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SPACELIKE FACTORABLE SURFACES IN FOUR-DIMENSIONAL MINKOWSKI SPACE

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Abstract. In the current work, we study factorable surfaces in Minkowski four space. We describe such surfaces in terms of their Gaussian and mean curvature functions. We classify flat and minimal spacelike factorable surfaces in $\mathbb{E}_1^4$.

1. Introduction

In $\mathbb{E}_1^4$, the Lorentzian inner product is defined by

$$\langle u, v\rangle = -u_0v_0 + u_1v_1 + u_2v_2 + u_3v_3$$

for all $u, v \in \mathbb{E}_1^4$. A surface $M : F = F(s,t) : (s,t) \in D$ ($D \subset \mathbb{E}^2$) in $\mathbb{E}_1^4$ is said to be spacelike if $\langle , \rangle$ induces a Riemannian metric on M. Therefore, we know the following decomposition at each point p of a spacelike surface M;

$$\mathbb{E}_1^4 = T_pM \oplus T_p^{\perp}M.$$

The Levi-Civita connections on M and $\mathbb{E}_1^4$ are represented by ∇ and $\widetilde{\nabla}$, respectively. Let X_1 and X_2 be tangent vector fields and η be a normal vector field of M. $\widetilde{\nabla}_{X_1}\eta$ and $\widetilde{\nabla}_{X_1}X_2$ are separated into tangential and normal components by the Weingarten and Gauss formulas;

$$\begin{aligned} \widetilde{\nabla}_{X_1}\eta &= -A_\eta X_1 + D_{X_1}\eta, \\ \widetilde{\nabla}_{X_1}X_2 &= \nabla_{X_1}X_2 + h(X_1, X_2). \end{aligned}$$

Thus, these formulas introduce the second fundamental tensor h and the shape operator A_η corresponding to η [4].

Denote H the mean curvature vector field of M, then $H = \frac{1}{2}trh$. Consequently, we have $H = \frac{1}{2}\left((h(X_1, X_1) + h\left(X_2, X_2\right)\right)$ with respect to a local orthonormal frame $\{X_1, X_2\}$.

Let $M : F = F(s,t) : (s,t) \in D$ ($D \subset \mathbb{E}^2$) be a local parametrization on a spacelike surface in Minkowski 4−space. In accordance with $\langle F_s, F_s\rangle > 0$, $\langle F_t, F_t\rangle >$

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0, $T_pM = span\{F_s, F_t\}$ is the tangent space at any point p on M. The first fundamental form is given by

$$I(s,t) = es^2 + 2fst + gt^2, \; s,t \in IR \tag{1.1}$$

where $e = \langle F_s, F_s \rangle$, $f = \langle F_s, F_t \rangle$, $g = \langle F_t, F_t \rangle$ [5]. As the surface M is spacelike, we denote $W = \sqrt{eg - f^2}$. We choose unit normal vector fields such that η_1 is timelike, η_2 is spacelike. We use the denotations Γ_{ij}^k and c_{ij}^k, $i,j,k = 1,2$ for the Cristoffel symbols and coefficients of the second fundamental form, respectively. Then, the covariant derivatives can be written as linear combinations of the vector fields F_s, F_t, η_1, η_2;

$$\begin{aligned} \widetilde{\nabla}_{F_s} F_s &= F_{ss} = \Gamma_{11}^1 F_s + \Gamma_{11}^2 F_t - c_{11}^1 \eta_1 + c_{11}^2 \eta_2, \\ \widetilde{\nabla}_{F_s} F_t &= F_{st} = \Gamma_{12}^1 F_s + \Gamma_{12}^2 F_t - c_{12}^1 \eta_1 + c_{12}^2 \eta_2, \\ \widetilde{\nabla}_{F_t} F_t &= F_{tt} = \Gamma_{22}^1 F_s + \Gamma_{22}^2 F_t - c_{22}^1 \eta_1 + c_{22}^2 \eta_2, \end{aligned} \tag{1.2}$$

where $\{F_s, F_t, \eta_1, \eta_2\}$ is positively oriented in $\mathbb{E}_1^4$, (see, [6]). c_{ij}^k, $i,j,k = 1,2$ are given by

$$\begin{aligned} c_{11}^1 &= \langle F_{ss}, \eta_1 \rangle, \quad c_{12}^1 = \langle F_{st}, \eta_1 \rangle, \quad c_{22}^1 = \langle F_{tt}, \eta_1 \rangle, \\ c_{11}^2 &= \langle F_{ss}, \eta_2 \rangle, \quad c_{12}^2 = \langle F_{st}, \eta_2 \rangle, \quad c_{22}^1 = \langle F_{tt}, \eta_2 \rangle, \end{aligned} \tag{1.3}$$

(see, [6]).

The second fundamental tensor h of M defined as (see [6])

$$\begin{aligned} h(F_s, F_s) &= -c_{11}^1 \eta_1 + c_{11}^2 \eta_2, \\ h(F_s, F_t) &= -c_{12}^1 \eta_1 + c_{12}^2 \eta_2, \\ h(F_t, F_t) &= -c_{22}^1 \eta_1 + c_{22}^2 \eta_2. \end{aligned} \tag{1.4}$$

Moreover, the second fundamental tensor can be written as

$$h(X_1, X_2) = -\langle A_{\eta_1}(X_1), X_2 \rangle \eta_1 + \langle A_{\eta_2}(X_1), X_2 \rangle \eta_2. \tag{1.5}$$

The $k-$th component of H denoted by H_k is obtained by $H_k = \langle H, \eta_k \rangle = \frac{tr(A_{\eta_k})}{2}$ [7]. Hence we get

$$H_k = \frac{c_{11}^k g - 2c_{12}^k f + c_{22}^k e}{2(eg - f^2)}.$$

According to the normal basis, the mean curvature vector field H becomes

$$H = -H_1 \eta_1 + H_2 \eta_2.$$

Mean curvature function of M is the norm of the vector H.

Gaussian curvature of a surface $M : F(s,t)$ can be calculated by using the shape operator matrices as

$$K = \frac{-\det(A_{\eta_1}) + \det(A_{\eta_2})}{W^2} = \frac{-c_{11}^1 c_{22}^1 + c_{11}^2 c_{22}^2 + \left(c_{12}^1\right)^2 - \left(c_{12}^2\right)^2}{eg - f^2}.$$

A surface is said to be minimal (flat) if its mean curvature vector (Gaussian curvature) vanishes [3].

Factorable surfaces (also known as homotethical surfaces) in Euclidean and Minkowski 3$-$spaces can be parametrized locally as $F(s,t) = (s, t, f(s)g(t))$, where f and g are differentiable functions [10, 11]. Some authors have considered factorable surfaces in Euclidean space and in semi-Euclidean spaces [8, 9, 11, 12]. In

[10], Van de Woestyne showed that minimal factorable surfaces in $\mathbb{L}^3$ are helicoids and planes.

In [1], Yu. A. Aminov introduced the surface M in $\mathbb{E}^4$ given by

$$F(s,t) = (s, t, z(s,t), w(s,t)), \tag{1.6}$$

where z and w are differentiable functions. The representation (1.6) is called a Monge patch. Also, in [2], the authors investigated the curvature properties of these type of surfaces.

In the present study, we consider a spacelike factorable surface in Minkowski 4-space, which can locally be written as a monge patch

$$F(s,t) = (s, t, f_1(s)g_1(t), f_2(s)g_2(t)),$$

for some differentiable functions, $f_i(s), g_i(t)$, $i = 1, 2$. We characterize such surfaces in terms of their Gaussian curvature and mean curvature functions.

2. Spacelike factorable surfaces in $\mathbb{E}^4_1$

Definition 2.1. Let M be a surface in 4-dimensional Minkowski space $\mathbb{E}^4_1$. If the surface is given by an explicit form $z(s,t) = f_1(s)g_1(t)$ and $w(s,t) = f_2(s)g_2(t)$ where $s, t,\ z, w$ are Cartesian coordinates in $\mathbb{E}^4_1$ and f_i, g_i $i \in \{1, 2\}$ are smooth functions , then the surface is called a factorable surface in $\mathbb{E}^4_1$. Thus, the factorable surface can be written as a monge patch

$$F(s,t) = (s, t, f_1(s)g_1(t), f_2(s)g_2(t)). \tag{2.1}$$

Let M be a spacelike factorable surface with the parametrization (2.1). We determine a normal frame $\{\eta_1, \eta_2\}$ such that $\langle \eta_1, \eta_1 \rangle = -1$, $\langle \eta_2, \eta_2 \rangle = 1$, and $\{F_s, F_t, \eta_1, \eta_2\}$ is positively oriented frame in $\mathbb{E}^4_1$.

The tangent space of M is spanned by the vector fields

$$\begin{aligned} F_s &= (1, 0, f_1'(s)g_1(t), f_2'(s)g_2(t)), \\ F_t &= (0, 1, f_1(s)g_1'(t), f_2(s)g_2'(t)). \end{aligned}$$

Thus the coefficients of the first fundamental form of the surface can be expressed as

$$\begin{aligned} e &= \langle F_s, F_s \rangle = -1 + (f_1' g_1)^2 + (f_2' g_2)^2, \\ f &= \langle F_s, F_t \rangle = f_1' f_1 g_1' g_1 + f_2' f_2 g_2' g_2, \\ g &= \langle F_t, F_t \rangle = 1 + (f_1 g_1')^2 + (f_2 g_2')^2, \end{aligned} \tag{2.2}$$

where $\langle , \rangle$ is the Lorentzian inner product in $\mathbb{E}^4_1$. As the surface M is spacelike, then $W = \sqrt{eg - f^2}$.

The second partial derivatives of $F(s,t)$ are

$$\begin{aligned} F_{ss} &= (0, 0, f_1''(s)g_1(t), f_2''(s)g_2(t)), \\ F_{st} &= (0, 0, f_1'(s)g_1'(t), f_2'(s)g_2'(t)), \\ F_{tt} &= (0, 0, f_1(s)g_1''(t), f_2(s)g_2''(t)). \end{aligned} \tag{2.3}$$

Further, the normal space of $M : F(s,t)$ is spanned by the orthonormal vector fields

$$\eta_1 = \frac{1}{\sqrt{|A|}}(f_1'(s)g_1(t), -f_1(s)g_1'(t), 1, 0), \tag{2.4}$$

$$\eta_2 = \frac{1}{\sqrt{|AD|}}(Af_2'(s)g_2(t) - Bf_1'(s)g_1(t), Bf_1(s)g_1'(t) - Af_2(s)g_2'(t), -B, A),$$

where

$$\begin{aligned} A &= 1 - \left(f_1^{'}g_1\right)^2 + \left(f_1g_1^{'}\right)^2, \\ B &= -f_1'f_2'g_1g_2 + f_1f_2g_1^{'}g_2', \\ C &= 1 - \left(f_2^{'}g_2\right)^2 + \left(f_2g_2^{'}\right)^2, \\ D &= AC - B^2. \end{aligned} \tag{2.5}$$

Since M is spacelike surface in $\mathbb{E}_1^4$ with respect to choosen orthonormal frame, A and D are negative definite. Using (2.3) and (2.4), one can find the coefficient functions of the second fundamental form as follows;

$$\begin{aligned} c_{11}^1 &= \frac{f_1^{''}g_1}{\sqrt{|A|}}, \quad c_{22}^1 = \frac{f_1g_1^{''}}{\sqrt{|A|}}, \\ c_{12}^1 &= \frac{f_1'g_1'}{\sqrt{|A|}}, \quad c_{12}^2 = \frac{Af_2'g_2' - Bf_1'g_1'}{\sqrt{|AD|}}, \\ c_{11}^2 &= \frac{Af_2^{''}g_2 - Bf_1^{''}g_1}{\sqrt{|AD|}}, \\ c_{22}^2 &= \frac{Af_2g_2^{''} - Bf_1g_1^{''}}{\sqrt{|AD|}}. \end{aligned} \tag{2.6}$$

Using Gram-Schmidt orthonormalization method for the spacelike vector fields F_s and F_t, we get orthonormal tangent vectors

$$\begin{aligned} X_1 &= \frac{F_s}{\sqrt{e}}, \\ X_2 &= \frac{\sqrt{e}}{W}\left(F_t - \frac{f}{e}F_s\right). \end{aligned} \tag{2.7}$$

By the use of (1.3), (1.4), (1.5) and (2.7) the second fundamental tensors A_{η_k} become

$$A_{\eta_1} = \frac{1}{e\sqrt{|A|}}\begin{pmatrix} f_1''g_1 & \frac{f_1'g_1'e - f_1''g_1f}{W} \\ \frac{f_1'g_1'e - f_1''g_1g}{W} & \frac{f_1g_1''e^2 - 2f_1'g_1'ef + f_1''g_1f^2}{W^2} \end{pmatrix},$$

and

$$A_{\eta_2} = \frac{1}{e\sqrt{|AD|}}\begin{pmatrix} \lambda & \frac{\mu e - \lambda f}{W} \\ \frac{\mu e - \lambda f}{W} & \frac{\delta e^2 - 2\mu ef + \lambda f^2}{W^2} \end{pmatrix},$$

where

$$\begin{aligned}\lambda &= Af_2''g_2 - Bf_1''g_1, \\ \mu &= Af_2'g_2' - Bf_1'g_1', \\ \delta &= Af_2g_2'' - Bf_1g_1''.\end{aligned}$$

2.1. **Flat factorable surfaces.**

Theorem 2.2. *Let M be a spacelike factorable surface in $\mathbb{E}_1^4$. Then the Gaussian curvature of the surface is given by*

$$K = \frac{\left(f_1''f_1g_1''g_1 - f_1'^2g_1'^2\right)C - \left(f_1''f_2g_1g_2'' + f_1f_2''g_1''g_2 - 2f_1'f_2'g_1'g_2'\right)B + \left(f_2''f_2g_2''g_2 - f_2'^2g_2'^2\right)A}{DW^2}.$$

Corollary 2.3. *Let M be a spacelike factorable surface in Minkowski 4−space. If M is given by one of the following parametrizations, then it is a flat surface:*

$$\begin{aligned}
&(1)\ F(s,t) = (s,t,a_1g_1(t),a_2g_2(t)), \\
&(2)\ F(s,t) = (s,t,b_1f_1(s),b_2f_2(s)), \\
&(3)\ F(s,t) = (s,t,a_1g_1(t),a_2f_2(s)), \\
&(4)\ F(s,t) = (s,t,b_1f_1(s),b_2g_2(t)), \\
&(5)\ F(s,t) = (s,t,a_1b_1,\exp(a_2s+b_2)\exp(a_3t+b_3)), \\
&(6)\ F(s,t) = \left(s,t,a_1b_1,(a_2s+b_2)^{\frac{1}{1-\lambda}}(a_3t+b_3)^{\frac{\lambda}{\lambda-1}}\right), \\
&(7)\ F(s,t) = \left(s,t,\exp(a_1s+b_1)\exp(a_2t+b_2),\exp(a_3s+b_3)\exp(a_3\frac{a_i}{a_j}t+b_4)\right), \\
&(8)\ F(s,t) = (s,t,f_1(s)\cos t,f_1(s)\sin t),
\end{aligned}$$

the function $f_1(s)$ satisfies

$$s = \pm\int\sqrt{\frac{a_1f_1^2(s)+1}{f_1^2(s)+1}}df_1(s)$$

where $i,j=1,2,\ i\neq j$ and $a_k,b_k,\ k=1,...,4$ are real constants.

Proof. Let M be a spacelike factorable surface given with the parametrization (2.1) in $\mathbb{E}_1^4$.

If $f_1'(s)=0$, $f_2'(s)=0$ or $g_1'(t)=0$, $g_2'(t)=0$ or $f_1'(s)=0$, $g_2'(t)=0$ ($f_2'(s)=0$, $g_1'(t)=0$) , then we obtain the cases (1), (2), (3) and (4).

If $f_1'(s)=0$, $g_1'(t)=0$, then we have

$$f_2''f_2g_2''g_2 - f_2'^2g_2'^2 = 0. \tag{2.8}$$

Let $p(s)=\frac{df_2}{ds}$ and $q(t)=\frac{dg_2}{dt}$. By the use of (2.8), we can write

$$f_2(s)p(s)\frac{dp}{df_2}g_2(t)q(t)\frac{dq}{dg_2} - (p(s)q(t))^2 = 0. \tag{2.9}$$

If $p(s)\neq 0$, $q(t)\neq 0$, from (2.9), we get

$$f_2(s)\frac{dp}{df_2}g_2(t)\frac{dq}{dg_2} = p(s)q(t).$$

Then we have differential equation

$$\frac{f_2(s)\frac{dp}{df_2}}{p(s)} = \frac{q(t)}{g_2(t)\frac{dq}{dg_2}} = \lambda, \tag{2.10}$$

where λ is constant.

(1) If $\lambda = 1$, from (2.10) we have

$$\begin{aligned} f_2(s) &= \exp(a_2 s + b_2), \\ g_2(t) &= \exp(a_3 t + b_3), \end{aligned} \tag{2.11}$$

which gives the case (5).

(2) If $\lambda \neq 1$,from (2.10) we have

$$\begin{aligned} f_2(s) &= (a_2 s + b_2)^{\frac{1}{1-\lambda}}, \\ g_2(t) &= (a_3 t + b_3)^{\frac{\lambda}{\lambda-1}}, \end{aligned} \tag{2.12}$$

which gives the case (6).

Further, we assume $f_i'' f_i g_i'' g_i - f_i'^2 g_i'^2 = 0$ holds for $i = 1$ and $i = 2$. Then we get

$$\begin{aligned} f_1(s) &= \exp(a_1 s + b_1), \qquad f_2(s) = \exp(a_3 s + b_3), \\ g_1(t) &= \exp(a_2 t + b_2), \qquad g_2(t) = \exp(a_4 t + b_4). \end{aligned} \tag{2.13}$$

Substituting these functions into $B = 0$ and $f_1'' f_2 g_1 g_2'' + f_1 f_2'' g_1'' g_2 - 2 f_1' f_2' g_1' g_2' = 0$, we have $a_4 = \frac{a_3 a_i}{a_j}$, $i, j = 1, 2$ $(i \neq j)$ which vanish the Gaussian curvature of the surface. Thus, we obtain the case (7).

Also, if $f_1(s) = f_2(s)$ and $g_1(t) = \cos t$, $g_2(t) = \sin t$, then by the use of the previous theorem, for a flat surface we get

$$-f_i''(s) f_i(s) \left(f_i^2(s) + 1\right) + (f_i'(s))^2 \left((f_i'(s))^2 - 1\right) = 0.$$

By the solution of this differential equation we obtain the case (8). □

2.2. **Minimal factorable surfaces.**

Theorem 2.4. *Let M be a spacelike factorable surface in $\mathbb{E}_1^4$. Then the mean curvature vector of the surface is given by*

$$\vec{H} = -\frac{f_1'' g_1 g + f_1 g_1'' e - 2 f_1' g_1' f}{2\sqrt{|A|} W^2} \eta_1 + \frac{A\left(f_2'' g_2 g + f_2 g_2'' e - 2 f_2' g_2' f\right) - B\left(f_1'' g_1 g + f_1 g_1'' e - 2 f_1' g_1' f\right)}{2\sqrt{|AD|} W^2} \eta_2.$$

Theorem 2.5. *Let M be a spacelike factorable surface in $\mathbb{E}_1^4$. Then M is a minimal surface if and only if*

$$f_i'' g_i g + f_i g_i'' e - 2 f_i' g_i' f = 0, \qquad i = 1, 2. \tag{2.14}$$

Proof. Let M be a spacelike factorable surface with the parametrization (2.1) in $\mathbb{E}_1^4$. We can write the mean curvature vector as $H = -H_1 \eta_1 + H_2 \eta_2$, for a minimal surface, $H_1 = 0$, $H_2 = 0$. By the use of the previous theorem, we get (2.14). The converse statement is trivial. □

Corollary 2.6. *Let M be a spacelike factorable surface in Minkowski $4-$space. If M is given by one of the following parametrizations, then it is a minimal surface:*

(1) $F(s,t) = (s, t, (a_1 s + a_2) b_1, (a_3 s + a_4) b_2)$,

(2) $F(s,t) = (s, t, a_1 (b_1 t + b_2), a_2 (b_3 t + b_4))$,

(3) $F(s,t) = (s, t, (a_1 s + a_2) b_1, a_3 (b_3 t + b_4))$,

(4) $F(s,t) = \left(s, t, a_1 b_1, (s + a_2) \frac{-1 - \exp(b_2 t + b_3)}{-1 + exp(b_2 t + b_3)}\right)$,

(5) $F(s,t) = (s, t, a_1 b_1, \tan(a_2 s + a_3)(t + b_2))$,

(6) $F(s,t)=\left(s,t,\frac{-1-a_1^2+\exp(\pm 2a_1(a_1s+a_2))}{2a_1\exp(\pm 2a_1(a_1s+a_2))}\cos t,\frac{-1-a_1^2+\exp(\pm 2a_1(a_1s+a_2))}{2a_1\exp(\pm 2a_1(a_1s+a_2))}\sin t\right)$,

(7) $F(s,t)=\left(s,t,(s+a_1)\frac{-1-\exp(b_1t+b_2)}{-1+\exp(b_1t+b_2)},(s+a_1)\frac{-1-\exp(b_1t+b_2)}{-1+\exp(b_1t+b_2)}\right)$,

(8) $F(s,t)=(s,t,\tan(a_1s+a_2)\,(t+b_1)\,,\tan(a_1s+a_2)\,(t+b_1))$,

(9) $F(s,t)=(s,t,a_1b_1,f_2(s)g_2(t))$,

(10) $F(s,t)=(s,t,f_1(s)g_1(t),f_1(s)g_1(t))$,
the functions $f_i(s)$, $g_i(t)$, $i=1,2$ *satisfy the equations*

$$s=\int\frac{df_i(s)}{\sqrt{2m\ln f_i(s)+a_1}},\ t=\int\frac{dg_i(t)}{\sqrt{a_2g_i^4(t)-\frac{n}{2}}},$$

or

$$s=\int\frac{df_i(s)}{\sqrt{a_1f_i^4(s)-\frac{m}{2}}},\ t=\int\frac{dg_i(t)}{\sqrt{2n\ln g_i(t)+a_2}},$$

or

$$s=\int\frac{df_i(s)}{\sqrt{a_1f_i^{2(1+c)}(s)-a_2}},\ t=\int\frac{dg_i(t)}{\sqrt{a_3g_i^{2(1-c)}(t)-a_4}},$$

where $c,m,n,a_k,b_k,\ k=1,..,4$ *are real constants and* $c\neq\pm 1$.

Proof. Let M be a spacelike factorable surface with the parametrization (2.1) in $\mathbb{E}_1^4$. By the use of (2.14) with (2.2),

$$f_i''g_i\left(1+f_1^2g_1'^2+f_2^2g_2'^2\right)+f_ig_i''\left(-1+f_1'^2g_1^2+f_2'^2g_2^2\right)-2f_i'g_i'\left(f_1'f_1g_1'g_1+f_2'f_2g_2'g_2\right)=0, \tag{2.15}$$

holds for $i=1,\ 2$. If $g_1'(t)=0,\ g_2'(t)=0$ or $f_1'(s)=0,\ f_2'(s)=0$, we obtain the cases (1) and (2), respectively.

If $f_2'(s)=0,\ g_1'(t)=0,\ i,j=1,2,\ i\neq j$, then

$$f_1''g_1\left(1+f_1^2g_1'^2+f_2^2g_2'^2\right) = 0, \tag{2.16}$$
$$f_2g_2''\left(-1+f_1'^2g_1^2+f_2'^2g_2^2\right) = 0. \tag{2.17}$$

Since the first fundamental forms e and g are positive, then we get $f_1''(s)=0$ and $g_2''(t)=0$ which congruent the case (3).

If $f_1'(s)=0,\ g_1'(t)=0$, from the equality (2.15) for $i=2$, we get

$$\frac{f_2''(s)}{f_2(s)}-\frac{g_2''(t)}{g_2(t)}+\left(f_2''(s)f_2(s)-f_2'^2(s)\right)g_2'^2(t)+\left(g_2''(t)g_2(t)-g_2'^2(t)\right)f_2'^2(s)=0. \tag{2.18}$$

If $f_2''(s)=0$ or $g_2''(t)=0$ in (2.18), we obtain the cases (4) and (5).

If $f_2''(s)g_2''(t)\neq 0$ in (2.18), differentiating (2.18) with respect to s and t, we have

$$\frac{\left(f_2''(s)f_2(s)-f_2'^2(s)\right)'}{\left(f_2'^2(s)\right)'}=-\frac{\left(g_2''(t)g_2(t)-g_2'^2(t)\right)'}{\left(g_2'^2(t)\right)'}=c. \tag{2.19}$$

Thus, we can write

$$\begin{array}{c} f_2''(s)f_2(s)-(1+c)f_2'^2(s)=m, \\ g_2''(t)g_2(t)-(1-c)g_2'^2(t)=n. \end{array} \tag{2.20}$$

If $c = 1$, $c = -1$ and $c \neq \pm 1$, then from the solution of (2.20), we obtain the case (9).

If $f_1(s) = f_2(s)$ and $g_1(t) = \cos t$, $g_2(t) = \sin t$, then we get

$$f_i''(s)\left(1 + f_i^2(s)\right) - f_i(s)\left(1 + (f_i'(s))^2\right) = 0.$$

By the solution of this differential equation we obtain the case (6).

If $f_1(s) = f_2(s)$, $g_1(t) = g_2(t)$ in (2.15), then for $i = 1$ or $i = 2$, we find

$$\frac{f_i''(s)}{f_i(s)} - \frac{g_i''(t)}{g_i(t)} + \left(f_i''(s)f_i(s) - f_i'^2(s)\right)2g_i'^2(t) + \left(g_i''(t)g_i(t) - g_i'^2(t)\right)2f_i'^2(s) = 0. \quad (2.21)$$

If $f_i''(s) = 0$ or $g_i''(t) = 0$ in (2.21), we obtain the cases (7) and (8). Also, if $f_i''(s)g_i''(t) \neq 0$, we obtain the case (10), which completes the proof. □

Example 2.7. By selecting $a_1 = 1$, $b_1 = 2$, $b_2 = 0$ for the case (7) in Corollary 2.6, we can plot the projection of this surface with mapple command:

$$plot3d([s, t, z + w], s = a..b, t = c..d). \quad (2.22)$$

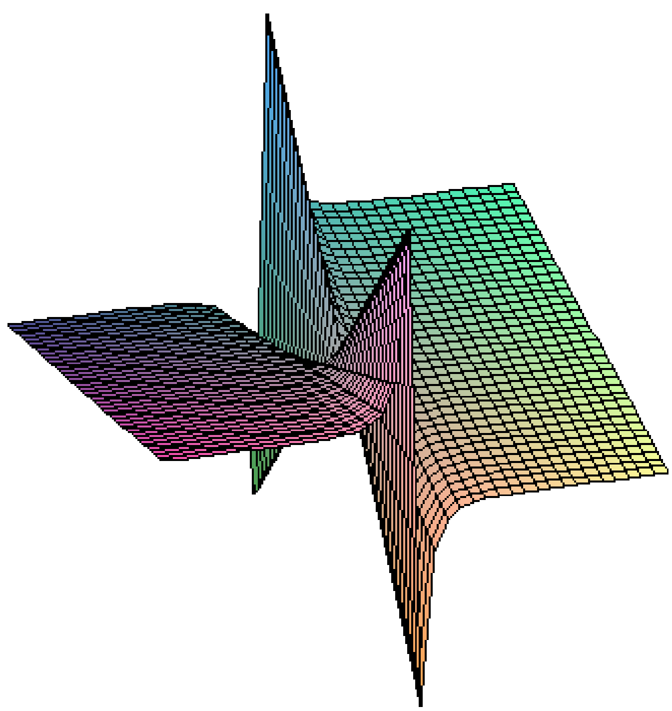

FIGURE 1. 3D Model of the surface given by the case (7) in Corollary 2.6

References

[1] Y. A. Aminov, *Surfaces in $\mathbb{E}^4$ with a Gaussian curvature coinciding with a Gaussian torsion up to sign*, Mathematical Notes **56** (1994) 5–6.
[2] B. Bulca, K. Arslan, *Surfaces Given with the Monge Patch in $\mathbb{E}^4$*, Journal of Mathematical Physics, Analysis, Geometry **9 4** (2013) 435–447.
[3] B. Y. Chen, *Geometry of Submanifolds*, Dekker, New York, 1973.
[4] B. Y. Chen, J. Van der Veken, *Marginally trapped surfaces in Lorentzian space forms with positive relative nullity*, Class. Quantum Grav. **24** (2007) 551–563.
[5] G. Gancgev, V. Milousheva, *An invariant theory of Marginally Trapped Surfaces in four-dimensional Minkowski space*, Journal of Mathematical Physics **53 3** (2012) 033705.
[6] G. Gancgev, V. Milousheva, *An invariant theory of spacelike surfaces in the four-dimensional Minkowski space*, Pliska Stud. Math. Bulgar. **21** (2012) 177–200.
[7] S. Heasen, M. Ortega, *Marginally trapped surfaces in Minkowski $4-$space invariant under a rotation subgroup of the Lorentz group*, Gen Relativ Gravit **41** (2009) 1819–1834.
[8] R. Lopez, M. Moruz, *Translation and Homothetical Surfaces in Euclidean Spaces with Constant Curvature*, J. Korean Math. Soc. **52 3** (2015) 523–535.
[9] H. Meng, H. Liu, *Factorable Surfaces in $3-$Minkowski Space*, Bull. Korean Math. Soc. **46 1** (2009) 155–169.
[10] I. Van de Woestyne, *A new characterization of helicoids*, Geometry and topology of submanifolds V, World Sci. Publ. River Edge, NJ, 1993.
[11] I. Van de Woestyne, *Minimal homothetical hypersurfaces of a semi-Euclidean space*, Results Math. **27 3** (1995) 333–342.
[12] Y. Yu, H. Liu, *The factorable minimal surfaces*, Proceedings of The Eleventh International Workshop on Diff. Geom. **11** (2007) 33–39.

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