BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **36**(3) (2013), 757–765

A Note on F-Weak Multiplication Modules

[1]HAMID AGHA TAVALLAEE, [2]MANSOUREH MAHTABI OGHANI,
AND [3]ROBABEH MAHTABI OGHANI

[1]Department of Mathematics, Tehran-North Branch, Islamic Azad University, Tehran, Iran
[2]Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran
[3]Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran
[1]tavallaee@iust.ac.ir, [2]mansourehmahtabi@gmail.com, [3]r.mahtabi@gmail.com

Abstract. In this paper the definition of an F-weak multiplication module is given and we prove some results for such a module. Then, using the definition of a semiprime submodule of a module, we characterize these submodules for F-weak multiplication modules. Finally, we show that any F-weak multiplication module satisfies the semi-radical formula.

2010 Mathematics Subject Classification: 13C05, 13C13, 13E15, 13C99

Keywords and phrases: F-weak multiplication module, semiprime submodule, envelope of a submodule, radical and semi-radical of a submodule, modules satisfying the semi-radical formula.

1. Introduction

In this paper all rings are commutative with identity and all modules over rings are unitary. If K and N are submodules of an R-module M, we recall that $(N :_R K) = (N : K) = \{r \in R \mid rK \subseteq N\}$, which is an ideal of R. A proper submodule N of an R-module M is said to be prime if for $r \in R,\ x \in M\ ;\ rx \in N$ implies that $x \in N$ or $r \in (N : M)$. In such a case $p = (N : M)$ is a prime ideal of R and N is said to be p-prime. The set of all prime submodules of M is denoted by $\operatorname{Spec}(M)$ and for a submodule N of M, $\operatorname{rad} N = \bigcap_{L\in \operatorname{Spec}(M), N\subseteq L} L$. If no prime submodule of M contains N, we write $\operatorname{rad} N = M$. Also the set of all maximal submodules of M is denoted by $\operatorname{Max}(M)$ and $\operatorname{Rad} M = \bigcap_{P\in \operatorname{Max}(M)} P$. For an ideal I of R, $\operatorname{rad} I = \bigcap_{p\in \operatorname{Spec}(R), I\subseteq p} p$. The ideal I of R is called a radical ideal if $\operatorname{rad} I = I$. Similarly, we say that a submodule N of an R-module M is a radical submodule if $\operatorname{rad} N = N$.

In Section 2, we recall the definition of F-weak multiplication module and we state and prove some properties of these modules. Then in Section 3, after recalling the definition of semiprime submodules and semi-radical formula, we find the semiprime submodules of F-weak multiplication modules.

Communicated by Lee See Keong.
Received: February 17, 2011; *Revised:* October 15, 2011.

2. Some basic definitions and results

Definition 2.1. *We recall that an R-module M is called weak multiplication if* $\mathrm{Spec}(M) = \emptyset$ *or for every prime submodule N of M we have* $N = IM$ *where I is an ideal of R. Also, if M is a weak multiplication, then* $N = (N : M)M$ *for every prime submodule N of M.*

Now we introduce our main definition, which was first stated in [12].

Definition 2.2. *An R-module M is F-weak multiplication, if:*

(1) *M is weak multiplication;*
(2) *For every* $p \in \mathrm{Spec}(R)$*, pM is a prime submodule of M and* $(pM : M) = p$.

We recall that an R-module M is called a multiplication R-module, if for any submodule N of M there exists an ideal I of R such that $N = IM$*. For example one can show that the R-module M is F-weak multiplication in the following cases:*

(i) *M is a finitely generated multiplication R-module such that* $\mathrm{Ann}_R(M) \subseteq p$ *for every* $p \in \mathrm{Spec}(R)$*;*
(ii) *In (i) we assume* $\mathrm{Ann}_R(M) = 0$*, that is, M is faithful.*

In the following example, we show that an F-weak multiplication module is not necessarily a multiplication module.

Example 2.1. Let K be a field and $A = K[x_1, x_2, x_3, ...]$ denote the polynomial ring in a countably infinite set of indeterminates $x_1, x_2, x_3, ...$ Let $\underline{a} = (x_1 - x_1^2, x_2 - x_2^2, x_3 - x_3^2, ...)$ and $B = A/\underline{a}$. Then the prime ideals of the ring B are as follows: $p = (y_1, y_2, y_3, ...)/(x_1 - x_1^2, x_2 - x_2^2, x_3 - x_3^2, ...)$ where $y_j = x_j$ or $y_j = 1 - x_j$ for every $j = 1, 2, 3, ...$ Obviously the ring B has infinitely many prime ideals and $\dim B = 0$.

Now, let $M = \prod_{p_i \in \mathrm{Spec}(B)} B/p_i = B/p_1 \times B/p_2 \times B/p_3 \times ....$ We show that M is a non-finitely generated F-weak multiplication B-module which is not a multiplication B-module. Let $p \in \mathrm{Spec}(B)$ be arbitrary, then $M/(pM) \cong B/p$ and since B/p is simple, then $M/(pM)$ is simple and so $pM \neq M$. Now by [7, Proposition 2], $pM \in \mathrm{Spec}(M)$ and $(pM : M) = p$. Since the only prime submodules of M are the set $\{pM \mid p \in \mathrm{Spec}(B)\}$ hence M is a weak multiplication B-module and therefore M is an F-weak multiplication B-module.

Now, let $p_1 = (x_1, x_2, x_3, ...)/(x_1 - x_1^2, x_2 - x_2^2, x_3 - x_3^2, ...)$ and $p_2 = (1 - x_1, 1 - x_2, 1 - x_3, ...)/(x_1 - x_1^2, x_2 - x_2^2, x_3 - x_3^2, ...)$ be two prime ideals of B and let $N = 0_{B/p_1} \times 0_{B/p_2} \times B/p_3 \times B/p_4 \times ...$ be a submodule of M. Then $p_1 p_2 = \underline{a}/\underline{a} = 0_B$ is the only ideal of B which kills both B/p_1 and B/p_2, but $p_1 p_2 M = 0_B M = 0 \neq N$. So there exists no ideal I of B such that $N = IM$, hence M is not a multiplication B-module.

Proposition 2.1. *Let R be a non-trivial ring and M an F-weak multiplication R-module. Then M has a maximal submodule.*

Proof. See [12, Proposition 2.4]. ∎

Here by a *pure submodule* of M we mean a proper submodule N such that $rM \cap N = rN$ for every $r \in R$.

Lemma 2.1. *Let R be an integral domain and M be an F-weak multiplication R-module. Then M is a torsion-free module. Consequently the only proper pure submodule of M is zero.*

Proof. For the first part of lemma, see [12, Proposition 2.6].

Assume that N be an arbitrary pure submodule of M. Since M is torsion-free. Hence by [7, Result 2], N is a (0)-prime submodule of M. Now by the definition of F-weak multiplication modules $N = \langle 0 \rangle$. ∎

Theorem 2.1. *Let R be a local ring with the maximal ideal $\underline{m}$ and M be an R-module. If $\underline{m}M \neq M$ is a maximal submodule of M then M is a cyclic R-module.*

Proof. We know that $M/(\underline{m}M)$ is a vector space over the field $R/\underline{m}$. Since $M/(\underline{m}M)$ is a simple $R/\underline{m}$-module, then $M/(\underline{m}M)$ is cyclic and so:

$$\exists y \in M - \underline{m}M \ ; \ \frac{M}{\underline{m}M} = \langle y + \underline{m}M \rangle.$$

On the other hand, since $\underline{m}M$ is a maximal submodule of M and $y \notin \underline{m}M$ then $\langle y \rangle + \underline{m}M = M$. Now we have:

$$\frac{M}{\underline{m}M} = \frac{\langle y \rangle + \underline{m}M}{\underline{m}M} \cong \frac{\langle y \rangle}{\langle y \rangle \cap \underline{m}M}. \tag{2.1}$$

Obviously $\langle y \rangle \cap \underline{m}M = \underline{m}\langle y \rangle$ and then by (2.1), $M/(\underline{m}M) \cong \langle y \rangle/(\underline{m}\langle y \rangle)$. But $M/(\underline{m}M)$ is a cyclic $R/\underline{m}$-module, hence $\langle y \rangle/(\underline{m}\langle y \rangle)$ is a cyclic $R/\underline{m}$-module and we have:

$$\frac{\langle y \rangle}{\underline{m}\langle y \rangle} = \langle ry + \underline{m}\langle y \rangle \rangle.$$

Since $r \in R - \underline{m}$ is a unit element, without loose of the generality we set $r = 1$ and hence $\langle y \rangle/(\underline{m}\langle y \rangle) = \langle y + \underline{m}\langle y \rangle \rangle$. Then we have

$$\frac{M}{\underline{m}M} = \langle y + \underline{m}M \rangle \cong \langle y + \underline{m}\langle y \rangle \rangle.$$

On the other hand, $y + \underline{m}\langle y \rangle \subseteq y + \underline{m}M$ therefore $\langle y + \underline{m}\langle y \rangle \rangle \subseteq \langle y + \underline{m}M \rangle$. Also since $\langle y + \underline{m}\langle y \rangle \rangle$ is an $R/\underline{m}$-module hence $\langle y + \underline{m}\langle y \rangle \rangle$ is an $R/\underline{m}$-submodule of $\langle y + \underline{m}M \rangle$. But $\langle y + \underline{m}M \rangle$ is a simple $R/\underline{m}$-module, hence

$$\langle y + \underline{m}\langle y \rangle \rangle = 0_{\frac{M}{\underline{m}M}} \ \textit{or} \ \langle y + \underline{m}\langle y \rangle \rangle = \langle y + \underline{m}M \rangle.$$

But $y \notin \underline{m}M$, hence $\langle y + \underline{m}\langle y \rangle \rangle = \langle y + \underline{m}M \rangle$.

Now we show that $M = \langle y \rangle$. Let $ry + \underline{m}\langle y \rangle \in \langle y + \underline{m}\langle y \rangle \rangle$ where $r \in R - \underline{m}$ be an arbitrary element, then there exists $r' \in R - \underline{m}$ such that $ry + \underline{m}\langle y \rangle = r'y + \underline{m}M$. Then,

$$(r - r')y + \underline{m}\langle y \rangle = \underline{m}M \implies \underline{m}M \subseteq \langle y \rangle.$$

But $\langle y \rangle \neq \underline{m}M$ and $\underline{m}M$ is maximal. Therefore $M = \langle y \rangle$ and the proof is now completed. ∎

Corollary 2.1. *Let R be a local ring with the maximal ideal $\underline{m}$ and let M be an F-weak multiplication R-module then M is a cyclic R-module.*

Proof. Since $\underline{m}M$ is the only maximal submodule of M then by Theorem 2.1, there exists $m \in M - \underline{m}M$ such that $M = \langle m \rangle$. ∎

Theorem 2.2. *Let R be a non-trivial ring and let M be an F-weak multiplication R-module. Then M_p is an F-weak multiplication R_p-module for every $p \in \operatorname{Spec}(R)$.*

Proof. Let M be an F-weak multiplication R-module then by [3, Lemma 2.3], M_p is weak multiplication R_p-module for every $p \in \mathrm{Spec}(R)$. Let $p \in \mathrm{Spec}(R)$ be arbitrary. First we show that $(pR_p)M_p = (pM)_p \neq M_p$. If not, $(pM)_p = M_p$. Then

$$\forall m \in M - pM, \quad \frac{m}{1} \in (pM)_p \quad \Longrightarrow \quad \exists t \in R - p, \quad tm \in pM$$

But $pM \in \mathrm{Spec}(M)$ hence $m \in pM$, a contradiction. Therefore $(pM)_p \neq M_p$. We assume $Q \in \mathrm{Spec}(R_p)$ then there exists $I \in \mathrm{Spec}(R)$ such that $I \cap (R - p) = \emptyset$ and $Q = IR_p$. Now we must show that $QM_p \in \mathrm{Spec}(M_p)$ and $(QM_p : M_p) = Q$. Since $QM_p = (IM)_p \subseteq (pM)_p$ and by above $(pM)_p \neq M_p$ then $(IM)_p \neq M_p$. Now let $r/s.m/s' \in (IM)_p$ where $r/s \in R_p$, $m/s' \in M_p$. Then $(rm)/(ss') \in (IM)_p$ and so there exists $t \in R - p$ such that $trm \in IM$. But $IM \in \mathrm{Spec}(M)$ hence $tr \in I$ *or* $m \in IM$. Thus $r \in I$ *or* $m \in IM$, so $(IM)_p \in \mathrm{Spec}(M_p)$.

Next we show that $((IM)_p : M_p) = IR_p$. We know

$$IR_p = (IM : M)_p \subseteq ((IM)_p : M_p). \tag{2.2}$$

Let $r/s \in ((IM)_p : M_p)$ be arbitrary. Then for any $m/s' \in M_p$ where $m \notin IM$ we have:

$$\frac{r}{s}.\frac{m}{s'} = \frac{rm}{ss'} \in (IM)_p \quad \Longrightarrow \quad \exists t \in R - p, \quad trm \in IM$$

But $IM \in \mathrm{Spec}(M)$ then $r \in I$ and so $r/s \in IR_p$. Now by (2.2) , $((IM)_p : M_p) = IR_p$. The proof is now completed. ∎

Corollary 2.2. *Let R be a non-trivial ring such that every non-zero prime ideal of R is a maximal ideal. Let M be an R-module. Then M is an F-weak multiplication R-module if and only if $M_{\underline{m}}$ is an F-weak multiplication $R_{\underline{m}}$-module for every $\underline{m} \in \mathrm{Max}(R)$.*

Proof. $(\Longrightarrow)$. By Theorem 2.2, is clear.

$(\Longleftarrow)$. Let $M_{\underline{m}}$ be an F-weak multiplication $R_{\underline{m}}$-module for every $\underline{m} \in \mathrm{Max}(R)$. We show that M is an F-weak multiplication R-module. First by [3, Lemma 2.3], M is a weak multiplication R-module. We prove that $(\underline{m}M : M) = \underline{m}$ for any $\underline{m} \in \mathrm{Max}(R)$. We know that,

$$\underline{m}R_{\underline{m}} \subseteq (\underline{m}M : M)_{\underline{m}} \subseteq ((\underline{m}M)_{\underline{m}} : M_{\underline{m}}). \tag{2.3}$$

By Corollary 2.1, $M_{\underline{m}}$ is cyclic. But by [8, Theorem 2 (4)] and [5, Theorem 2.5 (ii)], $M_{\underline{m}}$ has the only maximal submodule $(\underline{m}M)\underline{m}$. Hence

$$((\underline{m}M)_{\underline{m}} : M_{\underline{m}}) = \underline{m}R_{\underline{m}}. \tag{2.4}$$

So by (2.3) and (2.4), $(\underline{m}M : M)_{\underline{m}} = \underline{m}R_m$ and hence $(\underline{m}M : M) \neq R$. Therefore $(\underline{m}M : M) = \underline{m}$ and also by [7, Proposition 2], $\underline{m}M \in \mathrm{Spec}(M)$. The proof is now completed. ∎

Let us recall that a module M over a ring R is "*locally cyclic*" if $M_{\underline{m}}$ is a cyclic $R_{\underline{m}}$-module for all maximal ideals $\underline{m}$ of R.

Lemma 2.2. *F-weak multiplication modules are locally cyclic.*

Proof. Let M be an F-weak multiplication R-module and $\{\underline{m}_i\}_{i \in I} = \mathrm{Max}(R)$. Then by Theorem 2.2, $M_{\underline{m}}$ is an F-weak multiplication $R_{\underline{m}}$-module for every $\underline{m} \in \mathrm{Max}(R)$. Therefore by Corollary 2.1, M is locally cyclic. ∎

Theorem 2.3. *Let R be a non-trivial ring and let M be an F-weak multiplication R-module. Then every proper submodule of M is contained in a maximal submodule of M.*

Proof. If not, we assume that there exists a proper submodule N of M such that N is not contained in any maximal submodule of M. But we know by Proposition 2.1, that for every $\underline{m} \in \operatorname{Max}(R)$, $\underline{m}M$ is a maximal submodule of M, then:

$$\begin{aligned} N \not\subseteq \underline{m}M, \quad \forall \underline{m} \in \operatorname{Max}(R) &\implies N + \underline{m}M = M, \quad \forall \underline{m} \in \operatorname{Max}(R) \\ &\implies (N + \underline{m}M)_{\underline{m}} = N_{\underline{m}} + (\underline{m}M)_{\underline{m}} = M_{\underline{m}}, \quad \forall \underline{m} \in \operatorname{Max}(R). \end{aligned}$$

By Lemma 2.2, M is locally cyclic and so each $M_{\underline{m}}$ is cyclic. Now by [2, Corollary 2.7], $N_{\underline{m}} = M_{\underline{m}}$ for every $\underline{m} \in \operatorname{Max}(R)$. But $(M/N)_{\underline{m}} \cong M_{\underline{m}}/N_{\underline{m}}$, then $(M/N)_{\underline{m}} \cong 0$ for every $\underline{m} \in \operatorname{Max}(R)$.

By [2, Proposition 3.8], $M/N = 0$ and so $N = M$, a contradiction. Therefore there exists $\underline{m} \in \operatorname{Max}(R)$ such that $N \subseteq \underline{m}M$. ∎

Corollary 2.3. *Let M be an F-weak multiplication R-module and let N be a submodule of M such that $M = N + \operatorname{Rad} M$. Then $M = N$.*

Proof. If not $M \neq N$. Since M is F-weak multiplication, then N is contained in a maximal submodule of M, say $\underline{m}M$, where $\underline{m} \in \operatorname{Max}(R)$. Then,

$$M = N + \operatorname{Rad} M \subseteq \underline{m}M + \operatorname{Rad} M \subseteq \underline{m}M.$$

So, $M \subseteq \underline{m}M$, a contradiction. Therefore $M = N$. ∎

Definition 2.3. *An element u of an R-module M is said to be a unit provided that u does not belong to any maximal submodule of M.*

Theorem 2.4. *Let M be an F-weak multiplication R-module. Then $u \in M$ is a unit if and only if $\langle u \rangle = M$.*

Proof. Let $u \in M$ be a unit element, then we have:

$$\forall \underline{m} \in \operatorname{Max}(R), u \in M - \underline{m}M.$$

So, $\langle u \rangle \leq M$ and $\langle u \rangle \not\subseteq \underline{m}M$ for any $\underline{m} \in \operatorname{Max}(R)$. Thus, $\langle u \rangle = M$ or $\langle u \rangle$ is a maximal submodule of M. But $\langle u \rangle \neq \underline{m}M$ for every $\underline{m} \in \operatorname{Max}(R)$ and every maximal submodule of M is of the form $\underline{m}M$ for some $\underline{m} \in \operatorname{Max}(R)$. Therefore $\langle u \rangle = M$.

Conversely, let $\langle u \rangle = M$. We show that $u \in M$ is a unit element. If not, $\langle u \rangle$ is a proper submodule of M and then:

$$\exists \underline{m} \in \operatorname{Max}(R) \; ; \; \langle u \rangle \subseteq \underline{m}M$$

Hence $M = \underline{m}M$, a contradiction. Therefore $u \in M$ is a unit. ∎

Corollary 2.4. *If M is an F-weak multiplication R-module then for every proper submodule N of M,* $\operatorname{rad} N \neq M$.

Proof. The proof is clear by Theorem 2.3. ∎

Lemma 2.3. *Let M be a non-zero faithful multiplication R-module, then M is an F-weak multiplication R-module.*

Proof. Let M be a multiplication R-module then by [4, Lemma 2 (i)], M_p is a multiplication R_p-module for every $p \in \operatorname{Spec}(R)$. We show that $(pR_p)M_p \neq M_p$ for every $p \in \operatorname{Spec}(R)$.

Since M_p is a multiplication R_p-module hence by [8, Theorem 2 (4)], $\operatorname{Max}(M_p) \neq \emptyset$. Now let Q be a maximal submodule of M_p, then since R_p is a local ring with the maximal ideal pR_p hence by [5, Theorem 2.5 (ii)], $Q = (pR_p)M_p = (pM)_p$ is the only maximal submodule of M_p and so $(pR_p)M_p = (pM)_p \neq M_p$ and $((pM)_p : M_p) = pR_p$.

Therefore $(pR_p)M_p = (pM)_p \neq M_p$ for every $p \in \mathrm{Spec}(R)$ and so $pM \neq M$. Since $\mathrm{Ann}_R(M) \subseteq p$ for every $p \in \mathrm{Spec}(R)$ then by [5, Corollary 2.11], $pM \in \mathrm{Spec}(M)$.

Now, we show that $(pM : M) = p$. Let $r \in (pM : M)$ be arbitrary, then $rM \subseteq pM$ and hence $(rM)_p \subseteq (pM)_p$. Thus $r/1M_p \subseteq (pM)_p$ and then $r/1 \in ((pM)_p : M_p)$. By the above, $r/1 \in pR_p$ and hence $r \in p$. Therefore $(pM : M) \subseteq p$ and so $p = (pM : M)$. The proof is now completed. ∎

We recall that if $N = I_1M$ and $K = I_2M$ (I_1 and I_2 are ideals of R) are submodules of a multiplication R-module M then the *product* of N and K, denoted by NK, is defined by $NK = I_1I_2M$. It is clear that NK is a submodule of M and $NK \subseteq N \cap K$.

Proposition 2.2. *Let $M_1, \ldots, M_n$ be arbitrary submodules of a multiplication R-module M. Let P be a proper submodule of M. Then P is prime submodule of M if and only if $\prod_{i=1}^n M_i \subseteq P$ implies that $M_i \subseteq P$ for some $i = 1, ..., n$.*

Proof. Use [1, Theorem 3.16] and induction on n. ∎

Proposition 2.3. *Let $M_1, \ldots, M_n$ be submodules of a multiplication R-module M and let N be a prime submodules of M such that $\bigcap_{i=1}^n M_i \subseteq N$. Then $M_i \subseteq N$ for some $i = 1, ..., n$. Also, if $N = \bigcap_{i=1}^n M_i$, then $N = M_i$ for some $i = 1, ..., n$.*

Proof. Let $\bigcap_{i=1}^n M_i \subseteq N$. Since $\prod_{i=1}^n M_i \subseteq \bigcap_{i=1}^n M_i \subseteq N$, the result follows by the above proposition. ∎

Lemma 2.4. *Let M be an F-weak multiplication R-module and $M_1, \ldots, M_n$ be submodules of M and let N be a prime submodule of M such that $\bigcap_{i=1}^n M_i \subseteq N$. Then $M_i \subseteq N$ for some M_i $(1 \leq i \leq n)$. Also, if $N = \bigcap_{i=1}^n M_i$, then $N = M_i$ for some M_i $(1 \leq i \leq n)$.*

Proof. Since $N \in \mathrm{Spec}(M)$ hence $N = pM$ for some $p \in \mathrm{Spec}(R)$. Now, let $\bigcap_{i=1}^n M_i \subseteq N$ then $(\bigcap_{i=1}^n M_i)_p \subseteq N_p$ and hence $\bigcap_{i=1}^n (M_i)_p \subseteq N_p$. By Corollary 2.1 and Theorem 2.2, M_p is multiplication. So by Proposition 2.3, $(M_i)_p \subseteq N_p$ for some $(M_i)_p (1 \leq i \leq n)$. We show that $M_i \subseteq N$. Let $x \in M_i$ hence $x/1 \in (M_i)_p$ and so $x/1 \in N_p$. Then there exists $t \in R - p$ such that $tx \in N$. But $N \in \mathrm{Spec}(M)$ hence $x \in N$. Therefore $M_i \subseteq N$, and the proof is now completed. ∎

Lemma 2.5. *Let R be a non-trivial ring and let M be a multiplication R-module. Then $IM \neq M$ for any proper ideal I of R.*

Proof. Let I be an arbitrary proper ideal of R, then there exists a maximal ideal $\underline{m}$ of R such that $I \subseteq \underline{m}$. We show that $\underline{m}M \neq M$. By [4, Lemma 2 (i)], $M_{\underline{m}}$ is a multiplication $R_{\underline{m}}$-module and also by [8, Theorem 2 (4)], $\mathrm{Max}(M_{\underline{m}}) \neq \emptyset$. Now, let W be a maximal submodule of $M_{\underline{m}}$, then since $R_{\underline{m}}$ is a local ring with the maximal ideal $\underline{m}R_{\underline{m}}$, by [5, Theorem 2.5 (ii)], $W = (\underline{m}R_{\underline{m}})M_{\underline{m}} = (\underline{m}M)_{\underline{m}}$ and so $((\underline{m}M)_{\underline{m}} : M_{\underline{m}}) = \underline{m}R_{\underline{m}}$. But $\underline{m}R_{\underline{m}} \subseteq (\underline{m}M : M)_{\underline{m}} \subseteq ((\underline{m}M)_{\underline{m}} : M_{\underline{m}}) = \underline{m}R_{\underline{m}}$, so $(\underline{m}M : M)_{\underline{m}} = \underline{m}R_{\underline{m}}$, and therefore $\underline{m}M \neq M$. Now since $IM \subseteq \underline{m}M \neq M$, we have $IM \neq M$ for every proper ideal I of R. ∎

Corollary 2.5. *Let R be a non-trivial ring and let M be a non-zero multiplication R-module. Let every prime ideal of R be a maximal ideal of R. Then $pM \in \mathrm{Spec}(M)$ for any $p \in \mathrm{Spec}(R)$.*

Proof. It is clear by Lemma 2.5 and [7, Proposition 2]. ∎

Let M be a multiplication R-module. Then:

(i) If R is a ring with $\dim R = 0$, then Corollary 2.5 is satisfied for M.

(ii) If R is an integral domain with $\dim R = 1$, then for each non-zero prime ideal of R Corollary 2.5 is satisfied for M.

3. Semiprime submodules of F-weak multiplication modules

We recall the following definitions from [10].

Definition 3.1. *A proper submodule N of an R-module M is said to be semiprime in M, if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Since the ring R is an R-module over itself, a proper ideal I of R is semiprime if for every ideals J and K of R, $J^2K \subseteq I$ implies that $JK \subseteq I$.*

Remark 3.1. There exists another definition of semiprime submodules in [6] as follows:

A proper submodule N of the R-module M is semiprime if whenever $r^k m \in N$ for some $r \in R$, $m \in M$ and positive integer k, then $rm \in N$.

By [11, Remark 2.6], we see that this definition is equivalent to Definition 3.1.

Definition 3.2. *Let M be an R-module and $N \leq M$. The envelope of the submodule N is denoted by $E_M(N)$ or simply by $E(N)$ and is defined as $E(N) = \{x \in M \mid \exists r \in R,\ a \in M;\ x = ra \text{ and } r^n a \in N \text{ for some positive integer } n\}$.*

The envelope of a submodule is not a submodule in general.

Let M be an R-module and $N \leq M$. If there exists a semiprime submodule of M which contains N, then the intersection of all semiprime submodules containing N is called the *semi-radical* of N and is denoted by $S-\mathrm{rad}_M(N)$, or simply $S-\mathrm{rad}(N)$. If there is no semiprime submodule containing N, then we define $S-\mathrm{rad}(N) = M$, in particular $S-\mathrm{rad}(M) = M$.

We say that M *satisfies the radical formula*, or M (s.t.r.f) if for every $N \leq M$, $\mathrm{rad}N = \langle E(N)\rangle$. Also we say that M *satisfies the semi-radical formula*, or M (s.t.s.r.f) if for every $N \leq M$, $S-\mathrm{rad}(N) = \langle E(N)\rangle$. Now let $x \in E(N)$ and P be a semiprime submodule of M containing N. Then $x = ra$ for some $r \in R$, $a \in M$ and for some positive integer n, $r^n a \in N$. But $r^n a \in P$ and since P is semiprime we have $ra \in P$. Hence $E(N) \subseteq P$. We see that $E(N) \subseteq \bigcap P$ (P is a semiprime submodule containing N). So $E(N) \subseteq S-\mathrm{rad}(N)$. On the other hand, since every prime submodule of M is clearly semiprime, we have $S-\mathrm{rad}(N) \subseteq \mathrm{rad}N$. We conclude that $\langle E(N)\rangle \subseteq S-\mathrm{rad}(N) \subseteq \mathrm{rad}N$ and as a result if M (s.t.r.f) then it is also (s.t.s.r.f).

Remark 3.2. We define the $S-\mathrm{rad}$ of an ideal I of the ring R as the intersection of all semiprime ideals of R containing I.

Definition 3.3. *A submodule N of M is called an $S-\mathrm{rad}$ submodule if $S-\mathrm{rad}(N) = N$.*

Theorem 3.1. *Let M be an F-weak multiplication R-module, then M (s.t.s.r.f).*

Proof. By Lemma 2.2, M is locally cyclic. Hence $M_{\underline{m}}$ is a cyclic $R_{\underline{m}}$-module for every $\underline{m} \in \mathrm{Max}(R)$ and so by [10, Proposition 4.9, Theorem 4.10], M (s.t.s.r.f). ∎

Corollary 3.1. *If M is an F-weak multiplication R-module, then every proper submodule of M is semiprime.*

Proof. By Theorem 3.1, M (s.t.s.r.f) hence by [10, Proposition 4.1], every proper submodule of M is semiprime. ∎

Lemma 3.1. *Let M be an F-weak multiplication R-module. Then for any proper submodule N of M we have:*

$$\mathrm{rad}(N_{\underline{m}}) = (\mathrm{rad}\,N)_{\underline{m}}\ ;\ \forall \underline{m} \in \mathrm{Max}(R).$$

Proof. By [10, Theorem 3.15],

$$(\mathrm{rad}\,N)_{\underline{m}} \subseteq \mathrm{rad}(N_{\underline{m}}), \tag{3.1}$$

for any $N \leq M$. Also by Lemma 2.2, M is locally cyclic ,that is, $M_{\underline{m}}$ is a cyclic $R_{\underline{m}}$-module for any $\underline{m} \in \mathrm{Max}(R)$. So by [9, Theorem 4], $M_{\underline{m}}$ (s.t.r.f) and hence (s.t.s.r.f). Thus $\langle E(H)\rangle = S-\mathrm{rad}(H) = \mathrm{rad}\,H$ for every submodule H of $M_{\underline{m}}$. But by [10, Proposition 4.1], $S-\mathrm{rad}(H) = H$ for any submodule H of $M_{\underline{m}}$.

$$\mathrm{rad}\,N_{\underline{m}} = N_{\underline{m}}\ ;\ \forall N \leq M. \tag{3.2}$$

Since $N_{\underline{m}} \subseteq (\mathrm{rad}\,N)_{\underline{m}}$ then by (3.2),

$$\mathrm{rad}\,N_{\underline{m}} \subseteq (\mathrm{rad}\,N)_{\underline{m}}. \tag{3.3}$$

Now by (3.1) and (3.3),

$$\mathrm{rad}(N_{\underline{m}}) = (\mathrm{rad}\,N)_{\underline{m}}\ ;\ \forall \underline{m} \in \mathrm{Max}(R). \quad ∎$$

Lemma 3.2. *If M is an F-weak multiplication R-module and N is a proper submodule of M. Then M/N (s.t.s.r.f).*

Proof. By Theorem 3.1, M (s.t.s.r.f). Let H/N be an arbitrary proper submodule of M/N. Then by [10, Proposition 3.16], $S-\mathrm{rad}_{M/N}(H/N) = (S-\mathrm{rad}_M(H))/N = H/N$. Therefore every proper submodule H/N of M/N is semiprime and so by [10, Proposition 4.1], M/N (s.t.s.r.f). ∎

Lemma 3.3. *Let R be a ring and M an F-weak multiplication R-module. Then the only primary submodules of M are those submodules which are prime.*

Proof. Let M be an F-weak multiplication module. Let N be an arbitrary primary submodule of M. By Corollary 3.1, N is a semiprime submodule of M and by [11, Proposition 2.4], $(N : M)$ is a semiprime ideal of R. Now by [11, Lemma 3.1], N is a prime submodule of M. The proof is now completed. ∎

It should be noted that, Lemma 3.3 is not necessarily true if $M = R$, the ring itself. Because according to [10, Theorem 4.4], R (s.t.s.r.f) if we have one of the following.

(i) For every free R-module F, F (s.t.s.r.f).
(ii) For every faithful R-module C, C (s.t.s.r.f).

Acknowledgement. We would like to thank the referees for their comments and suggestions on the manuscript.

References

[1] R. Ameri, On the prime submodules of multiplication modules, *Int. J. Math. Math. Sci.* **2003**, no. 27, 1715–1724.

[2] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Co., Reading, MA, 1969.

[3] A. Azizi, Weak multiplication modules, *Czechoslovak Math. J.* **53(128)** (2003), no. 3, 529–534.

[4] A. Barnard, Multiplication modules, *J. Algebra* **71** (1981), no. 1, 174–178.

[5] Z. A. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16** (1988), no. 4, 755–779.

[6] J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1992), no. 12, 3593–3602.

[7] C.-P. Lu, Prime submodules of modules, *Comment. Math. Univ. St. Paul.* **33** (1984), no. 1, 61–69.

[8] C.-P. Lu, Spectra of modules, *Comm. Algebra* **23** (1995), no. 10, 3741–3752.

[9] R. L. McCasland and M. E. Moore, On radicals of submodules of finitely generated modules, *Canad. Math. Bull.* **29** (1986), no. 1, 37–39.

[10] H. A. Tavallaee, Modules satisfying the semi-radical formula, *Hadronic J.* **32** (2009), no. 4, 407–423.

[11] H. A. Tavallaee and S. Ghalandarzadeh, Semiprime submodules and envelope of modules, *IUST-International Journal of Engineering Sciences*, **14 (4)**, (2003), 131–142.

[12] H. A. Tavallaee and R. M. Oghani, Some remarks on F-weak multiplication modules, *Thai J. Math.* **9** (2011), no. 2, 439–448.