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SHILOV PARABOLIC SYSTEMS

M. KOSTIĆ

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A b s t r a c t. We apply semigroups of growth order $r > 0$ in the study of Shilov parabolic systems, and improve results obtained by the use of C-regularized semigroups. In the final part of paper, we consider some concrete examples and directly compute matrix exponentials with a view to strengthening the estimates obtained by abstract methods.

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1. *Introduction and Preliminaries*

The main purpose of the present paper is to apply (analytic) semigroups of growth order $r > 0$ in the study of Shilov parabolic systems of abstract differential equations. In the case of L^2-type spaces, the pioneering results in this direction were obtained by S. G. Krein [19, Chapter 1, §8], T. Ushijima

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[29, Section 10] and J. Wiener, L. Debnath [33]. It is also worth noting that H. Sunouchi [27] analyzed Shilov parabolic systems within the frame of $(0, A)$-semigroups ([24]). Concerning the applications of C-regularized semigroups to systems of abstract differential equations, the references [5]-[6], [14] and [36]-[38] are crucially important. In this paper, we refine results on the well-posedness of abstract Shilov parabolic systems obtained by Q. Zheng, Y. Li [36] and Q. Zheng [38]. In order to do that, we use the theory of C-regularized semigroups as an auxiliary tool.

Henceforth $(X, ||\cdot||)$ denotes a non-trivial complex Banach space and $L(X)$ denotes the Banach algebra of all linear continuous mappings from X into X. The range of a closed, linear operator B acting on X is denoted by $R(B)$, and by K is denoted an absolute positive constant appearing in our estimations and formulae. In the sequel, we basically follow the notation given in [5]. Given $\gamma \in (0, \pi]$ and $s \in \mathbb{R}$, set $\Sigma_\gamma := \{z \in \mathbb{C} : z \neq 0,\ |\arg(z)| < \gamma\}$ and $\lfloor s \rfloor := \inf\{k \in \mathbb{Z} : s \leq k\}$.

We need the following definitions.

Definition 1.1. ([4]) Let $r > 0$. An operator family $(T(t))_{t>0}$ in $L(X)$ is said to be a *semigroup of growth order* $r > 0$ iff the following conditions hold:

(i) $T(t+s) = T(t)T(s),\ t,\ s > 0$,

(ii) for every $x \in X$, the mapping $t \mapsto T(t)x,\ t > 0$ is continuous,

(iii) $||t^r T(t)|| = O(1),\ t \to 0+$,

(iv) $T(t)x = 0$ for all $t > 0$ implies $x = 0$, and

(v) $X_0 = \bigcup_{t>0} R(T(t))$ is dense in X.

The *infinitesimal generator* of $(T(t))_{t>0}$ is defined by

$$G = \Big\{(x, y) \in X \times X : \lim_{t\to 0+} \frac{T(t)x - x}{t} = y\Big\}.$$

The infinitesimal generator G of $(T(t))_{t>0}$ is a closable linear operator and its closure $\overline{G}$ is called the *complete infinitesimal generator*, in short c.i.g., of $(T(t))_{t>0}$.

Definition 1.2. ([28]) Let $(T(t))_{t>0}$ be a semigroup of growth order $r > 0$, let $\gamma \in (0, \frac{\pi}{2}]$ and let $(T(t))_{t>0}$ possess an analytic extension to the sector Σ_γ, denoted by the same symbol. Suppose, further, that there exists $\omega \in \mathbb{R}$ such that, for every $\delta \in (0, \gamma)$, there exists $M_\delta > 0$ satisfying $||z^r T(z)|| \leq M_\delta e^{\omega \Re z},\ z \in \Sigma_\delta$. Then the operator family $(T(t))_{t \in \Sigma_\gamma}$ is called an *analytic semigroup of growth order* r.

We refer the reader to [22] and [32] for the notions of a C-regularized semigroup (cosine function) and its subgenerator (integral generator).

Definition 1.3. ([18]) Let $(S(t))_{t \in [0,\tau)}$ be a (local) C-regularized semigroup having A as a subgenerator and let the mapping $t \mapsto S(t),\ t \in (0, \tau)$ be infinitely differentiable in the uniform operator topology. Then it is said that $(S(t))_{t \in [0,\tau)}$ is *ρ-hypoanalytic,* $1 \leq \rho < \infty$, iff for every compact set $L \subseteq (0, \tau)$ there exists $h_L > 0$ such that

$$\sup_{t \in L,\ l \in \mathbb{N}_0} \Big\| \frac{h_L^l \frac{d^l}{dt^l} S(t)}{l!^\rho} \Big\| < \infty.$$

Let $n \in \mathbb{N}$ and let $iA_j,\ 1 \leq j \leq n$ be commuting generators of bounded C_0-groups on X. Put $|A|^2 := \sum_{j=1}^{n} A_j^2$ and

$$(1 + |A|^2)^{-r} := (1 + |x|^2)^{-r}(A),\ r > 0, \tag{1}$$

where the right hand side of (1) is defined by means of the functional calculus for commuting generators of bounded C_0-groups (cf. [2] and [12, Section XII] for further information).

Let $P(x) = [p_{ij}(x)]_{m \times m},\ x \in \mathbb{R}^n$ be an $m \times m$ polynomial matrix and let $\lambda_j(x),\ 1 \leq j \leq m$ be the eigenvalues of $P(x),\ x \in \mathbb{R}^n$; see [36] for the definition of a closable operator $P(A)$. Set $k := 1 + \lfloor \frac{n}{2} \rfloor$, $\Lambda(P(x)) := \sup_{1 \leq j \leq m} \Re \lambda_j(x),\ x \in \mathbb{R}^n,\ N := \max(\mathrm{dg}(p_{ij}(x)))$ and assume that $r \in (0, N]$. Then it is said that $P(x)$ is *Shilov r-parabolic* [12] iff there exist $\omega > 0$ and $\omega' \in \mathbb{R}$ such that $\Lambda(P(x)) \leq -\omega|x|^r + \omega',\ x \in \mathbb{R}^n$; in the case $r = N$, it is also said that $P(x)$ is *Petrovskii parabolic.* Define $\pi_1(r) := \min_{1 \leq j \leq m,\ |x|=r} |\lambda_j(x)|,\ r > 0$, $\pi_2(r) := \max_{1 \leq j \leq m,\ |x|=r} |\lambda_j(x)|,\ r > 0$ and $S(P) := \{\lambda_j(x) : x \in \mathbb{R}^n,\ 1 \leq j \leq m\}$. Let $P(x)$ be Shilov r-parabolic for some $r \in (0, N]$. By a corollary of Seidenberg-Tarski theorem (cf. [13] and [29, Lemma 10.2]), we know that there exist real numbers $a_1,\ a_2,\ \alpha_1$ and α_2 such that $\pi_1(r) = a_1 r^{\alpha_1}(1 + o(1))$ as $r \to \infty$ and $\pi_2(r) = a_2 r^{\alpha_2}(1 + o(1))$ as $r \to \infty$. Obviously, $r \leq \alpha_1 \leq \alpha_2 \leq N$ and, by the proof of [29, Proposition 10.4], there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \Re\lambda \geq -\alpha|\Im\lambda|^{r/\alpha_2} + \beta\} \cap S(P) = \emptyset$.

2. *Shilov Parabolic Systems*

We start this section with the following theorem. Notice that the proof of this theorem strongly depends on the estimate (4), which can be trivially shown only in the case $m = 1$.

Theorem 2.1. *Let $P(x)$ be Shilov r-parabolic for some $r \in (0, N)$. Put $\kappa := \frac{(N-r)(m-1+\frac{n}{2})}{r}$.*

(a) *Then the operator $\overline{P(A)}$ is the c.i.g. of a semigroup $(T_0(t))_{t>0}$ of growth order κ which additionally satisfies that the mapping $t \mapsto T_0(t)$, $t > 0$ is infinitely differentiable in the uniform operator topology and that:*

$$\Bigl\|\frac{d^l}{dt^l}T_0(t)\Bigr\| \leq K^{l+1}(1+t)^{m-1+\frac{n}{2}}e^{\omega t}l!^{N/r}(1+t^{-\kappa-\frac{Nl}{r}}),\ t>0,\ l \in \mathbb{N}_0, \tag{2}$$

where $\omega \equiv \sup_{x\in\mathbb{R}^n} \Lambda(P(x))$.

(b) *Suppose that there exist $\alpha \in (0, \frac{\pi}{2}]$ and $\omega \in \mathbb{R}$ such that $\sigma(P(x)) \subseteq \omega + (\mathbb{C} \setminus \Sigma_{\frac{\pi}{2}+\alpha})$, $x \in \mathbb{R}^n$. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order κ.*

P r o o f. By the well known result of A. Friedman [11, p. 171] and the definition of Shilov parabolic systems, it follows that for each $\omega_0 < \omega$ there exists $\omega' > 0$ such that:

$$\|e^{tP(x)}\| \leq K(1+t+t|x|^N)^{m-1}e^{t(-\omega'|x|^r+\omega_0)},\ t \geq 0,\ x \in \mathbb{R}^n. \tag{3}$$

Using (3), [36, (3.2), p. 190] and the product rule, we have that there exists $\omega'' > 0$ such that, for every $t \geq 0$, $x \in \mathbb{R}^n$, $s \in \mathbb{R}$, $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$:

$$\begin{aligned}\Bigl\|D_x^\alpha(P(x)^l e^{tP(x)}(1+|x|^2)^s)\Bigr\| &\leq K^{l+1}(1+t)^{|\alpha|+m-1}\\ &\times(1+|x|)^{(N-r-1)|\alpha|+(N-r)(m-1)+2s+Nl}e^{t(-\omega''|x|^r+\omega)}.\end{aligned} \tag{4}$$

This simply implies that, for every $s \in \mathbb{R}$, $t > 0$, $l \in \mathbb{N}_0$ and $p \in \mathbb{N}$, $P(x)^l e^{tP(x)}(1+|x|^2)^s \in BC^p(\mathbb{R}^n)$. Using [5, Proposition 12.3], we get that, for every $s \in \mathbb{R}$, $t > 0$ and $p \in \mathbb{N}$, $e^{tP(x)}(1+|x|^2)^s \in \mathcal{A} \equiv \{f \in C_0(\mathbb{R}^n) : \mathcal{F}f \in L^1(\mathbb{R}^n)\}$, where $\mathcal{F}$ denotes the Fourier transform. Set $T_{r'}(t) := (e^{tP(x)}(1+|x|^2)^{-r'})(A)$, $t > 0$, $r' \geq 0$ and notice that [9, Lemma 2.3(c)] implies that, for every $r' > 0$, $T_{r'}(0) := (1+|A|^2)^{-r'} \in L(X)$. Put $T_0(0) := I$.

By [5, (12.2)], it follows that $T_{r'}(t+s)(1+|A|^2)^{-r'}=T_{r'}(t)T_{r'}(s),\ t\geq 0,\ s\geq 0,\ r'>0,\ T_{r'}(t)(1+|A|^2)^{-s}=T_{r'+s}(t),\ t\geq 0,\ s>0,\ r'\geq 0$ and $T_0(t+s)=T_0(t)T_0(s),\ t\geq 0,\ s\geq 0$. Keeping in mind (4), one can simply prove that the mapping $t\mapsto T_{r'}(t),\ t>0$ is infinitely differentiable in the uniform operator topology for all $r'\geq 0$, and that

$$\frac{d^l}{dt^l}T_{r'}(t)=(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'})(A),\ t>0,\ r'\geq 0,\ l\in\mathbb{N}_0. \tag{5}$$

Let $\epsilon>0$ and $s>n/2$ be fixed. Then Stirling's formula, (4) and the inequality

$$\xi^{\sigma}e^{-t\xi^r}\leq(\frac{\sigma}{tr})^{\sigma/r},\ t>0,\ \xi>0,\ \sigma>0, \tag{6}$$

taken together imply that, for every $t>0,\ l\in\mathbb{N}_0,\ x\in\mathbb{R}^n,\ r'\geq 0$, and for every $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\leq k$:

$$\begin{aligned}
&\Big\|D_x^{\alpha}(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'})\Big\|\\
&\leq K^{l+1}(1+t)^{m-1+|\alpha|}(1+|x|)^{(N-r)|\alpha|+(N-r)(m-1)+Nl-2r'}\\
&\quad e^{t(-\omega''|x|^r+\omega)}(1+|x|)^{-|\alpha|}\\
&\leq K^{l+1}(1+t)^{m-1+|\alpha|}l!^{N/r}e^{t\omega}(1+t^{-\frac{(N-r)(m-1)+Nl-2r'}{r}})\\
&\quad(1+t^{-\frac{N-r}{r}})^{|\alpha|}(1+|x|)^{(\epsilon-1)|\alpha|},
\end{aligned}$$

with K being independent of $t,\ x,\ \epsilon$ and r'. The preceding estimate combined with [9, Theorem 2.7(2)] yields that:

$$\begin{aligned}\Big\|(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'-\frac{\epsilon}{2}s})(A)\Big\|\leq\ & K^{l+1}(1+t)^{m-1+\frac{n}{2}}e^{t\omega}l!^{N/r}\\ &\Big(1+t^{-\frac{(N-r)(m-1+\frac{n}{2})+Nl-2r'}{r}}\Big),\end{aligned} \tag{7}$$

provided $t>0,\ l\in\mathbb{N}_0$ and $r'\geq 0$. Invoking the concrete representation of $D_x^{\alpha}e^{tP(x)}$ and an elementary argumentation, we infer that $\lim_{\epsilon\to 0+}(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'-\frac{\epsilon}{2}s})=P(x)^l e^{tP(x)}(1+|x|^2)^{-r'}$ in $H^k(\mathbb{R}^n)$ $(t>0,\ l\in\mathbb{N}_0,\ r'\geq 0)$, which implies by Bernstein's lemma [9, Lemma 2.3(d)] and (7) that:

$$\begin{aligned}\Big\|(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'})(A)\Big\|\leq\ & K^{l+1}(1+t)^{m-1+\frac{n}{2}}e^{t\omega}l!^{N/r}\\ &\Big(1+t^{-\frac{(N-r)(m-1+\frac{n}{2})+Nl-2r'}{r}}\Big).\end{aligned} \tag{8}$$

Hence, $(T_0(t))_{t>0}$ is a semigroup of growth order κ. Thanks to (5) and (8), we have that for every $t>0,\ l\in\mathbb{N}_0$ and $r'\geq 0$:

$$\Bigl\|\frac{d^l}{dt^l}T_{r'}(t)\Bigr\|\leq K^{l+1}(1+t)^{m-1+\frac{n}{2}}e^{t\omega}l!^{N/r}\Bigl(1+t^{-\frac{(N-r)(m-1+\frac{n}{2})+Nl-2r'}{r}}\Bigr), \quad (9)$$

and that (2) holds. Since $T_{r'}(t)(1+|A|^2)^{-s}=T_{r'+s}(t),\ t\geq 0,\ s>0$ and $(T_{r'}(t))_{t\geq 0}$ is a global exponentially bounded $(1+|A|^2)^{-r'}$-regularized semigroup generated by $\overline{P(A)}$ for all $r'>\frac{1}{2}(N-r)(m-1+\frac{n}{2})$ (cf. [36, Theorem 3.1]), we immediately obtain that the c.i.g. $\overline{G}$ of $(T_0(t))_{t>0}$ is contained in $\overline{P(A)}$. To prove that $\overline{P(A)}\subseteq\overline{G}$, one can use [36, Lemma 1.1(a)] and the inclusions $D(P(A)_{|E^m})\subseteq(1+|A|^2)^{-r'}(D(P(A)))\subseteq D(\overline{G})$, where $E\equiv\{\phi(A)x:\phi\in\mathcal{S},\ x\in X\}$ and $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. This completes the proof of (a). The proof of (b) follows from (a) and the arguments given in the proof of [38, Theorem 1.1]. $\square$

Remark 2.2.

(a) Let $r'\geq 0$. Then the proof of Theorem 2.1 implies that $L(X)\supseteq(T_{r'}(t))_{t>0}$ satisfies:

(a1) $T_{r'}(t+s)(1+|A|^2)^{-r'}=T_{r'}(t)T_{r'}(s),\ t,\ s>0,$

(a2) for every $x\in E$, the mapping $t\mapsto T_{r'}(t)x,\ t>0$ is continuous,

(a3) $T_{r'}(t)x=0$ for all $t>0$ implies $x=0$,

(a4) $X_{0,r'}=\bigcup_{t>0}R(T_{r'}(t))$ is dense in X, and

(a5) $\|t^{\frac{(N-r)(m-1+\frac{n}{2})-2r'}{r}}T_{r'}(t)\|=O(1),\ t\in(0,1]$.

The properties (a1)-(a5) betoken that $(T_{r'}(t))_{t>0}$ is a $(1+|A|^2)^{-r'}$-*regularized semigroup of growth order* $\frac{(N-r)(m-1+\frac{n}{2})-2r'}{r}$, provided $r'\in(0,\frac{1}{2}(N-r)(m-1+\frac{n}{2}))$; the results established in this paper do not represent the real basis for further theoretical research of C-regularized semigroups of growth order $\xi>0$. In our concrete situation, we have additionally that the mapping $t\mapsto T_{r'}(t),\ t>0$ is infinitely differentiable in the uniform operator topology.

(b) By the proof of Theorem 2.1, the inequality

$$\Bigl\|\frac{d^l}{dt^l}T_{r'}(t)\Bigr\|\leq K^{l+1}(1+t)^{m-1+\frac{n}{2}}e^{\omega t},\ t>0,\ l\in\mathbb{N}_0,$$

holds provided $(N-r)(m-1)+Nl-2r' \leq -(N-r)k$, and the inequality

$$\Bigl\|\frac{d^l}{dt^l}T_{r'}(t)\Bigr\| \leq \begin{cases} K^{l+1}l!^{N/r}t^{-\frac{(N-r)(m-1+\frac{n}{2})+Nl-2r'}{r}}, \ t\in(0,1], \ l\in\mathbb{N}_0, \\ K^{l+1}l!^{N/r}(1+t)^{m-1+\frac{n}{2}}e^{\omega t}, \ t>1, \ l\in\mathbb{N}_0, \end{cases}$$

holds provided $(N-r)(m-1)+Nl-2r' \in (-(N-r)k,0)$.

(c) Let $l\in\mathbb{N}$. Then, for every $r' > \frac{1}{2}(N-r)(m-1+\frac{n}{2})$, the estimate (9), in regard to the decay rate of derivatives of $(T_{r'}(t))_{t>0}$ in a neighborhood of zero, sharpens the corresponding one given in the formulation of [36, Theorem 3.1(a)].

(d) Now we will explain how one can improve ρ-hypoanalyticity of C-regularized semigroups constructed in [36, Theorem 3.1(a)], provided that $X=L^2(\mathbb{R}^n)$. By [29, Proposition 10.5] (cf. also [29, Remark 2, p. 124] and [36, Lemma 1.2]), we easily infer that there exist numbers $\xi>0, \ a, \ b, \ c>0$ and $\varpi\in\mathbb{R}$ such that:

$$\Bigl\|\frac{d^l}{dt^l}(t^{\xi}T_0(t))\Bigr\| \leq ab^l l!^{\frac{\alpha_2}{r}}e^{\varpi t}t^{-\frac{\alpha_2}{r}(l+c)}, \ l\in\mathbb{N}_0, \ t>0. \tag{10}$$

By the product rule, it follows, that for every compact set $L\subseteq(0,\infty)$, there exists $h_L>0$ such that $\sup_{t\in L,\ l\in\mathbb{N}_0}\|\frac{h_L^l\frac{d^l}{dt^l}T_0(t)}{l!^{\frac{\alpha_2}{r}}}\|<\infty$. It could be of interest to know whether the estimate (10) holds in the case of a general space X.

Now we pay our attention to the *numerical range* of $P(x)$, defined by $\text{n.r.}(P(x)) := \{(P(x)y,y) : y\in\mathbb{R}^n, \ ||y||=1\}, \ x\in\mathbb{R}^n$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{C}^n$ and $||y|| \equiv (y,y)^{1/2}$. Set $\tilde{\Lambda}(P(x)) := \sup\{\Re z : z\in \text{n.r.}(P(x))\}, \ x\in\mathbb{R}^n$.

Theorem 2.3. *Let $r\in(0,N)$, $\omega'>0$ and $\omega''>0$. Put $\kappa_{n.r.} := \frac{n(N-r)}{2r}$.*

(a) *Suppose*

$$\tilde{\Lambda}(P(x)) \leq -\omega'|x|^r+\omega'', \ x\in\mathbb{R}^n. \tag{11}$$

Then the operator $\overline{P(A)}$ is the c.i.g. of a semigroup $(T_0(t))_{t>0}$ of growth order $\kappa_{n.r.}$ which additionally satisfies that the mapping $t\mapsto$

$T_0(t),\ t>0$ is infinitely differentiable in the uniform operator topology and that:

$$\Big\|\frac{d^l}{dt^l}T_0(t)\Big\| \leq K^{l+1}(1+t)^{\frac{n}{2}}e^{\omega t}l!^{N/r}(1+t^{-\frac{(N-r)\frac{n}{2}+Nl}{r}}),\ t>0,\ l\in\mathbb{N}_0, \tag{12}$$

where $\omega \equiv \sup_{x\in\mathbb{R}^n}\tilde{\Lambda}(P(x))$.

(b) *Let $\alpha\in(0,\frac{\pi}{2}]$, $\omega\in\mathbb{R}$, $n.r.(P(x))\subseteq \omega+(\mathbb{C}\setminus\Sigma_{\frac{\pi}{2}+\alpha})$, $x\in\mathbb{R}^n$ and let $P(x)$ be Shilov r-parabolic. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order $\frac{nN}{2r}$.*

(c) *Let $\alpha\in(0,\frac{\pi}{2}]$, $\omega\in\mathbb{R}$, $n.r.(P(x))\subseteq \omega+(\mathbb{C}\setminus\Sigma_{\frac{\pi}{2}+\alpha})$, $x\in\mathbb{R}^n$ and let (11) hold. Then the operator $\overline{P(A)}$ is the c.i.g. of an analytic semigroup $(T_0(t))_{t\in\Sigma_\alpha}$ of growth order $\kappa_{n.r.}$.*

P r o o f. Suppose that (11) holds. Then the Lumer-Phillips theorem implies that there exist $\omega_1>0$ and $L\geq 1$ such that $||e^{tP(x)}||\leq Ke^{-2\omega_1 t|x|^r}e^{\omega t},\ t\geq 0,\ |x|\geq L$ and $||e^{tP(x)}||\leq Ke^{\omega t},\ t\geq 0,\ |x|\leq L$. Making use of this fact, we obtain inductively from [15, (8.10)] that there exists $a\in(0,2\omega_1)$ such that, for every $t\geq 0$, $x\in\mathbb{R}^n$, $r'\geq 0$, $l\in\mathbb{N}_0$, and for every multi-index $\alpha\in\mathbb{N}_0^n$ such that $|\alpha|\leq k$:

$$\|D_x^\alpha e^{tP(x)}\| \leq Ke^{\omega t}(1+t)^{|\alpha|}(1+|x|)^{(N-r)|\alpha|}e^{-at|x|^r}(1+|x|)^{-|\alpha|},\ t>0. \tag{13}$$

By (13) and the product rule, we reveal that, for every $t\geq 0$, $x\in\mathbb{R}^n$, $r'\geq 0$, $l\in\mathbb{N}_0$, and for every $\alpha\in\mathbb{N}_0^n$ such that $|\alpha|\leq k$:

$$\Big\|D_x^\alpha(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'})\Big\| \leq K^{l+1}e^{\omega t}(1+t)^{|\alpha|}(1+|x|)^{(N-r)|\alpha|+Nl-2r'}e^{-at|x|^r}(1+|x|)^{-|\alpha|},\ t>0. \tag{14}$$

In the case $t>0$, Stirling's formula combined with (6) implies that the later does not exceed

$$K^{l+1}e^{\omega t}(1+t)^{|\alpha|}l!^{N/r}(1+t^{-\frac{Nl-2r'}{r}})(1+t^{-\frac{N-r}{r}})^{|\alpha|}(1+|x|)^{-|\alpha|},\ t>0.$$

By the proof of Theorem 2.1, one yields that:

$$\Big\|(P(x)^l e^{tP(x)}(1+|x|^2)^{-r'})(A)\Big\| \leq K^{l+1}e^{\omega t}(1+t)^{\frac{n}{2}}l!^{N/r}(1+t^{-\frac{Nl-2r'}{r}})(1+t^{-\frac{(N-r)n}{2r}}),\ t>0,\ r'\geq 0,\ l\in\mathbb{N}_0. \tag{15}$$

The estimate (12) becomes apparent now, and the remnant of proof follows from the proofs of Theorem 2.1 and [38, Theorem 1.3]. □

Remark 2.4.

(a) The decay rate of derivatives of $(T_{r'}(t))_{t\geq 0}$ in a neighborhood of zero (cf. (15)) improves the corresponding one given in the formulation of [36, Theorem 3.2]. Further on, observe that the $(1+|A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geq 0}$ constructed in the proof of [36, Theorem 3.1], resp. [36, Theorem 3.2], is norm continuous provided $r' > \frac{1}{2}(N-r)(m-1+\frac{n}{2})$, resp. $r' > \frac{1}{4}n(N-r)$. Now we will prove that, in the situation in which the assumptions of [36, Theorem 3.1] are fulfilled, the operator $\overline{P(A)}$ generates an exponentially bounded $(1+|A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geq 0}$ for $r' = \frac{1}{2}(N-r)(m-1+\frac{n}{2})$. By Remark 2.2(a), the only non-trivial thing that should be explained is the strong continuity of $(T_{r'}(t))_{t\geq 0}$ at $t=0$. Using the denseness of E in X, and the boundedness of $||T_{r'}(t)||$ for $0 \leq t \leq 1$, it suffices to show that the mapping $t \mapsto T_{r'}(t)\vec{x}$, $t \geq 0$ is continuous at $t=0$ for every fixed $\vec{x} \in E^m$. Let $L \geq 1$ be sufficiently large. Then

$$\begin{aligned} \left\|e^{tP(x)} - I\right\| &\leq t\|P(x)\| \sum_{n=0}^{\infty} \frac{t^n ||P(x)||^n}{(n+1)!} \\ &\leq t\|P(x)\| \sum_{n=0}^{\infty} \frac{t^n (K(1+L)^N)^n}{(n+1)!}, \ t \geq 0, \ |x| \leq L. \end{aligned} \tag{16}$$

Given $x \in \mathbb{R}^n$, denote by $H(\lambda_1(x), \cdots, \lambda_m(x))$ the convex hull of the set $\{\lambda_1(x), \cdots, \lambda_m(x)\}$. Then, for every $z \in H(\lambda_1(x), \cdots, \lambda_m(x))$, there exist non-negative scalars $\alpha_1, \cdots, \alpha_m$ such that $\alpha_1 + \cdots + \alpha_m = 1$ and $z = \alpha_1\lambda_1(x) + \cdots + \alpha_m\lambda_m(x)$, which implies that there exists $\omega''' > 0$ with $\Re z \leq -\omega'''|x|^r + \omega \leq -|\omega|$, $|x| \geq L$, $z \in H(\lambda_1(x), \cdots, \lambda_m(x))$. By [11, Theorem 2, p. 169], we obtain that for every $t \geq 0$ and $|x| \geq L$,

$$\begin{aligned} &\left\|e^{tP(x)} - I\right\| \leq t\|P(x)\| \left\|\sum_{n=0}^{\infty} \frac{t^n P(x)^n}{(n+1)!}\right\| \\ &\leq Kt(1+|x|)^{mN} \sup_{z \in H(\lambda_1(x), \cdots, \lambda_m(x))} [|f_t(z)| + \cdots + |f_t^{(m-1)}(z)|], \end{aligned} \tag{17}$$

where $f_t(z) \equiv \sum_{n=0}^{\infty}(t^n z^n)/(n+1)!$, $t \geq 0$, $z \in \mathbb{C}$. It is clear that $|f_t(z)| = |\frac{e^{tz}-1}{tz}| \leq t|z|e^{t\Re z}$, $t > 0$, $z \in \mathbb{C} \setminus \{0\}$. This yields the existence of a number $\sigma > 0$ such that $|f_t(z)| + \cdots + |f_t^{(m-1)}(z)| \leq K(1+t)^{\sigma} e^{t\Re z} \leq$

$K(1+t)^{\sigma}e^{-|\omega|t}$, $t>0$, $|x|\geq L$, $z\in H(\lambda_1(x),\cdot\cdot\cdot,\lambda_m(x))$. In view of (17), the above implies:

$$\left\|e^{tP(x)}-I\right\|\leq Kt(1+|x|)^{mN}(1+t)^{\sigma}e^{-|\omega|t},\ t\geq 0,\ |x|\geq L. \tag{18}$$

On the other hand, the concrete representation of $D_x^{\alpha}e^{tP(x)}$ for $0<|\alpha|\leq k$ combined with the estimate (3) indicates that there exists $\sigma_1>0$ such that:

$$\left\|D_x^{\alpha}(e^{tP(x)}-I)\right\|\leq Kt(1+t)^{\sigma_1}(1+|x|)^{\sigma_1}e^{\omega t},\ t\geq 0,\ x\in\mathbb{R}^n,\ 0<|\alpha|\leq k. \tag{19}$$

Now the claimed assertion follows from (16), (18)-(19) and Bernstein's lemma. Notice finally that, in the situation of [36, Theorem 3.2], the operator $\overline{P(A)}$ generates an exponentially bounded $(1+|A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geq 0}$ for $r'=\frac{1}{4}n(N-r)$, and that the above assertions can be simply reformulated for exponentially bounded, analytic $(1+|A|^2)^{-r'}$-regularized semigroups.

(b) Suppose $1<p<\infty$, $X=L^p(\mathbb{R}^n)$ and set $n_X:=n|\frac{1}{2}-\frac{1}{p}|$. Then the growth order of $(T_0(t))_{t>0}$ and the final estimates for the derivatives of $(T_{r'}(t))_{t>0}$ $(r'\geq 0)$, in Theorem 2.1 and Theorem 2.3, can be slightly refined (cf. [9, p. 134], [14, p. 375] and [36, p. 193]):

Theorem 2.1 and its proof: One can replace κ, resp. the term $\frac{n}{2}$, appearing in (9), with $\frac{(N-r)(m-1+n_X)}{r}$, resp. n_X.

Theorem 2.3 and its proof: In the definitions of $\kappa_{n.r.}$ and the growth order of the semigroup appearing in the formulation of (b) of this theorem, one can replace the term $\frac{n}{2}$ with n_X. The same comment can be applied to the derivatives of $(T_{r'}(t))_{t>0}$ $(r'\geq 0)$.

(c) The definition of an (analytic) semigroup of growth order $r>0$ has been recently reconsidered in [26] and [18] by removing the density assumptions. Having this notion in mind, one can prove (cf. [21]), with some obvious technical modifications, that the assertions of Theorem 2.1 and Theorem 2.3 remain true in the case $X=C_b(\mathbb{R}^n)$ or $X=L^{\infty}(\mathbb{R}^n)$.

(d) Suppose that $P(x)$ is Shilov r-parabolic for some $r\in(0,N)$, and denote by $\Omega(T_0)$ the continuity set of the semigroup $(T_0(t))_{t>0}$ given in Theorem 2.1, resp. Theorem 2.3; that is $\Omega(T_0):=\{\vec{x}\in X^m:$

$\lim_{t\to 0+} T_0(t)\vec{x} = \vec{x}\}$. Then $\Omega(T_0)$ contains $R((1+|A|^2)^{-r'})$ for all $r' \geq \frac{1}{2}(N-r)(m-1+\frac{n}{2})$, resp. $r' \geq \frac{1}{4}n(N-r)$, and the abstract Cauchy problem

$$(ACP): \begin{cases} \vec{u} \in C([0,\infty) : X^m) \cap C^\infty((0,\infty) : X^m), \\ \frac{d}{dt}\vec{u}(t) = \overline{P(A)}\vec{u}(t),\ t > 0, \\ \vec{u}(0) = \vec{x}, \end{cases}$$

has a unique solution for all $\vec{x} \in \Omega(T_0)$, improving the corresponding result of Q. Zheng and Y. Li (cf. [36, Lemma 1.2(b)]). As Remark 2.7(c) shows, $R((1+|A|^2)^{-r'})$ can be strictly contained in $\Omega(T_0)$.

(e) Semigroups of growth order $r > 0$ can be also applied in the analysis of time-dependent Shilov parabolic systems ([20], [36]). Suppose that T, $P(\cdot,\cdot)$, T_Δ and $\overline{T_\Delta}$ possess the same meaning as in [36, Section 4]. Then a two-parameter operator family $(W(t,s))_{(t,s)\in\overline{T_\Delta}}$ is called an *evolution system of growth order* $r > 0$ iff the following holds:

(e1) $W(t,r)W(r,s) = W(t,s),\ 0 \leq s \leq r \leq t \leq T$,

(e2) $W(t,t) = I,\ 0 \leq t \leq T$,

(e3) the mapping $(t,s) \mapsto W(t,s)x,\ (t,s) \in T_\Delta$ is continuous for every fixed $x \in X$, and

(e4) $||(t-s)^r W(t,s)|| = O(1),\ (t,s) \in \overline{T_\Delta}$.

Let $\Lambda(P(t,x))$ (resp. $\tilde{\Lambda}(P(t,x))) \leq -\delta|x|^r + \omega,\ t \in [0,T],\ x \in \mathbb{R}^n$, for some $\delta > 0$ and $\omega \in \mathbb{R}$. Using the proofs of [36, Theorem 4.1-Theorem 4.3], Theorem 2.1 and Theorem 2.3, we are in a position to conclude that there exists a unique evolution system of growth order κ (resp. $\kappa_{n.r.}$) satisfying the conditions (a) and (b) of [36, Theorem 4.1]. In such a way, we improve results on the well-posedness of the abstract Cauchy problem [36, (4.4), p. 197]; the growth order of evolution system can be additionally refined in the case of L^p type spaces $(1 < p < \infty)$.

Now we will consider some concrete examples of polynomial operator matrices. The one-dimensional equation describing sound propagation in a viscous gas ([5], [12]) has the form

$$u_{tt} = 2u_{txx} + u_{xx} \tag{20}$$

and after the usual matrix reduction becomes

$$\frac{d}{dt}\vec{u}(t) = P(D)\vec{u}(t), \; t \geq 0,$$

where $D \equiv -i\frac{d}{dx}$ and $P(x) \equiv \begin{bmatrix} 0 & 1 \\ -x^2 & -2x^2 \end{bmatrix}$. Hence, $P(x)$ is Petrovskii correct (cf. [6] and [14] for the notion). In the subsequent theorem, we consider the C-wellposedness of the equation (20); the obtained results are better compared with those clarified in [17, Example 1.2.9(vi)].

Theorem 2.5. *Let $r > \frac{1}{2}$. Then the operator $\overline{P(A)}$ generates an exponentially bounded, analytic $(1+|A|^2)^{-r}$-regularized semigroup $(T_r(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$. Furthermore, the semigroup $(T_r(t))_{t\in\Sigma_{\frac{\pi}{2}}}$ can be extended to $\overline{\Sigma_{\frac{\pi}{2}}}$, $||T_r(t)|| \leq K(1+|t|)^{\frac{3}{2}} e^{\frac{1}{2}t\sin(|\arg(t)|)}, \; t \in \overline{\Sigma_{\frac{\pi}{2}}} \setminus \{0\}$ and the following holds:*

(a) *The mapping $t \mapsto T_r(t), \; t \in \overline{\Sigma_{\frac{\pi}{2}}}$ is continuous.*

(b) *$(T_r(it))_{t\in\mathbb{R}}$ is an exponentially bounded, norm continuous $(1+|A|^2)^{-r}$-regularized group generated by $i\overline{P(A)}$.*

Proof. We exploit the following elementary fact: Suppose $a, \; b, \; c, \; d \in \mathbb{C}$, $\alpha, \; \beta$ are the eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\alpha \neq \beta$ and $n \in \mathbb{N}$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n = \frac{1}{\alpha-\beta} \begin{bmatrix} (\alpha-d)\alpha^n - (\beta-d)\beta^n & b(\alpha^n-\beta^n) \\ c(\alpha^n-\beta^n) & (\alpha-a)\alpha^n - (\beta-a)\beta^n \end{bmatrix}.$$

Let $\sqrt{\cdot}$ be taken as the principle branch. Put $\alpha(x) := -x^2+\sqrt{x^4-x^2}$, $x \in \mathbb{R}$, $\beta(x) := -x^2 - \sqrt{x^4-x^2}$, $x \in \mathbb{R}$ and $\kappa(x) := \frac{2x^3-x}{\sqrt{x^4-x^2}}$, $x \in \mathbb{R} \setminus \{-1,0,1\}$. Fix, for a moment, $\gamma \in [-\pi/2, \pi/2]$. Noticing that $e^{i\gamma}\alpha(x)$, $e^{i\gamma}\beta(x)$ are the eigenvalues of $e^{i\gamma}P(x), \; x \in \mathbb{R}$, we easily infer that:

$$e^{te^{i\gamma}P(x)} = \frac{1}{2\sqrt{x^4-x^2}} \begin{bmatrix} A_{11}(t,x,\gamma) & A_{12}(t,x,\gamma) \\ A_{21}(t,x,\gamma) & A_{22}(t,x,\gamma) \end{bmatrix}, \; t \geq 0, \; x \in \mathbb{R}\setminus\{-1,0,1\},$$

where:

$$A_{11}(t,x,\gamma) = -\beta(x)e^{te^{i\gamma}\alpha(x)} + \alpha(x)e^{te^{i\gamma}\beta(x)}, \; A_{12}(t,x,\gamma) = e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)},$$

$$A_{21}(t,x,\gamma) = -x^2A_{12}(t,x,\gamma) \text{ and } A_{22}(t,x,\gamma) = \alpha(x)e^{te^{i\gamma}\alpha(x)} - \beta(x)e^{te^{i\gamma}\beta(x)}.$$

Clearly, $\mathrm{Re}(e^{i\gamma}(-x^2 \pm \sqrt{x^4-x^2})) \leq 0$ provided $x \in (-\infty, -1] \cup [1, \infty)$, and $\mathrm{Re}(e^{i\gamma}(-x^2 \pm \sqrt{x^4-x^2})) \leq -x^2\cos\gamma + |x|\sqrt{1-x^2}\sin\gamma \leq \frac{1}{2}\sin|\gamma|$ provided $x \in (-1, 1)$. Applying the Darboux inequality, one gets:

$$\begin{aligned} &|e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)}| \\ &\quad \leq 2|\sqrt{x^4-x^2}|t \max_{\xi \in [e^{i\gamma}\alpha(x), e^{i\gamma}\beta(x)]} e^{t\mathrm{Re}\xi} \\ &\quad \leq 2|\sqrt{x^4-x^2}|te^{\frac{1}{2}t\sin|\gamma|},\ t \geq 0,\ x \in \mathbb{R}. \end{aligned} \tag{21}$$

One obtains similarly that the following estimate holds:

$$|e^{te^{i\gamma}\alpha(x)} + e^{te^{i\gamma}\beta(x)}| \leq 2(1 + e^{\frac{1}{2}t\sin|\gamma|}),\ t \geq 0,\ x \in \mathbb{R}. \tag{22}$$

Using (21)-(22), it follows that:

$$\|e^{te^{i\gamma}P(x)}\| \leq K(1+t)e^{\frac{1}{2}t\sin|\gamma|},\ t \geq 0,\ x \in \mathbb{R}. \tag{23}$$

Taking into account (23) and the Wilcox identity $\frac{d}{dx}e^{tP(x)} = \int_0^t e^{(t-s)P(x)} \frac{d}{dx}P(x)e^{sP(x)}ds$, $t \geq 0$, $x \in \mathbb{R}$, we get that:

$$\Big\|\frac{d}{dx}e^{te^{i\gamma}P(x)}\Big\| \leq K(1+t)^2(1+|x|)e^{\frac{1}{2}t\sin|\gamma|},\ t \geq 0,\ x \in \mathbb{R}. \tag{24}$$

By (23)-(24) and [9, Theorem 2.7(2)], one gets that, for every $t \geq 0$ and $i,\ j \in \{1, 2\}$, the mapping $x \mapsto (1+x^2)^{-r}A_{ij}(t, x, \gamma)/2\sqrt{x^4-x^2}$, $x \in \mathbb{R}$ belongs to the space $\mathcal{A}$ and that:

$$\Big\|\Big((1+x^2)^{-r}A_{ij}(t, x, \gamma)/2\sqrt{x^4-x^2}\Big)(A)\Big\| \leq K(1+t)^{\frac{3}{2}}e^{\frac{1}{2}t\sin|\gamma|},\ t \geq 0. \tag{25}$$

Set, for every $t \geq 0$, $T_{r,\gamma}(t) := (e^{te^{i\gamma}P(x)}(1+x^2)^{-r})(A)$. Then $T_{r,\gamma}(t+s)(1+|A|^2)^{-r} = T_{r,\gamma}(t)T_{r,\gamma}(s)$, $t,\ s \geq 0$, $T_{r,\gamma}(0) = (1+|A|^2)^{-r}$ and $T_{r,\gamma}(t)(1+|A|^2)^{-s} = T_{r+s,\gamma}(t)$, $s > 0$, $t \geq 0$. Then the proof of [38, Theorem 1.2] combined with the above equalities and [17, Corollary 2.4.11] indicates that, for a sufficiently large $s > 0$, $(T_{r,\gamma}(t)(1+|A|^2)^{-s})_{t \geq 0}$ is an exponentially bounded $(1+|A|^2)^{-r-s}$-regularized semigroup generated by $e^{i\gamma}\overline{P(A)}$ ($\gamma \in (-\pi/2, \pi/2)$). With (25) in view, we obtain that, for every $\gamma \in (-\pi/2, \pi/2)$, $(T_{r,\gamma}(t))_{t \geq 0}$ is an exponentially bounded, norm continuous $(1+|A|^2)^{-r}$-regularized semigroup generated by $e^{i\gamma}\overline{P(A)}$. An application of [16, Theorem 2.3] gives that $\overline{P(A)}$ is the integral generator of an exponentially bounded, analytic $(1+|A|^2)^{-r}$-regularized semigroup

$(T_r(t) \equiv (e^{tP(x)}(1+x^2)^{-r})(A))_{t\in\Sigma_{\frac{\pi}{2}}}$. By the foregoing, $(T_r(t))_{t\in\Sigma_{\frac{\pi}{2}}}$ can be extended to $\overline{\Sigma_{\frac{\pi}{2}}}$ and $||T_r(t)|| \leq K(1+|t|)^{\frac{3}{2}}e^{\frac{1}{2}t\sin(|\arg(t)|)}$, $t \in \overline{\Sigma_{\frac{\pi}{2}}} \setminus \{0\}$. Assume $t,\ s \in \overline{\Sigma_{\frac{\pi}{2}}}$ and $ts \neq 0$. Denote by $\mathcal{F}^{-1}$ the inverse Fourier transform. Then [5, (12.2)] implies that there exists $M > 0$ such that:

$$\Big\|T_r(t) - T_r(s)\Big\| \leq$$

$$M \sum_{1\leq i,j\leq 2} \Big\|\mathcal{F}^{-1}\Big(\frac{1}{\sqrt{x^4-x^2}}(A_{ij}(|t|,x,\arg(t)) - A_{ij}(|s|,x,\arg(s)))(1+x^2)^{-r}\Big)\Big\|_{L^1(\mathbb{R})},$$

which immediately implies (a) by [35, Lemma 5.2, p. 20]. The proof of (b) is simple and therefore omitted. □

Before proceeding further, we would like to observe that the linearized Fitz-Hugh-Nagumo equations (cf. [31, p. 39]), which describe the propagation of nerve impulse, can be also treated with C-regularized semigroups since the corresponding matrix $P(x) \equiv \begin{bmatrix} -x^2+a & -1 \\ b & 0 \end{bmatrix}$ $(a,\ b \in \mathbb{C},\ x \in \mathbb{R})$ is Petrovskii correct. The isothermal motion of a one-dimensional body with small viscosity and capillarity ([3], [10], [38]) is described, in the simplest situation, by the system:

$$\begin{cases} u_t = 2au_{xx} + bv_x - cv_{xxx}, \\ v_t = u_x, \\ u(0) = u_0,\ v(0) = v_0, \end{cases}$$

where $a,\ b,\ c > 0$. The associated polynomial matrix

$$P(x) = \begin{bmatrix} -2ax^2 & ibx + icx^3 \\ ix & 0 \end{bmatrix}$$

is Shilov 2-parabolic. It is well known ([10]) that $P(D)$ does not generate a strongly continuous semigroup in $L^1(\mathbb{R}) \times L^1(\mathbb{R})$. Set

$$e^{te^{i\gamma}P(x)} := \begin{bmatrix} B_{11}(t,x,\gamma) & B_{12}(t,x,\gamma) \\ B_{21}(t,x,\gamma) & B_{22}(t,x,\gamma) \end{bmatrix},\ \gamma \in (0,\frac{\pi}{2}],\ t \geq 0,\ x \in \mathbb{R},$$

and, for every $x \in \mathbb{R} \setminus \{0, \pm\sqrt{\frac{b}{a^2-c}}\}$, $\alpha(x) := -ax^2 + \sqrt{(a^2-c)x^4 - bx^2}$, $\beta(x) := -ax^2 - \sqrt{(a^2-c)x^4 - bx^2}$ and $\chi(x) := \frac{4x^3(a^2-c)-2bx}{\alpha(x)-\beta(x)}$. Then

$$\begin{bmatrix} B_{11}(t,x,\gamma) & B_{12}(t,x,\gamma) \\ B_{21}(t,x,\gamma) & B_{22}(t,x,\gamma) \end{bmatrix} = \frac{1}{\alpha(x)-\beta(x)} \begin{bmatrix} A_{11}(t,x,\gamma) & A_{12}(t,x,\gamma) \\ A_{21}(t,x,\gamma) & A_{22}(t,x,\gamma) \end{bmatrix},$$

where, for every $t \geq 0$, $\gamma \in (0, \frac{\pi}{2}]$ and $x \in \mathbb{R} \setminus \{0, \pm\sqrt{\frac{b}{a^2-c}}\}$:

$$A_{11}(t,x,\gamma) = \alpha(x)e^{te^{i\gamma}\alpha(x)} + (\alpha(x) + 2ax^2)e^{te^{i\gamma}\beta(x)},$$

$$A_{12}(t,x,\gamma) = (ibx + icx^3)(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)}),$$

$$A_{21}(t,x,\gamma) = ix(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)}) \text{ and}$$

$$A_{22}(t,x,\gamma) = (\alpha(x) + 2ax^2)(e^{te^{i\gamma}\alpha(x)} + e^{te^{i\gamma}\beta(x)}).$$

In the case $a^2 \neq c$, the following theorem improves the results of Q. Zheng, Y. Li [36] and Q. Zheng [37]-[38].

Theorem 2.6.

(a) *Let $a^2 - c < 0$ and $r' > \frac{1}{2}$. Then $\overline{P(A)}$ is the integral generator of an analytic $(1 + |A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geq 0}$ of angle $\theta :=$ $\arctan \frac{a}{\sqrt{c-a^2}}$ and there exists a function $p : (-\theta, \theta) \to (0, \infty)$ such that:*

$$\|T_{r'}(z)\| \leq K(1 + |z|)^{\frac{3}{2}} e^{p(\arg(z))\sin(|\arg(z)|)|z|}, \; z \in \Sigma_\theta.$$

Furthermore, the mapping $z \mapsto T_{r'}(z)$, $z \in \overline{\Sigma_{\theta_0}}$ is continuous for every $\theta_0 \in (0, \theta)$.

(b) *Let $a^2 - c > 0$ and $r' > \frac{1}{2}$. Then $\overline{P(A)}$ generates an analytic $(1 + |A|^2)^{-r'}$-regularized semigroup $(T_{r'}(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$, and the following holds:*

$$\|T_{r'}(z)\| \leq K(1 + |z|)^{\frac{3}{2}} e^{\sqrt{b}\sin(|\arg(z)|)|z|}, \; z \in \Sigma_{\frac{\pi}{2}}.$$

Furthermore, the mapping $z \mapsto T_{r'}(z)$, $z \in \overline{\Sigma_{\theta_0}}$ is continuous for every $\theta_0 \in (0, \frac{\pi}{2})$.

(c) *Let $a^2 - c > 0$ and $r' > \frac{3}{4}$. Then $\overline{P(A)}$ generates an exponentially bounded, norm continuous $(1 + |A|^2)^{-r'}$-regularized cosine function $(C_{r'}(t))_{t\geq 0}$.*

P r o o f. Notice that, for every $t \geq 0$, $x \in \mathbb{R} \setminus \{0, \pm\sqrt{\frac{b}{a^2-c}}\}$ and

$\gamma \in (-\pi, \pi]$:

$$2\frac{d}{dx}B_{11}(t,x,\gamma)$$
$$= \frac{\frac{1}{2}(\alpha(x)-\beta(x))(-2ax+\chi(x)) - \alpha(x)\chi(x)}{(a^2-c)x^4 - bx^2} e^{te^{i\gamma}\alpha(x)}$$
$$+ \frac{2\alpha(x)}{\alpha(x)-\beta(x)} te^{i\gamma}(-2ax+\chi(x))e^{te^{i\gamma}\alpha(x)}$$
$$+ \frac{\frac{1}{2}(\alpha(x)-\beta(x))(2ax+\chi(x)) - (\alpha(x)+2ax^2)\chi(x)}{(a^2-c)x^4 - bx^2} e^{te^{i\gamma}\beta(x)}$$
$$- \frac{2\alpha(x)+4ax^2}{\alpha(x)-\beta(x)} te^{i\gamma}(2ax+\chi(x))e^{te^{i\gamma}\beta(x)},$$

$$2\frac{d}{dx}B_{12}(t,x,\gamma)$$
$$= \frac{\frac{1}{2}(ib+3icx^2)(\alpha(x)-\beta(x)) - (ibx+icx^3)\chi(x)}{(a^2-c)x^4 - bx^2}(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)})$$
$$+ \frac{2(ibx+icx^3)}{\alpha(x)-\beta(x)} te^{i\gamma}[(-2ax+\chi(x))e^{te^{i\gamma}\alpha(x)} + (2ax+\chi(x))e^{te^{i\gamma}\beta(x)}],$$

$$2\frac{d}{dx}B_{21}(t,x,\gamma)$$
$$= \frac{\frac{1}{2}i(\alpha(x)-\beta(x)) - ix\chi(x)}{(a^2-c)x^4 - bx^2}(e^{te^{i\gamma}\alpha(x)} - e^{te^{i\gamma}\beta(x)})$$
$$+ \frac{2ixte^{i\gamma}}{\alpha(x)-\beta(x)}[(-2ax+\chi(x))e^{te^{i\gamma}\alpha(x)} + (2ax+\chi(x))e^{te^{i\gamma}\beta(x)}]$$

and

$$2\frac{d}{dx}B_{22}(t,x,\gamma) = 2\frac{d}{dx}B_{11}(t,x,\gamma) + 2a\frac{4x(\alpha(x)-\beta(x)) - x^2\chi(x)}{(a^2-c)x^4 - bx^2}.$$

We first consider the case $a^2 - c < 0$. If $\gamma \in (-\theta, \theta)$, then there exist $p(\gamma) > 0$ and $q(\gamma) > 0$ such that:

$$\max(\Re(e^{i\gamma}\alpha(x)), \Re(e^{i\gamma}\beta(x)))$$
$$\leq -ax^2\cos\gamma + b|x|\sqrt{b-(a^2-c)}\sin|\gamma| \leq -q(\gamma)x^2 + p(\gamma)\sin|\gamma|.$$

By (3), we obtain that, for every $t \geq 0$ and $x \in \mathbb{R}$:

$$\left\| e^{te^{i\gamma}P(x)} \right\| \leq K(1+t+t|x|^3)e^{-tq(\gamma)x^2+p(\gamma)t\sin|\gamma|} \leq K(1+t)(1+|x|)e^{tp(\gamma)\sin|\gamma|}. \tag{26}$$

Let $L \geq 1$ be sufficiently large. Then the inequality $\max(|txe^{-tq(\gamma)x^2}|, |tx^2e^{-tq(\gamma)x^2}|) \leq K,\ t \geq 0,\ |x| \geq L$ and the concrete representation of $\frac{d}{dx}e^{zP(x)},\ z \in \mathbb{C}$ imply that:

$$\left\| \frac{d}{dx} e^{te^{i\gamma}P(x)} \right\| \leq K,\ t \geq 0,\ |x| \geq L. \tag{27}$$

By (26), we easily infer that:

$$\left\| \frac{d}{dx} e^{te^{i\gamma}P(x)} \right\| \leq K(1+t)^2 e^{tp(\gamma)\sin|\gamma|},\ t \geq 0,\ |x| \leq L. \tag{28}$$

Using (26)-(28) and the product rule, it follows that:

$$\left\| \frac{d}{dx}(e^{te^{i\gamma}P(x)}(1+x^2)^{-r'}) \right\| \leq K(1+t)^2 e^{p(\gamma)t\sin|\gamma|}(1+|x|)^{-2r'},\ t \geq 0,\ x \in \mathbb{R}.$$

Now one can apply [9, Theorem 2.7(2)] in order to see that:

$$\left\| (e^{te^{i\gamma}P(x)}(1+x^2)^{-r'})(A) \right\| \leq K(1+t)^{\frac{3}{2}} e^{p(\gamma)t\sin|\gamma|},\ t \geq 0. \tag{29}$$

Put $T_{r',\gamma}(t) := (e^{te^{i\gamma}P(x)}(1+x^2)^{-r'})(A),\ t \geq 0,\ \gamma \in (-\theta, \theta)$ and $T_{r'}(z) := (e^{zP(x)}(1+x^2)^{-r'})(A),\ z \in \Sigma_\theta$. By (29) and the proof of [38, Theorem 1.1], one gets that, for every $\gamma \in (-\theta, \theta)$, $(T_{r',\gamma}(t))_{t\geq 0}$ is a global $(1+|A|^2)^{-r'}$-regularized semigroup generated by $e^{i\gamma}\overline{P(A)}$. By [16, Theorem 2.3], we have that $\overline{P(A)}$ generates an analytic $(1+|A|^2)^{-r'}$-regularized semigroup of angle θ. The inequality stated in (a) is a consequence of (29), while the continuity of mapping $z \mapsto T_{r'}(z)$, $z \in \overline{\Sigma_{\theta_0}}$ $(\theta_0 \in (0, \theta))$ follows from the estimate

$$\begin{aligned} &\left\| T_{r'}(t_1 e^{i\gamma_1}) - T_{r'}(t_2 e^{i\gamma_2}) \right\| \\ &\quad \leq K \sum_{1\leq i,j\leq 2} \left\| \mathcal{F}^{-1}\Big((B_{ij}(t_1, x, \gamma_1) - B_{ij}(t_2, x, \gamma_2))(1+x^2)^{-r'} \Big) \right\|_{L^1(\mathbb{R})}, \end{aligned}$$

for any $t_1,\ t_2 \geq 0, |\gamma_1|,\ |\gamma_2| \leq \theta_0$, and the proof of [35, Lemma 1.2]. In order to prove (b), one can use a similar argumentation and the estimate $\max(\Re(e^{i\gamma}\alpha(x)), \Re(e^{i\gamma}\beta(x))) \leq (\sqrt{a^2-c} - a)x^2\cos\gamma + \frac{b}{\sqrt{a^2-c}}\sin|\gamma|,\ \gamma \in$

$(-\frac{\pi}{2}, \frac{\pi}{2})$, which holds in the case $a^2 > c$. To prove (c), set $f(t,x) := \sum_{n=0}^{\infty} \frac{t^{2n}P(x)^n}{(2n)!},\ t \geq 0,\ x \in \mathbb{R}$. Then

$$f(t,x) = \frac{1}{\alpha(x) - \beta(x)} \begin{bmatrix} C_{11}(t,x) & C_{12}(t,x) \\ C_{21}(t,x) & C_{22}(t,x) \end{bmatrix},$$

where, for any $t \geq 0$ and $x \in \mathbb{R} \setminus \{0, \pm\sqrt{\frac{b}{a^2-c}}\}$:

$$C_{11}(t,x) = \alpha(x)\cosh(t\sqrt{\alpha(x)}) + (\alpha(x) + 2ax^2)\cosh(t\sqrt{\beta(x)}),$$
$$C_{12}(t,x) = (ibx + icx^3)(\cosh(t\sqrt{\alpha(x)}) - \cosh(t\sqrt{\beta(x)})),$$
$$C_{21}(t,x) = ix(\cosh(t\sqrt{\alpha(x)}) - \cosh(t\sqrt{\beta(x)})) \text{ and}$$
$$C_{22}(t,x) = (\alpha(x) + 2ax^2)(\cosh(t\sqrt{\alpha(x)}) + \cosh(t\sqrt{\beta(x)})).$$

Then $\max(|\cosh(t\sqrt{\alpha(x)})|, |\cosh(t\sqrt{\beta(x)})|) \leq 1,\ t \geq 0,\ |x| \geq L$. Using this estimate, the product rule as well as the representation formulae for $f(t,x)$ and $\frac{d}{dx}f(t,x)$, it follows that there exist two exponentially bounded positive functions $f_0(t)$ and $f_1(t)$ such that $||f(t,x)(1+x^2)^{-r'}|| \leq Kf_0(t)(1+|x|)^{1-2r'},\ t \geq 0,\ x \in \mathbb{R}$ and $||\frac{d}{dx}(f(t,x)(1+x^2)^{-r'})|| \leq K(f_0(t)+f_1(t))(1+|x|)^{1-2r'},\ t \geq 0,\ x \in \mathbb{R}$. Define $C_{r'}(t) := (f(t,x)(1+x^2)^{-r'})(A),\ t \geq 0$. Then it is obvious that $(C_{r'}(t))_{t\geq 0}$ is an exponentially bounded, norm continuous $(1+|A|^2)^{-r'}$-regularized cosine function generated by $\overline{P(A)}$. □

Remark 2.7.

(a) It is not clear how one can prove the assertion of Theorem 2.6 in the case $a^2 = c$. The main problem is that, for every fixed $t \geq 0$ and $\gamma \in [0, \frac{\pi}{2}]$, the function $x \mapsto |\frac{d}{dx}B_{12}(t,x,\gamma)|,\ x \in \mathbb{R}$ is no longer bounded as $|x| \to \infty$. Nevertheless, the operator $\overline{P(A)}$ generates an exponentially bounded, analytic $(1+|A|^2)^{-3/4}$-regularized semigroup $(T_{3/4}(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$; as before, one can additionally refine the results obtained in Theorem 2.5-Theorem 2.6, provided that $X = L^p(\mathbb{R})$ $(1 < p < \infty)$.

(b) With the exception of the continuity of mapping $z \mapsto T_{r'}(z),\ z \in \overline{\Sigma_{\theta_0}}$, the assertions of Theorem 2.6(a)-(b) continue to hold in the case $r' = \frac{1}{2}$ (cf. [17, Example 1.2.9(vii.1)-(vii.2)] and Remark 2.4(a)). Furthermore, Theorem 2.6(c) is a proper extension of the result stated in [17, Example 1.2.9(vii.3)].

(c) Let $a^2 = c$, $r' = \frac{1}{2}$ and $r'' = \frac{3}{4}$. Then $\lim_{t\to 0+}(B_{12}(t,x,0))(A)g = 0$ and $\lim_{t\to 0+}(B_{22}(t,x,0))(A)g = g$ for all $g \in R((1+|A|^2)^{-r''})$. Keeping in mind the proof of Theorem 2.6, we get that $||(B_{11}(t,x,0)(1+x^2)^{-\frac{1}{2}})(A)|| \leq K(1+t)^{\frac{3}{2}}$ and $||(B_{21}(t,x,0)(1+x^2)^{-\frac{1}{2}})(A)|| \leq K(1+t)^{\frac{3}{2}}$, $t \geq 0$, $x \in \mathbb{R}$. By the denseness of E in X, the above implies $\lim_{t\to 0+}(B_{11}(t,x,0))(A)f = f$ for all $f \in R((1+|A|^2)^{-r'})$. On the other hand, the representation formulae for $B_{21}(t,x,0)$ and $\frac{d}{dx}B_{21}(t,x,0)$, as well as the proofs of Theorem 2.1 and Theorem 2.6, taken together imply $\lim_{t\to 0+}||(B_{21}(t,x,0))(A)|| = 0$. Hence, $R((1+|A|^2)^{-r'}) \times R((1+|A|^2)^{-r''}) \subseteq \Omega(T_0)$.

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Faculty of Technical Science
University of Novi Sad
Trg Dositeja Obradovića 6
21000 Novi Sad
Serbia