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INEQUALITIES WHICH INCLUDE q-INTEGRALS[1]

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A b s t r a c t. The main problem in analyzing inequalities which include q-integrals is the fact that q-integral of a function over an interval $[a, b]$ $(0 < a < b)$ is defined by the difference of two infinite sums. Thus defined q-integral properties must include the points outside of interval of integration.

In this paper, we will signify to some directions for solving this problem and derive some inequalities which are analogues to well-known ones in standard integral calculus.

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1. *Introduction*

In the fundamental books about q-calculus [3],[4] the q-integral of the function f over the interval $[0, b]$ is defined by

$$I_q(f; 0, b) = \int_0^b f(x) d_q x = b(1-q) \sum_{n=0}^{\infty} f(bq^n) q^n \quad (0 < q < 1). \tag{1}$$

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If f is integrable over $[0,b]$, then

$$\lim_{q\nearrow 1} I_q(f;0,b)=\int_0^b f(x)\ dx = I(f;0,b).$$

Our attention was pulled up by the definition of integral over the interval $[a,b]$. Namely, generally accepted definition is

$$I_q(f;a,b)=\int_a^b f(x)d_qx=\int_0^b f(x)d_qx-\int_0^a f(x)d_qx \quad (0<q<1). \tag{2}$$

For example, in that case the values of q-integrals of the polynomials over $[a,b]$ are very similar to well-known ones in the standard integral calculus. But, problem is what will happen if f is defined in $[a,b]$ and if it is not defined in $[0,a]$.

In this paper we specify two ways to overcome the mentioned problem. The first one is the restriction of the q-integral over $[a,b]$ to a finite sum whose number of the elements directly depends on a, b and q (see [2]). The second one is indicated in [6] and it means introduction of the definition of the q-integral of the Riemann type.

2. *The q-integrals, correlations and properties*

Let $a,\ b$ and q be some real numbers such that $0<a<b$ and $q\in(0,1)$.

Beside the q-integrals defined by (1) and (2), we will consider two other types of the q-integrals.

In the paper [2], H. Gauchman has introduced *the restricted q-integral*

$$G_q(f;a,b)=\int_a^b f(x)\ d_q^G x=b(1-q)\sum_{k=0}^{n-1} f(bq^k)q^k \quad (a=bq^n). \tag{3}$$

Let us notice that lower bound of integral is $a=bq^n$, i.e., it is tied by chosen q,b and positive integer n.

In the paper [6], we have introduced *Riemann-type q-integral* by

$$R_q(f;a,b)=\int_a^b f(t)d_q^R t=(b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k. \tag{4}$$

This definition includes only points within the interval of the integration.

The different types of the q-integral defined by (1)–(4) can be denoted in the unique way by $J_q(\,\cdot\,; a_{(J)}, b)$, where J can be G, I or R. Interval of the integration $E_{(J)} = [a_{(J)}, b]$ of q-integral $J_q(\,\cdot\,; a_{(J)}, b)$ depends on its type:

$a_{(G)} = bq^n$, $n \in \mathbb{N}$, for $G_q(\,\cdot\,; a, b)$;

$a_{(I)} = 0$ for $I_q(\,\cdot\,; 0, b)$;

$a_{(I)}$ and $a_{(R)}$ are arbitrary numbers $a \in [0, b]$ for $I_q(\,\cdot\,; a, b)$ and $R_q(\,\cdot\,; a, b)$.

We can say that a real function f is *q-integrable* on $[0, b]$ or $[a, b]$ if the series in (1) and (2) converge. In the similar way, we say that f is *qR-integrable* on $[a, b]$ if the series in (4) converges. From now on, it will be assumed that the function f is q-integrable on $[0, b]$ (qR-integrable on $[a, b]$) whenever $I_q(f; 0, b)$ or $I_q(f; a, b)$ ($R_q(f; a, b)$) appears in the formula.

In this research it is convenient to define the operators

$$\begin{array}{ll} \widehat{\ } : f \mapsto \widehat{f}, & \widehat{f}(x) = f(a + (b-a)x), \\ \widetilde{\ } : f \mapsto \widetilde{f}, & \widetilde{f}(x) = bf(bx) - af(ax), \\ \breve{\ } : f \mapsto \breve{f}, & \breve{f}(x) = f(bx) - f(ax), \end{array}$$

such that associate the functions defined on $[0, 1]$ to the function defined on $[a, b]$. Notice that, for $x \in [0, 1]$, it is

$$\widehat{(fg)}(x) = \widehat{f}(x)\ \widehat{g}(x), \quad \widetilde{(fg)}(x) = \frac{1}{b-a}(\widetilde{f}(x)\widetilde{g}(x) - ab\ \breve{f}(x)\breve{g}(x)). \tag{5}$$

The correlations between the q-integrals defined by (1)–(4) are given in the following lemma.

Lemma 2.1. *If the real function f is q-integrable on $[0, b]$ or qR-integrable on $[a, b]$, $0 < a < b$, then it holds*

$$I_q(f; 0, b) = \lim_{n\to\infty} G_q(f; bq^n, b), \tag{6}$$

$$I_q(f; a, b) = I_q(\widetilde{f}; 0, 1), \tag{7}$$

$$R_q(f; a, b) = (b-a) I_q(\widehat{f}; 0, 1). \tag{8}$$

P r o o f. The relation (6) is evident because $G_q(f; bq^n, b)$, $n \in \mathbb{N}$, are the partial sums of the series $I_q(f; 0, b)$. The equality (7) is valid according to

$$I_q(f; a, b) = (1-q)\sum_{k=0}^{\infty} (bf(bq^k) - af(aq^k))q^k = I_q(\widetilde{f}; 0, 1).$$

Finally, for (8), according to the definition we have

$$R_q(f;a,b) = (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k = (b-a)I_q(\widehat{f};0,1). \quad \square$$

The mentioned connections can be used to derive the inequalities for all types of the q-integrals. By (6), the inequalities for the infinite sum $I_q(f;0,b)$ can be derived in the limit process from this one for $G_q(f;a,b)$, defined by the finite sum. Using (7) and (8), the integrals $I_q(f;a,b)$ and $R_q(f;a,b)$ can be considered as the q-integrals over $[0,1]$. Nevertheless, the results for $I_q(f;a,b)$ are quite rough because the points outside of the interval of the integration (i.e., points on $[0,a]$) are included.

According to (5) and Lemma 2.1, the following integral relations are valid:

$$R_q(fg;a,b) = (b-a)I_q(\widehat{(fg)};0,1) = (b-a)I_q(\widehat{f}\ \widehat{g};0,1), \tag{9}$$

$$I_q(fg;a,b) = I_q(\widetilde{(fg)};0,1) = \frac{1}{b-a}\Big(I_q(\widetilde{f}\ \widetilde{g};0,1) - ab\ I_q(\breve{f}\ \breve{g};0,1)\Big). \tag{10}$$

At last, let us remind on some definitions and terms from q-calculus.

The q-natural number is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}, \qquad n \in \mathbb{N}.$$

The function $f:[a,b]\to\mathbb{R}$ is called *q-increasing* (*q-decreasing*) on $[a,b]$ if $f(qx)\le f(x)$ ($f(qx)\ge f(x)$) whenever $x\in[a,b]$ and $qx\in[a,b]$. It is easy to see that if the function f is increasing (decreasing), then it is q-increasing (q-decreasing) for $0<q<1$.

3. *q-Chebyshev inequality*

In this section we give the q-analogues of Chebyshev inequality for the monotonic functions (see [5], pp. 239.). The discrete case of this inequality is used in [2] for the restricted q-integrals. We derive its variants for the rest of the q-integrals.

Theorem 3.1. *Let $f,g: E_{(J)}\to\mathbb{R}$ be two real functions, both q-decreasing or both q-increasing. If $J_q(\,\cdot\,;a_{(J)},b)$ is the q-integral defined by* (1), (3) *or* (4), *it holds*

$$J_q(fg;a_{(J)},b) \ge \frac{1}{b-a_{(J)}} J_q(f;a_{(J)},b)\ J_q(g;a_{(J)},b).$$

P r o o f. For $J_q(\,\cdot\,; a_{(J)}, b) = G_q(\,\cdot\,; a, b)$, $a = bq^n$, the inequality is proved in [2]. So, the inequalities

$$G_q(fg; bq^n, b) \geq \frac{1}{b - bq^n} G_q(f; bq^n, b)\ G_q(g; bq^n, b)$$

are valid for all $n = 1, 2, \ldots$. When $n \to \infty$, using (6) we get the desired inequality for $J_q(\,\cdot\,; a_{(J)}, b) = I_q(\,\cdot\,; 0, b)$. In the case $J_q(\,\cdot\,; a_{(J)}, b) = R_q(\,\cdot\,; a, b)$, from the q-monotonicity of the functions f and g on $[a, b]$ follows the q-monotonicity of the functions $\widehat{f}$ and $\widehat{g}$ on $[0, 1]$. Hence, we have

$$I_q(\widehat{f}\ \widehat{g}; 0, 1) \geq I_q(\widehat{f}; 0, 1)\ I_q(\widehat{g}; 0, 1).$$

According to (8) and (9) we get the required inequality. $\square$

The Chebyshev inequality in the source form is not valid for $I_q(\,\cdot\,; a, b)$, where $0 < a < b$.

Example 3.1 For $f(x) = x^3$ and $g(x) = x^4$ on the interval $[1, 2]$ we have

$$I_q(x^3 \cdot x^4; 1, 2) - I_q(x^3; 1, 2) I_q(x^4; 1, 2) = 255 \frac{1 - q}{1 - q^8} - 465 \frac{(1 - q)^2}{(1 - q^4)(1 - q^5)},$$

wherefrom we conclude that the inequality holds only for $q > 1/2$, but it has opposite sign for $q < 1/2$.

Lemma 3.2. *Let the function $f : [0, b] \to \mathbb{R}$ be increasing and $0 < a < b$. If there exist two positive constants l and L such that $a^2/b^2 \leq l/L$ and for every $x, y \in [0, b]$ the inequality*

$$l \leq \frac{f(x) - f(y)}{x - y} \leq L$$

is valid, then the function $\widetilde{f} : [0, 1] \to \mathbb{R}$ is increasing too.

P r o o f. Under the conditions of the Lemma, for every $0 \leq x < y \leq b$ we have

$$l(y - x) \leq f(y) - f(x) \leq L(y - x).$$

Then it holds

$$\begin{aligned} \widetilde{f}(y) - \widetilde{f}(x) &= b(f(by) - f(bx)) - a(f(ay) - f(ax)) \\ &\geq (b^2 l - a^2 L)(y - x) \geq 0. \end{aligned}$$

$\square$

Theorem 3.3. *Let $f, g : [0, b] \to \mathbb{R}$ be two real increasing functions. If there exist the constants l_f, L_f, l_g and L_g such that $a^2/b^2 \le l_f/L_f$, $a^2/b^2 \le l_g/L_g$ and*

$$l_f \le \frac{f(x)-f(y)}{x-y} \le L_f, \quad l_g \le \frac{g(x)-g(y)}{x-y} \le L_g$$

holds, then the inequalities are valid:

$$(a) \quad I_q(fg;a,b) \ge \frac{1}{b-a} I_q(f;a,b) I_q(g;a,b) - \frac{ab(b-a)}{[3]_q} L_f L_g$$

$$(b) \quad I_q(fg;a,b) \ge \frac{1}{b-a} I_q(f;a,b) I_q(g;a,b) - \frac{ab}{b-a} (f(b)-f(0))(g(b)-g(0)).$$

P r o o f. Suppose that f and g are both increasing on $[0, b]$. Then, according to Lemma 3.2, $\widetilde{f}$ and $\widetilde{g}$ are both increasing and hence q-increasing on $[0, 1]$. With respect to (10) we can write

$$I_q(fg;a,b) = \frac{1}{b-a}\Big(I_q(\widetilde{f}\ \widetilde{g};0,1) - ab\ I_q(\breve{f}\ \breve{g};0,1)\Big).$$

Using Theorem 3.1, we have

$$I_q(\widetilde{f}\ \widetilde{g};0,1) \ge I_q(\widetilde{f};0,1)\ I_q(\widetilde{g};0,1),$$

wherefrom

$$I_q(fg;a,b) \ge \frac{1}{b-a}\Big(I_q(f;a,b)\ I_q(g;a,b) - ab I_q(\breve{f}\ \breve{g};0,1)\Big). \tag{11}$$

(a) Under the conditions satisfied by the functions f and g on $[0, b]$, it holds

$$\begin{aligned} I_q(\breve{f}\ \breve{g};0,1) &= (1-q)\sum_{k=0}^{\infty} (f(bq^k)-f(aq^k))(g(bq^k)-g(aq^k))q^k \\ &\le (1-q)\sum_{k=0}^{\infty} L_f L_g (bq^k - aq^k)^2 q^k = L_f L_g (b-a)^2 \frac{1-q}{1-q^3} \end{aligned}$$

Substituting this estimation in (11), we get the first inequality.
(b) Since the functions f and g are increasing on $[0, b]$, it holds

$$I_q(\breve{f}\ \breve{g};0,1) \le (1-q)(f(b)-f(0))(g(b)-g(0)) \sum_{k=0}^{\infty} q^k = (f(b)-f(0))(g(b)-g(0)),$$

what, with (11), gives the second inequality. $\square$

4. *q-Chebyshev functional and inequalities*

Let us denote by

$$\mathcal{T}(f,g;a,b)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\ \int_a^b g(x)dx$$

the Chebyshev functional, i.e., the functional whose positivity for the monotonic functions proves Chebyshev inequality. Its q-analogues can be made for each of mentioned types of q-integrals:

$$\mathcal{T}_q^{(J)}(f,g;a_{(J)},b)=\frac{J_q(fg;a_{(J)},b)}{b-a_{(J)}}-\frac{J_q(f;a_{(J)},b)J_q(g;a_{(J)},b)}{(b-a_{(J)})^2},$$

where $J_q(\,\cdot\,;a_{(J)},b)$ is one of q-integrals defined by (1)–(4).

With respect to Lemma 2.1, it can be evince that

$$\mathcal{T}_q^{(I)}(f,g;0,b) = \lim_{n\to\infty}\mathcal{T}_q^{(G)}(f,g;bq^n,b), \tag{12}$$

$$\mathcal{T}_q^{(R)}(f,g;a,b) = \mathcal{T}_q^{(I)}(\widehat{f},\widehat{g};0,1), \tag{13}$$

$$\mathcal{T}_q^{(I)}(f,g;a,b) = \frac{1}{(b-a)^2}\Big(\mathcal{T}_q^{(I)}(\widetilde{f},\widetilde{g};0,1)-ab\ I_q(\breve{f}\,\breve{g};0,1)\Big). \tag{14}$$

In that way, the basic inequalities including the Chebyshev functional are derived for $\mathcal{T}_q^{(G)}(f,g;a,b)$, and then for the other variants.

Theorem 4.1. *Let $f,g:E_{(J)}\to\mathbb{R}$ be two real functions. If there exist the real constants m and M such that*

$$m(g(x)-g(y))\le f(x)-f(y)\le M(g(x)-g(y)),\qquad a_{(J)}\le x<y\le b,$$

is valid, then it holds the inequality

$$(m+M)\ \mathcal{T}_q^{(J)}(f,g;a_{(J)},b)\ge\mathcal{T}_q^{(J)}(f,f;a_{(J)},b)+mM\ \mathcal{T}_q^{(J)}(g,g;a_{(J)},b),$$

where $J_q(\,\cdot\,;a_{(J)},b)$ is the q-integral defined by (1), (3) *or* (4).

P r o o f. At the start, let $J_q(\,\cdot\,;a_{(J)},b)=G_q(\,\cdot\,;a,b)$, $a=bq^n$, $n\in\mathbb{N}$. Under the conditions satisfied by the functions f and g, the product

$$\Big(M(g(bq^i)-g(bq^j))-(f(bq^i)-f(bq^j))\Big)\Big((f(bq^i)-f(bq^j))-m(g(bq^i)-g(bq^j))\Big)$$

is nonnegative for $i, j = 0, 1, \ldots, n-1$ $(i < j)$, i.e.,

$$\begin{aligned}(f(bq^i) - f(bq^j))^2 \quad &+ mM(g(bq^i) - g(bq^j))^2 \\ &\le (m+M)(f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)).\end{aligned}$$

Multiplying by q^{i+j} $(i < j)$ and summing over i and j, we obtain

$$\begin{aligned}&\sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))^2 q^{i+j} + mM \sum_{\substack{i,j=0\\ i<j}}^{n-1} (g(bq^i) - g(bq^j))^2 q^{i+j} \\ &\le (m+M) \sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)) q^{i+j}.\end{aligned}$$

Since (see [1],[5]) $\mathcal{T}_q^{(G)}(f, g; a, b)$ can be presented in the form

$$\mathcal{T}_q^{(G)}(f, g; a, b) = \left(\frac{b(1-q)}{b-a}\right)^2 \sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)) q^{i+j},$$

we have the required inequality.

For $J_q(\,\cdot\,; a_{(J)}, b) = I_q(\,\cdot\,; 0, b)$ it is enough to put $n \to \infty$ in the proved inequality for the previous case.

Finally, let $J_q(\,\cdot\,; a_{(J)}, b) = R_q(\,\cdot\,; a, b)$. Under the condition satisfied by the functions f and g on $[a, b]$, the functions $\widehat{f}$ and $\widehat{g}$ satisfy the same conditions on $[0, 1]$. Applying (13) and the proved inequality for $\mathcal{T}_q^{(I)}(\widehat{f}, \widehat{g}; 0, 1)$, we get the statement. $\square$

The upper inequality can be presented in the form

$$J_q((f - mg)(Mg - f); a_{(J)}, b) \ge \frac{1}{b - a_{(J)}} J_q(f - mg; a_{(J)}, b) J_q(Mg - f; a_{(J)}, b),$$

what is Chebyshev inequality for the monotonic functions $f - mg$ and $Mg - f$.

Theorem 4.2. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two real functions. If there exist the real constants l and L such that*

$$\begin{aligned}l((b - a_{(J)})g(x) - J_q(g; a_{(J)}, b)) \quad &\le (b - a_{(J)})f(x) - J_q(f; a_{(J)}, b) \\ &\le L((b - a_{(J)})g(x) - J_q(g; a_{(J)}, b)),\end{aligned}$$

on $[a,b]$*, then it holds the inequality*

$$(l+L)\ \mathcal{T}_q^{(J)}(f,g;a_{(J)},b) \geq \mathcal{T}_q^{(J)}(f,f;a_{(J)},b) + lL\ \mathcal{T}_q^{(J)}(g,g;a_{(J)},b),$$

where $J_q(\,\cdot\,;a_{(J)},b)$ *is the* q*-integral defined by* (1), (3) *or* (4).

P r o o f. Let $J_q(\,\cdot\,;a_{(J)},b)$ be $G_q(\,\cdot\,;a,b)$, $a=bq^n$, or $I_q(\,\cdot\,;0,b)$. Then Chebyshev functional $\mathcal{T}_q^{(J)}$ can be represented (see [1]) in the form

$$\begin{aligned}&\mathcal{T}_q^{(J)}(f,g;a_{(J)},b)\\ &=\frac{b(1-q)}{b-a_{(J)}}\sum_{i=0}^{N_{(J)}}\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)q^i,\end{aligned}$$

where $N_{(G)}=n-1$ and $N_{(I)}=\infty$. Under the conditions satisfied by the functions f and g, the product

$$\begin{aligned}&\bigg(L\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)-\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\bigg)\\ &\qquad\times\bigg(\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)-l\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)\bigg)\end{aligned}$$

is nonnegative for $i=0,1,\dots,N_{(J)}$, i.e.,

$$\begin{aligned}&\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)^2+lL\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)^2\\ &\qquad\leq(l+L)\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big).\end{aligned}$$

For $J_q(\,\cdot\,;a_{(J)},b)=G_q(\,\cdot\,;a,b)$, the desired inequality is obtained only by multiplying the upper inequalities by q^i and summing over i, $i=0,1,\dots,n-1$. In the case $J_q(\,\cdot\,;a_{(J)},b)=I_q(\,\cdot\,;0,b)$ it is needed to put $n\to\infty$.

For $J_q(\,\cdot\,;a_{(J)},b)=R_q(\,\cdot\,;a,b)$ it should be noticed that under the conditions satisfied by f and g on $[a,b]$ the inequality

$$l(\widehat{g}(x)-I_q(\widehat{g};0,1))\leq\widehat{f}(x)-I_q(\widehat{f};0,1)\leq L(\widehat{g}(x)-I_q(\widehat{g};0,1))$$

is valid for all $x\in[0,1]$. Hence, according to (13) and the proved inequality for the previous case, we get desired inequality. $\square$

REFERENCES

[1] S. S. Dragomir, *"A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities"*, J. Inequalities in Pure and Applied Mathematics, **4**(3) Art. 63, 2003.

[2] H. Gauchman, *Integral Inequalities in q-Calculus*, Computers and Mathematics with Applications, vol. 47, (2004), 281–300.

[3] W. Hahn, *"Lineare Geometrische Differenzengleichungen"*, 169 Berichte der Mathematisch-Statistischen Section im Forschungszentrum Graz, 1981.

[4] R. Koekoek, R. F. Swarttouw, *"Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue"*, Report of Delft University of Technology No 98–117, 1998.

[5] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *"Classical and New Inequalities in Analysis"*, Kluwer Academic Publishers, 1993.

[6] P. M. Rajković, M. S. Stanković, S. D. Marinković, *The zeros of polynomials orthogonal with respect to q-integral on several intervals in the complex plane*, Proceedings of The Fifth International Conference on Geometry, Integrability and Quantization, 2003, Varna, Bulgaria (ed. I.M. Mladenov, A.C.Hirsshfeld), 178–188.

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