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$\phi\ (\mathbf{F_{11}}) = \mathbf{88}$

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Abstract

Here, we show that the numbers appearing in the title give the largest solution to the Diophantine equation

$$\phi(F_n) = a\frac{10^m - 1}{10 - 1} \qquad a \in \{1, \dots, 9\},$$

where ϕ is the Euler function and F_n is the nth Fibonacci number.
Key words and phrases: Fibonacci numbers, Euler's ϕ function, Diophantine equation.

Resumen

Aquí se muestra que los números que aparecen en el título dan la mayor solución a la ecuación diofántica

$$\phi(F_n) = a\frac{10^m - 1}{10 - 1} \qquad a \in \{1, \dots, 9\},$$

donde ϕ es la función de Euler y F_n es el n-ésimo número de Fibonacci.
Palabras y frases clave: número de Fibonacci, función ϕ de Euler, ecuación diofántica.

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For a positive integer n let $\phi(n)$ be its Euler function. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Recall that a positive integer is a *rep-digit* (in the decimal system) if it is of the form $a(10^m - 1)/9$ for some digit $a \in \{1, \ldots, 9\}$. Here, we prove the following result.

Theorem 1. *The largest positive integer solution (n, m, a) of the equation*

$$\phi(F_n) = a\frac{10^m - 1}{10 - 1} \qquad a \in \{1, \ldots, 9\} \tag{1}$$

is $(n, m, a) = (11, 2, 8)$.

Proof. For a positive integer k let $\mu_2(k)$ be the order at which 2 divides the positive integer k. Since $(10^m - 1)/9$ is always odd, we get that if (n, m, a) satisfy equation (1), then

$$\mu_2(\phi(F_n)) = \mu_2\left(a\frac{10^m - 1}{9}\right) = \mu_2(a) \leq 3.$$

One checks by hand that $n = 11$ gives the largest solution of equation (1) when $n \leq 24$. Assume now that $n > 24$. We show that there exists a prime factor p of F_n such that $p \equiv 1 \pmod 4$. Indeed, let $(L_k)_{k\geq 0}$ be the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and satisfying the same recurrence relation $L_{k+2} = L_{k+1} + L_k$ for all $k \geq 0$ as $(F_k)_{k\geq 0}$ does. It is well-known that

$$L_k^2 - 5F_k^2 = 4(-1)^k. \tag{2}$$

If there exists a prime $r \geq 5$ dividing n, then F_r is odd and $F_r \mid F_n$. Let p be any prime factor of F_r. Reducing relation (2) with $k = r$ modulo p, we get $L_r^2 \equiv -4 \pmod p$. Since p is odd, this leads to the conclusion that -1 is a quadratic residue modulo p; hence, $p \equiv 1 \pmod 4$. Assume now that the largest prime factor of n is ≤ 3. Note that 3^2 does not divide n since otherwise $F_9 \mid F_n$, therefore $\mu_2(\phi(F_n)) \geq \mu_2(\phi(F_9)) = \mu_2(\phi(34)) = \mu_2(16) = 4$, which is impossible. Finally, if $n = 2^\alpha \cdot 3$ or $n = 2^\alpha$, then, since $n > 24$, we get that either $12 \mid n$ or $32 \mid n$; hence,

$$\mu_2(\phi(F_n)) \geq \min\{\mu_2(\phi(F_{12})), \mu_2(\phi(F_{32})\} = \min\{\mu_2(\phi(2^4 \cdot 3^2)),$$

$$\phi(3 \cdot 7 \cdot 47 \cdot 2207)\} = 4,$$

which is again impossible. Thus, we have shown that there exists a prime $p \equiv 1 \pmod 4$ which divides F_n. Clearly, $p - 1 \mid \phi(F_n)$ and so $\mu_2(p - 1) \geq 2$.

This shows that either there exists one other odd prime factor of F_n, let's call it q, or p is the only odd prime factor of F_n.

Case 1. *There exists an odd prime factor $q \neq p$ of F_n.*

Assume that n is odd. Since n is odd, relation (2) with $k = n$ gives $L_n^2 - 5F_n^2 = -4$. Reducing the above equation modulo both p and q, we get that $L_n^2 \equiv -4 \pmod{p}$ and also $L_n^2 \equiv -4 \pmod{q}$. In particular, -1 is a quadratic residue modulo both p and q, which implies that both p and q are congruent to 1 modulo 4. Since $(p-1)(q-1) \mid \phi(F_n)$, we get that $4 \leq \mu_2((p-1)(q-1)) \leq \mu_2(\phi(F_n))$, which is a contradiction. Thus, $n = 2m$ is even. Write $F_n = 2^\alpha p^\beta q^\gamma$. It is clear that $\alpha \leq 1$, for if not, then $2(p-1)(q-1) \mid \phi(F_n)$, which leads again to the conclusion that $\mu_2(\phi(F_n)) \geq 4$. If $\alpha = 1$, then $3 \mid n$. In particular, $6 \mid n$. Thus, $F_6 \mid F_n$, which is impossible because $F_6 = 8 = 2^3$. Hence, $\alpha = 0$. Note now that $F_{2m} = F_m L_m$ and F_m and L_m are coprime because $L_m^2 - 5F_m^2 = \pm 4$ and both L_m and F_m are odd. Furthermore, it is easy to see that m is odd. Indeed, assume that $m = 2h$ is even. Then $F_n = F_{4h} = F_h L_h L_{2h}$. Relation (2) with $k = h$ together with the fact that F_h is odd implies that F_h and L_h are coprime. Further, $F_h L_h = F_{2h}$, and now relation (2) with $k = 2h$ together with the fact that F_{2h} is odd gives that F_{2h} and L_{2h} are also coprime. Since $h = n/4 > 6$, we get that $L_{2h} > F_{2h} > F_h > F_6 = 8$. This argument shows that F_n has at least three odd prime factors, and since at least one of them (namely p) is congruent to 1 modulo 4, we get that $\mu_2(\phi(F_n)) \geq 4$, which is a contradiction. Hence, m is odd, therefore each prime factor of F_m is 1 modulo 4. Since $p \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$, we get that $F_m = p^\beta$ and $L_m = q^\gamma$. Since $m = n/2 > 12$, it follows, from the known perfect powers in the Fibonacci and Lucas sequences [1], that $\beta = \gamma = 1$. Thus, $F_n = pq$. Since clearly $a = 8$, equation (1) becomes

$$8\frac{10^m - 1}{9} = \phi(F_n) = \phi(pq) = (p-1)(q-1) = pq + 1 - (p+q) = F_n + 1 - (p+q),$$

therefore

$$p + q = F_n + 1 - 8\frac{10^m - 1}{9}.$$

Since also $pq = F_n$, we get that p and q are the two roots of the quadratic equation

$$x^2 - \left(F_n + 1 - 8\frac{10^m - 1}{9}\right)x + F_n = 0.$$

In order for the last equation above to have integer solutions p and q, its discriminant Δ must be a perfect square. Computing Δ modulo 5 we immediately get

$$\begin{aligned}\Delta &\equiv \left(F_n + 1 - 8\frac{10^m - 1}{9}\right)^2 - 4F_n \pmod 5\\ &\equiv (F_n + 1 + 8 \cdot 9^{-1})^2 + F_n \pmod 5\\ &\equiv (F_n + 3)^2 + F_n = F_n^2 + 2F_n + 4 \pmod 5.\end{aligned}$$

Clearly, F_n is not a multiple of 5 because $n = 2m$, $m = n/2 > 12$ and F_m and L_m are both primes (note that $F_5 = 5$). The only value of¿ $b \in \{1, 2, 3, 4\}$ such that $b^2 + 2b + 4$ is a perfect square modulo 5 is $b = 3$. Thus, $F_n \equiv 3 \pmod 5$. The sequence $(F_k)_{k\geq 0}$ is periodic modulo 5 with period 20 and if $F_n \equiv 3 \pmod 5$, then $n \equiv 4, 6, 7, 13 \pmod{20}$. Since $n = 2m$ is even but not a multiple of 4, we get that $n \equiv 6 \pmod{20}$. Hence, $m \equiv 3 \pmod{10}$. Both $(F_k)_{k\geq 0}$ and $(L_k)_{k\geq 0}$ are periodic modulo 11 with period 10. Since $m \equiv 3 \pmod{10}$, we get that $p = F_m \equiv F_3 \equiv 2 \pmod{11}$ and $q = L_m \equiv L_3 \equiv 4 \pmod{11}$. Thus, $\phi(F_n) = (p-1)(q-1) \equiv 3 \pmod{11}$. Reducing now equation (1) modulo 11 we get

$$3 \equiv 8((-1)^m - 1)9^{-1} \pmod{11},$$

which leads to $27 \equiv 0, -16 \pmod{11}$, which is impossible. This takes care of Case 1.

Case 2. *p is the only odd prime factor of F_n.*

Write $F_n = 2^\alpha p^\beta$. If $2 \mid F_n$, then $3 \mid n$. Put $n = 3m$. Then $F_n = F_{3m} = F_m(5F_m^2 + 3)$. One checks easily that $\gcd(F_m, 5F_m^2 + 3) = 1$ or 3. If $3 \mid F_m$, then $3 \neq p$ (because $p \equiv 1 \pmod 4$), so F_n is divisible by two distinct primes, which is a contradiction. Thus, $3 \nmid F_m$, therefore F_m and $5F_m^2+3$ are coprime. Since $5F_m^2 + 3$ is odd, we get that $p \mid 5F_m^2 + 3$, which in turn leads to the conclusion that F_m is a power of 2, which is impossible because $m = n/2 > 12$ (the largest power of 2 in the Fibonacci sequence is $F_6 = 8$). Thus, $\alpha = 0$. By the known perfect powers in the Fibonacci sequence again, we get that $\beta = 1$. Hence, $F_n = p$, therefore $\phi(F_n) = p - 1 = F_n - 1$. We thus get the equation

$$F_n - 1 = a\frac{10^m - 1}{9}.$$

Furthermore, since $F_n - 1 = p - 1$ is a multiple of 4, we get that a is a multiple of 4. Thus, $a \in \{4, 8\}$. When $a = 4$, we get that

$$F_n = 4\frac{10^m - 1}{9} + 1 = \frac{4 \cdot 10^m + 5}{9}$$

is a multiple of 5; hence, not a prime. Thus, $a = 8$. We now show that m is even. Indeed, assume that m is odd. Then $10^m \equiv -1 \pmod{11}$ which leads to the conclusion that the right hand side of equation (1) is congruent to 8 modulo 11. Hence, $F_n \equiv 9 \pmod{11}$. The period of the Fibonacci sequence $(F_k)_{k\geq 0}$ modulo 11 is 10. Checking the first 10 values one concludes that there is no Fibonacci number F_n which is congruent to 9 modulo 11. Hence, m is even. Since $n > 24$, we get that $F_n > 10^2$, therefore $m \geq 3$. Rewriting equation (1) as

$$9F_n - 1 = 8 \cdot 10^m, \tag{3}$$

we get that $9F_n - 1 \equiv 0 \pmod{64}$. The Fibonacci sequence $(F_k)_{k\geq 0}$ is periodic modulo 64 with period 96. Further, checking the first 96 values one gets that $n \equiv 14, 37, 59 \pmod{96}$. Since n is odd, we get that $n \equiv \pm 37 \pmod{96}$. But 96 is also the period of the Fibonacci sequence modulo 47, and if $n \equiv \pm 37 \pmod{96}$, then $F_n \equiv 5 \pmod{47}$. Reducing now equation (3) modulo 47 we get $44 \equiv 8 \cdot 10^m \pmod{47}$, which is equivalent to $29 \equiv (10^{m/2})^2 \pmod{47}$. However, this last congruence is false because 29 is not a quadratic residue modulo 47 as it can be seen since

$$\left(\frac{29}{47}\right) = \left(\frac{47}{29}\right) = \left(\frac{18}{29}\right) = \left(\frac{2}{29}\right) = -1,$$

because $29 \equiv 5 \pmod{8}$. In the above calculations, we used $\left(\frac{p}{q}\right)$ for the Legendre symbol of p with respect to q (where $q > 2$ is prime) and its elementary properties. This takes care of Case 2 and completes the proof of Theorem 1. □

Remark. The argument from the beginning of the proof of Theorem 1 could be somewhat simplified using a result of McDaniel [3] who showed that if $n \notin \{0, 1, 2, 3, 4, 6, 8, 16, 24, 32, 48\}$, then F_n has a prime factor which is congruent to 1 modulo 4. Furthermore, some of the arguments from the proof could also be simplified if one appeals to the *Primitive Divisor Theorem* for the Fibonacci sequence [2], which says that if $n > 12$, then there exists a prime factor $p \mid F_n$ such that $p \nmid F_m$ for any positive integer $m < n$. We have however preferred to give a self-contained proof of our Theorem 1 up to the knowledge of perfect powers in the Fibonacci and Lucas sequence [1]. It would be interesting to give a completely elementary proof of Theorem 1 (i.e., without appealing to the results from [1]). We could not succeed in finding such an argument and we leave this as a challenge to the reader.

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