and in case where $0 < t < 1$,

$$st_{F\lambda} - \lim_n \frac{1}{\lambda_k - \lambda_{t_k}} \sum_{j=t_k+1}^{k} X_j = 0. \tag{2.5}$$

In what follows we will show some auxiliary lemmas which are needful in the sequel.

Lemma 2.3. *For the sequence of real numbers* $\lambda = (\lambda_n)$, *condition given by relation (2.1) is equivalent to this one:*

$$st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} > 1, \quad 0 < t < 1. \tag{2.6}$$

Proof. Let us suppose that relation (2.1) is valid, $0 < t < 1$ and $m = t_n = [t \cdot n]$, $n \in \mathbb{N}$. Then it follows that

$$\frac{1}{t} > 1 \Rightarrow \frac{m}{t} = \frac{[t \cdot n]}{t} \leq n$$

now from nondecreasing of the sequence $\lambda = (\lambda_n)$, we get:

$$\frac{\lambda_n}{\lambda_{t_n}} \geq \frac{\lambda_{\left[\frac{m}{t}\right]}}{\lambda_m} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_n}{\lambda_{t_n}} \geq st_\lambda - \liminf_n \frac{\lambda_{\left[\frac{m}{t}\right]}}{\lambda_{t_n}} > 1.$$

Conversely, let us suppose that relation (2.6) is valid. Let $t > 1$ and be given and let t_1 be chosen such that $1 < t_1 < t$. Set $m = t_n = [t \cdot n]$. From $0 < \frac{1}{t} < \frac{1}{t_1} < 1$, it follows that:

$$n \leq \frac{tn - 1}{t_1} < \frac{[tn]}{t_1} = \frac{m}{t_1},$$

provided $t_1 \leq t - \frac{1}{n}$, which is the case if n is large enough. Under this conditions we have:

$$\frac{\lambda_{t_n}}{\lambda_n} \geq \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} \Rightarrow st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_n} \geq st_\lambda - \liminf_n \frac{\lambda_{t_n}}{\lambda_{\left[\frac{m}{t_1}\right]}} > 1.$$

□

Lemma 2.4. *Let us suppose that $X = (X_k)$ is a sequence of fuzzy numbers which is $(V, \lambda)-$statistically convergent to a fuzzy number L. Then for every $t > 0$,*

$$st_{F\lambda} - \lim_n T_{t_n} = L,$$

where by $t_n = [t \cdot n]$, is denote the integral part of the product $t \cdot n$.

Proof. Let us consider that $t > 1$, then from construction of the sequence $\lambda = (\lambda_n)$, we get:

$$\lim_n (n - \lambda_n) = \lim_n (t_n - \lambda_{t_n}), \tag{2.7}$$

and for every $\epsilon > 0$ we have:

$$\{k \in I_{t_n} : D(T_{t_k}, L) \geq \epsilon\} \subset \{k \in I_n : D(T_k, L) \geq \epsilon\}$$
$$\cup \left\{ k \in I_n : \frac{1}{\lambda_k} \sum_{j=k-\lambda_k+1}^{k} X_j \neq \frac{1}{\lambda_{t_k}} \sum_{j=t_k-\lambda_{t_k}+1}^{t_k} X_j \right\}.$$

Now proof of the lemma in this case follows from relation (2.7) and $st_\lambda - \lim_n T_n = L$.

(II) In this case we have that $0 < t < 1$. For $t_n = [t \cdot n]$, for any natural number n, we can conclude that (T_{t_n}) does not appear more than $[1 + t^{-1}]$ times in the sequence (T_n). In fact if there exists a integers k, l such that

$$n \leq t \cdot k < t(k+1) < \cdots < t(k+l-1) < n+1 \leq t(k+l),$$

then

$$n + t(l-1) \leq t(k+l-1) < n+1 \Rightarrow l < 1 + \frac{1}{t},$$

and we have this estimation:

$$\frac{1}{\lambda_{t_n}} |\{k \in I_{t_n} : D(T_{t_k}, L) \geq \epsilon\}| \leq \left(1 + \frac{1}{t}\right) \frac{1}{\lambda_{t_n}} |\{k \in I_n : D(T_k, L) \geq \epsilon\}| \leq$$

$$2(1+t)\frac{1}{\lambda_n}\left|\{k \in I_n : D(T_k, L) \geq \epsilon\}\right|,$$

provided $\frac{\lambda_n}{\lambda_{t_n}} \leq 2t$, which is the case if n is large enough. From last relation it follows: $st_{F\lambda} - \lim_n T_{t_n} = L$. □

Lemma 2.5. *Let us suppose that $X = (X_k)$ is a sequence of fuzzy numbers which is $(V, \lambda)-$statistically convergent to fuzzy number L. Then for every $t > 1$,*

$$st_{F\lambda} - \lim_n (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} X_j = L; \tag{2.8}$$

and for every $0 < t < 1$,

$$st_{F\lambda} - \lim_n (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} X_j = L. \tag{2.9}$$

Proof. Let us suppose that $t > 1$ and n large enough in the sense that $\lambda_{t_n} > \lambda_n$, then

$$D\left(\frac{1}{\lambda_{t_n} - \lambda_n}\sum_{k=n+1}^{t_n} X_k, L\right) \leq D\left(\frac{1}{\lambda_{t_n} - \lambda_n}\sum_{k=n+1}^{t_n} X_k, T_n\right) + D\left(T_n, L\right). \tag{2.10}$$

Claim If X, Y and Z are three fuzzy real numbers, then is valid the following relation:

$$D(X+Y, Z) \leq c \cdot D(X, Z),$$

for some positive constant $c > 1$.
Case where $t > 1$. After some calculations we get

$$(\lambda_{t_n} - \lambda_n)^{-1}\sum_{j=n+1}^{t_n} X_j = T_n + \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) + (\lambda_{t_n} - \lambda_n)^{-1}$$

$$\sum_{j=n-\lambda_n+1}^{t_n} X_j - (\lambda_{t_n} - \lambda_n)^{-1}\sum_{j=t_n-\lambda_{t_n}+1}^{t_n} X_j,$$

respectively

$$(\lambda_{t_n} - \lambda_n)^{-1}\sum_{j=n+1}^{t_n} X_j = T_n + \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) + (\lambda_{t_n} - \lambda_n)^{-1}$$

$$\left(\sum_{j=n-\lambda_n+1}^{t_n} X_j - \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} X_j\right). \tag{2.11}$$

From last relation and above claim we have

$$D\left(\frac{1}{\lambda_{t_n} - \lambda_n}\sum_{j=n+1}^{t_n} X_j, T_n\right) =$$

$$D\left(T_n + \frac{\lambda_{t_n}}{\lambda_{t_n} - \lambda_n}(T_{t_n} - T_n) + \frac{1}{\lambda_{t_n} - \lambda_n}\left(\sum_{j=n-\lambda_n+1}^{t_n} X_j - \sum_{j=t_n-\lambda_{t_n}+1}^{t_n} X_j\right), T_n\right)$$

$$\leq c_1 \cdot D\left(\frac{\lambda_{t_n}}{\lambda_{t_n}-\lambda_n}(T_{t_n}-T_n), T_n\right) \leq \frac{c_1\lambda_{t_n}}{\lambda_{t_n}-\lambda_n} D(T_{t_n}, T_n), \tag{2.12}$$

where c_1 is a constant greater then one. From definition if the (λ_n), it follows that

$$\lim_n \frac{\lambda_{t_n}}{\lambda_{t_n}-\lambda_n} < \infty.$$

Now relation (2.8) follows from (2.10), (2.12), Lemma 2.4 and statistical convergence of T_n.

Case where $0 < t < 1$. In this case we have:

$$(\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} X_j = T_n + \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) + (\lambda_n - \lambda_{t_n})^{-1}$$

$$\sum_{j=n-\lambda_n+1}^{n} x_j - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=t_n-\lambda_{t_n}+1}^{n} X_j.$$

Following Lemma 2.4 and the conclusions like as in the previous case we get that relation (2.9) is valid. □

In what follows we will prove the Theorem 2.1.

Proof of Theorem 2.1

Proof. Necessity. Let us suppose that $st_{F\lambda} - \lim_k x_k = L$, and $st_{F\lambda} - \lim_k T_k = L$. For every $t > 1$ following Lemma 2.4 we get relation (2.2) and in case where $0 < t < 1$, again applying Lemma 2.4 we obtain relation (2.3).

Sufficient: Assume that $st_{F\lambda} - \lim_n T_n = L$, and conditions (2.1), (2.2) and (2.3) are satisfied. In what follows we will prove that $st_{F\lambda} - \lim_n X_n = L$. Or equivalently, $st_\lambda - \lim_n D(T_n, X_n) = 0$.

First we consider the case where $t > 1$. We will start from this estimation:

$$X_n - T_n = \lambda_{t_n}(\lambda_{t_n} - \lambda_n)^{-1}(T_{t_n} - T_n) - (\lambda_{t_n} - \lambda_n)^{-1} \sum_{j=n+1}^{t_n} (X_j - X_n).$$

For any $\epsilon > 0$, we obtain:

$$\{k \in I_n : D(X_k, T_k) \geq \epsilon\} \subset c \cdot \left\{k \in I_n : D\left((\lambda_{t_k} - \lambda_k)^{-1} \sum_{j=k+1}^{t_k} (X_j - X_k), X_k\right) \geq \epsilon\right\}, \tag{2.13}$$

for some positive constant $c > 1$.

From relation (2.2), it follows that for every $\gamma > 0$, exists a $t > 1$ such that

$$\limsup_n \frac{1}{\lambda_n} \left|\left\{k \in I_n : D\left(\frac{1}{\lambda_{t_k} - \lambda_k} \sum_{j=k+1}^{t_k} (X_j - X_k), X_k\right) \geq \epsilon\right\}\right| \leq \gamma.$$

By last relation and relation (2.13) we get:

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : D(X_k, T_k) \geq \epsilon\}| \leq c \cdot \gamma,$$

and γ is arbitrary, we conclude that for every $\epsilon > 0$,

$$\limsup_n \frac{1}{\lambda_n} |\{k \in I_n : D(X_k, T_k) \geq \epsilon\}| = 0. \tag{2.14}$$

Now we consider case where $0 < t < 1$. From above we get that:

$$X_n - T_n = \lambda_{t_n}(\lambda_n - \lambda_{t_n})^{-1}(T_n - T_{t_n}) + (\lambda_n - \lambda_{t_n})^{-1} \sum_{j=t_n+1}^{n} (X_n - X_j).$$

For any $\epsilon > 0$,

$$\{k \in I_n : D(X_k, T_k) \geq \epsilon\} \subset c \cdot \left\{ k \in I_n : D\left((\lambda_k - \lambda_{t_k})^{-1} \sum_{j=t_k+1}^{k} (X_k - X_j), X_k \right) \geq \epsilon \right\},$$

for some positive constant $c > 1$. Now proof of this case follows from above relation and relation (2.3). □

3. Tauberian theorems for level convergence

In this section we will introduce some level convergence of fuzzy numbers(see [7]) which is generalization of that given by [17] and also we will give some results related to the statistical limit inferior and superior for sequence of fuzzy numbers related to the new definition.

Let $X = (X_n)$ and $Y = (Y_n)$ be two real sequences of fuzzy numbers. We say that (X_k) is $\lambda-$ statistically bounded from above if there exists a fuzzy number X_0, such that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k > X_0\} \cup \{k \in I_n : X_k \nsim X_0\}| = 0.$$

Similarly, (X_k) is $\lambda-$ statistically bounded from bellow if there exists a fuzzy number Y_0, such that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k < Y_0\} \cup \{k \in I_n : X_k \nsim Y_0\}| = 0.$$

If the sequence $X = (X_k)$ is both $\lambda-$ statistically bounded from above and bellow we will say that it is $\lambda-$ statistically bounded.

Definition 3.1. *Let $X = (X_n)$ be a $\lambda-$statistically bounded sequences of fuzzy numbers. Then the $\lambda-$statistical limit inferior of the $X = (X_n)$ is given by*

$$st_{F\lambda} - \liminf_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{\lambda_n} |k \in I_n : X_k < X_0| \neq 0 \right\},$$

and, the $\lambda-$statistical limit superior of the $X = (X_n)$ is given by

$$st_{F\lambda} - \limsup_n X_n = \inf \left\{ X_0 \in E : \lim_n \frac{1}{\lambda_n} |k \in I_n : X_k > X_0| \neq 0 \right\}.$$

Theorem 3.2. *Let $X = (X_n)$ be a $\lambda-$statistically bounded sequences of fuzzy numbers. If $st_{F\lambda} - \lim_n \inf X_n = X_0$, then*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k < X_0 - \epsilon\}| = 0 \tag{3.1}$$

and

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k < X_0 + \epsilon\} \cup \{k \in I_n : X_k \nsim X_0 + \epsilon\}| \neq 0, \tag{3.2}$$

for every $\epsilon > 0$.

Proof. Proof of theorem is similar to Theorem 2, given in [2]. For this reason we omit it. □

The same results is valid for the $\lambda-$statistically limit superior, as is shown in the following

Theorem 3.3. *Let* $X = (X_n)$ *be a* $\lambda-$*statistically bounded sequences of fuzzy numbers. If* $st_{F\lambda} - \lim_n \sup X_n = X_0$, *then*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k > X_0 + \epsilon\}| = 0 \tag{3.3}$$

and

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k > X_0 - \epsilon\} \cup \{k \in I_n : X_k \nsim X_0 - \epsilon\}| \neq 0, \tag{3.4}$$

for every $\epsilon > 0$.

The level convergence of sequence of fuzzy numbers was given by [7], as follows. Let $X = (X_k)$ be a sequence of fuzzy numbers, it is level-convergent to fuzzy number X_0, if

$$\lim_{n\to\infty} X_n^-(\alpha) = X_0^-(\alpha), \lim_{n\to\infty} X_n^+(\alpha) = X_0^+(\alpha), \tag{3.5}$$

for any $\alpha \in [0, 1]$.

Remark 3.4. *The level convergence differs from the standard convergence and it is shown in the following example.*

Example 3.5. *Let*

$$\underline{u_n}(r) = \begin{cases} (r - \frac{1}{3})^n; & \frac{1}{3} < r \leq 1 \\ 0; & 0 \leq r \leq \frac{1}{3} \end{cases}, \quad \overline{u_n}(r) = 1,$$

and

$$\underline{u_0}(r) = \begin{cases} 1; & \frac{1}{3} < r \leq 1 \\ 0; & 0 \leq r \leq \frac{1}{3} \end{cases}, \quad \overline{u_0}(r) = 1.$$

From representation theorem in [17], *there exists a unique fuzzy number* u_n *and a unique fuzzy number* u_0 *such that* $[u_n]^r = [\underline{u_n}(r), \overline{u_n}(r)], [u_0]^r = [\underline{u_0}(r), \overline{u_0}(r)]$. *Obviously , the sequences* $\underline{u_n}(r), \overline{u_n}(r)$ *converges to* $\underline{u_0}(r), \overline{u_0}(r)$, *respectively at any* $r \in [0, 1]$, *as* $n \to \infty$. *And* $D(u_n, u_0) = \sup_{r\in(\frac{1}{3},1]} \left\{1 - (r - \frac{1}{3})^{\frac{1}{n}}\right\} = 1$, *for any natural number* n. *But* u_n *does not converge to* u_0.

Following it and our definition related to the generalized de la Vallée-Poussin mean-convergent, we give this

Definition 3.6. *The fuzzy sequence* $X = (X_k)$ *is generalized de la Vallée-Poussin mean-level convergent to a fuzzy number* X_0, *if*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} X_k^-(\alpha) = X_0^-(\alpha), \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} X_k^+(\alpha) = X_0^+(\alpha),$$

for all $\alpha \in [0, 1]$.

In what follows we will prove a kind of Tauberian theorem for the new defined concept of generalized de la Vallée-Poussin mean-level convergent.

Theorem 3.7. *Let $X = (X_k)$ be a bounded sequence of fuzzy numbers. Assume that (X_k) is generalized de la Vallée-Poussin mean-level convergent to fuzzy number X_0. Also we suppose that $st_{F\lambda} - \lim_n \sup X_k = X_0$ and there is a number $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$,*

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k \nsim X_0 - \epsilon\}| = 0, \lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k \nsim X_0 + \epsilon\}| = 0.$$

Then $st_{F\lambda} - lim_n X_n = X_0$.

Proof. We say that fuzzy numbers v and w are not comparable if neither $v \leq w$ nor $w \leq v$. And this fact we will denote by $v \nsim w$. From $st_{F\lambda} - \lim_n \sup X_k = X_0$ and Theorem 3.3, we have

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k > X_0 + \epsilon\}| = 0,$$

for every $\epsilon > 0$. Let us suppose for the moment that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : D(X_k, X_0) > \epsilon\}| \neq 0.$$

Then exists an $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : X_k < X_0 - \epsilon_1\}| \neq 0.$$

We will define the following sets of fuzzy numbers:

$$A_1 = \{k \in I_n : X_k < X_0 - \epsilon\},$$

$$A_2 = \{k \in I_n : X_0 - \epsilon < X_k < X_0 + \epsilon\},$$

$$A_3 = \{k \in I_n : X_k > X_0 + \epsilon\} \cup \{k \in I_n : X_k \nsim X_0 - \epsilon\} \cup \{k \in I_n : X_k \nsim X_0 + \epsilon\}.$$

From above definitions we get

$$\lim_n \frac{1}{\lambda_n} |A_3| = 0, \lim_n \frac{1}{\lambda_n} |A_1| \neq 0, \lim_n \frac{1}{\lambda_n} |A_2| = 1 - \lim_n \frac{1}{\lambda_n} |A_1|.$$

On the other side, from second relation in the expression we have that

$$\frac{1}{\lambda_n} |A_1| \geq a > 0,$$

for infinitely n and

$$T_n = \frac{1}{\lambda_n} \sum_{k \in I_n} X_k = \frac{1}{\lambda_n} \sum_{k \in A_1} X_k + \frac{1}{\lambda_n} \sum_{k \in A_2} X_k + \frac{1}{\lambda_n} \sum_{k \in A_3} X_k <$$

$$\frac{X_0 - \epsilon}{\lambda_n} |A_1| + \frac{X_0 + \epsilon}{\lambda_n} |A_2| + \frac{A}{\lambda_n} |A_3|,$$

for some fuzzy number A. There exists an $\alpha \in [0, 1]$ such that

$$T_n^-(\alpha) < X_0^-(\alpha) + \epsilon(1 - 2a) + 0(1).$$

since $\epsilon \in (0, \epsilon_0)$ is arbitrary, then we get

$$\liminf T_n^-(\alpha) \leq X_0^-(\alpha).$$

Hence $X = (X_n)$ is not generalized de la Vallée-Poussin mean-level convergent to fuzzy number X_0, which prove Theorem. □

Remark 3.8. *The above theorem is valid also for the $st_{F\lambda} - \lim \inf_n X_k = X_0$.*

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