A CHARACTERIZATION OF CAYLEY GRAPHS OF BRANDT SEMIGROUPS

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Abstract. In this paper, first we characterize Cayley graphs of finite Brandt semigroups, and we give a criterion to check whether a finite digraph is a Cayley graph of a finite Brandt semigroup. Also Kelarev and Praeger gave necessary and sufficient conditions for Cayley graphs of semigroups to be vertex-transitive. Then, some authors gave descriptions for all vertex-transitive Cayley graphs of some special classes of semigroups. In this note similar descriptions for all vertex-transitive Cayley graphs of Brandt semigroups are given.

1. Introduction

Let S be a semigroup and C be a subset of S. Recall that the *Cayley graph* $Cay(S, C)$ of S with the *connection set* C is defined as the digraph with vertex set S and arc set $E(Cay(S, C)) = \{(s, cs) : s \in S, c \in C\}$.

Cayley graphs of groups have been extensively studied and some interesting results have been obtained (see for example, [1]). Also, the Cayley graphs of semigroups have been considered by some authors (see for example, [2], [3], [6]-[17]).

It is known that the Cayley graphs of groups are *vertex transitive*; i.e. for every two vertices g_1, g_2 there exists a graph automorphism ϕ such that $\phi(g_1) = g_2$. In [10], Kelarev and Praeger characterized vertex transitive Cayley graphs $Cay(S, C)$ of semigroups S for which all principal left ideals of the subsemigroup generated by the connection set C are finite. Using this result, in [3], [14], [15] and [17], descriptions of vertex transitive Cayley graphs of some special classes of semigroups are given. In this paper we give similar descriptions for all vertex-transitive Cayley graphs of Brandt semigroups which form one of the most popular classes of semigroups. Sabidussi in [18] presented a criterion to check whether a digraph is a Cayley graph of a group. In [16] by presenting a characterization of the Cayley graphs of Clifford semigroups, a similar criterion for these Cayley graphs is obtained. Similarly in [15], a characterization of the Cayley graphs of rectangular groups is obtained. Also in this note, we present a characterization of Cayley graphs of finite Brandt semigroups and we give a criterion to check whether a finite digraph is a Cayley graph of a finite Brandt semigroup.

2. Preliminaries

A *digraph* (*directed graph*) Γ is a non-empty set $V = V(\Gamma)$ of *vertices*, together with a binary relation $E = E(\Gamma)$ on V. We denote the digraph Γ by $\Gamma = (V, E)$. A digraph is *symmetric* if the relation E is symmetric. Symmetric digraphs are more conveniently viewed as (undirected) graphs. The elements $a = (u, v)$ of E are called

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the *arcs* of Γ, u is said the *tail* of a and v is its *head*. An *empty digraph* is one with no arcs. Given a digraph Γ, the *underlying graph* of Γ which is denoted by $\bar{\Gamma}$, is the graph with the same vertices of Γ and $(u, v), (v, u) \in E(\bar{\Gamma})$ if (u, v) or (v, u) belongs to $E(\Gamma)$. A digraph Γ is said to be *connected* if its underlying graph is connected. If for each pair of vertices u, v of Γ, there exists a directed path from u to v, then Γ is said to be *strongly connected*. By a *connected component* of a digraph Γ we mean any component of the underlying graph of Γ. The *in-degree* $d^-_\Gamma(v)$ of a vertex v in a digraph Γ is the number of arcs with head v; the *out-degree* $d^+_\Gamma(v)$ of v is the number of arcs with tail v.

Let $\Gamma = (V, E)$ be a digraph. Suppose that V' is a nonempty subset of V. The subgraph of Γ whose vertex set is V' and whose arc set is the set of those arcs of Γ that have both ends in V' is called the *subgraph of* Γ *induced by* V' and is denoted by $\Gamma[V']$. The *union* of digraphs Γ_1 and Γ_2, written $\Gamma_1 \cup \Gamma_2$, is the digraph with vertex set $V(\Gamma_1) \cup V(\Gamma_2)$ and arc set $E(\Gamma_1) \cup E(\Gamma_2)$. If Γ_1 and Γ_2 are disjoint, we denote their union by $\Gamma_1 + \Gamma_2$. In this paper, the i-th projection map is denoted by π_i.

Let S be a semigroup, and C be a non-empty subset of S. The *Cayley digraph* $Cay(S, C)$ *of* S *relative to* C (which is simply called Cayley graph) is defined as the digraph with vertex set S and arc set $E(C)$ consisting of those ordered pairs (s, t) such that $cs = t$, for some $c \in C$. The set C is called the *connection set* of $Cay(S, C)$ (see [7]). Obviously, if C is an empty set, then $Cay(S, C)$ is an empty digraph.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be digraphs. A *graph (digraph) homomorphism* $\phi : \Gamma_1 \to \Gamma_2$ is a mapping $\phi : V_1 \to V_2$ such that $(u, v) \in E_1$ implies $(\phi(u), \phi(v)) \in E_2$, and is called a *graph (digraph) isomorphism* if it is bijective and both ϕ and ϕ^{-1} are graph homomorphisms. A graph homomorphism $\phi : \Gamma \to \Gamma$ is called an *endomorphism*, and a graph isomorphism $\phi : \Gamma \to \Gamma$ is said to be an *automorphism*. We denote the set of all endomorphisms on a digraph Γ by $End(\Gamma)$, and the set of all automorphisms on Γ by $Aut(\Gamma)$.

For a Cayley graph $Cay(S, C)$, we denote $End(Cay(S, C))$ by $End_C(S)$, and $Aut(Cay(S, C))$ by $Aut_C(S)$. An element $f \in End_C(S)$ is called a *color-preserving endomorphism* if $cx = y$ implies $cf(x) = f(y)$ for every $x, y \in S$ and $c \in C$. The set of all color-preserving endomorphisms of $Cay(S, C)$ is denoted by $ColEnd_C(S)$, and the set of all color-preserving automorphisms of $Cay(S, C)$ by $ColAut_C(S)$. Obviously $ColEnd_C(S) \subseteq End_C(S)$ and $ColAut_C(S) \subseteq Aut_C(S)$.

The following proposition, known as Sabidussi's Theorem, gives a criterion to check whether a digraph is a Cayley graph of a group (see also [16, Theorem 2.5]).

Proposition 2.1. ([18]) *A finite digraph* $\Gamma = (V, E)$ *is a Cayley graph of a group* G *if and only if the automorphism group of* Γ *contains a subgroup* Δ *isomorphic to* G *such that for every two vertices* $u, v \in V$ *there exists a unique* $\sigma \in \Delta$ *such that* $\sigma(u) = v$.

The Cayley graph $Cay(S, C)$ is said to be *automorphism-vertex transitive* or simply $Aut_C(S)$*-vertex-transitive* if, for every two vertices $x, y \in S$, there exists $f \in Aut_C(S)$ such that $f(x) = y$. The notions of $ColAut_C(S)$*-vertex-transitive*, $ColEnd_C(S)$*-vertex-transitive*, and $End_C(S)$*-vertex-transitive* for Cayley graphs are defined similarly.

A *right zero semigroup* (*left zero semigroup*) is a semigroup S satisfying the identity $xy = y$ ($xy = x$). Also, recall that a semigroup is said to be *left simple* (*right simple*) if it has no proper left (right) ideals. A semigroup is called a *left group* (*right group*) if it is left (right) simple and right (left) cancellative. It is known that a semigroup is a right (left) group if and only if it is isomorphic to the direct product of a group and a right (left) zero semigroup (see [5]). The following proposition describes all semigroups S and all subsets C of S, satisfying a certain finiteness condition, such that the Cayley graph $Cay(S,C)$ is $ColAut_C(S)$-vertex-transitive.

Proposition 2.2. ([10, Theorem 2.1]) *Let S be a semigroup, and C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal left ideals of $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S,C)$ is $ColAut_C(S)$-vertex-transitive if and only if the following conditions hold:*

(i) *$cS = S$, for all $c \in C$;*
(ii) *$\langle C\rangle$ is isomorphic to a right group;*
(iii) *$|\langle C\rangle s|$ is independent of the choice of $s \in S$.*

A semigroup is *completely simple* if it has no proper ideals and has an idempotent element which is minimal with respect to the partial order on idempotents $e \leq f \Leftrightarrow e = ef = fe$.

Proposition 2.3. ([10, Theorem 2.2]) *Let S be a semigroup, and C be a subset of S such that all principal left ideals of the subsemigroup $\langle C\rangle$ are finite. Then, the Cayley graph $Cay(S,C)$ is $Aut_C(S)$-vertex-transitive if and only if the following conditions hold:*

(i) *$CS = S$;*
(ii) *$\langle C\rangle$ is a completely simple semigroup;*
(iii) *the Cayley graph $Cay(\langle C\rangle, C)$ is $Aut_C(\langle C\rangle)$-vertex-transitive;*
(iv) *$|\langle C\rangle s|$ is independent of the choice of $s \in S$.*

Let G be a group and I_λ be a set of cardinality $\lambda > 0$. Now we define a semigroup operation on $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ as follows:

$$(i,g,j)(l,h,k) = \begin{cases} (i,gh,k), & if \quad j = l, \\ 0, & if \quad j \neq l; \end{cases}$$

and $(i,g,j)0 = 0(i,g,j) = 00 = 0$, for all $i,j,l,k \in I_\lambda$ and $g,h \in G$. Then the semigroup S is called a *Brandt semigroup* and is denoted by $B(G,\lambda)$.

Lemma 2.4. ([10, Lemma 6.1]) *Let S be a semigroup, and C be a subset of S.*

(i) *If $Cay(S,C)$ is $End_C(S)$-vertex-transitive, then $CS = S$.*
(ii) *If $Cay(S,C)$ is $ColEnd_C(S)$-vertex-transitive, then $cS = S$ for each $c \in C$.*

Lemma 2.5. ([10, Lemma 5.2, Corollary 5.3]) *Let S be a semigroup with a subset C such that $\langle C\rangle$ is completely simple, and $CS = S$. Then, every connected component of the Cayley graph $Cay(S,C)$ is strongly connected, and for every $v \in S$, the connected component containing v is equal to $\langle C\rangle v$. Also, if $\langle C\rangle$ is isomorphic to a right group, then the right $\langle C\rangle$-cosets are the connected components of $Cay(S,C)$.*

For more information on graphs, we refer to [4], and for semigroups see [5].

3. Characterization of Cayley graphs of Brandt semigroups

In this section, we suppose that every digraph is finite. To provide a criterion for Cayley graphs of finite Brandt semigroups, we present a characterization of Cayley graphs of finite Brandt semigroups. Let S be a finite Brandt semigroup and $C \subseteq S$.

Then it is obvious that if $0 \in C$, then each vertex of $Cay(S, C)$ is joined to 0. Also if $C = \emptyset$, then $Cay(S, C)$ is an empty digraph. Therefore in the sequel of this section we suppose that C is a non-empty set and $0 \notin C$.

Theorem 3.1. *A finite digraph D is a Cayley graph of a finite Brandt semigroup if and only if D consists of a vertex v_0, with a loop on it, and λ mutually disjoint subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that $v_0 \notin V(D_\alpha)$, for each α. Also the arc set of D satisfies the following conditions: there exists no arc between $V(D_\alpha)$ and $V(D_{\alpha'})$, for $1 \le \alpha, \alpha' \le \lambda$ and $\alpha \neq \alpha'$, and every D_α is isomorphic to a digraph denoting by $\Gamma = (V, E)$ such that*

1. *$V = \bigcup_{i=1}^{\lambda} V_i$, where V_i's are pairwise disjoint and have the same cardinality,*
2. *there exists a group G such that for every $1 \le i \le \lambda$, if $\Gamma_i = \Gamma[V_i]$, then $\Gamma_i \cong Cay(G, C_i)$, for some $C_i \subseteq G$,*
3. *there exists a family of graph isomorphisms $\{f_i\}_{i=1}^{\lambda}$, $f_i : Cay(G, C_i) \to \Gamma_i$, for $1 \le i \le \lambda$ such that if, for $x \in G$ and e the identity of G, $f_i(e)$ is joined to $f_j(x)$, then $f_i(g)$ is joined to $f_j(xg)$ for every $g \in G$. Also there is not any other arc from Γ_i to Γ_j. Let C_{ij} be the elements of G, say x, such that $f_i(e)$ is joined to $f_j(x)$,*

moreover let $\eta_\alpha : \Gamma \to D_\alpha$, where $1 \le \alpha \le \lambda$, be the isomorphism between Γ and D_α. For every $1 \le \alpha \le \lambda$, if $C_i \neq \emptyset$, for some $1 \le i \le \lambda$ or $C_{ij} \neq \emptyset$, for some $1 \le i, j \le \lambda$ and $i \neq j$, then all vertices in $\eta_\alpha(V \setminus V_i)$ are joined to v_0 in D.

Proof. ($\Rightarrow$) Let $D = Cay(S, C)$, where $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ is a finite Brandt semigroup and $C \subseteq S$. By the definition of Brandt semigroup we know that I_λ is a set of cardinality λ, G is a group, and 0 is the zero of S. Without loss of generality we can assume that $I_\lambda = \{1, 2, \dots, \lambda\}$. Let $v_0 = 0$. Also since for every $c \in C$, $c0 = 0$, there exists a loop on 0. We know that $S = (\bigcup_{1 \le i,j \le \lambda} \{(i, g, j) | g \in G\}) \cup \{0\}$. For every $1 \le i, j \le \lambda$, let $D_{ij} = D[\{(i, g, j) | g \in G\}]$ and $A_{ij} = \{(i, g, j) \in C | g \in G\}$. We claim that $D_{ij} \cong Cay(G, C_i)$, where $C_i = \{g \in G | (i, g, i) \in C\}$. To prove it, we define $\psi_{ij} : D_{ij} \to Cay(G, C_i)$, by $(i, g, j) \mapsto g$. Obviously ψ_{ij} is one-to-one and onto. So it is enough to check that ψ_{ij} preserves adjacency and non-adjacency. To prove ψ_{ij} preserves adjacency, let $v_1 = (i, g_1, j)$, $v_2 = (i, g_2, j) \in V(D_{ij})$ and $(v_1, v_2) \in E(D_{ij})$. So there exists $c \in C$ such that $v_2 = cv_1$. So $(i, g_2, j) = c(i, g_1, j)$. Thus $g_2 = \pi_2(c)g_1$, $\pi_1(c) = i$, and also since $(i, g_2, j) \neq 0$, $\pi_3(c) = i$. Hence $\pi_2(c) \in C_i$. Therefore $(g_1, g_2) \in E(Cay(G, C_i))$. So $(\psi_{ij}(v_1), \psi_{ij}(v_2)) \in E(Cay(G, C_i))$. To prove ψ_{ij} preserves non-adjacency, let $(\psi_{ij}(v_1), \psi_{ij}(v_2)) = (g_1, g_2) \in E(Cay(G, C_i))$. Then, there exists $h \in C_i$, such that $g_2 = hg_1$. Since $h \in C_i$, $(i, h, i) \in A_{ii}$. Also since $v_1, v_2 \in V(D_{ij})$ and $(i, g_2, j) = (i, h, i)(i, g_1, j)$, we conclude that $((i, g_1, j), (i, g_2, j)) = (v_1, v_2) \in E(D_{ij})$. Therefore

$$D_{ij} \cong Cay(G, C_i), \tag{3.1}$$

for each $1 \le i, j \le \lambda$.

Now we show that there exists no arc between $V(D_{ij})$ and $V(D_{i'j'})$, for $1 \le i, i' \le \lambda$, $1 \le j, j' \le \lambda$ and $j \neq j'$. On the contrary if there exists some arcs between $V(D_{ij})$ and $V(D_{i'j'})$ in D, there exist $(i, g, j) \in V(D_{ij})$ and $(i', g', j') \in V(D_{i'j'})$ such that $((i, g, j), (i', g', j')) \in E(D)$. Since $D = Cay(S, C)$, there exists $(l, h, k) \in C$ such that $(i', g', j') = (l, h, k)(i, g, j)$. Since $(i', g', j') \neq 0$, we get that $k = i$. Thus $(i', g', j') = (l, hg, j)$. Hence $j = j'$, which is a contradiction. Now we prove that D has λ subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that D_α's are pairwise disjoint and isomorphic to each other. Let $D_\alpha = D[\bigcup_{i=1}^{\lambda} V(D_{i\alpha})]$, for $1 \le \alpha \le \lambda$. Then the D_α's are pairwise disjoint and there exists no arc between D_α and $D_{\alpha'}$ if $\alpha \ne \alpha'$. Obviously, $V(D) = \bigcup_{\alpha=1}^{\lambda} V(D_\alpha) \cup \{0\}$. Now we prove that D_α's are isomorphic to each other. To prove it, for every arbitrary $1 \le \alpha, \alpha' \le \lambda$, we define $\psi : D_\alpha \to D_{\alpha'}$, by $\psi(i, g, \alpha) = (i, g, \alpha')$, for every $(i, g, \alpha) \in V(D_\alpha)$. Since $(i_1, g_1, \alpha) = (i_2, g_2, \alpha)$ if and only if $(i_1, g_1, \alpha') = (i_2, g_2, \alpha')$, we get that ψ is well-defined and one-to-one. Also it is obvious that ψ is onto. So it is enough to prove that ψ preserves adjacency and non-adjacency. To prove ψ preserves adjacency, let $(u, v) \in E(D_\alpha)$, $u = (i_1, g_1, \alpha)$ and $v = (i_2, g_2, \alpha)$. Hence there exists $c = (l, h, k) \in C$ such that $(i_2, g_2, \alpha) = (l, h, k)(i_1, g_1, \alpha)$. So $l = i_2$, $g_2 = hg_1$ and $k = i_1$. Thus $c = (i_2, h, i_1)$ and $(i_2, g_2, \alpha') = (i_2, h, i_1)(i_1, g_1, \alpha')$. Therefore $(\psi(u), \psi(v)) \in E(D_{\alpha'})$. Similarly if $(\psi(u), \psi(v)) = ((i_1, g_1, \alpha'), (i_2, g_2, \alpha')) \in E(D_{\alpha'})$, then $((i_1, g_1, \alpha), (i_2, g_2, \alpha)) \in E(D_\alpha)$, which proves that ψ preserves non-adjacency. Without loss of generality we can assume that $\Gamma = (V, E)$ is equal to D_1. Let $\eta_\alpha : D_1 \to D_\alpha$ by

$$\eta_\alpha(i, g, 1) = (i, g, \alpha), \tag{3.2}$$

where $(i, g, \alpha) \in V(D_\alpha)$ and $1 \le \alpha \le \lambda$.

Now we prove that conditions (1) and (2) are satisfied. Let $V_i = V(D_{i1})$ and $\Gamma_i = \Gamma[V_i]$, $1 \le i \le \lambda$. Therefore $\Gamma_i = D_{i1}$ and, by (3.1), we have $\Gamma_i = D_{i1} \cong Cay(G, C_i)$. Also we note that $V(D_1) = \bigcup_{i=1}^{\lambda} V(D_{i1})$ and so $V = \bigcup_{i=1}^{\lambda} V_i$. Since by (3.1), $D_{i1} \cong Cay(G, C_i)$, we get that $|V(D_{i1})| = |G|$. So V_i's have the same cardinality. Hence conditions (1) and (2) are satisfied.

To prove condition (3), for every $1 \le i \le \lambda$, we define $f_i : Cay(G, C_i) \to \Gamma_i$, for $1 \le i \le \lambda$, by $f_i(g) = (i, g, 1)$. It is easy to check that the f_i's are well-defined, one-to-one and onto. So it is enough to prove that f_i preserves adjacency and non-adjacency. To prove that f_i preserves adjacency for every arc $(g_1, g_2) \in E(Cay(G, C_i))$, we know that there exists $d \in C_i$ such that $g_2 = dg_1$. So $(i, d, i) \in A_{ii}$ and $f_i(g_2) = (i, g_2, 1) = (i, d, i)(i, g_1, 1) = (i, d, i)f_i(g_1)$. Hence $(f_i(g_1), f_i(g_2)) \in E(\Gamma_i)$. Therefore f_i preserves adjacency. To prove f_i preserves non-adjacency, let $(f_i(g_1), f_i(g_2)) \in E(\Gamma_i)$. There exists $c \in C$ such that $f_i(g_2) = cf_i(g_1)$, since $D = Cay(S, C)$. Let $c = (l, d, k)$. Similarly to the above, we conclude that $\pi_1(c) = i, \pi_3(c) = i$. Thus, $c = (i, d, i)$, $d \in C_i$ and $g_2 = dg_1$. Therefore $(g_1, g_2) \in E(Cay(G, C_i))$. Hence f_i preserves adjacency and non-adjacency. Therefore f_i is a graph isomorphism. Since $(i, e, 1)$ is joined to $(j, x, 1)$, where $x \in C_{ij}$, it follows that $(j, x, i) \in C$ and so $\{j\} \times C_{ij} \times \{i\} \subseteq C$. Thus, for every $g \in G$, $f_i(g)$ is joined to each vertex of $\{(j, d, i)(i, g, 1) | d \in C_{ij}\} = f_j(C_{ij}g)$. Now we prove that all arcs from Γ_i to Γ_j are arcs mentioned above. Let there exists an arc from a vertex $f_i(g) \in V_i = V(\Gamma_i)$, for some $g \in G$, to a vertex $f_j(g') \in V_j = V(\Gamma_j)$, where $g' \in G$. Since $D = Cay(S, C)$, there exists $(l, h, k) \in C$ such that $(j, g', 1) = (l, h, k)(i, g, 1)$. So $l = j$, $k = i$ and $g' = hg$. Since $(j, h, i)(i, e, 1) = (j, h, 1)$, it follows that $f_i(e)$ is joined to $f_j(h)$. Thus $h \in C_{ij}$, and so $g' \in C_{ij}g$. Therefore $f_j(g') \in f_j(C_{ij}g)$ and condition (3) is satisfied.

Now we prove that if $C_i \ne \emptyset$ or $C_{ij} \ne \emptyset$, then each vertex of $\eta_\alpha(V \setminus V_i)$ are joined to v_0 in D, where $1 \le \alpha \le \lambda$. If $C_i \ne \emptyset$, then there exists $d \in C_i$ such that $(i, d, i) \in C$. Thus, for every vertex $(i', g, 1) \in V \setminus V_i$, we have $i \ne i'$ and since $(i, d, i)(i', g, 1) = 0$, we conclude that $(i', g, 1)$ is joined to 0. Also since, for every $1 \le \alpha \le \lambda$, $(i, d, i)(i', g, \alpha) = 0$, we get that $\eta_\alpha(i', g, 1) = (i', g, \alpha)$ is joined to 0 in D.

If $C_{ij} \neq \emptyset$, then as we mentioned above $(j,h,i) \in C$, for $h \in C_{ij}$. For every vertex $(i',g,1) \in V \setminus V_i$, we have $i \neq i'$ and since $(j,d,i)(i',g,1) = 0$, we conclude that $(i',g,1)$ is joined to 0. Also since, for every $1 \leq \alpha \leq \lambda$, $(j,h,i)(i',g,\alpha) = 0$, we get that $\eta_\alpha(i',g,1)$ is joined to 0 in D.

($\Leftarrow$) Take a digraph $\Gamma = (V,E)$ with properties (1)-(3) and take a digraph D with the given properties. Then D consists of a vertex v_0 with a loop on it and λ mutually disjoint subgraphs $\{D_\alpha\}_{\alpha=1}^{\lambda}$ such that each D_α is isomorphic to $\Gamma = (V,E)$. We define a Brandt semigroup S as $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$, where G is the group given in part (2) and $I_\lambda = \{1,2,\ldots,\lambda\}$. Let

$$C = \Big(\bigcup_{i=1}^{\lambda} \{i\} \times C_i \times \{i\}\Big) \cup \Big(\bigcup_{\substack{1\leq i,j\leq\lambda \\ i\neq j}} \{j\} \times C_{ij} \times \{i\}\Big), \tag{3.3}$$

where C_i and C_{ij} are given in parts (2) and (3), respectively. Let $D' = Cay(S,C)$ and $D'_\alpha = D'[\{(i,g,\alpha) | g \in G, 1 \leq i \leq \lambda\}]$, for $1 \leq \alpha \leq \lambda$. Using the ($\Rightarrow$) part of the theorem, we conclude that $D' = Cay(S,C)$ consists of the vertex 0 with a loop on it and λ pairwise disjoint subgraphs D'_α which are isomorphic to a graph satisfying conditions (1)-(3) and there exists no arc between these subgraphs. We claim that D is isomorphic to $D' = Cay(S,C)$.

To prove D is isomorphic to D', first we prove that $\Gamma \cong D'_1$. Using (2), we know that $\Gamma_i = \Gamma[V_i] \cong Cay(G,C_i)$, for $1 \leq i \leq \lambda$, and by (3) there exists a graph isomorphism $f_i : Cay(G,C_i) \to \Gamma_i$. For every $v \in V = V(\Gamma)$, using (1) we get that there exists a unique $1 \leq i \leq \lambda$ such that $v \in V_i = V(\Gamma_i)$. To prove $\Gamma \cong D'_1$, we define $\psi : \Gamma \to D'_1$, by $\psi(v) = (i, f_i^{-1}(v), 1)$, where $v \in V_i = V(\Gamma_i)$. Now we prove that ψ is a graph isomorphism. Since f_i^{-1} is a graph isomorphism, we get that ψ is one-to-one and onto. So it is enough to show that ψ preserves adjacency and non-adjacency. Let $(u,v) \in E(\Gamma)$. There exists $1 \leq i,j \leq \lambda$ such that $u \in V_i = V(\Gamma_i)$ and $v \in V_j = V(\Gamma_j)$. Now we consider two cases. If $i = j$, then using (2) we get that there exists $d \in C_i$ such that $f_i^{-1}(v) = df_i^{-1}(u)$. So by the definition of C in (3.3), we conclude that $(i,d,i) \in C$. Now since $(i, f_i^{-1}(v), 1) = (i,d,i)(i, f_i^{-1}(u), 1)$, we conclude that $(\psi(u), \psi(v)) \in E(D'_1)$. If $i \neq j$, then there exist $g, g' \in G$ such that $f_i(g) = u$, $f_j(g') = v$. Using (3), we get that $f_i(g)$ is joined in Γ_j only to $f_j(C_{ij}g)$. Hence $g'g^{-1} \in C_{ij}$. By the definition of C in (3.3), we get that $(j, g'g^{-1}, i) \in C$. Hence $(j, f_j^{-1}(v), 1) = (j, g'g^{-1}, i)(i,g,1) = (j, g'g^{-1}, i)(i, f_i^{-1}(u), 1)$. Thus, $(\psi(u), \psi(v)) \in E(D'_1)$. Therefore ψ preserves adjacency. To prove ψ preserves non-adjacency, let $(\psi(u), \psi(v)) \in E(D'_1)$. Also let $\psi(u) = (i,g,1)$ and $\psi(v) = (i',g',1)$. Therefore $g = f_i^{-1}(u)$ and $g' = f_{i'}^{-1}(v)$. By definition of Cayley graph, there exists $(i_c, g_c, j_c) \in C$ such that $(i',g',1) = (i_c,g_c,j_c)(i,g,1)$. So $i_c = i'$, $j_c = i$, and $g' = g_c g$. If $i = i'$, then by the definition of C in (3.3), we get that $g_c \in C_i$. Since $i = i'$, we have $g = f_i^{-1}(u)$ and $g' = f_i^{-1}(v)$. Since f_i is a graph isomorphism and $(g,g') \in E(Cay(G,C_i))$, $(f_i(g), f_i(g')) = (u,v) \in E(\Gamma_i) \subseteq E(\Gamma)$. If $i \neq i'$, then $(i', g_c, i) \in C$ and so $g_c \in C_{ii'}$. Using (3), each vertex $f_i(g'')$, $g'' \in G$, is joined to $f_{i'}(g_c g'')$. Thus $f_i(g)$ is joined to $f_{i'}(g_c g) = f_{i'}(g')$. Hence u is joined to v. So $(u,v) \in E(\Gamma)$. Therefore ψ preserves non-adjacency. Hence $\Gamma \cong D'_1$.

Now we prove that $D \cong D' = Cay(S,C)$. By assumption, $D' = Cay(S,C)$ is a Cayley graph of a Brandt semigroup. Therefore as we mentioned in the necessary part of the proof, for each $1 \leq \alpha \leq \lambda$, there exists a graph isomorphism $\eta'_\alpha : D'_1 \to D'_\alpha$, where $\eta'_\alpha(i,g,1) = (i,g,\alpha)$ (see 3.2). To prove $D \cong D' = Cay(S,C)$, we define $\mu : D \to D'$ by $\mu(v_0) = 0$ and $\mu(v) = \eta'_\alpha \psi \eta_\alpha^{-1}(v)$ if $v \in V(D_\alpha)$, for some $1 \leq \alpha \leq \lambda$. It is easy to check that μ is bijection since η'_α, ψ and η_α^{-1} are bijection and v_0 does not belong to any $V(D_\alpha)$, for $1 \leq \alpha \leq \lambda$. Hence to prove μ is a graph isomorphism, it is enough to prove that μ preserves adjacency and non-adjacency. For this purpose let $v_1, v_2 \in V(D)$ and $(v_1, v_2) \in E(D)$. Since in the graph D there does not exist any arc from v_0 to any other vertex of D, we have three following cases.

Case (1) Let $v_1 = v_2 = v_0$. Since we know that there is a loop on v_0 in D, and there is a loop on $\mu(v_0) = 0$ in D', we conclude that $(\mu(v_1), \mu(v_2)) = (0, 0) \in E(D')$.

Case (2) Let $v_1 \neq v_0$ and $v_2 \neq v_0$. Since there does not exist any arc between D_α and $D_{\alpha'}$, for $1 \leq \alpha, \alpha' \leq \lambda$ and $\alpha \neq \alpha'$, we conclude that there exists some $1 \leq \alpha \leq \lambda$ such that $v_1, v_2 \in V(D_\alpha)$. Since η'_α, ψ and η_α^{-1} are graph isomorphisms, we get that $(\mu(v_1), \mu(v_2)) = (\eta'_\alpha \psi \eta_\alpha^{-1}(v_1), \eta'_\alpha \psi \eta_\alpha^{-1}(v_2)) \in E(D'_\alpha) \subseteq E(D')$.

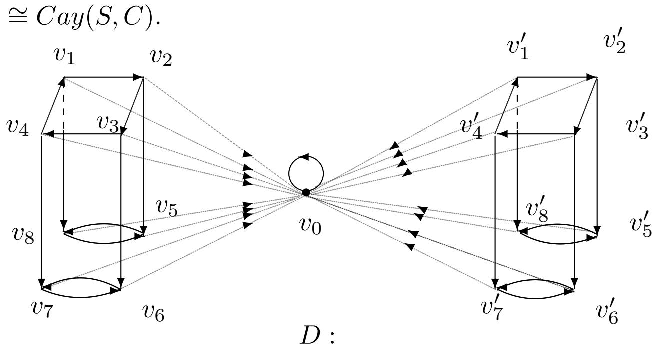
Case (3) Let $v_1 \neq v_0$ and $v_2 = v_0$. Then $v_1 \in V(D_\alpha)$, for some $1 \leq \alpha \leq \lambda$. By the hypothesis, v_1 is joined to v_0. Therefore $C_i \neq \emptyset$, for some $1 \leq i \leq \lambda$, or $C_{ij} \neq \emptyset$, for some $1 \leq i, j \leq \lambda$, $i \neq j$ and $\eta_\alpha^{-1}(v_1) \in V \setminus V_i$. Let $\eta_\alpha^{-1}(v_1) \in V_{i'} = V(\Gamma_{i'})$, for some $1 \leq i' \leq \lambda$, where $i' \neq i$. By the definition of ψ, we know that $\psi(\eta_\alpha^{-1}(v_1)) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), 1)$. Therefore $\mu(v_1) = \eta'_\alpha(\psi(\eta_\alpha^{-1}(v_1))) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha) \in V(D'_\alpha)$. If $C_i \neq \emptyset$, then there exists $d \in C_i$ and so $(i, d, i) \in C$. Then $(i, d, i)(i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha) = 0$ shows that $\mu(v_1)$ is joined to $\mu(v_0) = 0$. Similarly if $C_{ij} \neq \emptyset$ and $d \in C_{ij}$, then by the definition of C, $(j, d, i) \in C$. Similarly to the above, we conclude that $\mu(v_1) = \eta'_\alpha \psi \eta_\alpha^{-1}(v_1) = (i', f_{i'}^{-1}(\eta_\alpha^{-1}(v_1)), \alpha)$ is joined to $\mu(v_2) = 0$ in D'.

Thus $\mu(v_1)$ is joined to $\mu(v_2)$ in D'. Therefore μ preserves adjacency. Similarly we can conclude that μ preserves non-adjacency. Hence μ is a graph isomorphism. Thus $D \cong D' = Cay(S, C)$. Therefore D is isomorphic to a Cayley graph of a finite Brandt semigroup. □

In the next example we show that the following digraph is not a Cayley graph of a Brandt semigroup, because condition (3) of the above theorem is not satisfied.

Example 3.2. Let D be the following digraph. By Theorem 3.1, we show that D is not a Cayley graph of a Brandt semigroup. Throughout of the proof, we use the notations of Theorem 3.1. On the contrary suppose that D is a Cayley graph of a Brandt semigroup. Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup and $C \subseteq S$

such that $D \cong Cay(S,C)$.



$D:$

Since $|S| = \lambda^2|G| + 1 = 17$, we get that $\lambda \in \{1, 2, 4\}$. In any case $v_0 = 0$. If $\lambda = 1$, then $S \cong G^0$. So, by conditions (1) and (2) of Theorem 3.1 we conclude that $D[V \setminus \{0\}]$ must be isomorphic to a Cayley graph of a group. By Proposition 2.1, we know that every Cayley graph of a group is vertex-transitive. Also we know that in a finite vertex-transitive graph the in-degree is the same for each vertex, and is equal to its out-degree. Now we note that D is not vertex-transitive because $d^-_{D[V\setminus\{0\}]}(v_3) = 1$ and $d^-_{D[V\setminus\{0\}]}(v_6) = 2$. Since $D[V \setminus \{0\}]$ is not vertex-transitive, we get that $D[V \setminus \{0\}]$ can not be isomorphic to a Cayley graph of a group, which is a contradiction. Hence $\lambda > 1$. Then there exist λ mutually disjoint subgraphs, $\{D_i\}_{i=1}^{\lambda}$ such that there exists no arc between them. Let $v_1 \in V(D_1)$. Since there does not exist any arc between D_i's, we get that $v_2, v_4, v_8 \in V(D_1)$. Since $v_2, v_4, v_8 \in V(D_1)$, similarly to the above we conclude that $v_3, v_5, v_6, v_7 \in V(D_1)$, too. Similarly we conclude that there exists D_i, where $2 \le i \le \lambda$, such that $v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8$ belong to $V(D_i)$. This implies that $\lambda = 2$. Without loss of generality, we can assume that $I_\lambda = \{1, 2\}$. We choose $D_1 = D[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}]$ and $D_2 = D[\{v'_1, v'_2, v'_3, v'_4, v'_5, v'_6, v'_7, v'_8\}]$. It is obvious that D_1 and D_2 are isomorphic to each other and up to isomorphism the choices of D_1 and D_2 are unique. Without loss of generality, we can assume that $\Gamma = D_1$. By condition (1), we get that $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} = \bigcup_{i=1}^{2} V_i$ such that $|V_1| = |V_2| = 4$ and $\Gamma[V_i]$ is isomorphic to a Cayley graph of a group, for $i = 1, 2$. Without loss of generality let $v_1 \in V_1$. Now we consider the following four cases.

Case (1) Let $v_2 \in V_1$ and $v_8 \in V_1$. We claim that this case can not occur. Since $v_2, v_8 \in V_1$, $d^+_{\Gamma_1}(v_1) = 2$. But $d^-_{\Gamma_1}(v_1) \le d^-_{\Gamma}(v_1) = 1$, which is a contradiction because Γ_1 is vertex-transitive.

Case (2) Let $v_2 \notin V_1$ and $v_8 \in V_1$. Since Γ_1 is vertex-transitive, we get that $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 1$. So $v_4 \in V_1$ and $d^-_{\Gamma_1}(v_4) = d^+_{\Gamma_1}(v_4) = 1$. Therefore $v_3 \in V_1$ and $d^-_{\Gamma_1}(v_3) = d^+_{\Gamma_1}(v_3) = 1$ which implies that $v_2 \in V_1$, and this is a contradiction.

Case (3) Let $v_2 \in V_1$ and $v_8 \notin V_1$. Then $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 1$ and so $v_4 \in V_1$. Now similar to the above cases, we conclude that $v_3 \in V_1$. Therefore

$$V_1 = \{v_1, v_2, v_3, v_4\}, \qquad V_2 = \{v_5, v_6, v_7, v_8\}.$$

So $\Gamma[V_1] \cong Cay(\mathbb{Z}_4, \{c\})$, where $c = \bar{1}$ or $c = \bar{3}$, and $\Gamma[V_2] \cong Cay(\mathbb{Z}_4, \{\bar{2}\})$ (we note that since $\Gamma[V_1]$ is a square, then c must be an element of order 4 and so G can be only $\mathbb{Z}_4$). Hence $S = (I_2 \times \mathbb{Z}_4 \times I_2) \cup \{0\}$. Let $f_1 : Cay(\mathbb{Z}_4, \{c\}) \to \Gamma[V_1]$, where $c \in \{\bar{1}, \bar{3}\}$ and $f_2 : Cay(\mathbb{Z}_4, \{\bar{2}\}) \to \Gamma[V_2]$. Now we claim that condition (3) of Theorem 3.1 can not be satisfied. To prove it we note that $v_1 = f_1(g_1)$ is joined to $v_2 = f_1(g_2) \in V_1$ and $v_8 = f_2(g') \in V_2$, for some $g_1, g_2, g' \in \mathbb{Z}_4$. Since f_1 is a graph isomorphism, $(g_1, g_2) \in E(Cay(\mathbb{Z}_4, \{c\}))$ and so $g_2 = g_1 + c$. We note that $v_1 = f_1(g_1)$ is joined to $v_8 = f_2(g')$. Hence $f_1(e)$ is joined to $f_2(g' - g_1)$. By condition (3) of Theorem 3.1, since $v_2 = f_1(g_2) = f_1(g_1 + c)$ is joined to v_5, we get that $v_5 = f_2(g' - g_1 + g_1 + c)$. Therefore $v_5 = f_2(g' + c)$. Since f_2 is a graph isomorphism and $(v_5, v_8) \in E(\Gamma_2)$, we get that $(f_2^{-1}(v_5), f_2^{-1}(v_8)) \in E(Cay(\mathbb{Z}_4, \{\bar{2}\}))$ and so $f_2^{-1}(v_8) = f_2^{-1}(v_5) + \bar{2}$. Thus $g' = g' + c + \bar{2}$. Hence $c = \bar{2}$, which is a contradiction because $c \in \{\bar{1}, \bar{3}\}$. Therefore in this case the graph D can not be a Cayley graph of a Brandt semigroup.

Case (4) Let $v_2 \notin V_1$ and $v_8 \notin V_1$. Then $d^-_{\Gamma_1}(v_1) = d^+_{\Gamma_1}(v_1) = 0$. So $v_4 \in V_2$. Also $d^-_{\Gamma_2}(v_2) = d^+_{\Gamma_2}(v_2) = 0$ implies that $v_3, v_5 \in V_1$. Finally $d^-_{\Gamma_2}(v_4) = 0$ and so $v_7 \in V_1$. Therefore

$$V_1 = \{v_1, v_3, v_5, v_7\}, \qquad V_2 = \{v_2, v_4, v_6, v_8\}.$$

Also we note that by condition (3) of Theorem 3.1, each vertex of Γ_1 is joined to exactly $|C_{12}|$ vertices of Γ_2. Now v_1 is joined to v_2 and v_8 in $V_2 = V(\Gamma_2)$ but v_7 is joined only to v_6 in $V_2 = V(\Gamma_2)$, which is a contradiction. Therefore in this case the graph D can not be a Cayley graph of a Brandt semigroup.

So D is not a Cayley graph of a finite Brandt semigroup.

4. Vertex-transitive Cayley graphs of Brandt semigroups

In this section, we describe Cayley graphs of Brandt semigroups which are vertex transitive. Throughout this section, we assume that S is a Brandt semigroup and C is a nonempty subset of S.

Theorem 4.1. *Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup. Let C be a subset of S which generates a subsemigroup $\langle C\rangle$ such that all principal left ideals of $\langle C\rangle$ are finite. Then the following statements are equivalent:*

(i) *$Cay(S, C)$ is $ColAut_C(S)$-vertex-transitive;*
(ii) *$Cay(S, C)$ is $Aut_C(S)$-vertex-transitive;*
(iii) *$Cay(S, C)$ is $ColEnd_C(S)$-vertex-transitive;*
(iv) *$|I_\lambda| = 1$, $S \cong G^0$ and $C = \{(i, e_G, i)\}$, where $I_\lambda = \{i\}$;*
(v) *$Cay(S, C) \cong |S|\vec{K}_1$.*

Proof. (i)⇒(iv) By Proposition 2.2, we get that $cS = S$, for every $c \in C$. Let $c = (i_0, g_0, j_0) \in C$. For every $s = (i, g, j) \in S$, since $cS = S$, there exists $s' = (j_0, g', j) \in S$ such that $(i, g, j) = (i_0, g_0, j_0)(j_0, g', j)$. Since s is arbitrary, for every $i \in I_\lambda$, $i = i_0$. Therefore $|I_\lambda| = 1$. Let $I_\lambda = \{i\}$. Now we define $\psi : (\{i\} \times G \times \{i\}) \cup \{0\} \to G^0$, by $(i, g, i) \mapsto g$ and $0 \mapsto 0$. Obviously, ψ is a semigroup isomorphism. Hence $S \cong G^0$. Since for every $c \in C$, $cS = S$, we get that $0 \notin C$. So $C \subseteq \{i\} \times G \times \{i\}$.

By Proposition 2.2, we conclude that $\langle C\rangle$ is isomorphic to a right group. By Lemma 2.5, we conclude that for every $v \in S$ the connected component containing v is equal to $\langle C\rangle v$. Since $|\langle C\rangle 0| = |\{0\}| = 1$, by Proposition 2.2, we conclude that for every $v \in S$, $|\langle C\rangle v| = 1$. So the cardinality of all connected components of $Cay(S,C)$ are 1. Since C is not empty, all connected components of $Cay(S,C)$ are isomorphic to $\vec{K}_1$. Since $C \subseteq \{i\} \times G \times \{i\}$ and all connected components of $Cay(S,C)$ are isomorphic to $\vec{K}_1$, $C = \{(i, e_G, i)\}$.

(iv)⇒ (v) Since $C = \{(i, e_G, i)\}$ and for every (i, g, i) in S, $(i, e_G, i)(i, g, i) = (i, g, i)$, it follows that each vertex is joined only to itself. Therefore every connected component of $Cay(S,C)$ is isomorphic to $\vec{K}_1$. Hence $Cay(S,C) \cong |S|\vec{K}_1$.

(v)⇒ (i) It is routine to verify that the digraph $|S|\vec{K}_1$ is $ColAut_C(S)$-vertex-transitive.

(ii)⇔ (v) It is routine to verify that the digraph $|S|\vec{K}_1$ is $Aut_C(S)$-vertex-transitive. Conversely let $Cay(S,C)$ be an $Aut_C(S)$-vertex-transitive Cayley graph. First we claim that $0 \notin C$. On the contrary let $0 \in C$. So all vertices of $Cay(S,C)$ are joined to 0. Also we know that 0 is not adjacent to any other vertex of $Cay(S,C)$. Since $Cay(S,C)$ is $Aut_C(S)$-vertex-transitive, for a non-zero vertex v, we conclude that there exists $f \in Aut_C(S)$ such that $f(v) = 0$. Since $(v, 0) \in E(Cay(S,C))$, we get that $(f(v), f(0)) = (0, f(0)) \in E(Cay(S,C))$. Since 0 is not adjacent to any other vertex of $Cay(S,C)$, we conclude that $f(0) = 0$ which is a contradiction since $f(0) = 0 = f(v)$, $f \in Aut_C(S)$ and $v \neq 0$. Therefore $0 \notin C$. On the other hand, by Proposition 2.3 we know that $|\langle C\rangle s|$ is independent of $s \in S$. Since $|\langle C\rangle 0| = |\{0\}| = 1$, and $C \neq \emptyset$, by Lemma 2.5 we conclude that all connected components of $Cay(S,C)$ are isomorphic to $\vec{K}_1$. Therefore $Cay(S,C) \cong |S|\vec{K}_1$.

(iii)⇔ (v) It is routine to verify that the digraph $|S|\vec{K}_1$ is $ColEnd_C(S)$-vertex-transitive. Conversely let $Cay(S,C)$ be a $ColEnd_C(S)$-vertex-transitive Cayley graph. By Lemma 2.4, we get that $cS = S$, for every $c \in C$. Now similar to the proof of (i)⇒ (iv) we get that $|I_\lambda| = 1$, $0 \notin C$, and $S \cong G^0$. Let $I_\lambda = \{i\}$. Since $Cay(S,C)$ is $ColEnd_C(S)$-vertex-transitive and there exists a loop on the vertex 0, there exists a loop on each vertex of $Cay(S,C)$. Hence $(i, e_G, i) \in C$, since $C \subseteq \{i\} \times G \times \{i\}$. Since $Cay(S,C)$ is $ColEnd_C(S)$-vertex-transitive, for every vertex $v \neq 0$, there exists a $\psi \in ColEnd_C(S)$ such that $\psi(0) = v$. Since for every $c \in C$, $c0 = 0$, we get that $v = \psi(0) = \psi(c0) = c\psi(0) = cv$. So $(i, \pi_2(v), i) = (i, \pi_2(c), i)(i, \pi_2(v), i)$. Since $\pi_2(v) = \pi_2(c)\pi_2(v)$ and c is an arbitrary element of C, we conclude that $C = \{(i, e_G, i)\}$. So we get (iv) and we proved that (iv) and (v) are equivalent. □

Remark 4.2. Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup, and let C be a subset of S. By the proof of Theorem 4.1 we conclude that the following statements are equivalent:

(i) Cay(S,C) is $ColEnd_C(S)$-vertex-transitive;
(ii) $|I_\lambda| = 1$, $S \cong G^0$ and $C = \{(i, e_G, i)\}$ where $I_\lambda = \{i\}$;
(iii) $Cay(S,C) \cong |S|\vec{K}_1$.

Now we present a necessary and sufficient condition for Cayley graphs of Brandt semigroups to be endomorphism-vertex-transitive.

Theorem 4.3. *Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup, and let C be a subset of S such that all principal left ideals of the subsemigroup $\langle C \rangle$ are finite. Then the following statements are equivalent:*

(i) *Cay(S,C) is $End_C(S)$-vertex-transitive;*
(ii) *there exists a loop on each vertex;*
(iii) *$(i, e_G, i) \in C$, for every $i \in I_\lambda$.*

Proof. (i)⇒(ii) Since $C \neq \emptyset$, there exists a loop on vertex 0. Also since $Cay(S,C)$ is $End_C(S)$-vertex-transitive, there exists a loop on each vertex of $Cay(S,C)$.

(ii)⇒ (i) For every $s \in S$, we consider the map $\psi_s(v) = s$, which maps every vertex of $Cay(S,C)$ to s. Since there exists a loop on each vertex of $Cay(S,C)$, every ψ_s is a digraph endomorphism, for $s \in S$. Hence for every vertices $s,t \in S$, $\psi_s(t) = s$ and so $Cay(S,C)$ is $End_C(S)$-vertex-transitive.

(ii)⇒(iii) For every $(i,g,j) \in S \setminus \{0\}$, there exists $(i_c, g_c, j_c) \in C$ such that $(i,g,j) = (i_c,g_c,j_c)(i,g,j)$. So $j_c = i$, $i_c = i$ and $g_c g = g$. Hence $g_c = e_G$. Therefore for every $i \in I_\lambda$, $(i, e_G, i) \in C$.

(iii)⇒ (ii) It is obvious. □

Theorem 4.4. *Let $S = (I_\lambda \times G \times I_\lambda) \cup \{0\}$ be a Brandt semigroup and $C \subseteq S$. Then $\Gamma = Cay(S,C)$ is symmetric if and only if*

(i) $|I_\lambda| = 1$;
(ii) $\pi_2(C) = (\pi_2(C))^{-1}$;
(iii) $0 \notin C$.

Proof. (⇒) We claim that $|I_\lambda| = 1$. On the contrary suppose that $|I_\lambda| > 1$. Since C is not empty, there exists $(i_c, g_c, j_c) \in C$. Since $|I_\lambda| > 1$, there exists $i \in I_\lambda$ such that $i \neq j_c$. So every vertex $(i,g,j) \in S$ is joined to 0, which is a contradiction since there does not exist any arc from 0 to (i,g,j) and we know that $Cay(S,C)$ is symmetric. So $|I_\lambda| = 1$. Let $I_\lambda = \{i\}$. If $0 \in C$, then every vertex of Γ is joined to 0 and similarly we get a contradiction. Let $c \in C$. Since $I_\lambda = \{i\}$, we get that $c = (i,t,i)$, where $t \in G$. Therefore $(i,t,i)(i,g,i) = (i,tg,i)$ implies that $Cay(S,C) \cong Cay(G, \pi_2(C)) + \vec{K}_1$. To prove $\pi_2(C) = (\pi_2(C))^{-1}$, let $c \in C$. Then $c = (i,t,i)$, for some $t \in G$. For every $(i,g,i) \in S$, since $((i,g,i),(i,t,i)(i,g,i)) \in E(Cay(S,C))$, then $((i,t,i)(i,g,i),(i,g,i)) \in E(Cay(S,C))$. So there exists $(i,g',i) \in C$ such that $(i,g,i) = (i,g',i)(i,t,i)(i,g,i)$. Hence $t^{-1} = g' \in \pi_2(C)$. Therefore $\pi_2(C) = \pi_2(C)^{-1}$.

(⇐) Since $|I_\lambda| = 1$, $S \cong G^0$. Also since $0 \notin C$, then as we mentioned above it follows that $Cay(S,C) \cong Cay(G, \pi_2(C)) + \vec{K}_1$. On the other hand we know that if $\pi_2(C) = (\pi_2(C))^{-1}$, then $Cay(G, \pi_2(C))$ is symmetric. Therefore $Cay(S,C)$ is symmetric. □

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