Bull. Malays. Math. Sci. Soc. Vol. x, 201x, no. xx, xxx - xxx

Relative Projective Dimensions

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Abstract

In (n,d)-ring and n-coherent ring theory, n-presented modules plays an important role. In this paper, we firstly give some new characterizations of n-presented modules and n-coherent rings. Then, we introduce the concept of $(n,0)$-projective dimension, which measures how far away a finitely generated module is from being n-presented and how far away a ring is from being Noetherian, for modules and rings. This dimension has nice properties when the ring in question is n-coherent. Some known results are extended or obtained as corollaries.

2010 Subject Classification:16D10, 16E40

Key words and phrases: Noetherian rings; n-coherent rings; $(n,0)$-projective module (dimension); $(n,0)$-injective module; n-presented module

1 Introduction

Throughout this paper all rings are associative with identity and modules are unitary. $rD(R)$ stands for the right global dimension of a ring R. $pd(M)$, $id(M)$ and $fd(M)$ denote the projective, injective and flat dimension of an R-module M, respectively.

Let $n \geq 0$ be an integer. Following [2; 3; 11], we call a right R-module P *n-presented* if there exists an exact sequence of right R-modules

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to P \to 0$$

where each F_i is finitely generated free (equivalently projective), $i = 0, 1, \cdots, n$. An R-module is 0-presented (resp. 1-presented) if and only if it is finitely generated (resp. finitely presented). Every m-presented R-module is n-presented for $m \geq n$. A ring R is called *right n-coherent* [3] in case every

n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent (resp. 1-coherent) if and only if R is right Noetherian (resp. coherent), and every n-coherent ring is m-coherent for $m \geq n$.

Let n and d be non-negative integers and M a right R-module. M is called (n,d)-*injective* [12] if ${Ext_R}^{d+1}(N,M) = 0$ for any n-presented right R-module N. M is said to be (n,d)-*projective* [8] if $Ext_R^{d+1}(M,N) = 0$ for any (n,d)-injective R-module N. It is easy to see that both (n,d)-injective modules and (n,d)-projective modules are closed under direct summands and finite direct sums. $(1,0)$-injective (resp. $(1,0)$-projective) modules are also called FP-*injective* (resp. FP-*projective*) modules. It is clear that every (n,d)-injective (resp. (m,d)-projective) module is (m,d)-injective (resp. (n,d)-projective) for $m \geq n$.

In (n,d)-ring and n-coherent ring theory (see [2; 3; 8; 12]), n-presented modules plays an important role. For modules and rings, Mao and Ding [7] defined a dimension, called an FP-projective dimension; Ng [15] introduced the concept of finitely presented dimension. In this paper, we introduce a kind of n-presented dimension of modules and rings.

Let $n \geq 1$ be a fixed integer. In Section 2, we introduce the concept of $(n,0)$-*projective dimension* $npd(M)$ for a right R-module M, and the concept of *right* $(n,0)$-*projective dimension* for a ring R, which measures how far away a finitely generated right R-module M is from being n-presented, and how far away a ring is from being right Noetherian, respectively. It is shown that a finitely generated right R-module M is n-presented if and only if it is $(n,0)$-projective if and only if $npD(M) = 0$ (Theorem 2.3); R is an n-coherent ring if and only if every $(n,0)$-injective right R-module is $(n,1)$-injective if and only if every $(n,1)$-projective right R-module is $(n,0)$-projective (Theorem 2.6); R is a right Noetherian ring if and only if $rnpD(R) = 0$ if and only if every right R-module is $(n,0)$-projective if and only if for a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if both B and C are finitely generated, then A is also finitely generated (Corollary 2.7).

Let $n \geq 1$ be a fixed integer and R a right n-coherent ring. In Section 3, we prove that $rnpD(R) = \sup\{npd(M)$: M is a cyclic right R-module$\}$ $= \sup\{id(M)$: M is an $(n,0)$-injective right R-module$\}$ (Theorem 3.4). As corollaries we obtain that R is right Noetherian if and only if $rnpD(R) < \infty$ and every injective right R-module is $(n,0)$-projective if and only if every $(n,0)$-injective right R-module has an $(n,0)$-projective cover with the unique mapping property if and only if every $(n,0)$-injective right R-module has an injective envelope with the unique mapping property (Corollary 3.6). If $rnpD(R) \leq m$, then we have that R is a right m-coherent ring (Proposition 3.9). Let S and T be rings. If $S \oplus T$ is an right n-coherent ring, then we get that $rnpD(S \oplus T) = \sup\{rnpD(S), rnpD(T)\}$ (Theorem 3.14). Let R be a commutative n-coherent ring and P any prime ideal of R, then $npD(R_{\mathrm{P}}) \leq npD(R)$, where R_P is the localization of R at P (Theorem 3.18).

2 Definition and General Results

Let R be a ring and $m \geq 0$ an integer . Mao and Ding [7] defined the *FP-projective dimension* $fpd(M)$ of a right R-module M as inf{m: $Ext_R^{m+1}(M,N) = 0$ for any FP-injective right R-module N}, if no such m exists, set $fpd(M) = \infty$; and the *right FP-projective dimension* $rfpD(R)$ of R as $\sup\{fpd(M)$: M is a finitely generated right R-module$\}$. We generalize it as follows.

Definition 2.1 *Let $m \geq 0$, $n \geq 1$ be integers, and R a ring. For a right R-module M, set $npd(M) = inf\{m$: $Ext_R^{m+1}(M,N) = 0$ for any $(n,0)$-injective right R-module $N\}$, called the $(n,0)$-projective dimension of M. If no such m exists, set $npd(M) = \infty$.*

Put $rnpD(R) = \sup\{npd(M)$: M is a finitely generated right R-module$\}$, and call $rnpD(R)$ the *right $(n,0)$-projective dimension* of R. The left (n,0)-projective dimension $lnpD(R)$ of R may be defined similarly. If R is a commutative ring, we drop the unneeded letters r and l.

We list the following lemma proved in [8; Lemma 3.3] for convenient using.

Lemma 2.2 *([8; Lemma 3.3]) Let R be a ring, $n \geq 0$ an integer and $0 \to A \to B \to C \to 0$ a short exact sequence of right R-modules. If C is $(n+1,0)$-projective and B is $(n,0)$-projective, then A is $(n,0)$-projective.*

It is clear that an n-presented right R-module is $(n,0)$-projective. In general, the converse is not true. Glaz (see [4; Theorem 2.1.10]) proved that a finitely generated right R-module is finitely presented if and only if it is FP-projective. We generalize it as the following

Theorem 2.3 *Let $n \geq 0$ be a fixed integer and R a ring. Then the following are equivalent for a finitely generated right R-module P.*

(1) *P is n-presented.*

(2) *P is $(n,0)$-projective.*

(3) *$npd(P) = 0$.*

Proof. (1) $\Rightarrow$ (2) is obvious, and (2) $\Leftrightarrow$ (3) holds by definition.

(2) $\Rightarrow$ (1). We use induction on n. The case $n = 0$ is clear, and the case $n = 1$ has been proven in [4; Theorem 2.1.10]. Assume $n > 1$, and P is $(n,0)$-projective. Then P is $(n-1,0)$-projective. So P is $(n-1,0)$-presented by the induction hypothesis. Therefore there exists an exact sequence of right R-modules

$$F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to P \to 0$$

where each F_i is finitely generated projective (hence $(m, 0)$-projective, for any non-negative integer m), $i = 0, 1, \ldots, n-1$. Write $K_1 = \ker(F_0 \to P)$, $K_m = \ker(F_{m-1} \to F_{m-2})$, $m = 2, 3, \ldots, n-1$. Then we have the following short exact sequences

$$\begin{gathered} 0 \longrightarrow K_1 \longrightarrow F_0 \longrightarrow P \longrightarrow 0, \\ 0 \longrightarrow K_2 \longrightarrow F_1 \longrightarrow K_1 \longrightarrow 0, \\ \vdots \\ 0 \longrightarrow K_{n-1} \longrightarrow F_{n-2} \longrightarrow K_{n-2} \longrightarrow 0. \end{gathered}$$

Note that P is $(n, 0)$-projective and F_0 is $(n-1, 0)$-projective, we obtain K_1 is $(n-1, 0)$-projective by Lemma 2.2. It follows that K_2 is $(n-2, 0)$-projective again by Lemma 2.2. Continuing this way, we see that K_{n-1} is $(1, 0)$-projective. Clearly, K_{n-1} is finitely generated. Thus K_{n-1} is finitely presented by [4; Theorem 2.1.10], and hence there exists an exact sequence $F_n^{'} \to F_{n-1}^{'} \to K_{n-1} \to 0$ with $F_n^{'}$ and $F_{n-1}^{'}$ finitely generated projective. So we get an exact sequence

$$F_n^{'} \to F_{n-1}^{'} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to P \to 0.$$

It follows that P is n-presented, as required.

The following corollary is well-known.

Corollary 2.4 *Let $n \geq 0$ be a fixed integer and R a ring. Then the following statements hold:*

(1) *Every finitely generated projective right R-module is n-presented.*

(2) *For a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if both A and C are n-presented, then B is also n-presented.*

(3) *If $B \cong A \oplus C$, then B is n-presented if and only if both A and C are n-presented.*

Proof. (1). Note that every projective right R-module is $(n, 0)$-projective. Thus (1) follows from Theorem 2.3.

(2). Since A and C are n-presented, we have both A and C are finitely generated and $(n, 0)$-projective. Hence B is also finitely generated and $(n, 0)$-projective. Therefore B is n-presented by Theorem 2.3.

(3). If $B \cong A \oplus C$, then it is easy to see that B is finitely generated and $(n, 0)$-projective if and only if both A and C are finitely generated and $(n, 0)$-projective. Thus (3) holds by Theorem 2.3, and we complete the proof.

Corollary 2.5 *Let R be a ring, $n \geq 0$ an integer and $0 \to K \to P \to M \to 0$ a short exact sequence of right R-modules, where P is finitely generated projective. Then K is n-presented if and only if M is $(n+1, 0)$-presented.*

Proof. If K is n-presented, then clearly M is $(n+1)$-presented. Conversely, if M is $(n+1)$-presented (hence $(n+1,0)$-projective), then it is easy to see that K is finitely generated. On the other hand, K is $(n,0)$-projective by Lemma 2.2. It follows that K is n-presented from Theorem 2.3.

Theorem 2.6 *Let R be a ring, and $n \geq 0$ a fixed integer. Then the following are equivalent:*

(1) *R is a right n-coherent ring.*

(2) *Every $(n+1,0)$-injective right R-module is $(n,0)$-injective.*

(3) *Every $(n,0)$-projective right R-module is $(n+1,0)$-projective.*

(4) *For a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules with B finitely generated projective, if C is n-presented, then A is also n-presented.*

(5) *For a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if both B and C are n-presented, then A is also n-presented.*

If $n \geq 1$, then the above conditions are also equivalent to:

(6) *Every $(n,0)$-injective right R-module is $(n,1)$-injective*

(7) *Every $(n,1)$-projective right R-module is $(n,0)$-projective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). are obvious.

(3) $\Rightarrow$ (1). Let M be an n-presented right R-modules. Then M is finitely generated and $(n,0)$-projective by Theorem 2.3. Note that M is $(n+1,0)$-projective by (3). Thus M is $(n+1)$-presented again by Theorem 2.3.

(4) $\Rightarrow$ (1). Let M be any n-presented right R-module. Then there exits a short exact sequence $0 \to K \to P \to M \to 0$ of right R-modules with P finitely generated projective and K n-presented by (4). Hence M is $(n+1)$-presented by Corollary 2.5, and (1) follows.

(1) $\Rightarrow$ (5). If C is n-presented, then C is $(n+1)$-presented by (1). The rest proof is similar to that of Corollary 2.5.

(5) $\Rightarrow$ (4). By (5), it suffices to show that B is n-presented. But this follows from Corollary 2.4.

Now suppose $n \geq 1$.

(4) $\Rightarrow$ (6). Let M be an $(n,0)$-injective right R-module and C any n-presented right R-module. Then we get a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules with B finitely generated projective. By (4), A is n-presented. Thus,

$$Ext_R^2(C, M) \cong Ext_R^1(A, M) = 0.$$

Therefore, M is $(n,1)$-injective.

(6) $\Rightarrow$ (7) is easy.

(7) $\Rightarrow$ (1). Let P be an n-presented right R-module. We get a short exact sequence $0 \to K \to F \to P \to 0$ of right R-modules with F finitely generated

projective and K finitely generated. For any $(n,1)$-injective right R-module M, we have

$$Ext^1_R(K,M) \cong Ext^2_R(P,M) = 0.$$

So K is $(n,1)$-projective and hence $(n,0)$-projective by (7). Thus, K is n-presented by Theorem 2.3. Therefore, P is $(n+1)$-presented and (1) holds.

It is well known that a ring R is right Noetherian if and only if every right R-module is FP-projective if and only if $rfpD(R) = 0$ (see [7; Proposition 2.6]). Now, we have the following

Corollary 2.7 *Let $n \geq 1$ be a fixed integer. Then the following are equivalent for a ring R:*

(1) *R is right Noetherian.*

(2) $rnpD(R) = 0$.

(3) *Every finitely generated right R-module is n-presented.*

(4) *Every $(n,0)$-injective right R-module is injective.*

(5) *Every right R-module is $(n,0)$-projective.*

(6) *Every finitely generated right R-module is $(n,0)$-projective.*

(7) *Every cyclic right R-module is $(n,0)$-projective.*

(8) *For a short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if both B and C are finitely generated, then A is also finitely generated.*

If R is right n-coherent , then the above conditions are also equivalent to:

(9) *Every $(n,0)$-injective right R-module is $(n,0)$-projective.*

Proof. (1) $\Leftrightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial.

(4) $\Rightarrow$ (5) Let M be any right R-module and N any $(n,0)$-injective right R-module. Then $Ext^1_R(M,N) = 0$ since N is injective by (4). Hence M is $(n,0)$-projective.

(7) $\Rightarrow$ (4). Let N be any $(n,0)$-injective right R-module, and I any right ideal of R. By (7), R/I is $(n,0)$-projective. So $Ext^1_R(R/I,N) = 0$. That is, N is injective.

(2) $\Leftrightarrow$ (6) holds by definition, (3) $\Leftrightarrow$ (6) holds by Theorem 2.3, (1) $\Leftrightarrow$ (8) holds by Theorem 2.6, and (4) $\Leftrightarrow$ (9) has been proven in [8; Proposition 4.10].

Corollary 2.8 *Let $n \geq 1$ be an integer and R a ring. If $rnpD(R) \leq 1$, then $rnpD(R) = rfpD(R)$.*

Proof. This follows from the fact that $rnpD(R) = 0$ if and only if $rfpD(R) = 0$ by Corollary 2.7 and [7; Proposition 2.6].

Remark 2.9 (1) *From Theorem 2.3 and Corollary 2.7, we see that $npd(M)$ measures how far away a finitely generated right R-module M is from being*

n-presented, and $rnpD(R)$ measures how far away a ring is from being right Noetherian.

(2) *It is clear that $fpd(M) \leq npd(M) \leq pd(M)$, and $rfpD(R) \leq rnpD(R) \leq rD(R)$. Since $rfpD(R) = rD(R)$ if and only if R is von Neumann regular [7; Remarks 2.2], we have $rfpD(R) = rnpD(R) = rD(R)$ if and only if R is von Neumann regular. It is also easy to see that $rnpD(R) = rD(R)$ if and only if R is a right $(n, 0)$-ring (see [12; Definition 2.5]).*

(3) *It is known that a right Noetherian ring need not be left Noetherian, so $rnpD(R) \neq lnpD(R)$ in general.*

(4) *The equivalence of (1) through (3) in Theorem 2.6 has been proven in [8; Theorem 4.1]. Here we prove the equivalence in a different way.*

(5) *If $n = 1$, then Theorem 2.6 is just some characterizations of coherent rings.*

Recall that a ring R is called right self-$(n, 0)$-injective in case R_R is $(n, 0)$-injective. Stenström proved that if R is right coherent and right self-FP-injective, then every flat right R-module is FP-injective (see [9; Lemma 4.1]). We generalize it as the following

Proposition 2.10 *Let $n \geq 1$ be a fixed integer. If R is a right n-coherent and right self-$(n, 0)$-injective ring, then every flat right R-module is $(n, 0)$-injective.*

Proof. Let M be a flat right R-module. Then, by [16; Theorem 4.85], we get a pure short exact sequence $0 \to K \to F \to M \to 0$ where $F \cong \bigoplus_I R$ for a set I. Since R is right n-coherent and right self-$(n, 0)$-injective, we have F is $(n, 0)$-injective by [12; Lemma 2.9]. Hence we obtain the following exact sequence

$$0 \to Hom_R(N, K) \to Hom_R(N, F) \to Hom_R(N, M) \to Ext^1_R(N, K) \to Ext^1_R(N, F) = 0$$

for any n-presented (hence finitely presented) right R-module N. It follows that $Ext^1_R(N, K) = 0$, and so K is $(n, 0)$-injective. Note that R is right n-coherent, we have M is $(n, 0)$-injective by [8; Theorem 4.1], as desired.

3 $(n, 0)$-Projective Dimensions over n-Coherent Rings

Proposition 3.1 *Let $n \geq 1$, $m \geq 0$ be integers. If R is a right n-coherent ring, then the following are equivalent for a right R-module M:*

(1) *$npd(M) \leq m$.*

(2) *$Ext^{m+1}_R(M, N) = 0$ for any $(n, 0)$-injective right R-module N.*

(3) $Ext_R^{m+j}(M,N) = 0$ *for any* $(n,0)$*-injective right* R*-module* N *and* $j \geq 1$.

(4) *There exists an exact sequence* $0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, *where each* P_i *is* $(n,0)$*-projective.*

(5) *If* $\cdots \to P_{m-1} \to P_{m-2} \to \cdots \to P_1 \to P_0 \to M \to 0$ *is a projective resolution of* M, *then* $ker(P_{m-1} \to P_{m-2})$ *is* $(n,0)$*-projective.*

Proof. (1) $\Rightarrow$ (2). We use induction on m. The case $m = 0$ is clear. Let $m \geq 1$. If $npd(M) = m$, then (2) holds by definition. Suppose $npd(M) \leq m - 1$. For any $(n,0)$-injective right R-module N, the short exact sequence $0 \to N \to E \to L \to 0$ with E injective induces an exact sequence

$$Ext_R^m(M,L) \to Ext_R^{m+1}(M,N) \to Ext_R^{m+1}(M,E) = 0.$$

Since R is n-coherent, we get L is $(n,0)$-injective by [8; Theorem 4.1]. So $Ext_R^m(M,L) = 0$ by the induction hypothesis. It follows that $Ext_R^{m+1}(M,N) = 0$, as desired.

(2) $\Rightarrow$ (3). Using induction on j, the proof is similar to that of (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1), and (2) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2). Write $K_1 = \ker(P_0 \to M)$, $K_i = \ker(P_{i-1} \to P_{i-2})$, $i = 2, 3, \ldots, m-1$. Then we have the following short exact sequences

$$\begin{array}{c} 0 \longrightarrow K_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0, \\ 0 \longrightarrow K_2 \longrightarrow P_1 \longrightarrow K_1 \longrightarrow 0, \\ \vdots \\ 0 \longrightarrow P_m \longrightarrow P_{m-1} \longrightarrow K_{m-1} \longrightarrow 0. \end{array}$$

From the bottom exact sequence, we get the exactness of the sequence

$$0 = Ext_R^1(P_m, N) \to Ext_R^2(K_{m-1}, N) \to Ext_R^2(P_{m-1}, N)$$

for any $(n,0)$-injective right R-module N. Since P_{m-1} is $(n,0)$-projective, using an argument similar to that of (1) $\Rightarrow$ (2), we get $Ext_R^2(P_{m-1}, N) = 0$. Hence $Ext_R^2(K_{m-1}, N) = 0$. Continuing this way, we obtain $Ext_R^{m+1}(M,N) = 0$. Thus (2) holds.

Proposition 3.2 *Let* R *be a right* n*-coherent ring* $(n \geq 1)$ *and* $0 \to A \to B \to C \to 0$ *a short exact sequence of right* R*-modules. Then the following are true:*

(1) *If two of* $npd(A)$, $npd(B)$ *and* $npd(C)$ *are finite, so is the third.*

(2) $npd(A) \leq sup\{npd(B),\ npd(C) - 1\}$.

(3) $npd(B) \leq sup\{npd(A),\ npd(C)\}$.

(4) $npd(C) \leq sup\{npd(B),\ npd(A) + 1\}$.

(5) *If* B *is* $(n,0)$*-projective and* $0 < npd(A) < \infty$, *then* $npd(C) = npd(A) + 1$.

Proof. Easy to verify by Proposition 3.1.

Corollary 3.3 *Let R be a right n-coherent ring ($n \geq 1$), A, B and C right R-modules. If $B \cong A \oplus C$, then $npd(B) = sup\{npd(A), npd(C)\}$.*

Proof. Since $B \cong A \oplus C$, we get two short exact sequences $0 \to A \to B \to C \to 0$ and $0 \to C \to B \to A \to 0$. By Proposition 3.2 (3), it is enough to show that $npd(B) \geq \sup\{npd(A), npd(C)\}$. Suppose $npd(B) < \sup\{npd(A), npd(C)\}$, then $npd(B) < npd(A)$ or $npd(B) < npd(C)$. We may assume $npd(B) < npd(A)$. By Proposition 3.2 (2), $npd(C) \leq \sup\{npd(B), npd(A)-1\}$. So $npd(C) \leq npd(A)-1$, that is, $npd(C) < npd(A)$. In addition, also by Proposition 3.2 (2), we have $npd(A) \leq \sup\{npd(B), npd(C)-1\}$. Hence $npd(A) \leq npd(C) - 1$, since $npd(B) < npd(A)$, and so $npd(A) < npd(C)$, a contradiction.

Let M be a right R-module. Recall that a a homomorphism $\phi : M \to F$ where F is a right $(n, 0)$-injective R-module, is called an $(n, 0)$-injective *preenvelope* [5] of M if for any homomorphism $f : M \to F'$ with F' is $(n, 0)$-injective, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if the only such g are automorphism of F when $F' = F$ and $f = \phi$, then the $(n, 0)$-injective preenvelope ϕ is called an $(n, 0)$-injective *envelope*. A monomorphic $(n, 0)$-injective preenvelope ϕ is said to be *special* [6; Definition 7.1.6] if cokerϕ is $(n, 0)$-projective. $(n, 0)$-projective (pre)covers and special $(n, 0)$-projective precovers can be defined dually. It is proved that every right R-module has a special $(n, 0)$-projective precover and a special $(n, 0)$-injective preenvelope (see [8; Theorem 3.9]).

Theorem 3.4 *Let R be a right n-coherent ring ($n \geq 1$), then the following are identical:*

(1) *$rnpD(R)$*
(2) *$sup\{npd(M)$: M is a cyclic right R-module$\}$*
(3) *$sup\{npd(M)$: M is any right R-module$\}$*
(4) *$sup\{npd(M)$: M is an $(n, 0)$-injective right R-module$\}$*
(5) *$sup\{id(M)$: M is an $(n, 0)$-injective right R-module$\}$*

Proof. (1) $\leq$ (2). We may assume $\sup\{npd(M)$: M is a cyclic right R-module$\}= m < \infty$. Let A be any finitely generated right R-module. We use induction on the number of generators of A. If A has l generators, let A' be a submodule generated by one of these generators. Then both A/A' and A' are finitely generated on less then l generators. Let N be any $(n, 0)$-injective right R-module. Consider the short exact sequence $0 \to A' \to A \to A/A' \to 0$ which induces an exact sequence

$$Ext_R^{m+1}(A/A', N) \to Ext_R^{m+1}(A, N) \to Ext_R^{m+1}(A', N)$$

where

$$Ext_R^{m+1}(A/A', N) = Ext_R^{m+1}(A', N) = 0$$

by induction hypothesis. Thus $Ext_R^{m+1}(A, N) = 0$. So $npd(A) \leq m$.

(2) $\leq$ (3) is clear.

(3) $\leq$ (4). We may assume $\sup\{npd(M)$: M is an $(n,0)$-injective right R-module$\}= m < \infty$. Let A be any right R-module, then A has a special $(n,0)$-injective preenvelope by [8; Theorem 3.9], that is, there exists a short exact sequence $0 \to A \to E \to L \to 0$ with E $(n,0)$-injective and L $(n,0)$-projective. Therefore, $npd(A) \leq npd(E) \leq m$ by Proposition 3.2.

(4) $\leq$ (5). We may assume $\sup\{id(M)$: M is an $(n,0)$-injective right R-module$\}= m < \infty$. Let A and B be any $(n,0)$-injective right R-modules. Then $Ext_R^{m+1}(A, B) = 0$ since $id(B) \leq m$. So $npd(A) \leq m$ by Proposition 3.1.

(5) $\leq$ (1). We may assume $rnpD(R)= m < \infty$. Let M be an $(n,0)$-injective right R-module. Then $Ext_R^{m+1}(R/I, M) = 0$ for any right ideal I of R since $npd(R/I) \leq m$ by hypothesis. Hence $id(M) \leq m$, this completes the proof.

Corollary 3.5 *Let $n \geq 1$ be a fixed integer. Then the following are equivalent for a right n-coherent ring R:*

(1) *$rnpD(R) \leq m$.*

(2) *$npd(M) \leq m$ for any $(n,0)$-injective right R-module M.*

(3) *$npd(M) \leq m$ for any injective right R-module M, and $rnpD(R) < \infty$.*

(4) *$id(M) \leq m$ for any $(n,0)$-injective right R-module M.*

(5) *$id(M) \leq m$ for all right R-module M that are both $(n,0)$-injective and $(n,0)$-projective, and $rnpD(R) < \infty$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) holds by Theorem 3.4. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (4). Let M be any $(n,0)$-injective right R-module. By (5) and Theorem 3.4 (4), $npd(M) = m$ for a non-negative integer m. Note that every right R-module has a special $(n,0)$-projective precover by [8; Theorem 3.9], we obtain an exact sequence

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_t is both $(n,0)$-projective and $(n,0)$-injective, $t = 0, 1, \ldots, m$. Hence $id(P_t) \leq m$ by (5), $t = 0, 1, \ldots, m$. So $id(M) \leq m$.

(3) $\Rightarrow$ (2). Let M be any $(n,0)$-injective right R-module. By (3) and Theorem 3.4 (5), $id(M) = t$ for a non-negative integer t. Hence we get an injective resolution of M:

$$0 \to M \to E^0 \to E^1 \cdots \to E^{t-1} \to E^t \to 0.$$

By (3), $npd(E^i) \leq m$, $i = 0, 1, \ldots, t$. Hence we have $npd(M) \leq m$ by Proposition 3.2, as desired.

Recall that an injective envelope $\phi : M \to E(M)$ of M has the *unique mapping property* [13] if for any homomorphism $f : M \to A$ with A injective, there is a unique homomorphism $g : E(M) \to A$ such that $g\phi = f$. The concept of an $(n, 0)$-projective cover with the unique mapping property can be defined similarly.

Corollary 3.6 *Let $n \geq 1$ be a fixed integer. Then the following are equivalent for a right n-coherent ring R:*

(1) *R is right Noetherian.*

(2) *$rnpD(R) < \infty$ and every injective right R-module is $(n, 0)$-projective.*

(3) *Every $(n, 0)$-injective right R-module is $(n, 0)$-projective.*

(4) *Every $(n, 0)$-injective right R-module has an $(n, 0)$-projective cover with the unique mapping property.*

(5) *Every $(n, 0)$-injective right R-module has an injective envelope with the unique mapping property.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) holds by Corollary 3.5 and Corollary 2.7.

(1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5). Let M be any $(n, 0)$-injective right R-module. Then M is $(n, 0)$-projective and injective, since R is right Noetherian by (1). Thus (4) and (5) follows.

(4) $\Rightarrow$ (3). For any $(n, 0)$-injective right R-module M, let $g : P \to M$ be the $(n, 0)$-projective cover of M with the unique mapping property, where P is $(n, 0)$-projective. Write $K = \ker g$. Then K is $(n, 0)$-injective by [6; Corollary 7.2.3] and [8; Theorem 3.9]. Hence there exists an $(n, 0)$-projective cover $f : P' \to K$ of K by (4). So, we obtain the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & P' & & & & \\ & & & \overset{f}{\swarrow} & \downarrow if & \searrow 0 & & & \\ 0 & \longrightarrow & K & \overset{i}{\longrightarrow} & P & \overset{g}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

Since $g(if) = 0$, we have $if = 0$ by (4). Whence $K = \mathrm{Im} f \subseteq \ker(i) = 0$, that is, M is $(n, 0)$-projective.

(5) $\Rightarrow$ (1). Let M be any $(n, 0)$-injective right R-module. By Corollary 2.7, we need only to show that M is injective. Let $f : M \to E$ be the injective envelope of M with the unique mapping property. Write $L = \mathrm{coker} f$. Since R is n-coherent, L is $(n, 0)$-injective by [8; Theorem 4.1]. So there exists an injective envelope $g : L \to E'$ of L by (5). Therefore we get the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \overset{f}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & L & \longrightarrow & 0 \\ & & & \overset{0}{\searrow} & \downarrow g\pi & \swarrow g & & & \\ & & & & E' & & & & \end{array}$$

Since $(g\pi)f = 0$, we have $g\pi = 0$ by (5). Hence $L = \mathrm{Im}\pi \subseteq \ker(g) = 0$. So M is injective. This completes the proof.

Recall that a short exact sequence $0 \to A \to B \to C \to 0$ is said to be *n-pure* [8] if $Hom(M, B) \to Hom(M, C) \to 0$ is exact for any n-presented module M. A submodule N of M is called an n-pure submodule if the sequence $0 \to N \to M \to M/N \to 0$ is n-pure.

Proposition 3.7 *Let $n \geq 1$ be a fixed integer and R a right n-coherent ring. Observe the following statements:*

(1) $rnpD(R) \leq 1$.

(2) *For any n-pure submodule N of an injective right R-module E, the quotient E/N is injective (i.e., $id(N) \leq 1$).*

(3) *Every submodule of an $(n, 0)$-projective right R-module is $(n, 0)$-projective.*

(4) *Every right ideal of R is $(n, 0)$-projective.*

(5) *For any pure submodule N of an injective right R-module E, the quotient E/N is injective.*

(6) *Every submodule of an FP-projective right R-module is FP-projective.*

(7) *Every right ideal of R is FP-projective.*

Then: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) *and* (2) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7).

Proof. (1) $\Rightarrow$ (2). Let N be an n-pure submodule of an injective right R-module E. Then it is easy to see that N is $(n, 0)$-injective. Hence $id(N) \leq 1$ by Theorem 3.4 (5). So the short exact sequence $0 \to N \to E \to E/N \to 0$ implies that E/N is injective.

(2) $\Rightarrow$ (3). Let L be any $(n, 0)$-injective right R-module. Then it is clear that L is an n-pure submodule of its injective envelope $E(L)$, and hence $id(L) \leq 1$ by (2). If N is a submodule of an $(n, 0)$-projective right R-module M, then the exactness of the sequence

$$0 = Ext^1_R(M, L) \to Ext^1_R(N, L) \to Ext^2_R(M/N, L) = 0$$

implies that $Ext^1_R(N, L) = 0$, and so N is $(n, 0)$-projective.

(4) $\Rightarrow$ (1). Let I be an ideal of R. The exact sequence $0 \to I \to R \to R/I \to 0$ implies that $npd(R/I) \leq 1$ by Proposition 3.1. So (1) holds by Theorem 3.4 (2).

(2) $\Rightarrow$ (5). It is easy to verify that every pure right R-module is n-pure. So (5) follows.

(5) $\Rightarrow$ (6) is similar to that of (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (6) $\Rightarrow$ (7) are trivial.

It is known that if R is a right coherent ring, then $fd(M) = pd(M)$ for any finitely present right R-module M (see [10; Lemma 5]). Mao and Ding (see [7; Proposition4.1]) proved that if R is also self-FP-injective, then $fd(M) = pd(M)$ for any FP-projective right R-module M. Here we have the following

Proposition 3.8 *Let n be a fixed positive integer. If R is a right n-coherent and right self-$(n, 0)$-injective ring, then $fd(M) = pd(M)$ for any $(n, 0)$-projective right R-module M.*

Proof. It is enough to show that $fd(M) \geq pd(M)$. We may assume that $fd(M) = m < \infty$. Then there exists an exact sequence

$$0 \to F_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with P_0, P_1, $\cdots$, P_{m-1} projective and F_m flat. Consider the short exact sequence $0 \to K \to P \to F_m \to 0$ where P is projective. By [16; Theorem 4.85], the short exact sequence above is pure, and hence n-pure. By Proposition 2.10, P is $(n, 0)$-injective. So K is $(n, 0)$-injective by [8; Proposition 3.6]. Since M is $(n, 0)$-projective, so is F_m. Thus the exactness of the sequence

$$0 \to Hom_R(F_m, K) \to Hom_R(P, K) \to Hom_R(K, K) \to Ext^1_R(F_m, K) = 0$$

implies that the sequence $0 \to K \to P \to F_m \to 0$ is split exact, and so F_m is projective, that is, $pd(M) \leq m$. This completes the proof.

Proposition 3.9 *Let $n \geq 1$ be a fixed integer and R a right n-coherent ring. If $rnpD(R) \leq m$, then R is a right m-coherent ring.*

Proof. The case $m = 0$ holds by Corollary 2.7. Suppose $m \geq 1$. Let M be an m-presented right R-module, then M has a free resolution

$$F_m \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

with each F_i finitely generated free. Write $K_m = \ker(F_{m-1} \to F_{m-2})$, then

$$Ext^1_R(K_m, N) \cong Ext^{m+1}_R(M, N) = 0$$

for any FP-injective right R-module N, since $rnpD(R) \leq m$ and every FP-injective right R-module is $(n, 0)$-injective. Note that K_m is finitely generated. We obtain K_m is finitely presented by Theorem 2.3. This implies that M is $(m+1)$-presented, and so R is a right m-coherent ring.

To prove the next main result, we need four lemmas.

Lemma 3.10 *Let $f\colon R \to S$ be a surjective ring homomorphism. If M_S is a right S-module (hence a right R-module) and A_R is a right R-module, then the following statements hold:*

(1) *$M \otimes_R S_S \cong M_S$.*

(2) *If A_R is a finitely generated right R-module, then $A \otimes_R S_S$ is a finitely generated right S-module.*

(3) *M_S is a finitely generated right S-module if and only if M_R is a finitely generated right R-module.*

Proof. (1). Easy.

(2). Clearly, S is a cyclic R-module. Suppose x_1, x_2, $\cdots$, x_n are generators of A. Then it is easy to verify that $x_1 \otimes 1_S$, $x_2 \otimes 1_S$, $\cdots$, $x_n \otimes 1_S$ are generators of $A \otimes_R S_S$, where 1_S denotes the identity of S. Thus $A \otimes_R S_S$ is a finitely generated right S-module.

(3). If M_S is a finitely generated right S-module, and suppose x_1, x_2, $\cdots$, x_n are generators of M, then $M = x_1 S + x_2 S + \cdots + x_n S$. So $M = x_1 R + x_2 R + \cdots + x_n R$ since $f\colon R \to S$ is surjective. Hence M_R is a finitely generated right R-module. The converse holds by (1) and (2).

Lemma 3.11 *Let $f\colon R \to S$ be a surjective ring homomorphism, n a non-negative integer, and M a right S-module. If both S_R and ${}_RS$ are projective, then M_S is an n-presented right S-module if and only if M_R is an n-presented right R-module. (Note that the case $n = 1$ has been proven in [7; Lemma 3.13].)*

Proof. The case $n = 0$ follows by Lemma 3.10. So next we assume $n > 0$.

"$\Rightarrow$". Suppose M is an n-presented right S-module. Then there exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of right S-modules with K finitely generated, and P_i finitely generated projective, $i = 0, 1, \cdots, n-1$. By Lemma 3.10, each P_i and K are finitely generated right R-modules. Since S_R is projective, we have each P_i is a projective right R-module. So, M is an n-presented right R-module.

"$\Leftarrow$". Assume M is an n-presented right R-module. Then there exists an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of right R-modules with K finitely generated, and P_i finitely generated projective, $i = 0, 1, \cdots, n-1$. Since ${}_RS$ is projective, the sequence

$$0 \to K \otimes_R S_S \to P_{n-1} \otimes_R S_S \to \cdots \to P_1 \otimes_R S_S \to P_0 \otimes_R S_S \to M \otimes_R S_S \to 0$$

is exact. By Lemma 3.10, $M \otimes_R S_S \cong M_S$, and both $K \otimes_R S_S$ and each $P_i \otimes_R S_S$ are finitely generated S-modules. Since each P_i is a projective right R-module, we have each $P_i \otimes_R S_S$ is a projective right S-module. So M is an n-presented right S-module.

Let n and d be non-negative integers. Recall that a left R-module A is called (n, d)-*flat* [12], in case $Tor^R_{d+1}(B, A) = 0$ for any n-presented right R-module B.

Lemma 3.12 *Let $f\colon R \to S$ be a surjective ring homomorphism, M_S a right S-module and ${}_SA$ a left S-module. If both S_R and ${}_RS$ are projective, then the following statements hold for any non-negative integers n and d:*

(1) *M_S is an (n,d)-injective right S-module if and only if M_R is an (n,d)-injective right R-module.*

(2) *${}_SA$ is an (n,d)-flat left S-module if and only if ${}_RA$ is an (n,d)-flat left R-module.*

(3) *If R is a right n-coherent ring, then S is a right n-coherent ring.*

Proof. (1). "⇒". Suppose M_S is an (n,d)-injective right S-module. Let N_R be any n-presented right R-module. Then, using an argument similar to that in Lemma 3.11, we get that $N \otimes_R S_S$ is an n-presented right S-module. By [14; Theorem 11.65], we have

$$Ext_R^{d+1}(N_R, M_R) \cong Ext_S^{d+1}(N \otimes_R S_S, M_S) = 0.$$

Therefore M_R is an (n,d)-injective right R-module.

"⇐". Assume M_R is an (n,d)-injective right R-module. Let N_S be any n-presented right S-module. Then $N \otimes_R S_S \cong N_S$ by Lemma 3.10 and N_R is an n-presented right R-module by Lemma 3.11. Again by [14; Theorem 11.65], we have

$$Ext_S^{d+1}(N_S, M_S) \cong Ext_S^{d+1}(N \otimes_R S_S, M_S) \cong Ext_R^{d+1}(N_R, M_R) = 0.$$

Therefore M_S is an (n,d)-injective right S-module.

(2). "⇒". If ${}_SA$ is an (n,d)-flat left S-module. Let B_R be any n-presented right R-module. Then $B \otimes_R S_S$ is an n-presented right S-module. By [14; Corollary 11.63] , we have

$$Tor_{d+1}^R(B_R, {}_RA) \cong Tor_{d+1}^S(B \otimes_R S_S, {}_SA) = 0.$$

Therefore ${}_RA$ is an (n,d)-flat left R-module.

"⇐". If ${}_RA$ is an (n,d)-flat left R-module. Let B_S be any n-presented right R-module. Then $B \otimes_R S_S \cong B_S$ by Lemma 3.10 and B_R is an n-presented right R-module by Lemma 3.11. By [14; Corollary 11.63] , we have

$$Tor_{d+1}^S(B_S, {}_SA) \cong Tor_{d+1}^S(B \otimes_R S_S, {}_SA) \cong Tor_{d+1}^R(B_R, {}_RA) = 0.$$

Therefore ${}_SA$ is an (n,d)-flat left S-module.

(3). Let M_S be an n-presented right R-module, then M_R is an n-presented right R-module by Lemma 3.11. Thus M_R is an $(n+1)$-presented right R-module since R is a right n-coherent ring. Therefore M_S is an $(n+1)$-presented right S-module again by Lemma 3.11, and so S is a right n-coherent ring.

We list the following lemma proved in [7; Lemma 3.14] for convenient using.

Lemma 3.13 *([7; Lemma 3.14]). Let R and S be rings. Every right $(R \oplus S)$-module has a unique decomposition that $M = A \oplus B$, where $A = M(R, 0)$ is a right R-module and $B = M(0, S)$ is a right S-module via $xr = x(r, 0)$ for $x \in A$, $r \in R$, and $ys = y(0, s)$ for $y \in B$, $s \in S$.*

We are now in a position to prove the following main result.

Theorem 3.14 *Let S and T be rings, and $n \geq 1$ a fixed integer. If $S \oplus T$ is a right n-coherent ring, then*

$$rnpD(S \oplus T) = sup\{rnpD(S), rnpD(T)$$

.

Proof. For convenience, we write $R = S \oplus T$. Since R is a right n-coherent ring, we have both S and T are right n-coherent rings by Lemma 3.12.

We first show that $rnpD(R) \leq \sup\{rnpD(S),\ rnpD(T)\}$. We may assume $\sup\{rnpD(S),\ rnpD(T)\} = m < \infty$. Let M be a right (R)-module and N any $(n, 0)$-injective right (R)-module. Then $N = A \oplus B$, where A is a right S-module and B is a right T-module by Lemma 3.13. Note that both A and B are $(n, 0)$-injective right (R)-modules. Hence A is an $(n, 0)$-injective right S-module and B is an $(n, 0)$-injective right T-module by Lemma 3.12. By [14; Theorem 11.65], we have

$$\begin{aligned} Ext_R^{m+1}(M, N) &\cong Ext_R^{m+1}(M, A) \oplus Ext_R^{m+1}(M, B) \\ &\cong Ext_S^{m+1}(M \otimes_R S_S, A) \oplus Ext_T^{m+1}(M \otimes_R T_T, B) \\ &= 0, \end{aligned}$$

and hence $rnpD(R) \leq \sup\{rnpD(S),\ rnpD(T)\}$.

Next we prove that $rnpD(R) \geq \sup\{rnpD(S),\ rnpD(T)\}$. We may assume $rnpD(R) = m < \infty$. Let M be a right S-module and N any $(n, 0)$-injective right S-module. Then N is an $(n, 0)$-injective right (R)-module by Lemma 3.12. By Lemma 3.10, $M \otimes_R S_S \cong M_S$. Again by [14; Theorem 11.65], we have

$$Ext_S^{m+1}(M, N) \cong Ext_S^{m+1}(M \otimes_R S_S, N) \cong Ext_R^{m+1}(M, N) = 0.$$

Therefore $rnpD(R) \geq rnpD(S)$. Similarly for $rnpD(R) \geq rnpD(T)$, and hence $rnpD(R) \geq \sup\{rnpD(S),\ rnpD(T)\}$. This completes the proof.

Remark 3.15 *Let R_1, R_2, $\cdots$, R_m be rings and n a positive integer. The theorem above shows that $rnpD(\bigoplus_{i=1}^m R_i) = sup\{rnpD(R_1),\ rnpD(R_2),\ \cdots,\ rnpD(R_m)\}$ if $\bigoplus_{i=1}^m R_i$ is an n-coherent ring. In particular, we obtain the known result that $\bigoplus_{i=1}^m R_i$ is right Noetherian if and only if each R_i is right Noetherian. But in general $rnpD(\bigoplus_{i=1}^\infty R_i) \neq sup_{i \geq 1}\{rnpD(R_i)\}$. For example, Z_2 is a field of two elements, but $\bigoplus_{i=1}^\infty Z_2$ is not Noetherian .*

Lemma 3.16 *Assume n and d are non-negative integers, R is a commutative ring, and P is any prime ideal of R. Let R_P denote the localization of R at P, M is an R_P-module (M may be viewed as an R-module), and A is an R-module. Then the following statements hold:*

(1) *If A is an n-presented R-module, then A_P is an n-presented R_P-module.*

(2) *If M is an (n,d)-injective R_P-module, then M is an (n,d)-injective R-module.*

(3) *If M is an (n,d)-flat R_P-module, then M is an (n,d)-flat R-module.*

(4) *If A is an (n,d)-projective R-module, then A_P is an (n,d)-projective R_P-module.*

Proof. (1). Suppose A is an n-presented R-module. Then there exists an exact sequence of R-modules

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to A \to 0$$

where each F_i is finitely generated projective, $i = 0, 1, \cdots, n$. It gives rise to the exactness of the sequence

$$(F_n)_P \to (F_{n-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to A_P \to 0$$

of R_P-modules. By [6; Remark 2.2.5], each $(F_i)_P$ is a finitely generated projective R_P-module, $i = 0, 1, \cdots, n$. Hence A_P is an n-presented R_P-module.

(2). Assume M is an (n,d)-injective R_P-module. Let N be any n-presented R-module, then N_P is an n-presented R_P-module by (1). Note that R_P is a flat R-module and $R_P \otimes_R N \cong N_P$. By [14; Theorem 11.65], we have

$$Ext_R^{d+1}(N, M) \cong Ext_{R_P}^{d+1}(R_P \otimes_R N, M) \cong Ext_{R_P}^{d+1}(N_P, M) = 0.$$

Therefore M is an (n,d)-injective R-module.

(3). Similar to that of (2).

(4). Suppose A is an (n,d)-projective R-module. Let B be any (n,d)-injective R_P-module, then B is an (n,d)-injective R-module by (2). Note that $A_P \cong R_P \otimes_R A$. By [14; Theorem 11.65], we have

$$Ext_{R_P}^1(A_P, B) \cong Ext_{R_P}^1(R_P \otimes_R A, B) \cong Ext_R^1(A, B) = 0.$$

Therefore A_P is an (n,d)-projective R_P-module.

Corollary 3.17 *Let R be a commutative ring and P any prime ideal of R. If M is an R_P-module, then the following statements hold:*

(1) *M is an injective R_P-module if and only if M is an injective R-module.*

(2) *M is a flat R_P-module if and only if M is a flat R-module.*

Proof. (1). If M is an injective R_P-module, then M is an injective R-module by Lemma 3.16. If M is an injective R-module, then M_P is an injective R_P-module by [14; Theorem 3.76]. Note that $M \cong M_P$ as R_P-modules. Thus (1) follows.

(2). Similar to that of (1).

Theorem 3.18 *Let $n \geq 1$ be a fixed integer and R a commutative n-coherent ring. If P is any prime ideal of R, then $npD(R_P) \leq npD(R)$.*

Proof. We may assume $npD(R) = t < \infty$. Let M be any R_P-module. Note that M may be viewed as an R-module. Thus $npd(M_R) \leq t$. If $t = 0$, then M is an $(n, 0)$-projective R-module. Since $M \cong M_P$ as R_P-modules, we have M is an $(n, 0)$-projective R_P-module by Lemma 3.16, and so the theorem follows. Next we assume $t \geq 1$. By Proposition 3.1 (5), There exists an exact sequence

$$0 \to K \to F_{t-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

of R-modules, where each F_i is a projective R-module, $i = 1, 2, \cdots, t-1$, and K is an $(n, 0)$-projective R-module. The above sequence induces an R_P-module exact sequence

$$0 \to K_P \to (F_{t-1})_P \to \cdots \to (F_1)_P \to (F_0)_P \to M_P \to 0.$$

By [6; Remark 2.2.5], each $(F_i)_P$ is a projective R_P-module, $i = 1, 2, \cdots, t-1$. Note that K_P is an $(n, 0)$-projective R_P-module by Lemma 3.16. Thus, for any $(n, 0)$-injective R_P-module N, we have

$$Ext^{t+1}_{R_P}(M_P, N) \cong Ext^1_{R_P}(K_P, N) = 0$$

and so $npd(M_P)_{R_P} \leq t$ by definition. Since $M \cong M_P$ as R_P-modules, $npd(M) \leq t$. Therefore $npD(R_P) \leq npD(R)$, and we complete the proof.

Remark 3.19 (1) *The theorem above shows the well-known result that any localization of a Noetherian ring is again Noetherian. But in general $npD(R) \neq sup\{npD(R_P)$: P is a prime ideal of $R\}$. For example, take R to be the direct product of countably many copies of $\mathbf{Z}_2$, then R is not Noetherian. Thus $npD(R) > 0$. However, $npD(R_P) = 0$ for any prime ideal of R.*

(2) *Let R be a commutative ring and P any prime ideal of R. Corollary 3.17 shows that if M is an R_P-module, then M is a flat (resp. injective) R_P-module if and only if M is a flat (resp. injective) R-module. But, in general, a projective R_P-module need not be a projective R-module. For example, R_P is a projective R_P-module, but R_P need not be a projective R-module.*

Acknowledgement. This research was supported by the Scientific Research Fund of Education Department of Guangxi Province(No.201203YB224).

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