Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 9 Issue 4(2017), Pages 42-57.

# FIXED POINT THEOREMS OF INTUITIONISTIC FUZZY MAPPINGS IN QUASI-PSEUDO METRIC SPACES

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Abstract. The main focus in this article is to establish some fixed point theorems for intuitionistic fuzzy mappings in left $K$-sequentially complete quasi-pseudo metric spaces as well as right $K$-sequentially complete quasi-pseudo metric spaces in association with $(\alpha, \beta) - cut$ set of an intuitionistic fuzzy set. Our main results generalize and unify several well-known fixed point results for intuitionistic fuzzy maps. Some examples are presented to prove the validity of hypothesis of our main results.

## 1. Introduction

Heilpern [15] first bring about the idea of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings on complete metric linear spaces which is an extension of the fixed point theorem for multivalued mappings of Nadler [20]. The notion of fixed point for fuzzy mappings was initiated by Weiss [27] and Butnariu [13]. Subsequently, several authors (see, [1, 6, 7, 8, 9, 19, 25]) generalized and studied the existence of fixed points of fuzzy mappings satisfying a contractive condition.
In 1963, Kelly [18] introduced the concept of Cauchy sequence for a quasi-pseudo-metric space and gave a generalization of the Baire category theorem. This definition was further extended by Reilly et al. [22] in seven different notions. Afterwards, in 1999, Gregori [14] established a fixed point theorem for fuzzy contraction mappings in left $K$-sequentially complete quasi-pseudo metric space to extend the result of Heilpern [15]. Later on, in 2003, Telci [26] proved a fixed point theorem for fuzzy mappings in quasi-metric spaces. In 2005, Sahin et al. [23] presented common fixed point theorem for a pair of fuzzy mappings in left (right) $K$-sequentially complete quasi-pseudo metric spaces.
In 2013, Azam et al. [9] established some local versions of fixed point theorems for fuzzy contraction mappings in left (right) $K$-sequentially complete quasi-pseudo metric spaces to generalize the result of Gregori [14]. Recently, Rashid et al. [10] generalized the results [9, 14] by introducing $L$-fuzzy contraction mappings in left (right) $K$-sequentially complete quasi-pseudo metric spaces.

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2000 *Mathematics Subject Classification.* 46S40, 47H10, 54H25.
*Key words and phrases.* Fixed point, intuitionistic fuzzy mapping, quasi-pseudo metric space, left $K$-sequentially complete, right $K$-sequentially complete.

Submitted August 16, 2017. Published November 27, 2017.
Communicated by N. Hussain.

On the other hand, intuitionistic fuzzy set $(IFS)$, developed by Atanassov (see, [3, 4, 5]) in 1983, is a powerful tool to deal with vagueness. A prominent characteristic of $(IFS)$ is that it assigns to each element a membership degree and a non-membership degree and thus, the $(IFS)$ constitutes an extension of Zadeh's fuzzy set [28], which only assigns to each element a membership degree. Moreover, fuzzy set is a special case of $(IFS)$ by assigning each element a degree of non-membership. Thus, $(IFS)$ would be better to deal with the imprecise and uncertain information than fuzzy sets. Afterwards, several researchers explored on the extension of the notion of intuitionistic fuzzy set see, ([2, 11, 12, 16, 17, 21, 24, 25, 26]). In 2012, Shen et al. [24] introduced the concept of intuitionistic fuzzy mappings and defined some operations of intuitionistic fuzzy mappings to develop a soft algebra and proposed a relationship with the intuitionistic fuzzy relation.
Thus, motivated and inspired by the researcher's work presented in [24], fixed point theorems for intuitionistic fuzzy mappings with $(\alpha, \beta) - cut$ set of an intuitionistic fuzzy set in left (right) $K$-sequentially complete quasi-pseudo metric spaces have been established. Moreover, our results are based on the fact that mappings are not contractive on the whole space but they may be contractive on its subset.
Using same techniques of this paper, a variety of fixed point results of literature can be improved/generalized in connection with the idea of $(\alpha, \beta) - cut$ set for intuitionistic fuzzy mappings.

## 2. Preliminaries

This section explores the some basic definitions and known results needed in the sequel.

**Definition 2.1** [9] Let $X$ be a nonempty set. Then the function $d : X \times X \to [0, \infty)$ is called quasi-pseudo metric on $X$, if for all $x,\ y,\ z \in X$, satisfying the following
$(i)$ $d(x, x) = 0$
$(ii)$ $d(x, y) \leq d(x, z) + d(z, y)$.
If $d$ is a quasi-pseudo metric on $X$, then a pair $(X, d)$ is said to be quasi-pseudo metric space.
Each quasi-pseudo metric $d$ on $X$ induces a topology $\tau(d)$ which has as a base the family of all $d$-balls $B_\epsilon(x)$, where

$$B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}.$$

If $d$ is a quasi-pseudo metric on $X$, then the function $d^{-1} : X \times X \to [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ is also quasi-pseudo metric on $X$. Here, $d \wedge d^{-1} = \min\{d, d^{-1}\}$ and $d \vee d^{-1} = \max\{d, d^{-1}\}$.

**Definition 2.2** [22] Let $(X, d)$ be a quasi-pseudo metric space. A sequence $\{x_n\}$ where $n \in N$ in $X$ is said to be
$(i)$ left $K$-Cauchy if for each $\epsilon > 0$, there exists $k \in N$ such that $d(x_n, x_m) < \epsilon$ for all $m \in N$ with $m \geq n \geq k$.
$(ii)$ right $K$-Cauchy if for each $\epsilon > 0$, there exists $k \in N$ such that $d(x_n, x_m) < \epsilon$ for all $m \in N$ with $n \geq m \geq k$.
A quasi-pseudo metric space $(X, d)$ is said to be left(right) $K$-sequentially complete if each left(right) $K$-Cauchy sequence in $(X, d)$ converges to some point in $X$.

**Definition 2.3** [9] Let $(X, d)$ be a quasi-pseudo metric space and let $A$ and $B$ be two nonempty closed subsets of $X$ w.r.t $\tau(d)$. Then the Hausdorff distance between subsets $A$ and $B$ is defined as

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where

$$d(a, B) = \inf\{d(a, x) : x \in B\}.$$

Clearly, $H$ is the usual Hausdorff distance if $d$ is a metric on $X$.

**Definition 2.4** [9] A fuzzy set $A$ on a nonempty set $X$ is a function $A : X \to [0, 1]$. If $A$ is a fuzzy set and $x \in X$, then the function values $A(x)$ is called the grade of membership of $x$ in $A$.
The $\alpha - level$ set of $A$ is denoted by $[A]_\alpha$ and is defined as:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\} \quad \text{for each } \alpha \in (0, 1], \\ [A]_0 &= cl(\{x : A(x) > 0\}). \end{aligned}$$

Here, $cl(B)$ denotes the closure of the set $B$.

**Definition 2.5** [14] Let $X$ be an arbitrary set and $Y$ be a metric space. A mapping $T$ is called fuzzy mapping if $T$ is a mapping from $X$ into $I^Y$.

The classes $W^*(X)$ and $W^/(X)$ of fuzzy sets on quasi-pseudo metric space $(X, d)$ and $W(V)$ of fuzzy sets on a metric linear space $(V, d_V)$ are defined as follows:

$$\begin{aligned} W^*(X) &= \left\{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d^{-1}\text{-compact}\right\} \\ W^/(X) &= \left\{A \in I^X : A_1 \text{ is nonempty } d\text{-closed and } d\text{-compact}\right\} \\ W(V) &= \left\{\begin{array}{c} A \in I^V : A_\alpha \text{ is compact and convex in } V \\ \text{for each } \alpha \in (0, 1] \\ \text{with } \sup\limits_{x\in V} \mu_A(x) = 1 \end{array}\right\}. \end{aligned}$$

**Definition 2.6** [23] Let $(X, d)$ be a quasi-pseudo metric space and let $A, B \in W^*(X)$ $\left(or\ W^/(X)\right)$ and $\alpha \in [0, 1]$. Then following notions are defined as:

$$D_\alpha(A, B) = H([A]_\alpha, [B]_\alpha),$$

where the Hausdorff metric $H$ is deduced from the quasi-pseudo metric $d$ on $X$.

$$d_\infty(A, B) = \sup\{D_\alpha(A, B) : \alpha \in (0, 1]\}.$$

**Lemma 2.7** [15] Suppose that $K \neq \varphi$ is compact in the quasi-pseudo metric space $\left(X, d^{-1}\right)$ or $(X, d)$. If $z \in X$, then there exists $k_0 \in K$ such that

$$d(z, K) = d(z, k_0) \quad \text{or} \quad d(K, z) = d(k_0, z).$$

## 3. Fixed point theorems for intuitionistic fuzzy contractive maps

In this section, on the basis of $(IFS)$, some definitions and lemmas are discussed.

**Definition 3.1** [4] Let $X$ be a nonempty set. An Intuitionistic fuzzy set $E$ in $X$ is an object having the form

$$E = \{\langle x, \mu_E(x), \nu_E(x)\rangle : x \in X\},$$

where the functions $\mu_E(x) : X \to [0,1]$ and $\nu_E(x) : X \to [0,1]$ define respectively, the degree of membership and degree of non-membership of $x$ to $E$, satisfying the condition

$$0 \leq \mu_E(x) + \upsilon_E(x) \leq 1 \text{ for each } x \in X.$$

Furthermore, we have

$$\pi_E(x) = 1 - \mu_E(x) - \upsilon_E(x)$$

called the hesitation margin of $x$ to $E$. Clearly

$$0 \leq \pi_E(x) \leq 1 \text{ for all } x \in X.$$

We denote the collection of all intuitionistic fuzzy sets on $X$ by $(IFS)^X$.

**Definition 3.2** [3] Let $E$ is an intuitionistic fuzzy set and $x \in X$, then $\alpha - level$ set of an intuitionistic fuzzy set $E$ is denoted by $[E]_\alpha$ and defined as

$$[E]_\alpha = \{x \in X : \mu_E(x) \geq \alpha \text{ and } \upsilon_E(x) \leq 1 - \alpha\} \text{ if } \alpha \in [0,1].$$

**Definition 3.3** [16] Let $E$ is an intuitionistic fuzzy set on $X$, then $(\alpha, \beta) - cut$ set of $E$ is defined as

$$E_{(\alpha,\beta)} = \{x \in X : \mu_E(x) \geq \alpha \text{ and } \upsilon_E(x) \leq \beta\}, \text{ where } \alpha + \beta \leq 1,$$

which is generalization of $\alpha - level$ set.
We denote $E_{(M,m)}$ by $[E]^*$ and is defined as

$$E_{(M,m)} = [E]^* = \{x \in X : \mu_E(x) = M \text{ and } \upsilon_E(x) = m\},$$

where

$$\begin{aligned} M &= \max_{x \in X} \mu_E(x), \\ m &= \min_{x \in X} \upsilon_E(x). \end{aligned}$$

**Definition 3.4** [17] Any crisp set $A$ can be represented as an intuitionistic fuzzy set by its intuitionistic characteristic function $\langle \Phi_A, \Psi_A \rangle$ defined as:

$$\Phi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases} \qquad \Psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A. \end{cases}$$

For convenience, we use the following notations

$$\Omega_{A^\alpha_\beta}(x) = \begin{cases} \alpha & x \in A \\ \beta & x \notin A, \end{cases}$$

which gives the following comparison

$$\Phi_A(x) = \Omega_{A^1_0}(x) = \chi_A(x) \text{ and } \Psi_A(x) = \Omega_{A^0_1}(x) = 1 - \chi_A(x).$$

Now we are defining the classes for intuitionistic fuzzy sets on the basis of classes defined in [13].

**Definition 3.5.** Let $(X,d)$ be a quasi-pseudo metric space and $(V,d_V)$ be a metric linear space. The families $W^*_{IF}(X)$ and $W^{/}_{IF}(X)$ of intuitionistic fuzzy sets on $(X,d)$ and $W_{IF}(V)$ of intuitionistic fuzzy sets on $(V,d_V)$ are defined by

$$\begin{aligned} W^*_{IF}(X) &= \left\{E \in (IFS)^X : [E]^* \text{ is nonempty } d\text{-closed and } d^{-1}\text{-compact}\right\} \\ W^{/}_{IF}(X) &= \left\{E \in (IFS)^X : [E]^* \text{ is nonempty } d\text{-closed and } d\text{-compact}\right\} \\ W_{IF}(V) &= \left\{ \begin{array}{c} E \in (IFS)^V : E_{(\alpha,\beta)} \text{ is compact and convex in } V \\ \text{for each } (\alpha,\beta) \in (0,1] \times [0,1) \\ \text{with } \sup_{x\in V} \mu_E(x) = 1 \text{ and } \inf_{x \in V} \upsilon_E(x) = 0 \end{array} \right\}. \end{aligned}$$

**Definition 3.6** [24] Let $X$ be an arbitrary set and $Y$ be a quasi-pseudo metric space. $S$ is said to be intuitionistic fuzzy mapping if $S$ is a mapping from $X$ into $W^*_{IF}(Y)$ or $\left(W^{/}_{IF}(Y)\right)$.

**Definition 3.7.** A point $x^* \in X$ is called fixed point of an intuitionistic fuzzy mapping $S : X \to (IFS)^X$ if $x^* \in [Sx^*]^*$, i.e.

$$\text{for all } x \in X,\ \mu_{S(x^*)}(x) \le \mu_{S(x^*)}(x^*) \text{ and } \upsilon_{S(x^*)}(x) \ge \upsilon_{S(x^*)}(x^*).$$

**Definition 3.8.** Let $(X,d)$ be a quasi-pseudo metric space and let $E$, $G \in W^*_{IF}(X)$ or $W^{/}_{IF}(X)$ and $(\alpha,\beta) \in (0,1] \times [0,1)$. Then it can be defined as

$$D_{(\alpha,\beta)}(E,G) = H\left([E]_{(\alpha,\beta)}, [G]_{(\alpha,\beta)}\right),$$

where the Hausdorff metric $H$ is deduced from the quasi-pseudo metric $d$ on $X$,

$$d_{(\infty,\infty)}(E,G) = \sup\left\{D_{(\alpha,\beta)}(E,G) : (\alpha,\beta) \in (0,1] \times [0,1)\right\}.$$

For $A$, $B \in (IFS)^X$, $A \subset B$ implies that $\mu_A(x) \le \mu_B(x)$ and $\upsilon_A(x) \ge \upsilon_B(x)$ for each $x \in X$.
Now we are modifying the following Lemmas (see, [9, 14]) for intuitionistic fuzzy mappings.

**Lemma 3.9.** Let $(X,d)$ be a quasi-pseudo metric space, $x \in X$ and $E \in W^*_{IF}(X)$ or $W^{/}_{IF}(X)$. Then $x \in [E]^*$

$$\begin{aligned} \text{if and only if } d\left(x, [E]^*\right) &= 0 \quad \text{or} \\ d\left([E]^*, x\right) &= 0. \end{aligned}$$

**Lemma 3.10.** Let $(X,d)$ be a quasi-pseudo metric space and $E \in W^*_{IF}(X)$ or $W^{/}_{IF}(X)$. Then for any $x$, $y \in X$ and $(\alpha,\beta) \in (0,1] \times [0,1)$, we have

$$d\left(x, [E]_{(\alpha,\beta)}\right) \le d(x,y) + d\left(y, [E]_{(\alpha,\beta)}\right) \quad \text{or}$$

$$d\left([E]_{(\alpha,\beta)}, x\right) \le d\left([E]_{(\alpha,\beta)}, y\right) + d(y,x).$$

**Lemma 3.11.** Let $(X, d)$ be a quasi-pseudo metric space and $E$, $G \in W^{*}_{IF}(X)$ or $W^{/}_{IF}(X)$. Then for $x_0 \in [E]_{(\alpha,\beta)}$ and $(\alpha, \beta) \in (0, 1] \times [0, 1)$, we have

$$d\left(x_0, [G]_{(\alpha,\beta)}\right) \leq D_{(\alpha,\beta)}(E, G) \quad \text{or}$$

$$d\left([G]_{(\alpha,\beta)}, x_0\right) \leq D_{(\alpha,\beta)}(G, E).$$

## 4. Fixed point theorems for intuitionistic fuzzy contractive maps

In this section, fixed point results for intuitionistic fuzzy contraction mappings in a left (right) $K$-sequentially complete quasi-pseudo metric space are presented.

**Theorem 4.1.** Let $(X, d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^{*}_{IF}(X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < 1$ such that for each $x$, $y \in \overline{B}_d(x_0, \epsilon)$, we have

$$d_{(\infty,\infty)}(Sx, Sy) \leq q\left(d^{-1} \wedge d\right)(x, y)$$

and

$$d\left(x_0, [Sx_0]^{*}\right) < (1 - q)\,\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $z \in [Sz]^{*}$.

**Proof.** Since $K = [Sx_0]^{*}$ is nonempty $d^{-1}$-compact and $x_0 \in X$, It follows from Lemma 2.7, there exists $x_1 \in [Sx_0]^{*}$ such that

$$\begin{aligned} d(x_0, x_1) &= d\left(x_0, [Sx_0]^{*}\right) \\ &< (1 - q)\,\epsilon. \end{aligned}$$

This also implies that $x_1 \in \overline{B}_d(x_0, \epsilon)$.
By Lemma 2.7, there exists $x_2 \in [Sx_1]^{*}$ for $x_1 \in X$ such that

$$\begin{aligned} d(x_1, x_2) &= d\left(x_1, [Sx_1]^{*}\right) \\ &\leq D_{(M,m)}(Sx_0, Sx_1) \\ &\leq d_{(\infty,\infty)}(Sx_0, Sx_1) \\ &\leq q\left(d^{-1} \wedge d\right)(x_0, x_1) \\ &\leq q d(x_0, x_1) \\ &< q(1 - q)\,\epsilon. \end{aligned}$$

As

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) \\ &< (1 - q)\,\epsilon + q(1 - q)\,\epsilon \\ &< (1 - q)\left(1 + q + q^2 + \cdots\right)\epsilon = \epsilon. \end{aligned}$$

This implies that $x_2 \in \overline{B}_d(x_0, \epsilon)$.
By Lemma 2.7, there exists $x_3 \in [Sx_2]^*$ for $x_2 \in X$ such that

$$\begin{aligned} d(x_2, x_3) &= d\left(x_2, [Sx_2]^*\right) \\ &\leq D_{(M,m)}(Sx_1, Sx_2) \\ &\leq d_{(\infty,\infty)}(Sx_1, Sx_2) \\ &\leq q\left(d^{-1} \wedge d\right)(x_1, x_2) \\ &\leq qd(x_1, x_2) \\ &< q^2 d(x_0, x_1) \\ &< q^2 (1-q)\,\epsilon. \end{aligned}$$

Now,

$$\begin{aligned} d(x_0, x_3) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) \\ &< (1-q)\,\epsilon + q(1-q)\,\epsilon + q^2(1-q)\,\epsilon \\ &< (1-q)\left(1 + q + q^2 + \cdots\right)\epsilon = \epsilon. \end{aligned}$$

This implies that $x_3 \in \overline{B}_d(x_0, \epsilon)$. Continuing in this way, we can obtain a sequence $x_n \in [Sx_{n-1}]^*$ such that

$$d(x_{n-1}, x_n) < q^{n-1} d(x_0, x_1) < q^{n-1}(1-q)\,\epsilon \quad \text{for } n = 2, 3, 4, \cdots.$$

and $x_n \in \overline{B}_d(x_0, \epsilon)$.
Hence, it can be verified that $\{x_n\}$ is a left $K$-Cauchy sequence. For $n < m$, we obtain

$$d(x_n, x_m) \leq \sum_{j=0}^{m-n-1} d(x_{n+j}, x_{n+j+1}) < \sum_{j=n}^{m-1} q^j d(x_0, x_1) < \left(\frac{q^n}{1-q}\right) d(x_0, x_1).$$

Since $0 < q < 1$, it follows that $\{x_n\}$ is a left $K$-Cauchy sequence in the left $K$-sequentially complete quasi-pseudo metric space $(X, d)$. Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $\lim_{n\to\infty} x_n = z$.
Now by Lemma 3.10 and Lemma 3.11, we obtain

$$\begin{aligned} d\left(z, [Sz]^*\right) &\leq d(z, x_n) + d\left(x_n, [Sz]^*\right) \\ &\leq d(z, x_n) + D_{(M,m)}(Sx_{n-1}, Sz) \\ &\leq d(z, x_n) + d_{(\infty,\infty)}(Sx_{n-1}, Sz) \\ &\leq d(z, x_n) + q\left(d^{-1} \wedge d\right)(x_{n-1}, z) \\ &\leq d(z, x_n) + qd^{-1}(x_{n-1}, z) \\ &\leq d(z, x_n) + qd(z, x_{n-1}). \end{aligned}$$

It implies that

$$d(z, x_n) \text{ and } d(z, x_{n-1}) \text{ converges to zero as } n \to \infty.$$

Therefore,

$$d\left(z, [Sz]^*\right) = 0.$$

By Lemma 3.9, we obtain

$$z \in [Sz]^*.$$

This completes the proof.

**Theorem 4.2.** Let $(X,d)$ be a right $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W_{IF}^{/}(X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < 1$ such that for each $x$, $y \in \overline{B}_d(x_0, \epsilon)$, we obtain

$$d_{(\infty,\infty)}(Sx, Sy) \leq q\left(d^{-1} \wedge d\right)(x,y)$$

and

$$d\left([Sx_0]^*, x_0\right) < (1-q)\,\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $z \in [Sz]^*$.

**Proof.** The proof is similar to the proof of Theorem 4.1 and thus omitted.

**Corollary 4.3** [9] Let $(X,d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^*(X)$ be fuzzy mapping. If there exists $0 < q < 1$ such that for each $x$, $y \in \overline{B}_d(x_0, \epsilon)$, we have

$$d_\infty(Sx, Sy) \leq q\left(d^{-1} \wedge d\right)(x,y)$$

and

$$d\left(x_{0,}\, [Sx_0]_1\right) < (1-q)\,\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $\chi_{\{z\}} \subset Sz$.

**Example 4.4.** Let $X = R \cup \{\omega\}$, $\omega \notin R$. Define $d : X \times X \to [0, \infty)$ by $d(x,y) = |x-y|$, for all $x$, $y \in R$ and $d(\omega,\omega) = 0$.

$$d(x,\omega) = \left\{ \begin{array}{cc} -x+\frac{1}{3}, & x < -\frac{1}{3} \\ 2, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ x+2, & x > \frac{1}{3} \end{array} \right\}$$

and

$$d(\omega,x) = \left\{ \begin{array}{cc} -x-\frac{1}{3}, & x < -\frac{1}{3} \\ 0, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ x-\frac{1}{3}, & x > \frac{1}{3} \end{array} \right\}.$$

Then $(X,d)$ is a left $K$-sequentially complete quasi-pseudo metric space. Define intuitionistic fuzzy mapping $S = \langle \mu_S, \upsilon_S \rangle : X \to W_{IF}^*(X)$ as follows:

$$\begin{aligned} Sx &= \{\langle \mu_{Sx}, \upsilon_{Sx} \rangle\}. \\ &= \left\{ \begin{array}{cc} \left\langle \Omega_{\{-x-\frac{1}{3}\}_{0.08}^{0.9}}, \Omega_{\{-x-\frac{1}{3}\}_{0.9}^{0.08}} \right\rangle, & x < -\frac{1}{3} \\ \left\langle \Omega_{\{\frac{-x}{3}\}_{0.08}^{0.9}}, \Omega_{\{\frac{-x}{3}\}_{0.9}^{0.08}} \right\rangle, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ \left\langle \Omega_{\{-\frac{1}{3}+x\}_{0.08}^{0.9}}, \Omega_{\{-\frac{1}{3}+x\}_{0.9}^{0.08}} \right\rangle, & x > \frac{1}{3} \\ \left\langle \Omega_{\{0\}_{0.08}^{0.9}}, \Omega_{\{0\}_{0.9}^{0.08}} \right\rangle, & x = \omega \end{array} \right\}. \end{aligned}$$

Thus,

$$[Sx]^* = [Sx]_{(0.9,0.08)} = \left\{ \begin{array}{cc} t \in X : \Omega_{\{-x-\frac{1}{3}\}_{0.08}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\{-x-\frac{1}{3}\}_{0.9}^{0.08}}(t) = 0.08, & x < -\frac{1}{3} \\ t \in X : \Omega_{\{\frac{-x}{3}\}_{0.08}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\{\frac{-x}{3}\}_{0.9}^{0.08}}(t) = 0.08, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ t \in X : \Omega_{\{-\frac{1}{3}+x\}_{0.08}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\{-\frac{1}{3}+x\}_{0.9}^{0.08}}(t) = 0.08, & x > \frac{1}{3} \\ t \in X : \Omega_{\{0\}_{0.08}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\{0\}_{0.9}^{0.08}}(t) = 0.08, & x = \omega \end{array} \right\}.$$

This implies that

$$[Sx]^* = \left\{ \begin{array}{cc} -x - \frac{1}{3}, & x < -\frac{1}{3} \\ \frac{-x}{3}, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ -\frac{1}{3} + x, & x > \frac{1}{3} \\ 0, & x = \omega \end{array} \right\}.$$

Now

$$\left| \frac{x-y}{3} \right| < \frac{2}{3} |x - y|.$$

Therefore, for $q = \frac{2}{3}$ and for each $x$, $y \in \overline{B}_d \left(0, \frac{1}{3}\right)$ we have

$$d_{(\infty,\infty)} (Sx, Sy) \leq q \left(d^{-1} \wedge d\right) (x, y)$$

and

$$d \left(0, [S0]^*\right) < (1 - q).$$

This implies that $0 \in \overline{B}_d \left(0, \frac{1}{3}\right)$. Thus, all the assumptions of theorem 4.1 are satisfied to obtain $0 \in [S0]^*$.

It is noticed that an intuitionistic fuzzy mapping defined in the above example is not contractive on the whole space $X$. For example, when $x$ and $y$ are not in the interval $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

**Theorem 4.5.** Let $(X, d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^*_{IF} (X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x$, $y \in \overline{B}_d (x_0, \epsilon)$,

$$d_{(\infty,\infty)} (Sx, Sy) \leq q \max \left\{ \left(d^{-1} \wedge d\right) (x, y), d \left(x, [Sx]^*\right) + d \left(y, [Sy]^*\right) \right\}$$

and

$$d \left(x_0, [Sx_0]^*\right) < (1 - q) \epsilon.$$

Thus, there exists $z \in \overline{B}_d (x_0, \epsilon)$ such that $z \in [Sz]^*$.

**Proof.** Since $K = [Sx_0]^*$ is nonempty $d^{-1}$-compact and $x_0 \in X$, It follows from Lemma 2.7, there exists $x_1 \in [Sx_0]^*$ such that

$$\begin{aligned} d (x_0, x_1) &= d \left(x_0, [Sx_0]^*\right) \\ &< (1 - q) \epsilon. \end{aligned} \tag{1}$$

This also implies that $x_1 \in \overline{B}_d (x_0, \epsilon)$.

By Lemma 2.7, there exists $x_2 \in [Sx_1]^*$ for $x_1 \in X$ such that

$$\begin{aligned} d (x_1, x_2) &= d \left(x_1, [Sx_1]^*\right) \\ &\leq D_{(M,m)} (Sx_0, Sx_1) \\ &\leq d_{(\infty,\infty)} (Sx_0, Sx_1) \\ &\leq q \max \left\{ \left(d^{-1} \wedge d\right) (x_0, x_1), d \left(x_0, [Sx_0]^*\right) + d \left(x_1, [Sx_1]^*\right) \right\} \\ &\leq q \max \left\{ d (x_0, x_1), d (x_0, x_1) + d (x_1, x_2) \right\}. \end{aligned} \tag{2}$$

Now, we are considering the following cases:

**Case 1:** If $d (x_0, x_1)$ is taken as maximum in above inequality (2) and also using inequality (1), we have

$$d (x_1, x_2) \leq q d (x_0, x_1) < q (1 - q) \epsilon.$$

**Case 2.** If $d(x_0,x_1)+d(x_1,x_2)$ is taken as maximum in above inequality (2), we get

$$\begin{aligned} d(x_1,x_2) &\leq q\{d(x_0,x_1)+d(x_1,x_2)\} \\ &\leq \left(\frac{q}{1-q}\right) d(x_0,x_1). \end{aligned}$$

As $\left(\frac{q}{1-q}\right) < q$, then by inequality (1), we obtain

$$d(x_1,x_2) \leq qd(x_0,x_1) < q(1-q)\epsilon.$$

It follows from the above two cases that

$$d(x_1,x_2) < q(1-q)\epsilon.$$

Since

$$\begin{aligned} d(x_0,x_2) &\leq d(x_0,x_1)+d(x_1,x_2) \\ &< (1-q)\epsilon + q(1-q)\epsilon \\ &< (1-q)\left(1+q+q^2+\cdots\right)\epsilon = \epsilon. \end{aligned}$$

It implies that $x_2 \in \overline{B}_d(x_0,\epsilon)$.
Following in the same way, we obtain a sequence $x_n \in [Sx_{n-1}]^*$ such that

$$d(x_{n-1},x_n) < q^{n-1}d(x_0,x_1) < q^{n-1}(1-q)\epsilon \quad \text{for } n=2,3,4,\cdots$$

and $x_n \in \overline{B}_d(x_0,\epsilon)$.
Now, it can be shown that $\{x_n\}$ is a left $K$-Cauchy sequence. For $n<m$, we obtain

$$d(x_n,x_m) \leq \sum_{j=0}^{m-n-1} d(x_{n+j},x_{n+j+1}) < \sum_{j=n}^{m-1} q^j d(x_0,x_1) < \left(\frac{q^n}{1-q}\right) d(x_0,x_1).$$

Since $0<q<1$, it follows that $\{x_n\}$ is a left $K$-Cauchy sequence in the left $K$-sequentially complete quasi-pseudo metric space $(X,d)$. Thus, there exists $z \in \overline{B}_d(x_0,\epsilon)$ such that $\lim_{n\to\infty} x_n = z$. Now from Lemma 3.10 and Lemma 3.11, we obtain

$$\begin{aligned} d\left(z,[Sz]^*\right) &\leq d(z,x_n)+d\left(x_n,[Sz]^*\right) \\ &\leq d(z,x_n)+D_{(M,m)}(Sx_{n-1},Sz) \\ &\leq d(z,x_n)+d_{(\infty,\infty)}(Sx_{n-1},Sz) \\ &\leq d(z,x_n)+q\max\left\{\left(d^{-1}\wedge d\right)(x_{n-1},z), d\left(x_{n-1},[Sx_{n-1}]^*\right)+d\left(z,[Sz]^*\right)\right\} \\ &\leq d(z,x_n)+q\max\left\{d^{-1}(x_{n-1},z), d(x_{n-1},x_n)+d\left(z,[Sz]^*\right)\right\} \\ &\leq d(z,x_n)+q\max\left\{d(z,x_{n-1}), d(x_{n-1},x_n)+d\left(z,[Sz]^*\right)\right\}. \end{aligned}$$

As

$$d(z,x_n) \text{ and } d(z,x_{n-1}) \text{ and } d(x_{n-1},x_n) \text{ converges to zero as } n\to\infty.$$

Hence, we obtain

$$d\left(z,[Sz]^*\right) \leq q\max\left\{d\left(z,[Sz]^*\right)\right\},$$

it implies that

$$d\left(z,[Sz]^*\right)=0.$$

Thus by Lemma 3.9, we have

$$z \in [Sz]^*.$$

This completes the proof.

**Corollary 4.6** [9] Let $(X,d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^*(X)$ be a fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x,\, y \in \overline{B}_d(x_0,\epsilon)$,

$$d_\infty(Sx,Sy) \le q \max\left\{\left(d^{-1}\wedge d\right)(x,y),\, d\left(x,[Sx]_1\right)+d\left(y,[Sy]_1\right)\right\}$$

and

$$d\left(x_0,[Sx_0]\right) < (1-q)\,\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0,\epsilon)$ such that $\chi_{\{z\}} \subset Sz$.

**Example 4.7.** Let $(X,d)$ be a left $K$-sequentially complete quasi-pseudo metric space of Example 4.4.
Now $S : X \to W^*_{IF}(X)$ defined as:

$$\begin{aligned} Sx &= \{\langle \mu_{Sx}, \upsilon_{Sx}\rangle\} \\ &= \left\{\begin{array}{ll} \left\langle \Omega_{\left\{-x-\frac{1}{3}\right\}_{0.1}^{0.9}}, \Omega_{\left\{-x-\frac{1}{3}\right\}_{0.9}^{0.1}}\right\rangle, & x < -\frac{1}{3} \\ \left\langle \Omega_{\left\{\frac{-x}{6}\right\}_{0.1}^{0.9}}, \Omega_{\left\{\frac{-x}{6}\right\}_{0.9}^{0.1}}\right\rangle, & -\frac{1}{3} \le x \le \frac{1}{3} \\ \left\langle \Omega_{\left\{-\frac{1}{3}+x\right\}_{0.1}^{0.9}}, \Omega_{\left\{-\frac{1}{3}+x\right\}_{0.9}^{0.1}}\right\rangle, & x > \frac{1}{3} \\ \left\langle \Omega_{\{0\}_{0.1}^{0.9}}, \Omega_{\{0\}_{0.9}^{0.1}}\right\rangle, & x = \omega \end{array}\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} [Sx]^* &= [Sx]_{(0.9,0.1)} \\ &= \left\{\begin{array}{ll} t \in X : \Omega_{\left\{-x-\frac{1}{3}\right\}_{0.1}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\left\{-x-\frac{1}{3}\right\}_{0.9}^{0.1}}(t) = 0.1, & x < -\frac{1}{3} \\ t \in X : \Omega_{\left\{\frac{-x}{6}\right\}_{0.1}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\left\{\frac{-x}{6}\right\}_{0.9}^{0.1}}(t) = 0.1, & -\frac{1}{3} \le x \le \frac{1}{3} \\ t \in X : \Omega_{\left\{-\frac{1}{3}+x\right\}_{0.1}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\left\{-\frac{1}{3}+x\right\}_{0.9}^{0.1}}(t) = 0.1, & x > \frac{1}{3} \\ t \in X : \Omega_{\{0\}_{0.1}^{0.9}}(t) = 0.9 \text{ and } \Omega_{\{0\}_{0.9}^{0.1}}(t) = 0.1, & x = \omega \end{array}\right\}. \end{aligned}$$

It implies that

$$[Sx]^* = \left\{\begin{array}{ll} -x-\frac{1}{3}, & x < -\frac{1}{3} \\ \frac{-x}{6}, & -\frac{1}{3} \le x \le \frac{1}{3} \\ -\frac{1}{3}+x, & x > \frac{1}{3} \\ 0, & x = \omega \end{array}\right\}.$$

Now,

$$\left|\frac{x-y}{6}\right| \le \frac{2}{5}\left[\left|x-\frac{x}{6}\right| + \left|y-\frac{y}{6}\right|\right].$$

Therefore, for $q = \frac{2}{5}$ and for each $x,\, y \in \overline{B}_d\left(0,\frac{1}{3}\right)$ we have

$$d_{(\infty,\infty)}(Sx,Sy) \le q \max\left\{\left(d^{-1}\wedge d\right)(x,y),\, d\left(x,[Sx]^*\right)+d\left(y,[Sy]^*\right)\right\}$$

and

$$d\left(0,[S0]^*\right) < (1-q)\,.$$

It yields $0 \in \overline{B}_d\left(0,\frac{1}{3}\right)$. Thus, all the assumptions of theorem 4.5 are satisfied to obtain $0 \in [S0]^*$.

**Theorem 4.8.** Let $(X,d)$ be a right $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^{/}_{IF}(X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x$, $y \in \overline{B}_d(x_0,\epsilon)$,

$$d_{(\infty,\infty)}(Sx,Sy) \leq q\max\left\{\left(d^{-1}\wedge d\right)(x,y), d\left([Sx]^*, x\right) + d\left([Sy]^*, y\right)\right\}$$

and

$$d\left([Sx_0]^*, x_0\right) < (1-q)\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0,\epsilon)$ such that $z \in [Sz]^*$.

**Proof**. The proof is similar to the proof of Theorem (4.5) and so omitted.

**Theorem 4.9.** Let $(X,d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^*_{IF}(X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x$, $y \in \overline{B}_d(x_0,\epsilon)$,

$$d_{(\infty,\infty)}(Sx,Sy) \leq q\max\left\{\left(d^{-1}\wedge d\right)(x,y), d\left(x,[Sy]^*\right) + d\left(y,[Sx]^*\right)\right\}$$

and

$$d\left(x_0,[Sx_0]^*\right) < (1-q)\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0,\epsilon)$ such that $z \in [Sz]^*$.

**Proof.** Since $K = [Sx_0]^*$ is nonempty $d^{-1}$-compact and $x_0 \in X$, it follows from Lemma 2.7, there exists $x_1 \in [Sx_0]^*$ such that

$$\begin{aligned} d(x_0,x_1) &= d\left(x_0,[Sx_0]^*\right) \\ &< (1-q)\epsilon. \end{aligned} \tag{3}$$

This also implies that $x_1 \in \overline{B}_d(x_0,\epsilon)$.
By Lemma 2.7, there exists $x_2 \in [Sx_1]^*$ for $x_1 \in X$ such that

$$\begin{aligned} d(x_1,x_2) &= d\left(x_1,[Sx_1]^*\right) \\ &\leq D_{(M,m)}(Sx_0,Sx_1) \\ &\leq d_{(\infty,\infty)}(Sx_0,Sx_1) \\ &\leq q\max\left\{\left(d^{-1}\wedge d\right)(x_0,x_1), d\left(x_0,[Sx_1]^*\right) + d\left(x_1,[Sx_0]^*\right)\right\} \\ &\leq q\max\left\{d(x_0,x_1), d(x_0,x_1) + d(x_1,x_2)\right\}. \end{aligned} \tag{4}$$

Now, we are considering the following cases:
**Case 1:** If $d(x_0,x_1)$ is taken as maximum in above inequality (4) and also using inequality (3), we have

$$d(x_1,x_2) \leq qd(x_0,x_1) < q(1-q)\epsilon.$$

**Case 2.** If $d(x_0,x_1) + d(x_1,x_2)$ is taken as maximum in above inequality (4), we get

$$\begin{aligned} d(x_1,x_2) &\leq q\left\{d(x_0,x_1) + d(x_1,x_2)\right\} \\ &\leq \left(\frac{q}{1-q}\right)d(x_0,x_1). \end{aligned}$$

As $\left(\frac{q}{1-q}\right) < q$, then by inequality (3), we obtain

$$d(x_1,x_2) \leq qd(x_0,x_1) < q(1-q)\epsilon.$$

It follows from the above two cases that

$$d(x_1, x_2) < q(1-q)\epsilon.$$

Since

$$\begin{aligned} d(x_0, x_2) &\leq d(x_0, x_1) + d(x_1, x_2) < (1-q)\epsilon + q(1-q)\epsilon \\ &< (1-q)\left(1 + q + q^2 + \cdots\right)\epsilon = \epsilon. \end{aligned}$$

It implies that

$$x_2 \in \overline{B}_d(x_0, \epsilon).$$

Following in the same way, we obtain a sequence $x_n \in [Sx_{n-1}]^*$ such that

$$d(x_{n-1}, x_n) < q^{n-1} d(x_0, x_1) < q^{n-1}(1-q)\epsilon \quad \text{for } n = 2, 3, 4, \cdots$$

and $x_n \in \overline{B}_d(x_0, \epsilon)$.

Now, it can be shown that $\{x_n\}$ is a left $K$-Cauchy sequence, for $n < m$, we obtain

$$d(x_n, x_m) \leq \sum_{j=0}^{m-n-1} d(x_{n+j}, x_{n+j+1}) < \sum_{j=n}^{m-1} q^j d(x_0, x_1) < \left(\frac{q^n}{1-q}\right) d(x_0, x_1).$$

Since $0 < q < 1$, it follows that $\{x_n\}$ is a left $K$-Cauchy sequence in the left $K$-sequentially complete quasi-pseudo metric space $(X, d)$. Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $\lim_{n\to\infty} x_n = z$.

Now from Lemma 3.10 and Lemma 3.11, we obtain

$$\begin{aligned} d\left(z, [Sz]^*\right) &\leq d(z, x_n) + d\left(x_n, [Sz]^*\right) \\ &\leq d(z, x_n) + D_{(M,m)}(Sx_{n-1}, Sz) \\ &\leq d(z, x_n) + d_{(\infty,\infty)}(Sx_{n-1}, Sz) \\ &\leq d(z, x_n) + q \max\left\{\left(d^{-1} \wedge d\right)(x_{n-1}, z), d\left(x_{n-1}, [Sz]^*\right) + d\left(z, [Sx_{n-1}]^*\right)\right\} \\ &\leq d(z, x_n) + q \max\left\{d^{-1}(x_{n-1}, z), d\left(x_{n-1}, [Sz]^*\right) + d(z, x_n)\right\} \\ &\leq d(z, x_n) + q \max\left\{d(z, x_{n-1}), d\left(x_{n-1}, [Sz]^*\right) + d(z, x_n)\right\}. \end{aligned}$$

As

$$d(z, x_n),\ d(z, x_{n-1}) \text{ and } d(z, x_n) \text{ converges to zero as } n \to \infty.$$

Hence, we obtain

$$d\left(z, [Sz]^*\right) \leq q \max\left\{d\left(z, [Sz]^*\right)\right\},$$

this implies that

$$d\left(z, [Sz]^*\right) = 0.$$

Therefore, Lemma 3.9 yields

$$z \in [Sz]^*.$$

Hence this completes the proof.

**Corollary 4.10** [9] Let $(X, d)$ be a left $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^*(X)$ be fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x$, $y \in \overline{B}_d(x_0, \epsilon)$

$$D_1(Sx, Sy) \leq q \max\left\{\left(d^{-1} \wedge d\right)(x, y), d\left(x, [Sy]_1\right) + d\left(y, [Sx]_1\right)\right\}$$

and

$$d(x_0, [Sx_0]) < (1-q)\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $\chi_{\{z\}} \subset Sz$.

**Example 4.11.** Let $(X,d)$ be a left $K$-sequentially complete quasi-pseudo metric space of Example 4.5.
Now $S : X \to W^{*}_{IF}(X)$ defined as

$$\begin{aligned} Sx &= \{\langle \mu_{Sx}, \upsilon_{Sx}\rangle\}. \\ &= \left\{ \begin{array}{ll} \left\langle \Omega_{\{-x-\frac{1}{3}\}^{0.99}_{0.01}}, \Omega_{\{-x-\frac{1}{3}\}^{0.01}_{0.99}} \right\rangle, & x < -\frac{1}{3} \\ \left\langle \Omega_{\{\frac{-x}{6}\}^{0.99}_{0.01}}, \Omega_{\{\frac{-x}{6}\}^{0.01}_{0.99}} \right\rangle, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ \left\langle \Omega_{\{-\frac{1}{3}+x\}^{0.99}_{0.01}}, \Omega_{\{-\frac{1}{3}+x\}^{0.01}_{0.99}} \right\rangle, & x > \frac{1}{3} \\ \left\langle \Omega_{\{0\}^{0.99}_{0.01}}, \Omega_{\{0\}^{0.01}_{0.99}} \right\rangle, & x = \omega \end{array} \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} [Sx]^{*} &= [Sx]_{(0.99,0.01)} \\ &= \left\{ \begin{array}{ll} t \in X : \Omega_{\{-x-\frac{1}{3}\}^{0.99}_{0.01}}(t) = 0.99 \text{ and } \Omega_{\{-x-\frac{1}{3}\}^{0.01}_{0.99}}(t) = 0.01, & x < -\frac{1}{3} \\ t \in X : \Omega_{\{\frac{-x}{5}\}^{0.99}_{0.01}}(t) = 0.99 \text{ and } \Omega_{\{\frac{-x}{5}\}^{0.01}_{0.99}}(t) = 0.01, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ t \in X : \Omega_{\{-\frac{1}{3}+x\}^{0.99}_{0.01}}(t) = 0.99 \text{ and } \Omega_{\{-\frac{1}{3}+x\}^{0.01}_{0.99}}(t) = 0.01, & x > \frac{1}{3} \\ t \in X : \Omega_{\{0\}^{0.99}_{0.01}}(t) = 0.99 \text{ and } \Omega_{\{0\}^{0.01}_{0.99}}(t) = 0.01, & x = \omega \end{array} \right\}. \end{aligned}$$

It yields

$$[Sx]^{*} = \left\{ \begin{array}{ll} -x - \frac{1}{3}, & x < -\frac{1}{3} \\ \frac{-x}{5}, & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ -\frac{1}{3} + x, & x > \frac{1}{3} \\ 0, & x = \omega \end{array} \right\}.$$

Now

$$\left| \frac{x-y}{5} \right| \leq \frac{1}{4} \left[ \left| x - \frac{y}{5} \right| + \left| y - \frac{x}{5} \right| \right].$$

Thus, for $q = \frac{1}{4}$ and for each $x, y \in \overline{B}_d\left(0, \frac{1}{3}\right)$ we have

$$d_{(\infty,\infty)}(Sx, Sy) \leq q \max\left\{ \left(d^{-1} \wedge d\right)(x,y), d\left(x, [Sy]^{*}\right) + d\left(y, [Sx]^{*}\right) \right\}$$

and

$$d\left(0, [S0]^{*}\right) < (1-q).$$

This implies that $0 \in \overline{B}_d\left(0, \frac{1}{3}\right)$. Thus, all the assumptions of theorem 4.9 are satisfied to obtain $0 \in [S0]^{*}$.

**Theorem 4.12** Let $(X,d)$ be a right $K$-sequentially complete quasi-pseudo metric space, $x_0 \in X$, $\epsilon > 0$ and $S : X \to W^{/}_{IF}(X)$ is an intuitionistic fuzzy mapping. If there exists $0 < q < \frac{1}{2}$ such that for each $x, y \in \overline{B}_d(x_0, \epsilon)$,

$$d_{(\infty,\infty)}(Sx, Sy) \leq q \max\left\{ \left(d^{-1} \wedge d\right)(x,y), d\left([Sy]^{*}, x\right) + d\left([Sx]^{*}, y\right) \right\}$$

and

$$d\left([Sx_0]^{*}, x_0\right) < (1-q)\,\epsilon.$$

Thus, there exists $z \in \overline{B}_d(x_0, \epsilon)$ such that $z \in [Sz]^{*}$.

**Proof.** The proof is similar to the proof of Theorem 4.9 and so omitted.

**Remark 4.13** Main results of [14] and [9] can be obtained as corollaries of our theorems and examples 4.4, 4.7 and 4.11 can be considered as evidence of this statement. As a matter of fact, in examples 4.4, 4.7 and 4.11, $[Sx]_{(1,0)}$ is an empty set, therefore the results proved in [9, 10, 14] are not valid in this case.

**Acknowledgements.** The authors are very grateful to the referee for his useful comments and suggestions to improve the quality of this paper.

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