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PICARD OPERATORS IN b-METRIC SPACES VIA DIGRAPHS

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Abstract. In this paper we prove some fixed point theorems in b-metric spaces endowed with a graph which are generalizations of the Banach Contraction Principle. We also prove Edelstein theorem in the setting of b-metric spaces.

1. Introduction

The notion of a b-metric space was introduced by Bakhtin[1] and Czerwik[4]. This is a generalization of the usual notion of a metric space. Several authors reformulated many problems of fixed point theory in b-metric spaces. In 2005, Echenique[6] studied fixed point theory by using graphs. Afterwards, Espinola and Kirk[7] applied fixed point results in graph theory. Recently, Jachymski[9] proved a sufficient condition for a selfmap f of a metric space (X, d) to be a Picard operator and applied it to the Kelisky-Rivlin theorem on iterates of the Bernstein operators on the space $C[0, 1]$. Motivated by the idea given in[9], we reformulated some important fixed point results in metric spaces to b-metric spaces endowed with a graph. We also prove b-metric version of Edelstein theorem. Finally, an example is provided to support our main result.

2. Some basic concepts

We begin with some basic notations and definitions in b-metric spaces.

Definition 2.1. [4] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*;*
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

If $s = 1$, then the triangle inequality in a metric space is satisfied, however it does not hold true when $s > 1$.

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Definition 2.2. [2] *Let (X,d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) *(x_n) converges to x if and only if $\lim\limits_{n\to\infty} d(x_n,x) = 0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.*

(ii) *(x_n) is Cauchy if and only if $\lim\limits_{n,m\to\infty} d(x_n,x_m) = 0$.*

(iii) *(X,d) is complete if and only if every Cauchy sequence in X is convergent.*

Definition 2.3. *The sequences (x_n) and (y_n) in a b-metric space (X,d) are called Cauchy equivalent if each of them is a Cauchy sequence and $d(x_n,y_n) \to 0$ as $n \to \infty$.*

Definition 2.4. *Let (X,d) be a b-metric space. A mapping $f : X \to X$ is called a Picard operator (abbr., PO) if f has a unique fixed point $u \in X$ and $\lim\limits_{n\to\infty} f^n x = u$ for all $x \in X$.*

We next review some basic notions in graph theory.
Let (X,d) be a metric space. We assume that G is a directed graph (digraph) with the set $V(G)$ of its vertices coincides with X and a set of edges $E(G)$ contains all the loops, i.e., $E(G) \supseteq \Delta$, where $\Delta = \{(x,x) : x \in X\}$. We also assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. G may be considered as a weighted graph by assigning to each edge the distance between its vertices. By G^{-1} we denote the graph obtained from G by reversing the direction of edges i.e., $E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}$. We treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, $E(\tilde{G}) = E(G) \cup E(G^{-1})$. Our graph theory notations and terminology are standard and can be found in all graph theory books, like [3, 5, 8]. If x, y are vertices of the digraph G, then a path in G from x to y of length n ($n \in \mathbb{N}$) is a sequence $(x_i)_{i=0}^n$ of $n+1$ vertices such that $x_0 = x,\ x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \cdots, n$. A graph G is connected if there is a path between any two vertices of G. G is weakly connected if $\tilde{G}$ is connected. If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. We note that $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation R defined on $V(G)$ by the rule:

$$yRz \ if \ there \ is \ a \ path \ in \ G \ from \ y \ to \ z.$$

Clearly, G_x is connected.

Definition 2.5. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$ and let $G = (V(G), E(G))$ be a graph. A mapping $f : X \to X$ is called a Banach G-contraction or simply G-contraction if f preserves edges of G, i.e.,*

$$\forall x, y \in X, ((x,y) \in E(G) \Rightarrow (fx, fy) \in E(G)),$$

and f decreases weights of edges of G in the following way:
there exists $\alpha \in (0, \frac{1}{s})$ such that

$$d(fx, fy) \leq \alpha\, d(x,y)$$

for all $x, y \in X$ with $(x,y) \in E(G)$.

Any Banach contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$. But it is worth mentioning that a Banach G-contraction need not be a Banach contraction (see Remark 3.9).

Remark 2.6. *If f is a G-contraction, then f is both a G^{-1}-contraction and a $\tilde{G}$-contraction.*

Definition 2.7. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and let $f : X \to X$ be a given mapping. We say that f is continuous at $x_0 \in X$ if for every sequence (x_n) in X, we have $x_n \to x_0$ as $n \to \infty \Longrightarrow fx_n \to fx_0$ as $n \to \infty$. If f is continuous at each point $x_0 \in X$, then we say that f is continuous on X.*

Definition 2.8. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,*

$$f^{k_n}x \to y \text{ implies } f(f^{k_n}x) \to fy \text{ as } n \to \infty.$$

Definition 2.9. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is called G-continuous if given $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$,*

$$x_n \to x \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } fx_n \to fx.$$

Definition 2.10. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$. A mapping $f : X \to X$ is called orbitally G-continuous if for all $x, y \in X$ and any sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers,*

$$f^{k_n}x \to y \text{ and } (f^{k_n}x, f^{k_{n+1}}x) \in E(G) \text{ for } n \in \mathbb{N} \text{ imply } f(f^{k_n}x) \to fy.$$

It is easy to observe the following relations:
continuity $\Rightarrow$ orbital continuity $\Rightarrow$ orbital G-continuity;
continuity $\Rightarrow$ G-continuity $\Rightarrow$ orbital G-continuity.

3. Main Results

In this section we always assume that (X, d) is a b-metric space, and G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.
We begin with the following lemma.

Lemma 3.1. *Let (X, d) be a b-metric space with the coefficient $s \geq 1$ and $f : X \to X$ be a G-contraction with a constant $\alpha \in (0, \frac{1}{s})$. Then, given $x \in X$ and $y \in [x]_{\tilde{G}}$, there is $r(x, y) \geq 0$ such that*

$$d(f^n x, f^n y) \leq \alpha^n\, r(x, y),\ \forall n \in \mathbb{N}.$$

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\tilde{G}$ from x to y, i.e., $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, 2, \cdots, N$. Since f is a G-contraction, it is also a $\tilde{G}$-contraction. By mathematical induction, we have

$$(f^n x_{i-1}, f^n x_i) \in E(\tilde{G}) \text{ and } d\,(f^n x_{i-1}, f^n x_i) \leq \alpha^n\, d(x_{i-1}, x_i)$$

for all $n \in \mathbb{N}$ and $i = 1, 2, \cdots, N$.
Now,

$$\begin{aligned} d(f^n x, f^n y) &\leq s\, d(f^n x_0, f^n x_1) + s^2\, d(f^n x_1, f^n x_2) + \cdots \\ &\quad + s^{N-1}\, d(f^n x_{N-2}, f^n x_{N-1}) + s^{N-1}\, d(f^n x_{N-1}, f^n x_N) \\ &\leq \alpha^n \sum_{i=1}^{N} s^i d(x_{i-1}, x_i),\ \text{since } s \geq 1. \end{aligned}$$

If we set $r(x,y) = \sum_{i=1}^{N} s^i d(x_{i-1}, x_i)$, then

$$d(f^n x, f^n y) \leq \alpha^n \, r(x,y), \; \forall n \in \mathbb{N}.$$

$\square$

Theorem 3.2. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$, and let the triple (X,d,G) has the following property:*
$(*)$ For any sequence (x_n) in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence (x_{k_n}) of (x_n) such that $(x_{k_n}, x) \in E(G)$ for all $n \geq 1$.
Let $f : X \to X$ be a G-contraction, and $X_f = \{x \in X : (x, fx) \in E(G)\}$. Then,

(i) *for any $x \in X_f$, $f|_{[x]_{\tilde{G}}}$ is a $\tilde{G}_x$-contraction and $f|_{[x]_{\tilde{G}}}$ is a PO.*
(ii) *if $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*

Proof. (i) Let $x \in X_f$. Then $(x, fx) \in E(G)$ and so $fx \in [x]_{\tilde{G}}$. Consequently, it follows that $[x]_{\tilde{G}} = [fx]_{\tilde{G}}$.
We first show that $f|_{[x]_{\tilde{G}}}$ is a $\tilde{G}_x$-contraction.
Let $y \in [x]_{\tilde{G}}$. Then there exists a path $(x_i)_{i=0}^{p}$ from x to y where $x_0 = x$, $x_p = y$ and $(x_{i-1}, x_i) \in E(\tilde{G})$ for $i = 1, 2, \cdots, p$. Since f is a G-contraction, it is also a $\tilde{G}$-contraction. Then, $(x_{i-1}, x_i) \in E(\tilde{G})$ implies $(fx_{i-1}, fx_i) \in E(\tilde{G})$ for $i = 1, 2, \cdots, p$. This proves that $(fx_i)_{i=0}^{p}$ is a path in $\tilde{G}$ from fx to fy and hence $fy \in [fx]_{\tilde{G}} = [x]_{\tilde{G}}$. Thus, $y \in [x]_{\tilde{G}} \Rightarrow fy \in [x]_{\tilde{G}}$.

Let $(y,z) \in E(\tilde{G}_x)$. By our preceeding discussion, we have $fy, fz \in [x]_{\tilde{G}}$. Since $y \in [x]_{\tilde{G}}$, there exists a path $(y_i)_{i=0}^{q-1}$ in $\tilde{G}$ from x to y where $y_0 = x$, $y_{q-1} = y$. This combining with $(y,z) \in E(\tilde{G}_x)$, there is a path $(y_i)_{i=0}^{q}$ in $\tilde{G}$ from x to z where $y_q = z$. Let $(z_i)_{i=0}^{r}$ be a path in $\tilde{G}$ from x to fx where $z_0 = x = y_0$, $z_r = fx = fy_0$. As f preserves edges of $\tilde{G}$, $(x, z_1, z_2, \cdots, fx, fy_1, \cdots, fy_{q-1}, fy_q)$ is a path in $\tilde{G}$ from x to fz. In particular, $(fy_{q-1}, fy_q) \in E(\tilde{G}_x)$ i.e., $(fy, fz) \in E(\tilde{G}_x)$. Therefore, $f|_{[x]_{\tilde{G}}}$ is a $\tilde{G}_x$-contraction. Since $fx \in [x]_{\tilde{G}}$, by applying Lemma 3.1, we get

$$d(f^n x, f^{n+1} x) \leq \alpha^n \, r(x, fx), \;\; \forall n \in \mathbb{N}. \tag{3.1}$$

For $m, n \in \mathbb{N}$ with $m > n$, using condition (3.1), we have

$$\begin{aligned}
d(f^n x, f^m x) \quad &\leq \quad s\, d(f^n x, f^{n+1} x) + s^2\, d(f^{n+1} x, f^{n+2} x) + \cdots \\
&\quad\;\; + s^{m-n-1}\, d(f^{m-2} x, f^{m-1} x) + s^{m-n-1}\, d(f^{m-1} x, f^m x) \\
&\leq \quad \left[s\alpha^n + s^2 \alpha^{n+1} + \cdots + s^{m-n-1}\alpha^{m-2} + s^{m-n-1}\alpha^{m-1}\right] r(x, fx) \\
&\leq \quad s\alpha^n \left[1 + s\alpha + \cdots + (s\alpha)^{m-n-2} + (s\alpha)^{m-n-1}\right] r(x, fx) \\
&\leq \quad \frac{s\alpha^n}{1 - s\alpha}\, r(x, fx) \\
&\to 0 \quad \textit{as } m, n \to \infty.
\end{aligned}$$

Therefore, $(f^n x)$ is a Cauchy sequence in $[x]_{\tilde{G}}$.

If $y \in [x]_{\tilde{G}}$, then $fy \in [x]_{\tilde{G}} = [y]_{\tilde{G}}$. By an argument similar to that used above, $(f^n y)$ is a Cauchy sequence in $[x]_{\tilde{G}}$.

Again, by using Lemma 3.1,

$$d(f^n x, f^n y) \leq \alpha^n\, r(x,y) \to 0 \ as\ n \to \infty.$$

Hence, $(f^n x)$ and $(f^n y)$ are Cauchy equivalent. By completeness of X, $(f^n x)$ converges to some $u \in X$.

Now,

$$d(f^n y, u) \leq sd(f^n y, f^n x) + sd(f^n x, u)$$

gives that, $\lim_{n\to\infty} f^n y = u$. Thus, $\lim_{n\to\infty} f^n y = u$, for all $y \in [x]_{\tilde{G}}$.

As f is a G-contraction and $(x, fx) \in E(G)$, it follows that $(f^n x, f^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$. By property $(*)$, there exists a subsequence $\big(f^{k_n}x\big)$ of $(f^n x)$ such that $\big(f^{k_n}x, u\big) \in E(G)$. We note that $\big(x, fx, f^2x, \cdots, f^{k_1}x, u\big)$ is a path in G and hence it is also a path in $\tilde{G}$ from x to u. This proves that $u \in [x]_{\tilde{G}}$.

Furthermore,

$$\begin{aligned} d(u, fu) &\leq sd(u, f^{k_n+1}x) + sd(f^{k_n+1}x, fu) \\ &\leq sd(u, f^{k_n+1}x) + \alpha sd(f^{k_n}x, u) \\ &\to 0 \quad as\ n \to \infty. \end{aligned}$$

This implies that, $d(u, fu) = 0$ i.e., $fu = u$. Thus, $f\,|_{[x]_{\tilde{G}}}$ has a fixed point $u \in [x]_{\tilde{G}}$.

The next is to show that the fixed point is unique. Assume that there is another point $v \in [x]_{\tilde{G}}$ such that $fv = v$. Since $\lim_{n\to\infty} f^n y = u$, for all $y \in [x]_{\tilde{G}}$, we have $\lim_{n\to\infty} f^n v = u$ and so, $v = u$. Thus, $f\,|_{[x]_{\tilde{G}}}$ is a PO.

(ii) If G is weakly connected, then $[x]_{\tilde{G}} = X$. Therefore, it follows from (i) that f has a unique fixed point u in X and $\lim_{n\to\infty} f^n x = u$, for all $x \in X$. Thus, f is a PO. □

The following corollary is the b-metric version of Banach Contraction Principle.

Corollary 3.3. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and the mapping $f : X \to X$ be such that*

$$d(fx, fy) \leq \alpha\, d(x, y)$$

for all $x, y \in X$, where $\alpha \in (0, \frac{1}{s})$ is a constant. Then f has a unique fixed point u in X and $f^n x \to u$ for all $x \in X$.

Proof. The proof can be obtained from Theorem 3.2 by taking $G = G_0$, where G_0 is the complete graph $(X, X \times X)$. □

Corollary 3.4. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $\preceq$ be a partial ordering on X such that given $x, y \in X$, there is a sequence $(x_i)_{i=0}^N$ such that $x_0 = x$, $x_N = y$ and for all $i = 1, 2, \cdots, N$, x_{i-1} and x_i are comparable. Let $f : X \to X$ be such that f preserves comparable elements and*

$$d(fx, fy) \leq \alpha d(x, y)$$

for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$ and $\alpha \in (o, \frac{1}{s})$ is a constant. Assume that the triple $(X, d, \preceq)$ has the following property:

For any sequence (x_n) in X, if $x_n \to x$ and x_n, x_{n+1} are comparable for all $n \geq 1$, then there exists a subsequence (x_{k_n}) of (x_n) such that x_{k_n}, x are comparable for all $n \geq 1$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$, then f is a PO.

Proof. The proof can be obtained from Theorem 3.2 by taking $G = G_2 = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

Theorem 3.5. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$, and let $f : X \to X$ be a G-contraction such that f is orbitally G-continuous. Let $X_f = \{x \in X : (x, fx) \in E(G)\}$. Then,*

(i) *for any $x \in X_f$ and $y \in [x]_{\tilde{G}}$, $(f^n y)$ converges to a fixed point of f and $\lim_{n\to\infty} f^n y$ does not depend on y.*

(ii) *if $X_f \neq \emptyset$ and G is weakly connected, then f is a PO.*

Proof. (i) Let $x \in X_f$ i.e., $(x, fx) \in E(G)$. Let $y \in [x]_{\tilde{G}}$. Then proceeding as in Theorem 3.2, we can show that the sequences $(f^n x)$ and $(f^n y)$ are Cauchy equivalent. By completeness of X, $(f^n x)$ converges to some $u \in X$.

Now,

$$\begin{aligned} d(f^n y, u) \quad &\leq \quad sd(f^n y, f^n x) + sd(f^n x, u) \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which gives that, $\lim_{n\to\infty} f^n y = u$ for all $y \in [x]_{\tilde{G}}$.

We now show that u is a fixed point of f.
Since f preserves edges of G and $(x, fx) \in E(G)$, it follows that $\big(f^n x, f^{n+1} x\big) \in E(G)$ for all $n \in \mathbb{N}$. Again, f being orbitally G-continuous, we have $f(f^n x) \to fu$ which implies that $fu = u$ since, simultaneously, $f(f^n x) = f^{n+1} x \to u$. Thus, $(f^n y)$ converges to a fixed point u of f.
(ii) If $x \in X_f$ and G is weakly connected, then $[x]_{\tilde{G}} = X$ and so by (i), f is a PO. □

Corollary 3.6. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and let $\preceq$ be a partial ordering on X such that given $x, y \in X$, there is a sequence $(x_i)_{i=0}^N$ such that $x_0 = x$, $x_N = y$ and for all $i = 1, 2, \cdots, N$, x_{i-1} and x_i are comparable. Let $f : X \to X$ be an orbitally continuous function such that f preserves comparable elements and*

$$d(fx, fy) \leq \alpha d(x, y)$$

for all $x, y \in X$ with $x \preceq y$ or $y \preceq x$ and $\alpha \in (o, \frac{1}{s})$ is a constant. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$, then f is a PO.

Proof. The proof can be obtained from Theorem 3.5 by taking $G = G_2 = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$. □

The following theorem is the b-metric version of Edelstein theorem.

Theorem 3.7. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$ and ϵ-chainable for some $\epsilon > 0$, i.e., given $x, y \in X$, there is $N \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^N$ such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, N$. Let $f : X \to X$ be such that for all $x, y \in X$,*

$$d(x,y) < \epsilon \Rightarrow d(fx, fy)) < \alpha\, d(x,y) \tag{3.2}$$

where $\alpha \in (0, \frac{1}{s})$ is a constant. Then f is a PO.

Proof. It follows from condition (3.2) that f is continuous on X.

Let $x \in X$ be arbitrary. If $fx = x$, then a fixed point of f is assured. Therefore, we assume that $fx \neq x$. Since X is ϵ-chainable, there exists a sequence $(x_i)_{i=0}^N$ such that $x_0 = x$, $x_N = fx$ and $d(x_{i-1}, x_i) < \epsilon$ for $i = 1, 2, \cdots, N$.
By using condition (3.2), we have

$$d(fx_{i-1}, fx_i) < \alpha\, d(x_{i-1}, x_i) < \alpha\epsilon < \epsilon.$$

and therefore

$$\begin{aligned} d(f^2 x_{i-1}, f^2 x_i) &= d(f(fx_{i-1}), f(fx_i)) \\ &< \alpha\, d(fx_{i-1}, fx_i) \\ &< \alpha^2 \epsilon. \end{aligned}$$

In general, for any positive integer p, we get

$$d(f^p x_{i-1}, f^p x_i) < \alpha^p \epsilon, \ \ for\ i = 1,\ 2, \cdots, N.$$

Now,

$$\begin{aligned} d(f^p x, f^{p+1} x) &= d(f^p x, f^p(fx)) \\ &= d(f^p x_0, f^p x_N) \\ &\leq sd(f^p x_0, f^p x_1) + s^2 d(f^p x_1, f^p x_2) + \cdots \\ &\quad + s^{N-1} d(f^p x_{N-2}, f^p x_{N-1}) + s^{N-1} d(f^p x_{N-1}, f^p x_N) \\ &< (s + s^2 + \cdots + s^{N-1} + s^N)\alpha^p \epsilon \\ &= k\alpha^p \epsilon, \end{aligned} \tag{3.3}$$

where $k = (s + s^2 + \cdots + s^{N-1} + s^N)$.

For $m, n \in \mathbb{N}$ with $m > n$ and using condition (3.3), we obtain

$$\begin{aligned} d(f^n x, f^m x) &\leq sd(f^n x, f^{n+1} x) + s^2 d(f^{n+1} x, f^{n+2} x) + \cdots \\ &\quad + s^{m-n-1} d(f^{m-2} x, f^{m-1} x) + s^{m-n-1} d(f^{m-1} x, f^m x) \\ &< k\,\epsilon \left(s\alpha^n + s^2\alpha^{n+1} + \cdots + s^{m-n-1}\alpha^{m-2} + s^{m-n}\alpha^{m-1}\right) \\ &= k\,\epsilon s\, \alpha^n \left(1 + (s\alpha) + (s\alpha)^2 + \cdots + (s\alpha)^{m-n-1}\right) \\ &< k\,\epsilon s\, \alpha^n \frac{1}{1 - s\alpha},\ since\ s\alpha < 1 \\ &\to 0\ as\ n \to \infty. \end{aligned}$$

This shows that $(f^n x)$ is a Cauchy sequence in (X,d). Since (X,d) is complete, $(f^n x)$ converges to some point $u \in X$. Continuity of f implies that $f(f^n x) \to fu$. This gives that, $fu = u$ since, simultaneously, $f(f^n x) = f^{n+1} x \to u$. Thus, u is a fixed point of f.

We now show that u is the unique fixed point of f. If possible, suppose that there is another point $v(\neq u)$ in X such that $fv = v$. Then, by ϵ-chainability, there exists a sequence $(y_i)_{i=0}^{r}$ such that $y_0 = u$, $y_r = v$ and $d(y_{i-1}, y_i) < \epsilon$ for $i = 1, 2, \cdots, r$.
Then,

$$\begin{aligned} d(u,v) &= d(f^n u, f^n v) \\ &= d(f^n y_0, f^n y_r) \\ &\leq sd(f^n y_0, f^n y_1) + s^2 d(f^n y_1, f^n y_2) + \cdots \\ &\quad + s^{r-1} d(f^n y_{r-2}, f^n y_{r-1}) + s^{r-1} d(f^n y_{r-1}, f^n y_r) \\ &< (s + s^2 + \cdots + s^{r-1} + s^r)\alpha^n \epsilon \\ &= k_1 \alpha^n \epsilon, \ where \ k_1 = (s + s^2 + \cdots + s^{r-1} + s^r) \\ &\to 0 \ as \ n \to \infty, \end{aligned}$$

which is a contradiction. Therefore, $u = v$.

We now show that $\lim_{n\to\infty} f^n x = u$ for all $x \in X$.
If possible, suppose that $\lim_{n\to\infty} f^n y = w$ for some $y \in X$. Then, by our preceding discussion, it follows that w is a fixed point of f. Since u is the unique fixed point of f, we must have $u = w$ and hence $\lim_{n\to\infty} f^n x = u$ for all $x \in X$.
Thus, f is a PO. □

We conclude with some examples in favour of our main result.

Example 3.8. *Let $X = \mathbb{R}$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \mid x - y \mid^2$ for all x, $y \in X$. Then (X, d) is a complete b-metric space with the coefficient $s = 2$. Let G be a directed graph such that $V(G) = X$ and $E(G) = \Delta \cup \{(0, \frac{1}{8^n}) : n = 0, 1, 2, \cdots\}$. Any sequence (x_n) in X with the property $(x_n, x_{n+1}) \in E(G)$ must be a constant sequence. Consequently it follows that the triple (X, d, G) has the property $(*)$. Let $f : X \to X$ be defined by*

$$\begin{aligned} fx &= \frac{x}{8}, \ \ if \ \ x \neq \frac{7}{8} \\ &= 1, \ \ if \ \ x = \frac{7}{8}. \end{aligned}$$

For $(0, \frac{1}{8^n}) \in E(G)$, we have

$$d\left(f(0), f(\frac{1}{8^n})\right) = d\left(0, \frac{1}{8^{n+1}}\right) = \frac{1}{8^{2n+2}} = \frac{1}{64}.\frac{1}{8^{2n}} = \alpha d\left(0, \frac{1}{8^n}\right)$$

where $\alpha = \frac{1}{64} \in (0, \frac{1}{s})$ is a constant. Also, f preserves edges of G. Therefore, f is a Banach G-contraction. Clearly, $0 \in X_f$. Thus, we have all the conditions of Theorem 3.2 and $f\mid_{[0]_{\tilde{G}}}$ is a PO.

Remark 3.9. *In Example 3.8, f is a Banach G-contraction with constant $\alpha = \frac{1}{64}$ but it is not a Banach contraction. In fact, if $x = \frac{7}{8}$, $y = 1$, then*

$$d(fx, fy) = d(1, \frac{1}{8}) = \frac{49}{64} > \alpha.\frac{1}{64} = \alpha d(\frac{7}{8}, 1)$$

for any $\alpha \in (0, \frac{1}{s})$. So, f is not a Banach contraction.

The next example shows that the property $(*)$ in Theorem 3.2 is necessary.

Example 3.10. *Let $X = [0,1]$ and define $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \mid x - y \mid^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with the coefficient $s = 2$. Let G be a directed graph such that $V(G) = X$ and $E(G) = \{(0,0)\} \cup \{(x,y) : (x,y) \in (0,1] \times (0,1], x \geq y\}$. Let $f : X \to X$ be defined by*

$$\begin{aligned} fx &= \frac{x}{5}, \ \ if \ \ x \in (0,1] \\ &= 1, \ \ if \ \ x = 0. \end{aligned}$$

Clearly, f preserves edges of G. Moreover, for $(x,y) \in E(G)$, we have

$$d(fx, fy) = \frac{1}{25} d(x,y)$$

where $\alpha = \frac{1}{25} \in (0, \frac{1}{s})$ is a constant. Therefore, f is a Banach G-contraction. It is easy to verify that $X_f = (0,1]$ and $f^n x \to 0$ for all $x \in X$ but f has no fixed point. Consequently it follows that for any $x \in X_f$, $f \mid_{[x]_{\tilde{G}}}$ is not a PO. We observe that the property $(*)$ does not hold. In fact, (x_n) is a sequence in X with $x_n \to 0$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ where $x_n = \frac{1}{n}$. But there exists no subsequence (x_{k_n}) of (x_n) such that $(x_{k_n}, 0) \in E(G)$.

Remark 3.11. *In Example 3.10, the graph G is not weakly connected because there is no path in $\tilde{G}$ from 0 to 1. Moreover, f is a Banach G-contraction with constant $\alpha = \frac{1}{25}$ but it is not a Banach contraction. In fact, if $x = 0$, $y = 1$, then*

$$d(fx, fy) = d(1, \frac{1}{5}) = \frac{16}{25} > \alpha d(0,1)$$

for any $\alpha \in (0, \frac{1}{s})$. So, f is not a Banach contraction.

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