Divulgaciones Matemáticas Vol. 13 No. 2(2005), pp. 83–89

An Extension of Witzgall's Result on Convex Metrics

Una Extensión del Resultado de Witzgall sobre Métricas Convexas

Luca Guerrini (guerrini@rimini.unibo.it)
Università degli Studi di Bologna
Dipartimento di Matematica per le Scienze Economiche e Sociali
Viale Filopanti 5, 40126 - Bologna, Italy

Abstract

In [5] Witzgall proved that any weak metric defined on a real vector space, which is convex in each of the arguments, is determined by a weak gauge. In this paper we extend this result to any continuous weak metric defined on the positive cone in a totally ordered vector space, which is convex in each of the arguments.
Key words and phrases: Weak metric, weak gauge, convex metric.

Resumen

En [5] Witzgall probó que cualquier métrica débil definida sobre un espacio vectorial real que sea convexa en cada uno de sus argumento, está determinada por una gauge débil. En este trabajo se extiende ese resultado a cualquier métrica débil continua definida sobre el cono positivo en un espacio vectorial totalmente ordenado, que sea convexa en cada uno de sus argumentos.
Palabras y frases clave: métrica débil, gauge débil, métrica convexa.

1 Introduction

In continuous location theory the concept of distance is of fundamental importance and many different metrics may be of interest according to the applications. Witzgall was the first to point out the fact that practical distances are

Received 2004/05/19. Revised 2005/11/11. Accepted 2005/11/12.
MSC (2000): 46A40.

seldom symmetric [6]. Let E be a real vector space. A weak gauge on E is a real valued function $\gamma : E \to \mathbb{R}^+$ satisfying (G1) $\gamma(u) \geq 0$ for any $u \in E$, (G2) $\gamma(ru) = r\gamma(u)$ for any $r \geq 0$ and $u \in E$, (G3) $\gamma(u+v) \leq \gamma(u) + \gamma(v)$ for any $u, v \in E$. Any weak gauge γ defines a weak metric d in E by $d(x,y) = \gamma(x-y)$, $x, y \in E$, i.e. a map $d : E \times E \to \mathbb{R}^+$ such that $d(x,y) \geq 0$, $d(x,x) = 0$ and $d(x,z) \leq d(x,y) + d(y,z)$ hold for all $x, y, z \in E$. Since γ is a convex function, the derived distance d is a convex function. This implies that, for any $x \in E$, each of the functions $d(x,\cdot)$ and $d(\cdot,x)$ is convex on E. In [5] Witzgall proved that the converse holds (see e.g. [2], [3], [4] for asymmetric distance problems concerning distances derived from gauges). The aim of this paper is to show that this is also true for any continuous convex weak metric defined on the positive cone C in a totally ordered vector space E.

2 Main results

Let us recall some definitions [1]. An ordered set $(E, \leq)$ is a non-empty set E equipped with a relation $\leq$ which is reflexive, antisymmetric and transitive. If, in addition, for any two elements $x, y \in E$ either $x \leq y$ or $y \leq x$, then $(E, \leq)$ is called a totally ordered set. A lattice is an ordered set $(E, \leq)$ such that any two elements have a least upper bound and a greatest lower bound. Any totally ordered set is clearly a lattice. A real vector space E which is also an ordered set is called an ordered vector space if the order and the vector space structure are compatible. This means that if $x, y \in E$, $x \leq y$ implies $x + z \leq y + z$ for all $z \in E$ and $\alpha x \leq \alpha y$ for all real $\alpha \geq 0$. If, in addition, E is a lattice, then we speak of a Riesz space or a vector lattice. $\mathbb{R}$ is clearly an example of a totally ordered vector space. Another example is given by $\mathbb{R}^n$ $(n \geq 2)$ equipped with the so-called lexicographical order, i.e. $x = (x_1, \ldots, x_n) < (y_1, \ldots, y_n) = y$, if there exists $k \in \{0, 1, \ldots, n\}$ such that $x_1 = y_1, \ldots, x_k = y_k$ and $x_{k+1} < y_{k+1}$.

Let E be a totally ordered vector space and let $C = \{x \in E : x \geq 0\}$ be its positive cone. $C + C \subset C$ and $\alpha C \subset C$ for all $\alpha \geq 0$. Let $\gamma : E \to \mathbb{R}^+$ be a weak gauge. The function $d_\gamma : C \times C \to \mathbb{R}^+$ defined by $d_\gamma(x,y) = \gamma(x-y)$, $x, y \in C$, is a weak metric on C that is convex in each of the arguments. The next result says the converse also holds when the weak metric is continuous.

Main Theorem. *Let E be a topological totally ordered vector space and let C be its positive cone. Any continuous weak metric $d : C \times C \to \mathbb{R}^+$ on C that is convex in each of the arguments comes from a weak gauge $\gamma : E \to \mathbb{R}^+$.*

We need some preliminary results.

Proposition. *Let E be a totally ordered vector space and let C be its positive cone. Let d be a weak metric on C convex in each of the arguments.*

(*i*) *Let $z \in C$. For $w \geq -z$, $0 \leq \beta \leq 1$, and for $w \geq -z/\beta$, $\beta > 1$:*

$$d(z+\beta w, z) = \beta d(z+w, z); \tag{1}$$

$$d(z, z+\beta w) = \beta d(z, z+w). \tag{2}$$

In particular, for $w \geq 0$, $\beta \geq 0$:

$$d(\beta w, 0) = \beta d(w, 0),\ d(0, \beta w) = \beta d(0, w).$$

(*ii*) *Let $x^*, y^* \in C$. For $-min\{x^*, y^*\} \leq u^* \leq min\{x^*, y^*\}$:*

$$d(x^*, y^*) \leq d(x^* + u^*, y^* + u^*). \tag{3}$$

(*iii*) *Let $x, y \in C$. For $-min\{x, y\}/2 \leq u \leq min\{x, y\}$:*

$$d(x+u, y+u) = d(x, y). \tag{4}$$

Proof. (i) Let start proving (1). If $\beta = 0, 1$, the result is immediate. Let $\beta \in (0, 1)$. By the convexity of d in the first argument

$$\begin{aligned} d(z+\beta w, z) &= d((1-\beta)z + \beta(z+w), z) \\ &\leq (1-\beta)d(z,z) + \beta d(z+w, z) = \beta d(z+w, z) \end{aligned}$$

and by that in the second argument

$$\begin{aligned} d(z+w, z) &\leq d\left(z+w, \beta(z+w) + (1-\beta)z\right) + d(z+\beta w, z) \\ &\leq \beta d(z+w, z+w) + (1-\beta)d(z+w, z) + d(z+\beta w, z) \\ &= (1-\beta)d(z+w, z) + d(z+\beta w, z) \end{aligned}$$

i.e. the inequality in the opposite direction. Let $\beta \in (1, +\infty)$. As a consequence of the previous case

$$d(z+\beta w, z) = \beta\,\frac{1}{\beta}d(z+\beta w, z). = \beta d(z+w, z).$$

Similarly the proof of (2).

(ii) Let $x^* \geq y^*$. Let $\alpha \geq 1$. Taking $z = y^*$, $w = x^* - y^*$ and $\beta = \alpha$ in (1) yields

$$\begin{aligned} d(x^*, y^*) &= \frac{1}{\alpha} d(y^* + \alpha(x^* - y^*), y^*) \\ &\leq \frac{1}{\alpha} d(y^* + \alpha(x^* - y^*), y^* + u^*) + \frac{1}{\alpha} d(y^* + u^*, y^*). \end{aligned}$$

Next, again by (1), but with $z = u^* + y^*$, $w = x^* - y^* - u^*/\alpha$ and $\beta = 1/\alpha$, we see that

$$\begin{aligned} \frac{1}{\alpha} d(y^* + \alpha(x^* - y^*), y^* + u^*) &= \frac{1}{\alpha} d(y^* + u^* + \alpha(x^* - y^* - u^*/\alpha), y^* + u^*) \\ &= d(u^* + x^* - u^*/\alpha, y^* + u^*) \\ &\leq d(u^* + x^* - u^*/\alpha, x^* + u^*) + d(x^* + u^*, y^* + u^*) \\ &= \frac{1}{\alpha} d(x^*, x^* + u^*) + d(x^* + u^*, y^* + u^*), \end{aligned}$$

with the last equality following from (1) with $z = x^* + u^*$, $w = -u^*$ and $\beta = 1/\alpha$. In conclusion, we have showed that

$$d(x^*, y^*) \leq \frac{1}{\alpha} d(x^*, x^* + u^*) + d(x^* + u^*, y^* + u^*) + \frac{1}{\alpha} d(y^* + u^*, y^*),$$

and the statement now follows from the above as $\alpha \to +\infty$. The proof when $x^* \leq y^*$ and $\alpha \geq 1$ is analogous. In fact, (2) with $z = x^*$, $w = y^* - x^*$ and $\beta = \alpha$ yields

$$\begin{aligned} d(x^*, y^*) &= \frac{1}{\alpha} d(x^*, x^* + \alpha(y^* - x^*)) \\ &\leq \frac{1}{\alpha} d(x^*, x^* + u^*) + \frac{1}{\alpha} d(x^* + u^*, x^* + \alpha(y^* - x^*)). \end{aligned}$$

Now proceed as done before.
(iii) Let $x \geq y$. (3) with $x^* = x$, $y^* = y$ and $u^* = u$ becomes

$$d(x, y) \leq d(x + u, y + u), \; -y \leq u \leq y, \tag{5}$$

and with $x^* = x + u$, $y^* = y + u$, $-y \leq u \leq y$,

$$d(x + u, y + u) \leq d(x + u + u^*, y + u + u^*), \; -y - u \leq u^* \leq y + u. \tag{6}$$

We would like to choose $u^* = -u$ in (6). This can be done if $-y - u \leq -u \leq y + u$, i.e. if $-y \leq 0 \leq y + 2u$. Hence

$$d(x+u, y+u) \leq d(x,y), \ -y/2 \leq u. \tag{7}$$

The statement follows from (5) and (7). Similarly the case $x \leq y$. □

Remark. *In all of the proof of the previous Proposition we have left to the reader to check that with the inequalities assumed in the hypotheses all uses of $d(\cdot,\cdot)$ are only applied to elements of C. For example, if $x^* \geq y^*$, then as $-y^* \leq u^* \leq y^*$, it follows that $0 \leq x^* - y^* \leq x^* + u^*$. Hence, $x^* + u^* \in C$.*

Corollary. *Let the assumptions be as in the previous Proposition. Let $x, y \in C$. For each $n \geq 0$:*

$$d(x - y + y/2^{n+1}, y/2^{n+1}) = d(x,y), \ \textit{if } x \geq y; \tag{8}$$

$$d(x/2^{n+1}, y - x + x/2^{n+1}) = d(x,y), \ \textit{if } x \leq y. \tag{9}$$

Proof. Let $x \geq y$. The proof is by induction on n. The case $n = 0$ follows from (4) with $u = -y/2$. Let (8) be true for $n-1$. This and (4) with $u = -y/2^{n+1}$ imply

$$\begin{aligned} d(x,y) &= d(x - y + y/2^n, y/2^n) \\ &= d(x - y + y/2^n - y/2^{n+1}, y/2^n - y/2^{n+1}) \\ &= d(x - y + y/2^{n+1}, y/2^{n+1}). \end{aligned}$$

The statement in (9) is proved analogously. □

Proof of Main Theorem. Let $\gamma : E \to \mathbb{R}^+$ be the function defined by

$$\gamma(u) = \begin{cases} d(u,0), & \text{if } u \in C, \\ d(0,-u), & \text{if } u \notin C. \end{cases}$$

This map is a weak gauge on E. (G1) is immediate, (G2) is a consequence of 2.1 (i). (G3) is as follows. Let $u, v \in E$ and suppose $u \geq v$. Now $u + v \in C$ or $u + v \notin C$. Let $u + v \in C$. If $u \in C$, $v \in C$, by (3) with $x^* = u + v$, $y^* = v$ and $u^* = -v$:

$$\gamma(u+v) = d(u+v, 0) \leq d(u+v, v) + d(v, 0) \leq d(u,0) + d(v,0) = \gamma(u) + \gamma(v).$$

Instead, if $u \in C$, $v \notin C$, by (3) with x^*, y^* as before and $u^* = -u - v$:

$$\gamma(u+v) = d(u+v,0) \leq d(u+v,u) + d(u,0) \leq d(0,-v) + d(u,0) = \gamma(v) + \gamma(u).$$

The case $u \notin C$, $v \notin C$ cannot happen because then $u + v \notin C$. Similarly the case $u \notin C$, $v \in C$ because then $u \not\geq v$. Next, let $u + v \notin C$. If $u \in C$, $v \notin C$, for $x^* = -v$, $y^* = -u - v$ and $u^* = u + v$ in (3):

$$d(0,-u-v) \leq d(0,-v) + d(-v,-u-v) \leq d(0,-v) + d(u,0).$$

If $u \notin C$, $v \notin C$, by (3) with x^*, y^* as before and $u^* = v$:

$$d(0,-u-v) \leq d(0,-v) + d(-v,-u-v) \leq d(0,-v) + d(0,-u).$$

The case $u \in C$, $v \in C$ cannot happen because then $u + v \in C$, as well as the case $u \notin C$, $v \in C$ because then $u \not\geq v$. Finally, $d : C \times C \to \mathbb{R}^+$ is derived from γ. Indeed, as $n \to +\infty$ in (8) and (9), we can conclude by the continuity of d that

$$\gamma(x-y) = d(x-y,0) = d(x,y), \text{ if } x \geq y;$$

$$\gamma(x-y) = d(0,y-x) = d(x,y), \text{ if } x \leq y.$$

$\square$

Acknowledgements

I would like to express my gratitude to P. L. Papini for suggesting the problem and for time spent in discussions.

References

[1] Luxemburg, W. A., Zaanen, A. C., *Riesz spaces I*, North-Holland, Amsterdam, 1971.

[2] Plastria, F., *On the destination optimality in asymmetric distance Fermat-Weber problems*, Annals of Operations Research, vol. 40, 1992, 355–369.

[3] Plastria, F., *Continuous location problems*, chapter 11, 225 − 262, in Z. Drezner, ed., *Facility Location: A Survey of Applications and Methods*, Springer Series in Operations research, Springer Verlag, New York, 1995.

[4] Plastria, F., *Asymmetric distances, semidirected networks, and majority in Fermat-Weber problems*, Locator: ePublication of Location Analysis, vol. 2, No. 1, 2001, 15–62.

[5] Witzgall, C., *On convex metrics*, Journal of Research of the National Bureau of Standards B. Mathematics and Mathematical Physics, vol. 69B, No. 3, 1965, 175–177.

[6] Witzgall, C., *Optimal location of a central facility: mathematical models and concepts*, Technical Report 8388, National Bureau of Standards, Washington, D.C., USA, 1964.