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RELATION-THEORETIC CONTRACTION PRINCIPLE IN METRIC-LIKE SPACES

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Abstract. In this paper, we extend the Banach contraction principle to metric-like as well as partial metric spaces (not essentially complete) equipped with an arbitrary binary relation. Thereafter, we derive some fixed point results which are sharper versions of the corresponding known results of the existing literature. Finally, we use some examples to demonstrate the usability and generality of our main result.

1. Introduction

Metric fixed point theory continues to be an active area of research under the ambit of non-linear analysis. Banach contraction principle remains a source of inspiration for the researchers of this domain which was established by Banach [11] in 1922. In recent years, many researchers studied fixed point results in ordered metric spaces (e.g., [1–5, 15, 23, 25, 26, 28] and references cited therein). The most natural and much discussed idea of metric space has been generalized and improved by introducing several variants such as: metric-like space, partial metric space, symmetric space, pseudo metric space, b-metric space, 2-metric space, G-metric space and several others.

In 1994, Matthews [22] initiated the concept of partial metric space and also established Banach contraction principle in such spaces. In recent years, a multitude of metrical fixed point theorems were extended to partial metrics (e.g., [6, 8, 17, 19, 22, 24, 26]) and such research activity is still on.

Hitzler [13], proved an interesting extension of the Banach contraction principle by introducing dislocated metric spaces. Here, it can be pointed out that dislocated metric spaces are also sometimes referred as metric-like spaces (e.g., Amini-Harandi [7]). For further details on metric-like spaces one can consult [7, 9, 10, 13, 14, 16] and references cited therein.

The aim of this paper is to extend the Banach contraction principle to metric-like spaces (not essentially complete) equipped with an arbitrary binary relation. As consequences to our main result, we derive some fixed point results which are

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sharper versions of the corresponding known results of the existing literature. Finally, we furnish some examples to demonstrate the usability and generality of our main result.

Throughout this paper, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ respectively, stand for the set of non-negative real numbers, the set of natural numbers and the set of whole numbers.

2. Preliminaries

We begin with definitions of partial metric and metric-like spaces well followed by some of their relevant properties.

Definition 1. *[22] Let X be a non-empty set. Then a mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x, y, z \in X$,*

(p_1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space.

Definition 2. *[13] Let X be a non-empty set. Then a mapping $\sigma : X \times X \to \mathbb{R}^+$ is said to be a metric-like (or dislocated) on X if for all $x, y, z \in X$*

(σ_1) $\sigma(x,y) = 0 \Rightarrow x = y$,
(σ_2) $\sigma(x,y) = \sigma(y,x)$,
(σ_3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$.

The pair (X,σ) is called a metric-like (or dislocated) space. Here it can be pointed out that all the requirements of a metric are met out except $\sigma(x,x)$ may be positive for $x \in X$. For convenience, we also sometimes denote metric-like spaces (X,σ) merely by X.

Remark 1. *Every metric is a partial metric and every partial metric is a metric-like but converse implication is not true in general.*

Example 1. *Let us take $X = \{a,b,c\}$. Define $\sigma, p : X \times X \to \mathbb{R}^+$ by*

$$\begin{cases} \sigma(a,a) = \sigma(b,b) = 0, \\ \sigma(c,c) = \sigma(a,b) = \sigma(b,a) = 2, \\ \sigma(a,c) = \sigma(c,a) = \sigma(b,c) = \sigma(c,b) = 1; \end{cases}$$

and

$$p(x,y) = \begin{cases} 0, & x = y = a; \\ 1, & \textit{otherwise}. \end{cases}$$

Observe that (X,σ) is a metric-like space but not a partial metric space due to the fact that $\sigma(c,c) = 2 \nleq 1 = \sigma(c,a)$ while (X,p) a partial metric space but not a metric space as $p(b,b) \neq 0$.

The following terminologies are needed in our subsequent discussion.

Definition 3. *[7] Let $\{u_n\}$ be a sequence in a metric-like space (X,σ). Then we say that*

- *$\{u_n\}$ converges to a point u in X if and only if $\lim\limits_{n\to\infty} \sigma(u_n,u) = \sigma(u,u)$,*

- *$\{u_n\}$ is Cauchy in X if and only if $\lim_{n,m\to\infty} \sigma(u_n, u_m)$ (finitely) exists,*
- *the metric-like space (X, σ) is complete if every Cauchy sequence $\{u_n\}$ in X converges to a point u in X with respect to topology τ_σ generated by σ (denote as $u_n \xrightarrow{\tau_\sigma} u$) such that*

$$\lim_{n,m\to\infty} \sigma(u_n, u_m) = \sigma(u, u) = \lim_{n\to\infty} \sigma(u_n, u).$$

Next, we present some relevant relation-theoretic notions:

Recall that a binary relation $\mathcal{R}$ is a subset of $X \times X$ where X is a non-empty set. We say that "x is related to y under $\mathcal{R}$" if and only if $(x, y) \in \mathcal{R}$. In what follows, $\mathcal{R}$ stands for a non-empty binary relation.

Definition 4. *[21] A binary relation $\mathcal{R}$ on X is called complete if for all $x, y \in X$, either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$ which is denoted by $[x, y] \in \mathcal{R}$.*

Definition 5. *[3] Let f be a self-mapping defined on a non-empty set X. Then a binary relation $\mathcal{R}$ on X is called f-closed if $(fx, fy) \in \mathcal{R}$ whenever $(x, y) \in \mathcal{R}$, for all $x, y \in X$.*

Definition 6. *[3] Let $\mathcal{R}$ be a binary relation on X. Then a sequence $\{u_n\}$ in X is called $\mathcal{R}$-preserving if $(u_n, u_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}$.*

Motivated by Alam and Imdad [5], we introduce relation-theoretic variants of completeness and continuity in metric-like spaces.

Definition 7. *Let (X, σ) be a metric-like space equipped with a binary relation $\mathcal{R}$. We say that (X, σ) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence $\{u_n\}$ in X, there is some $u \in X$ such that*

$$\lim_{n,m\to\infty} \sigma(u_n, u_m) = \sigma(u, u) = \lim_{n\to\infty} \sigma(u_n, u).$$

Recall that the limit of a convergent sequence in metric-like spaces need not be unique.

Remark 2. *Every complete metric-like space is an $\mathcal{R}$-complete but not conversely. The notion of $\mathcal{R}$-completeness coincides with completeness if the relation $\mathcal{R}$ is universal.*

Example 2. *Let $X = (0, 1]$ and define $\sigma : X \times X \to \mathbb{R}^+$ by*

$$\sigma(x, y) = \begin{cases} 2x, & if\ x = y \\ max\{x, y\}, & otherwise \end{cases}$$

and a binary relation $\mathcal{R} = \{(x, y) \in X^2 \mid x \leq y\}$. Then (X, σ) is a metric-like space which is neither a partial metric space nor a metric space. Even the metric-like space (X, σ) is an $\mathcal{R}$-complete but not complete due to the fact that the Cauchy sequence $\{\frac{1}{n}\}$ in X converges to 0 whereas $0 \notin X$.

Definition 8. *A self-mapping f defined on a metric-like space (X, σ) is said to be a sequentially-continuous at u if for any sequence $\{u_n\}$ with $u_n \xrightarrow{\tau_\sigma} u$, we have $f(u_n) \xrightarrow{\tau_\sigma} f(u)$. As usual, f is said to be a sequentially-continuous if it is a sequentially-continuous at each point of X.*

Definition 9. *Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$. Then a mapping $f : X \to X$ is said to be an $\mathcal{R}$-sequentially-continuous at u if for any $\mathcal{R}$-preserving sequence $\{u_n\}$ with $u_n \xrightarrow{\tau_\sigma} u$, we have $f(u_n) \xrightarrow{\tau_\sigma} f(u)$. As usual, f is said to be an $\mathcal{R}$-sequentially-continuous if it is an $\mathcal{R}$-sequentially-continuous at each point of X.*

Remark 3. *On metric-like spaces, every continuous mapping is a sequentially-continuous and every sequentially-continuous mapping is an $\mathcal{R}$-sequentially-continuous but not conversely. The notion of $\mathcal{R}$-sequentially-continuity coincides with sequentially-continuity if the relation $\mathcal{R}$ is universal.*

Definition 10. *[3] Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$. Then $\mathcal{R}$ is said to be a σ-self-closed if for any $\mathcal{R}$-preserving sequence $\{u_n\}$ with $u_n \xrightarrow{\tau_\sigma} u$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $[u_{n_k}, u] \in \mathcal{R}$ for all $k \in \mathbb{N}$.*

Definition 11. *[27] Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$. Then a subset D of X is said to be an $\mathcal{R}$-directed if for every pair of points $x, y \in D$, there is z in X such that $(x,z) \in \mathcal{R}$ and $(y,z) \in \mathcal{R}$.*

Definition 12. *[20] Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$ and x, y a pair of points in X. Then a finite sequence $\{z_0, z_1, z_2, ..., z_l\}$ in X is said to be a path of length l (where $l \in \mathbb{N}$) from x to y in $\mathcal{R}$ if $z_0 = x, z_l = y$ and $(z_i, z_{i+1}) \in \mathcal{R}$ for each $i \in \{1, 2, 3, \cdots, l-1\}$.*

Here it can be pointed out that a path of length l involves $(l+1)$ elements of X which are not required to be distinct in general.

Remark 4. *In an ordered metric-like space $(X,\sigma,\preceq)$ with $\succeq$ as dual of the partial order relation $\preceq$. On setting $\mathcal{R} = \{(x,y) \in X^2 \mid x \prec\succ y\}$ (where $x \prec\succ y$ for $x, y \in X \iff$ either $x \preceq y$ or $x \succeq y$) in Definition 12, the path $\{z_0, z_1, z_2, ..., z_l\}$ reduces to $\prec\succ$-chain .*

In a metric-like space (X,σ), a self-mapping f on X and a binary relation $\mathcal{R}$ on X, we employ the following notations:

- $F(f)$: the set of all fixed points of f;
- $\Upsilon(x, y, \mathcal{R})$: the family of all paths from x to y in $\mathcal{R}$;
- $C(x, y, \prec\succ)$: the class of all chains between x and y.

3. Main results

The main result of this paper is the following one:

Theorem 3.1. *Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Suppose that the following conditions are satisfied:*

(A) : *there exists a subset $Y \subseteq X$ with $fX \subseteq Y$ such that (Y,σ) is $\mathcal{R}$-complete,*
(B) : *there exists u_0 such that $(u_0, fu_0) \in \mathcal{R}$,*
(C) : *$\mathcal{R}$ is f-closed,*
(D) : *either f is $\mathcal{R}$-sequentially-continuous or $\mathcal{R}|_Y$ is σ-self-closed,*
(E) : *there exists a constant $k \in [0,1)$ such that (for all $x, y \in X$ with $(x,y) \in \mathcal{R}$)*

$$\sigma(fx, fy) \leq k\sigma(x,y).$$

Then f has a fixed point. Moreover, if

(F) : *$\Upsilon(fx, fy, \mathcal{R}^s)$ is non-empty, for each $x, y \in X$,* *then f has a unique fixed point.*

Proof. Construct Picard iterate $\{u_n\}$ corresponding to u_0, *i.e.*, $u_n = f^n u_0$ for all $n \in \mathbb{N}_0$. Since $(u_0, fu_0) \in \mathcal{R}$ and $\mathcal{R}$ is f-closed, we find that

$$(fu_0, f^2u_0), (f^2u_0, f^3u_0), \cdots, (f^nu_0, f^{n+1}u_0), \cdots \in \mathcal{R},$$

so that

$$(u_n, u_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{3.1}$$

Hence $\{u_n\}$ is an $\mathcal{R}$-preserving sequence. Now we are required to show that $\{u_n\}$ is a Cauchy sequence. To establish this, using the condition (E), we have (for all $n \in \mathbb{N}_0$)

$$\sigma(u_{n+1}, u_{n+2}) = \sigma(fu_n, fu_{n+1}) \leq k\sigma(u_n, u_{n+1}),$$

which yields (by induction) that

$$\sigma(u_{n+1}, u_{n+2}) \leq k^{n+1}\sigma(u_0, fu_0) \ \forall\ n \in \mathbb{N}_0. \tag{3.2}$$

On using (3.2) and triangular inequality, we have (for all $n, m \in \mathbb{N}_0$ with $m > n$)

$$\begin{aligned}
\sigma(u_n, u_m) &\leq \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) + \cdots + \sigma(u_{m-1}, u_m) \\
&\leq (k^n + k^{n+1} + \cdots + k^{m-1})\sigma(u_0, fu_0) \\
&= k^n\sigma(u_0, fu_0) \sum_{j=0}^{m-n-1} k^j \\
&\leq \frac{k^n}{1-k}\sigma(u_0, fu_0) \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

which shows that $\{u_n\}$ is an $\mathcal{R}$-preserving Cauchy in Y. By $\mathcal{R}$-completeness of (Y, σ), there is $y \in Y$ such that the sequence $\{u_n\}$ converges to y with respect to topology τ_σ generated by σ *i.e.*,

$$\lim_{n\to\infty} \sigma(u_n, y) = \sigma(y, y) = \lim_{n,m\to\infty} \sigma(u_n, u_m) = 0. \tag{3.3}$$

Firstly, assume that f is $\mathcal{R}$-*sequentially-continuous*. Then $u_{n+1} = fu_n \xrightarrow{\tau_\sigma} fy$, so that

$$\lim_{n\to\infty} \sigma(u_{n+1}, fy) = \lim_{n\to\infty} \sigma(fu_n, fy) = \sigma(fy, fy) = \lim_{n,m\to\infty} \sigma(u_n, u_m) = 0. \tag{3.4}$$

On using triangular inequality, (3.3) and (3.4), we have $\sigma(y, fy) = 0$, so that y is a fixed point of f.

Alternately, if $\mathcal{R}|_Y$ is σ-self-closed, then due to the fact that $\{u_n\}$ is an $\mathcal{R}$-preserving sequence in Y and $u_n \xrightarrow{\tau_\sigma} y$, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with $[u_{n_k}, y] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$. In view of the condition (E) and the symmetry of the metric-like σ, we have

$$\sigma(u_{n_k+1}, fy) = \sigma(fu_{n_k}, fy) \leq k\sigma(u_{n_k}, y).$$

Taking the limit as $k \to \infty$ and using (3.3), we have

$$\lim_{k\to\infty} \sigma(u_{n_k+1}, fy) = 0. \tag{3.5}$$

On using triangular inequality, (3.3) and (3.5), we have $\sigma(y, fy) = 0$, so that, $fy = y$, *i.e.*, y is a fixed point of f.

Next, if $F(f)$ is singleton then result follows. Otherwise, take u_p, u_q be two arbitrary elements of $F(f)$, *i.e.*,

$$fu_p = u_p \text{ and } fu_q = u_q.$$

Owing to the condition (F), there exists a path (say $\{u_0, u_1, u_2, \cdots, u_l\}$) of finite length l in $\mathcal{R}^s$ from u_p to u_q such that

$$u_0 = u_p,\ u_l = u_q \text{ and } [u_i, u_{i+1}] \in \mathcal{R} \text{ for each } i \in \{1, 2, \cdots, l-1\}.$$

As $\mathcal{R}$ is f-closed, we have

$$[f^n u_i, f^n u_{i+1}] \in \mathcal{R} \text{ for each } i \in \{1, 2, \cdots, l-1\} \text{ and for each } n \in \mathbb{N}_0.$$

On using triangular inequality and hypothesis (E), we obtain

$$\begin{aligned}
\sigma(u_p, u_q) = \sigma(f^n u_0, f^n u_l) &\leq \sum_{i=0}^{l-1} \sigma(f^n u_i, f^n u_{i+1}) \\
&\leq k \sum_{i=0}^{l-1} \sigma(f^{n-1} u_i, f^{n-1} u_{i+1}) \\
&\leq k^2 \sum_{i=0}^{l-1} \sigma(f^{n-2} u_i, f^{n-2} u_{i+1}) \\
&\quad \vdots \\
&\leq k^n \sum_{i=0}^{l-1} \sigma(u_i, u_{i+1}) \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}$$

yielding thereby $u_p = u_q$. Hence f has a unique fixed point. □

On setting $Y = X$ in Theorem 3.1, we deduce the following:

Corollary 3.2. *Let (X, σ) be a metric-like space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Suppose that the conditions $(B), (C), (E), (F)$ together with following conditions are satisfied:*

$(G):$ *(X, σ) is $\mathcal{R}$-complete,*
$(H):$ *either f is $\mathcal{R}$-sequentially-continuous or $\mathcal{R}$ is σ-self-closed.*

Then f has a unique fixed point.

In view of Remarks 2 and 3, we deduce the following natural result:

Corollary 3.3. *Let (X, σ) be a metric-like space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Suppose that the conditions $(B), (C), (E), (F)$ together with following conditions are satisfied:*

$(I):$ *there exists a subset $Y \subseteq X$ with $fX \subseteq Y$ such that (Y, σ) is complete,*
$(J):$ *either f is continuous or $\mathcal{R}|_Y$ is σ-self-closed.*

Then f has a unique fixed point.

Employing '$\mathcal{R}^s$-directedness of fX' and 'the completeness of relation $\mathcal{R}$ on fX', we can have the following.

Corollary 3.4. *Theorem 3.1 remains true if we replace condition* (F) *by one of the following conditions besides retaining the rest of the hypotheses:*

$(K):$ fX *is* $\mathcal{R}^s$*-directed,*

$(L):$ $\mathcal{R}|_{fX}$ *is complete.*

Proof. Assume that (K) holds. Then for each pair of points a, b in fX, $\exists\ x \in X$ such that $[a, x] \in \mathcal{R}$ and $[b, x] \in \mathcal{R}$ so that the finite sequence $\{a, x, c\}$ is a path of length 2 from a to b in $\mathcal{R}^s$. Thus, for each $a, b \in fX$, $\Upsilon(a, b, \mathcal{R}^s)$ is non-empty and hence result follows from Theorem 3.1.

Secondly, if condition (L) holds, then for each pair of points $a, b \in fX$, $[a, b] \in \mathcal{R}$, which implies that $\{a, b\}$ is a path of length 1 from a to b in $\mathcal{R}^s$, so that $\Upsilon(a, b, \mathcal{R}^s)$ is non-empty, for each $a, b \in fX$. Finally, proceeding on the lines of the proof of Theorem 3.1, the conclusion can be established. □

4. Some Consequences

In this section, we derive some special cases corresponding to different type of binary relations.

Our first corollary is a natural result in metric-like spaces which is obtained by setting the relation $\mathcal{R} = X^2$. Indeed this corollary remains an improved version of Banach contraction principle on metric-like spaces obtained by Aydi and Karapinar [9, Corollary 3.6].

Corollary 4.1. *Let* (X, σ) *be a metric-like space and* f *a self-mapping on* X*. Suppose that there exists a subset* Y *of* X *with* $fX \subseteq Y \subseteq X$ *such that* (Y, σ) *is complete. If there is* $k \in [0, 1)$ *such that* $\sigma(fx, fy) \leq k\sigma(x, y)$*, for all* $x, y \in X$*, then* f *has a unique fixed point.*

The next corollary is a result in ordered metric-like spaces involving increasing mappings. Inspired by Alam and Imdad [4], we chalk out the following definitions.

Definition 13. *An ordered metric-like space* $(X, \sigma, \preceq)$ *is said to be an* $\overline{O}$*-complete (resp.* $\underline{O}$*-complete,* O*-complete), if every increasing (resp. decreasing, monotone) Cauchy sequence converges to a point of* X*.*

Definition 14. *Let* $(X, \sigma, \preceq)$ *be an ordered metric-like space. Then a mapping* $f : X \to X$ *is said to be an* $\overline{O}$*-sequentially-continuous (resp.* $\underline{O}$*-sequentially-continuous,* O*-sequentially-continuous) at* u *if every increasing (resp. decreasing, monotone) sequence* $\{u_n\}$ *with* $u_n \xrightarrow{\tau_\sigma} u$*, we have* $f(u_n) \xrightarrow{\tau_\sigma} f(u)$*.*

As usual, f is said to be an $\overline{O}$-sequentially-continuous (resp. $\underline{O}$-sequentially-continuous, O-sequentially-continuous) if it is an $\overline{O}$-sequentially-continuous (resp. $\underline{O}$-sequentially-continuous, O-sequentially-continuous) on X.

Definition 15. *An ordered metric-like space* $(X, \sigma, \preceq)$ *enjoys* σ*-ICC (*σ*-increasing-convergence-c-bound) [resp.* σ*-DCC,* σ*-TCC] property if every increasing [resp. decreasing, termwise monotone] convergence sequence* $\{u_n\}$ *in* X *(i.e., there is* $u \in X$ *with* $\lim\limits_{n\to\infty} \sigma(u_n, u) = \sigma(u, u)$*), admits a subsequence* $\{u_{n_k}\}$ *such that* $u_{n_k} \prec\succ u$*, for all* $k \in \mathbb{N}$.

Employing preceding definitions, we can have the following:

Corollary 4.2. *Let* $(X, \sigma, \preceq)$ *be an ordered metric-like space and* f *a self-mapping on* X*. Suppose that the following conditions are satisfied:*

(M) : *(Y,σ) is $\overline{O}$-complete where $fX \subseteq Y \subseteq X$,*
(N) : *there exists $u_0 \in X$ such that $u_0 \preceq fu_0$,*
(O) : *f is increasing,*
(P) : *either f is $\overline{O}$-sequentially-continuous or $(Y,\sigma,\preceq)$ enjoys σ-ICC property,*
(Q) : *there exists a constant $k \in [0,1)$ such that (for all $x,y \in X$ with $x \preceq y$)*

$$\sigma(fx,fy) \leq k\sigma(x,y).$$

Then f has a fixed point. Moreover, if

(R) : *$C(fx,fy,\prec\succ)$ is non-empty for all $x,y \in X$,*

then f has a unique fixed point.

We can also have the following corollary in ordered metric-like spaces involving comparable mappings

Corollary 4.3. *Let $(X,\sigma,\preceq)$ be an ordered metric-like space and f a self-mapping on X. Suppose that the following conditions are satisfied:*

(S) : *(Y,σ) is O-complete where $fX \subseteq Y \subseteq X$,*
(T) : *there exists $u_0 \in X$ such that $u_0 \prec\succ fu_0$,*
(U) : *f is comparable, i.e., for $x,y \in X$ such that $x \prec\succ y$, we have $fx \prec\succ fy$,*
(V) : *either f is O-sequentially-continuous or $(Y,\sigma,\preceq)$ enjoys σ-TCC property,*
(W) : *condition (Q) holds (for all $x,y \in X$ with $x \prec\succ y$).*

Then f has a fixed point. Moreover, if condition (R) also holds, then f has a unique fixed point.

Let Ω be the set of all mappings $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(i) ρ is a Lebesgue-integrable on each compact subset of $\mathbb{R}^+$, and
(ii) $\int_0^\varepsilon \rho(t) > 0,$ for all $\varepsilon > 0$.

Now, we predict the following generalized form of Theorem 3.1 employing the integral type contractive condition, as follows.

Theorem 4.4. *Let (X,σ) be a metric-like space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Suppose that conditions $(A),(B),(C),(D),(F)$ are satisfied (together with the following condition):*

(X) : *there exists $\rho \in \Omega$ such that (for all $x,y \in X$ with $(x,y) \in \mathcal{R}$)*

$$\int_0^{\sigma(fx,fy)} \rho(t)dt \leq k \int_0^{\sigma(x,y)} \rho(t)dt.$$

Then f has a unique fixed point.

Proof. Proceeding on the lines of the proof of Theorem 3.1 and Theorem 2.1 of Branciari [12], one can complete the proof. □

Under universal relation, Theorem 4.4 deduces to an improved version of Theorem 2.1 due to Branciari [12], as follows.

Corollary 4.5. *Let (X,σ) be a metric-like space and f a self-mapping on X. Suppose that there exists a subset Y of X with $fX \subseteq Y \subseteq X$ such that (Y,σ) is complete. If there exists $k \in [0,1)$ and $\rho \in \Omega$ such that (for all $x,y \in X$)*

$$\int_0^{\sigma(fx,fy)} \rho(t)dt \leq k \int_0^{\sigma(x,y)} \rho(t)dt,$$

then f has a unique fixed point.

Remark 5. *In view of Remark 1, the class of metric-like spaces is relatively larger than classes of partial metric spaces and metric spaces. Consequently, one can easily deduce the analogues results corresponding to Theorems 3.1, 4.4 and Corollaries 3.2– 3.4, 4.1–4.3, 4.5 in partial metric spaces as well as metric spaces.*

For the sake of convenience, one can present a result to the partial metric space as under:

Corollary 4.6. *Let (X, σ) be a partial metric space equipped with a binary relation $\mathcal{R}$ and f a self-mapping on X. Suppose that the conditions $(A), (B), (C), (D), (E)$ and (F) are satisfied. Then f has a unique fixed point.*

5. Illustrative Examples

Finally, we furnish some examples to demonstrate the usability and generality of Theorem 3.1.

Example 3. *Let us take $X = \{a, b, c\}$. Define $\sigma : X \times X \to \mathbb{R}^+$ as*

$$\begin{cases} \sigma(a,a) = \sigma(b,b) = 0, \ \sigma(c,c) = 3, \\ \sigma(a,b) = \sigma(b,a) = 1, \\ \sigma(a,c) = \sigma(c,a) = \sigma(b,c) = \sigma(c,b) = 2. \end{cases}$$

Then (X, σ) is a metric-like space which is neither a partial metric space nor a metric space. Now, we define a mapping $f : X \to X$ by

$$fa = b, \ fb = b \text{ and } fc = a.$$

Consider a binary relation $\mathcal{R} = \{(a,a), (b,b), (a,b)\}$ on X and $Y = \{a, b\}$. Then $\mathcal{R}$ is f-closed and Y is $\mathcal{R}$-complete. Take any $\mathcal{R}$-preserving sequence $\{u_n\}$ with

$$u_n \xrightarrow{\tau_\sigma} p \text{ and } (u_n, u_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0.$$

Notice that if $(u_n, u_{n+1}) \in \mathcal{R}|_Y$ for all $n \in \mathbb{N}_0$, then there exists an integer $N \in \mathbb{N}_0$ such that $u_n = p \in \{a, b\}$ $\forall n \geq N$. So, we can take a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ with $u_{n_k} = p$ $\forall k \in \mathbb{N}_0$, which amounts to saying that $[u_{n_k}, p] \in \mathcal{R}|_Y$ $\forall k \in \mathbb{N}_0$. Therefore, $\mathcal{R}|_Y$ is σ-self-closed.

In order to check the condition (E) of Theorem 3.1, it is sufficient to show that the condition (E) holds for $x \in \{a, b\}$ and $y = c$ (or, $x = c$ and $y \in \{a, b\}$), as in rest of the cases $\sigma(fx, fy) = 0$. If $x \in \{a, b\}$ and $y = c$, then $\sigma(fx, fy) = 1 \leq k2 = k\sigma(x, y)$ holds for all $k \in [\frac{1}{2}, 1)$. As $\mathcal{R}|_{fX}$ is complete, the condition (F) holds. Thus all the requirements of Theorem 3.1 are fulfilled. Hence f has a unique fixed point. Observe that f has a unique fixed point namely, 'b'.

Example 4. *Let $X = [0, 1]$ and define $\sigma : X \times X \to \mathbb{R}^+$ by*

$$\sigma(x,y) = \begin{cases} 2x, & if \ x = y \\ max\{x, y\}, & otherwise \end{cases}$$

and a binary relation $\mathcal{R} = \{(x, y) \in X^2 \mid x \leq y \text{ and } y < 1\}$. Then (X, σ) is a metric-like space which is neither a partial metric space nor a metric space. Also the metric-like space (X, σ) is an $\mathcal{R}$-complete. Now, define a mapping $f : X \to X$ as

$$f(x) = \begin{cases} \frac{x}{2}, \ if \ x \in [0, 1) \\ 0, \ if \ x = 1. \end{cases}$$

Then $\mathcal{R}$ is f-closed. Let $\{x_n\}$ be an arbitrary $\mathcal{R}$-presearving sequence such that $x_n \xrightarrow{\tau_\sigma} x$ (for some $x \in X$), i.e., $\{x_n\}$ is a sequence in $[0,1)$ such that $x_n \leq x_{n+1}$ $\forall n$ with $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x)$. Then $x \in [0,1)$ and

$$\begin{aligned}\sigma(fx,fx) = \sigma(\frac{x}{2},\frac{x}{2}) = x &= \frac{1}{2}(\sigma(x,x)) \\ &= \frac{1}{2}(\lim_{n\to\infty}\sigma(x_n,x)) \\ &= \frac{1}{2}\left(\lim_{n\to\infty}\begin{cases}2x_n, & if\ x_n = x\\ \max\{x_n,x\}, & otherwise\end{cases}\right) \\ &= \lim_{n\to\infty}\begin{cases}2\frac{x_n}{2}, & if\ x_n = x\\ \max\{\frac{x_n}{2},\frac{x}{2}\}, & otherwise\end{cases} \\ &= \lim_{n\to\infty}\sigma(fx_n,fx).\end{aligned}$$

This shows that $fx_n \xrightarrow{\tau_\sigma} fx$ and hence f is an $\mathcal{R}$-seqentially-continuous. Now, for any $(x,y) \in X^2$ with $(x,y) \in \mathcal{R}$, we have

$$\begin{aligned}\sigma(fx,fy) &= \sigma(\frac{x}{2},\frac{y}{2}) \\ &= \begin{cases}x, & if\ x = y\\ \max\{\frac{x}{2},\frac{y}{2}\}, & otherwise\end{cases} \\ &= \frac{1}{2}\begin{cases}2x, & if\ x = y\\ \max\{x,y\}, & otherwise\end{cases} \\ &= \frac{1}{2}\sigma(x,y).\end{aligned}$$

This shows that $\sigma(fx,fy) \leq k\sigma(x,y)$ for all $k \geq \frac{1}{2}$ and rest of the hupotheses of Theorem 3.1 are trivially satisfied. Hence f has a unique fixed point. Observe that the point $x = 0$ is only the fixed point of f.

Notice that the Banach contraction principle can not be used in the context of our examples (*i.e.*, Examples 3 and 4), while Theorem 3.1 is applicable which demonstrates the genuineness as well as utility of our result proved herein.

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