

Browder's theorems and the spectral mapping theorem

Los teoremas de Browder y el teorema de la aplicación espectral

Pietro Aiena (paiena@unipa.it)

Dipartimento di Matematica ed Applicazioni
Facoltà di Ingegneria, Università di Palermo
Viale delle Scienze, I-90128 Palermo (Italy)

Carlos Carpintero (ccarpi@cumana.sucra.udo.ve)

Departamento de Matemáticas, Facultad de Ciencias
Universidad UDO, Cumaná (Venezuela)

Ennis Rosas (erosas@cumana.sucra.udo.ve)

Departamento de Matemáticas, Facultad de Ciencias
Universidad UDO, Cumaná (Venezuela)

Abstract

A bounded linear operator $T \in L(X)$ on a Banach space X is said to satisfy Browder's theorem if two important spectra, originating from Fredholm theory, the Browder spectrum and the Weyl spectrum, coincide. This expository article also concerns with an approximate point version of Browder's theorem. A bounded linear operator $T \in L(X)$ is said to satisfy a -Browder's theorem if the upper semi-Browder spectrum coincides with the approximate point Weyl spectrum. In this note we give several characterizations of operators satisfying these theorems. Most of these characterizations are obtained by using a localized version of the single-valued extension property of T . This paper also deals with the relationships between Browder's theorem, a -Browder's theorem and the spectral mapping theorem for certain parts of the spectrum.

Key words and phrases: Local spectral theory, Fredholm theory, Weyl's theorem.

Resumen

Un operador lineal acotado $T \in L(X)$ sobre un espacio de Banach X se dice que satisface el teorema de Browder, si dos importantes espectros, en el contexto de la teoría de Fredholm, el espectro de Browder y el espectro de Weyl, coinciden. Este artículo expositivo trata con una versión puntual del teorema de Browder. Un operador lineal acotado $T \in L(X)$ sobre un espacio de Banach X se dice que satisface el teorema de α -Browder si el espectro superior semi-Browder coincide con el espectro puntual aproximado de Weyl. En esta nota damos varias caracterizaciones de operadores que satisfacen estos teoremas. La mayoría de estas caracterizaciones se obtienen de versiones localizadas de la propiedad de extensión univaluada de T . Este trabajo también considera las relaciones entre el teorema de Browder el teorema α -Browder y el teorema de transformación espectral para ciertas partes del espectro.

Palabras y frases clave: Teoría espectral local, teoría de Fredholm, teorema de Weyl.

1 Introduction and definitions

If X is an infinite-dimensional complex Banach space and $T \in L(X)$ is a bounded linear operator, we denote by $\alpha(T) := \dim \ker T$, the dimension of the null space $\ker T$, and by $\beta(T) := \text{codim } T(X)$ the codimension of the range $T(X)$. Two important classes in Fredholm theory are given by the class of all *upper semi-Fredholm* operators $\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$, and the class of all *lower semi-Fredholm* operators defined by $\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$. The class of all *semi-Fredholm operators* is defined by $\Phi_{\pm}(X) := \Phi_+(X) \cup \Phi_-(X)$, while $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ defines the class of all *Fredholm operators*. The *index* of $T \in \Phi_{\pm}(X)$ is defined by $\text{ind}(T) := \alpha(T) - \beta(T)$. Recall that a bounded operator T is said *bounded below* if it is injective and it has closed range. Define

$$W_+(X) := \{T \in \Phi_+(X) : \text{ind } T \leq 0\},$$

and

$$W_-(X) := \{T \in \Phi_-(X) : \text{ind } T \geq 0\}.$$

The set of *Weyl operators* is defined by

$$W(X) := W_+(X) \cap W_-(X) = \{T \in \Phi(X) : \text{ind } T = 0\}.$$

The classes of operators defined above generate the following spectra. The *Fredholm spectrum* (known in literature also as *essential spectrum*) is defined

by

$$\sigma_f(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

The *Weyl spectrum* is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\},$$

the *Weyl essential approximate point spectrum* is defined by

$$\sigma_{wa}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *Weyl essential surjectivity spectrum* is defined by

$$\sigma_{ws}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

Denote by

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

the *approximate point spectrum*, and by

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\},$$

the *surjectivity spectrum*.

The spectrum $\sigma_{wa}(T)$ admits a nice characterization: it is the intersection of all approximate point spectra $\sigma_a(T + K)$ of compact perturbations K of T , while, dually, $\sigma_{ws}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations K of T , see for instance [1, Theorem 3.65]. From the classical Fredholm theory we have

$$\sigma_{wa}(T) = \sigma_{ws}(T^*) \quad \text{and} \quad \sigma_{wa}(T^*) = \sigma_{ws}(T).$$

This paper concerns also with two other classical quantities associated with an operator T : the *ascent* $p := p(T)$, i.e. the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$, and the *descent* $q := q(T)$, i.e. the smallest non-negative integer q , such that $T^q(X) = T^{q+1}(X)$. If such integers do not exist we shall set $p(T) = \infty$ and $q(T) = \infty$, respectively. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3]. Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ if and only if λ belongs to the spectrum $\sigma(T)$ and is a pole of the function resolvent $\lambda \rightarrow (\lambda I - T)^{-1}$, see Proposition 50.2 of [18]. The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T) = q(T) < \infty\},$$

the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

while the class of all *lower semi-Browder operators* is defined

$$B_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

Obviously, $B(X) = B_+(X) \cap B_-(X)$ and

$$B(X) \subseteq W(X), \quad B_+(X) \subseteq W_+(X), \quad B_-(X) \subseteq W_-(X)$$

see [1, Theorem 3.4].

The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\},$$

the *upper semi-Browder spectrum* is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_+(X)\},$$

and analogously the *lower semi-Browder spectrum* is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B_-(X)\}.$$

Clearly,

$$\sigma_f(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T),$$

and

$$\sigma_{ub}(T) = \sigma_{lb}(T^*) \quad \text{and} \quad \sigma_{lb}(T) = \sigma_{ub}(T^*).$$

Furthermore, by part (v) of Theorem 3.65 [1] we have

$$\sigma_{ub}(T) = \sigma_{wa}(T) \cup \text{acc } \sigma_a(T), \tag{1}$$

$$\sigma_{lb}(T) = \sigma_{ws}(T) \cup \text{acc } \sigma_s(T), \tag{2}$$

and

$$\sigma_b(T) = \sigma_w(T) \cup \text{acc } \sigma(T), \tag{3}$$

where we write $\text{acc } K$ for the set of all cluster points of $K \subseteq \mathbb{C}$.

A bounded operator $T \in L(X)$ is said to be *semi-regular* if it has closed range and

$$\ker T^n \subseteq T(X) \quad \text{for all } n \in \mathbb{N}.$$

The *Kato spectrum* is defined by

$$\sigma_k(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

Note that $\sigma_k(T)$ is a non-empty compact subset of \mathbb{C} , since it contains the boundary of the spectrum, see [1, Theorem 1.75]. An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T -invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|_M$ is semi-regular and $T|_N$ is quasi-nilpotent. A relevant case is obtained if we assume in the definition above that $T|_N$ is nilpotent. In this case T is said to be of *Kato type*. If N is finite-dimensional then T is said to be *essentially semi-regular*. Every semi-Fredholm operator is essentially semi-regular, by the classical result of Kato, see Theorem 1.62 of [1]. Recall that $T \in L(X)$ is said to admit a *generalized inverse* $S \in L(X)$ if $TST = T$. It is well known that T admits a generalized inverse if and only if both subspaces $\ker T$ and $T(X)$ are complemented in X . It is well-known that every Fredholm operator admits a generalized inverse, see Theorem 7.3 of [1]. A "complemented" version of Kato operators is given by the *Saphar operators*: $T \in L(X)$ is said to be Saphar if T is semi-regular and admits a generalized inverse. The *Saphar spectrum* is defined by

$$\sigma_{sa}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Saphar}\}.$$

Clearly, $\sigma_k(T) \subseteq \sigma_{sa}(T)$, so $\sigma_{sa}(T)$ is nonempty; for other properties on Saphar operators see Müller [22, Chapter II, §13].

2 SVEP

There is an elegant interplay between Fredholm theory and the single-valued extension property, an important role that has a crucial role in local spectral theory. This property was introduced in the early years of local spectral theory by Dunford [13], [14] and plays an important role in the recent monographs by Laursen and Neumann [20], or by Aiena [1]. Recently, there has been a flurry of activity regarding a localized version of the single-valued extension property, considered first by [15] and examined in several more recent papers, for instance [21], [5], and [7].

Definition 2.1. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0 , the only analytic*

function $f : U \rightarrow X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0, \quad \text{for all } \lambda \in U$$

is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

The SVEP may be characterized by means of some typical tools of the local spectral theory, see [8] or Proposition 1.2.16 of [20]. Note that by the identity theorem for analytic function both T and T^* have SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, both T and the dual T^* have SVEP at the isolated points of $\sigma(T)$.

A basic result links the ascent, descent and localized SVEP:

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda,$$

see [1, Theorem 3.8].

Furthermore, from the definition of localized SVEP it is easy to see that

$$\sigma_a(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \quad (4)$$

while

$$\sigma_s(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.$$

An important subspace in local spectral theory is the *quasi-nilpotent part* of T , namely, the set

$$H_0(T) := \{x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Moreover, T is quasi-nilpotent if and only if $H_0(T) = X$, see [1, Theorem 1.68]. If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$, see [1] for informations on the subspaces $H_0(T)$, $K(T)$. The subspaces $H_0(T)$ and $K(T)$ are invariant under T and may be not closed. We have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda,$$

see [5].

In the following theorem we collect some characterizations of SVEP for operators of Kato type.

Theorem 2.2. *Suppose that $\lambda_0 I - T$ is of Kato type. Then the following statements are equivalent:*

- (i) T has SVEP at λ_0 ;
- (ii) $p(\lambda_0 I - T) < \infty$;
- (iii) $H_0(\lambda_0 I - T)$ is closed;
- (iv) $\sigma_a(T)$ does not cluster at λ .

If $\lambda_0 I - T$ is essentially semi-regular the statements (i) - (iv) are equivalent to the following condition:

- (v) $H_0(\lambda_0 I - T)$ is finite-dimensional.

If $\lambda_0 I - T$ is semi-regular the statements (i) - (v) are equivalent to the following condition:

- (vi) $\lambda_0 I - T$ is injective.

Dually, the following statements are equivalent:

- (vii) T^* has SVEP at λ_0 ;
- (viii) $q(\lambda_0 I - T) < \infty$;
- (ix) $\sigma_s(T)$ does not cluster at λ .

If $\lambda_0 I - T$ is essentially semi-regular the statements (vi) - (viii) are equivalent to the following condition:

- (x) $K(\lambda I - T)$ is finite-codimensional.

If $\lambda_0 I - T$ is semi-regular the statements (vii) - (x) are equivalent to the following condition:

- (xi) $\lambda_0 I - T$ is surjective.

Remark 2.3. Note that the condition $p(T) < \infty$ (respectively, $q(T) < \infty$) implies for a semi-Fredholm that $\text{ind } T \leq 0$ (respectively, $\text{ind } T \geq 0$), see [1, Theorem 3.4]. Consequently, if T has SVEP then $\lambda \notin \sigma_f(T)$ then $\text{ind}(\lambda I - T) \leq 0$, while if T^* has SVEP then $\text{ind}(\lambda I - T) \geq 0$.

Let λ_0 be an isolated point of $\sigma(T)$ and let P_0 denote the spectral projection

$$P_0 := \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - T)^{-1} d\lambda$$

associated with $\{\lambda_0\}$, via the classical Riesz functional calculus. A classical result shows that the range $P_0(X)$ is $N := H_0(\lambda_0 I - T)$, see Heuser [18, Proposition 49.1], while $\ker P_0$ is the analytic core $M := K(\lambda_0 I - T)$ of $\lambda_0 I - T$, see [24] and [21]. In this case, $X = M \oplus N$ and

$$\sigma(\lambda_0 I - T|N) = \{\lambda_0\}, \quad \sigma(\lambda_0 I - T|M) = \sigma(T) \setminus \{\lambda_0\},$$

so $\lambda_0 I - T|M$ is invertible and hence $H_0(\lambda_0 I - T|M) = \{0\}$. Therefore from the decomposition $H_0(\lambda_0 I - T) = H_0(\lambda_0 I - T|M) \oplus H_0(\lambda_0 I - T|N)$ we deduce that $N = H_0(\lambda_0 I - T|N)$, so $\lambda_0 I - T|N$ is quasi-nilpotent. Hence the pair (M, N) is a GKD for $\lambda_0 I - T$.

Corollary 2.4. *Let λ_0 be an isolated point of $\sigma(T)$. Then*

$$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$$

and the following assertions are equivalent:

- (i) $\lambda_0 I - T$ is semi-Fredholm;
- (ii) $H_0(\lambda_0 I - T)$ is finite-dimensional;
- (iii) $K(\lambda_0 I - T)$ is finite-codimensional.

Proof. Since for every operator $T \in L(X)$, both T and T^* have SVEP at any isolated point, the equivalence of the assertions easily follows from the decomposition $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$, and from Theorem 2.2. ■

3 Browder's theorem

In 1997 Harte and W. Y. Lee [16] have christened that *Browder's theorem* holds for T if

$$\sigma_w(T) = \sigma_b(T),$$

or equivalently, by (3), if

$$\text{acc } \sigma(T) \subseteq \sigma_w(T). \quad (5)$$

Let write *iso* K for the set of all isolated points of $K \subseteq \mathbb{C}$. To look more closely to Browder's theorem, let us introduce the following parts of the spectrum: For a bounded operator $T \in L(X)$ define

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : \lambda I - T \in \mathcal{B}(X)\},$$

the set of all *Riesz points* in $\sigma(T)$. Finally, let us consider the following set:

$$\Delta(T) := \sigma(T) \setminus \sigma_w(T).$$

Clearly, if $\lambda \in \Delta(T)$ then $\lambda I - T \in W(X)$ and since $\lambda \in \sigma(T)$ it follows that $\alpha(\lambda I - T) = \beta(\lambda I - T) > 0$, so we can write

$$\Delta(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W(X), 0 < \alpha(\lambda I - T)\}.$$

The set $\Delta(T)$ has been recently studied in [16], where the points of $\Delta(T)$ are called *generalized Riesz points*. It is easily seen that

$$p_{00}(T) \subseteq \Delta(T) \quad \text{for all } T \in L(X).$$

Our first result shows that Browder's theorem is equivalent to the localized SVEP at some points of \mathbb{C} .

Theorem 3.1. *For an operator $T \in L(X)$ the following statements are equivalent:*

- (i) $p_{00}(T) = \Delta(T)$;
- (ii) T satisfies Browder's theorem;
- (iii) T^* satisfies Browder's theorem;
- (iv) T has SVEP at every $\lambda \notin \sigma_w(T)$;
- (v) T^* has SVEP at every $\lambda \notin \sigma_w(T)$.

From Theorem 3.1 we deduce that the SVEP for either T or T^* entails that both T and T^* satisfy Browder's theorem. However, the following example shows that SVEP for T or T^* is a not necessary condition for Browder's theorem.

Example 3.2. Let $T := L \oplus L^* \oplus Q$, where L is the unilateral left shift on $\ell^2(\mathbb{N})$, defined by

$$L(x_1, x_2, \dots) := (x_2, x_3, \dots), \quad (x_n) \in \ell^2(\mathbb{N}),$$

and Q is any quasi-nilpotent operator. L does not have SVEP, see [1, p. 71], so also T and T^* do not have SVEP, see Theorem 2.9 of [1]. On the other hand, we have $\sigma_b(T) = \sigma_w(T) = \mathbf{D}$, where \mathbf{D} is the closed unit disc in \mathbb{C} , so that Browder's theorem holds for T .

A very clear spectral picture of operators for which Browder's theorem holds is given by the following theorem:

Theorem 3.3. [3] *For an operator $T \in L(X)$ the following statements are equivalent:*

- (i) T satisfies Browder's theorem;
- (ii) Every $\lambda \in \Delta(T)$ is an isolated point of $\sigma(T)$;
- (iii) $\Delta(T) \subseteq \partial\sigma(T)$, $\partial\sigma(T)$ the topological boundary of $\sigma(T)$;
- (iv) $\text{int } \Delta(T) = \emptyset$, $\text{int } \Delta(T)$ the interior of $\Delta(T)$;

- (v) $\sigma(T) = \sigma_w(T) \cup \text{iso } \sigma(T)$.
- (vi) $\Delta(T) \subseteq \sigma_k(T)$;
- (vii) $\Delta(T) \subseteq \text{iso } \sigma_k(T)$;
- (viii) $\Delta(T) \subseteq \sigma_{\text{sa}}(T)$;
- (ix) $\Delta(T) \subseteq \text{iso } \sigma_{\text{sa}}(T)$.

Other characterizations of Browder's theorem involve the quasi-nilpotent part and the analytic core of T :

Theorem 3.4. *For a bounded operator $T \in L(X)$ Browder's theorem holds precisely when one of the following statements holds;*

- (i) $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta(T)$;
- (ii) $H_0(\lambda I - T)$ is closed for all $\lambda \in \Delta(T)$;
- (iii) $K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta(T)$.

Define

$$\sigma_1(T) := \sigma_w(T) \cup \sigma_k(T).$$

We show now, by using different methods, some recent results of X. Cao, M. Guo, B. Meng [10]. These results characterize Browder's theorem through some special parts of the spectrum defined by means the concept of semi-regularity.

Theorem 3.5. *For a bounded operator the following statements are equivalent:*

- (i) T satisfies Browder's theorem;
- (ii) $\sigma(T) = \sigma_1(T)$;
- (iii) $\Delta(T) \subseteq \sigma_1(T)$,
- (iv) $\Delta(T) \subseteq \text{iso } \sigma_1(T)$.
- (v) $\sigma_b(T) \subseteq \sigma_1(T)$;

Proof. The equivalence (i) \Leftrightarrow (ii) has been proved in [10], but is clear from Theorem 3.3.

(i) \Leftrightarrow (iii) Suppose that T satisfies Browder's theorem or equivalently, by Theorem 3.3, that $\Delta(T) \subseteq \sigma_k(T)$. Then $\Delta(T) \subseteq \sigma_w(T) \cup \sigma_k(T) = \sigma_1(T)$. Conversely, if $\Delta(T) \subseteq \sigma_1(T)$ then $\Delta(T) \subseteq \sigma_k(T)$, since by definition $\Delta(T) \cap \sigma_w(T) = \emptyset$.

(iii) \Rightarrow (iv) Suppose that the inclusion $\Delta(T) \subseteq \sigma_1(T)$ holds. We know by the first part of the proof that this inclusion is equivalent to Browder's theorem, or also to the equality $\sigma(T) = \sigma_1(T)$. By Theorem 3.3 we then have

$$\Delta(T) \subseteq \text{iso } \sigma(T) = \text{iso } \sigma_1(T).$$

(iv) \Rightarrow (iii) Obvious.

(i) \Rightarrow (v) If T satisfies Browder's theorem then $\sigma_b(T) = \sigma_w(T) \subseteq \sigma_1(T)$.

(v) \Rightarrow (ii) Suppose that $\sigma_b(T) \subseteq \sigma_1(T)$. We show that $\sigma(T) = \sigma_1(T)$. It suffices only to show $\sigma(T) \subseteq \sigma_1(T)$. Let $\lambda \notin \sigma_1(T) = \sigma_w(T) \cup \sigma_k(T)$. Then $\lambda \notin \sigma_b(T)$, so λ is an isolated point of $\sigma(T)$ and $\alpha(\lambda I - T) = \beta(\lambda I - T)$. Since $\lambda \notin \sigma_k(T)$ then $\lambda I - T$ is semi-regular and the SVEP at λ implies by Theorem 2.2 that $\alpha(\lambda I - T) = \beta(\lambda I - T) = 0$, i.e. $\lambda \notin \sigma(T)$. ■

By passing we note that the paper by X. Cao, M. Guo, and B. Meng [10] contains two mistakes. The authors claim in Lemma 1.1 that $\text{iso } \sigma_k(T) \subseteq \sigma_w(T)$ for every $T \in L(X)$. This is false, for instance if λ is a Riesz point of T then $\lambda \in \partial\sigma(T)$, since λ is isolated in $\sigma(T)$, and hence $\lambda \in \sigma_k(T)$, see [1, Theorem 1.75], so $\lambda \in \text{iso } \sigma_k(T)$. On the other hand, $\lambda I - T$ is Weyl and hence $\lambda \notin \sigma_w(T)$.

Also the equivalence: Browder's theorem for $T \Leftrightarrow \sigma(T) \setminus \sigma_k(T) \subseteq \text{iso } \sigma_k(T)$, claimed in Corollary 2.3 of [10] is not corrected, the correct statement is the equivalence (i) \Leftrightarrow (vi) established in Theorem 3.3.

Denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$, let $f(T)$ be defined by means of the classical functional calculus. It should be noted that the spectral mapping theorem does not hold for $\sigma_1(T)$. In fact we have the following result.

Theorem 3.6. [10] *Suppose that $T \in L(X)$. For every $f \in \mathcal{H}(\sigma(T))$ we have $\sigma_1(f(T)) \subseteq f(\sigma_1(T))$. The equality $f(\sigma_1(T)) = \sigma_1(f(T))$ holds for every $f \in \mathcal{H}(\sigma(T))$ precisely when the spectral mapping theorem holds for $\sigma_w(T)$, i.e.,*

$$f(\sigma_w(T)) = \sigma_w(f(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)).$$

Note that the spectral mapping theorem for $\sigma_w(T)$ holds if either T or T^* satisfies SVEP, see also next Theorem 4.3. This is also an easy consequence of Remark 2.3.

Theorem 3.7. [10] *The spectral mapping theorem holds for $\sigma_1(T)$ precisely when $\text{ind}(\lambda I - T) \cdot \text{ind}(\mu I - T) \geq 0$ for each pair $\lambda, \mu \notin \sigma_f(T)$.*

In general, Browder's theorem for T does not entail Browder's theorem for $f(T)$. However, we have the following result.

Theorem 3.8. *Suppose that both $T \in L(X)$ and $S \in L(X)$ satisfy Browder's theorem, $f \in \mathcal{H}(\sigma(T))$ and p a polynomial. Then we have:*

- (i) [10] *Browder's theorem holds for $f(T)$ if and only if $f(\sigma_1(T)) = \sigma_1(f(T))$.*
- (ii) [10] *Browder's theorem holds for $T \oplus S$ if and only if $\sigma_1(T) \cup \sigma_1(S) = \sigma_1(T \oplus S)$.*
- (iii) [16] *Browder's theorem holds for $p(T)$ if and only if $p(\sigma_w(T)) \subseteq \sigma_w(p(T))$.*
- (iv) [16] *Browder's theorem holds for $T \oplus S$ if and only if $\sigma_w(T) \cup \sigma_w(S) \subseteq \sigma_w(T \oplus S)$.*

Browder's theorem survives under perturbation of compact operators K commuting with T . In fact, we have

$$\sigma_w(T + K) = \sigma_w(T) \quad \text{and} \quad \sigma_b(T + K) = \sigma_b(T); \quad (6)$$

the first equality is a standard result from Fredholm theory, while the second equality is due to V. Rakočević [23]. It is not difficult to extend this result to Riesz operators commuting with T (recall that $K \in L(X)$ is said to be a *Riesz operator* if $\lambda I - K \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$). Indeed, the equalities (6) hold also in the case where K is Riesz [23]. An analogous result holds if we assume that K is a commuting quasi-nilpotent operator, see [16, Theorem 11], since quasi-nilpotent operators are Riesz. These results may fail if K is not assumed to commute, see [16, Example 12]. Browder's theorem for T and S transfers successfully to the tensor product $T \otimes S$ [17, Theorem 6]. In [16] it is also shown that Browder's theorem holds for a Hilbert space operator $T \in L(H)$ if T is reduced by its finite dimensional eigenspaces.

Browder's theorem entails the continuity of some mappings. To see this, we need some preliminary definitions. Let (σ_n) be a sequence of compact subsets of \mathbb{C} and define canonically its *limit inferior* by

$$\liminf \sigma_n := \{\lambda \in \mathbb{C} : \text{there exists } \lambda_n \in \sigma_n \text{ with } \lambda_n \rightarrow \lambda\}.$$

Define the *limit superior* of (σ_n) by

$$\limsup \sigma_n := \{\lambda \in \mathbb{C} : \text{there exists } \lambda_{n_k} \in \sigma_{n_k} \text{ with } \lambda_{n_k} \rightarrow \lambda\}.$$

A mapping φ , defined on $L(X)$ whose values are compact subsets of \mathbb{C} is said to be *upper semi-continuous at T* (respectively, *lower semi-continuous at T*) provided that if $T_n \rightarrow T$, in the norm topology, then $\limsup \varphi(T_n) \subseteq \varphi(T)$ (respectively, $\varphi(T) \subseteq \liminf \varphi(T_n)$). If the map φ is both upper and lower

semi-continuous then φ is said to be *continuous* at T . In this case we write $\lim_{n \in \mathbb{N}} \varphi(T_n) = \varphi(T)$. In the following result we consider mappings that associate to an operator its Browder spectrum or its Weyl spectrum.

Theorem 3.9. [12] *If $T \in L(X)$ then the following assertions hold:*

- (i) *The map $T \in L(X) \rightarrow \sigma_b(T)$ is continuous at T_0 if and only if Browder's theorem holds for T_0 .*
- (ii) *If Browder's theorem holds for T_0 then the map $T \in L(X) \rightarrow \sigma(T)$ is continuous at T_0 .*

By contrast, we see now that Browder's theorem is equivalent to the discontinuity of some other mappings. Recall that *reduced minimum modulus* of a non-zero operator T is defined by

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\text{dist}(x, \ker T)}.$$

In the following result we use the concept of gap metric, see [19] for details.

Theorem 3.10. [3] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

- (i) *T satisfies Browder's theorem;*
- (ii) *the mapping $\lambda \rightarrow \ker(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric;*
- (iii) *the mapping $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta(T)$;*
- (iv) *the mapping $\lambda \rightarrow (\lambda I - T)(X)$ is not continuous at every $\lambda \in \Delta(T)$ in the gap metric.*

4 a -Browder's theorem

An approximation point version of Browder's theorem is given by the so-called a -Browder's theorem. A bounded operator $T \in L(X)$ is said to satisfy *a -Browder's theorem* if

$$\sigma_{\text{wa}}(T) = \sigma_{\text{ub}}(T),$$

or equivalently, by (1), if

$$\text{acc } \sigma_a(T) \subseteq \sigma_{\text{wa}}(T).$$

Define

$$p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{\text{ub}}(T) = \{\lambda \in \sigma_a(T) : \lambda I - T \in \mathcal{B}_+(X)\},$$

and let us consider the following set:

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{\text{wa}}(T).$$

Since $\lambda I - T \in W_a(X)$ implies that $(\lambda I - T)(X)$ is closed, we can write

$$\Delta_a(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in W_a(X), 0 < \alpha(\lambda I - T)\}.$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$. We have

$$p_{00}^a(T) \subseteq \pi_{00}^a(T) \quad \text{for all } T \in L(X),$$

and

$$p_{00}^a(T) \subseteq \Delta_a(T) \subseteq \sigma_a(T) \quad \text{for all } T \in L(X).$$

Theorem 4.1. *For a bounded operator $T \in L(X)$, a -Browder's theorem holds for T if and only if $p_{00}^a(T) = \Delta_a(T)$. In particular, a -Browder's theorem holds whenever $\Delta_a(T) = \emptyset$.*

A precise description of operators satisfying a -Browder's theorem may be given in terms of SVEP at certain sets.

Theorem 4.2. *If $T \in L(X)$ the following statements hold:*

- (i) *T satisfies a -Browder's theorem if and only if T has SVEP at every $\lambda \notin \sigma_{\text{wa}}(T)$.*
- (ii) *T^* satisfies a -Browder's theorem if and only if T^* has SVEP at every $\lambda \notin \sigma_{\text{ws}}(T)$.*
- (iii) *If T has SVEP at every $\lambda \notin \sigma_{\text{ws}}(T)$ then a -Browder's theorem holds for T^* .*
- (iv) *If T^* has SVEP at every $\lambda \notin \sigma_{\text{wa}}(T)$ then a -Browder's theorem holds for T .*

Since $\sigma_{\text{wa}}(T) \subseteq \sigma_{\text{w}}(T)$, from Theorem 4.2 and Theorem 3.1 we readily obtain:

$$a\text{-Browder's theorem for } T \Rightarrow \text{Browder's theorem for } T,$$

while

$$\text{SVEP for either } T \text{ or } T^* \Rightarrow a\text{-Browder's theorem holds for both } T, T^*. \quad (7)$$

Note that the reverse of the assertions (iii) and (iv) of Theorem 3.1 generally do not hold. An example of unilateral weighted shifts T on $\ell^p(\mathbb{N})$ for which a -Browder's theorem holds for T (respectively, a -Browder's theorem holds for T^*) and such that SVEP fails at some points $\lambda \notin \sigma_{\text{ws}}(T)$ (respectively, at some points $\lambda \notin \sigma_{\text{wa}}(T)$) may be found in [4].

The implication of (7) may be considerably extended as follows.

Theorem 4.3. [11], [2] *Let $T \in L(X)$ and suppose that T or T^* satisfies SVEP. Then a -Browder's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$, i.e. $\sigma_{\text{wa}}(f(T)) = \sigma_{\text{ub}}(f(T))$. Furthermore,*

$$\sigma_{\text{ws}}(f(T)) = \sigma_{\text{lb}}(f(T)), \quad \sigma_{\text{w}}(f(T)) = \sigma_{\text{b}}(f(T)),$$

and the spectral mapping theorem holds for all the spectra $\sigma_{\text{wa}}(T)$, $\sigma_{\text{ws}}(T)$ and $\sigma_{\text{w}}(T)$.

Theorem 4.3 is an easy consequence of the fact that $f(T)$ satisfies Browder's theorem and that the spectral mapping theorem holds for the Browder spectrum and semi-Browder spectra, see [1, Theorem 3.69 and Theorem 3.70]. In general, the spectral mapping theorems for the Weyl spectra $\sigma_{\text{w}}(T)$, $\sigma_{\text{wa}}(T)$ and $\sigma_{\text{ws}}(T)$ are liable to fail. Moreover, Browder's theorem and the spectral mapping theorem are independent. In [16, Example 6] is given an example of an operator T for which the spectral mapping theorem holds for $\sigma_{\text{w}}(T)$ but Browder's theorem fails for T . Another example [16, Example 7] shows that there exist operators for which Browder's theorem holds, while the spectral mapping theorem for the Weyl spectrum fails.

The following results are analogous to the results of Theorem 3.3, and give a precise spectral picture of a -Browder's theorem.

Theorem 4.4. [4], [10] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

- (i) T satisfies a -Browder's theorem;
- (ii) $\Delta_a(T) \subseteq \text{iso } \sigma_a(T)$;
- (iii) $\Delta_a(T) \subseteq \partial\sigma_a(T)$, $\partial\sigma_a(T)$ the topological boundary of $\sigma_a(T)$;
- (iv) $\sigma_a(T) = \sigma_{\text{wa}}(T) \cup \sigma_{\text{k}}(T)$;
- (v) $\Delta_a(T) \subseteq \sigma_{\text{k}}(T)$;
- (vi) $\Delta_a(T) \subseteq \text{iso } \sigma_{\text{k}}(T)$;
- (vii) $\Delta_a(T) \subseteq \sigma_{\text{sa}}(T)$;
- (viii) $\Delta_a(T) \subseteq \text{iso } \sigma_{\text{sa}}(T)$.

We also have:

Theorem 4.5. [3] $T \in L(X)$ satisfies a -Browder's theorem if and only if

$$\sigma_a(T) = \sigma_{wa}(T) \cup \text{iso } \sigma_a(T). \quad (8)$$

Analogously, a -Browder's theorem holds for T^* if and only if

$$\sigma_s(T) = \sigma_{ws}(T) \cup \text{iso } \sigma_s(T). \quad (9)$$

The results established above have some nice consequences.

Corollary 4.6. Suppose that T^* has SVEP. Then $\Delta_a(T) \subseteq \text{iso } \sigma(T)$.

Proof. We can suppose that $\Delta_a(T)$ is non-empty. If T^* has SVEP then a -Browder's theorem holds for T , so by Theorem 4.4 $\Delta_a \subseteq \text{iso } \sigma_a(T)$. Moreover, by Corollary 3.19 of [1] for all $\lambda \in \Delta_a(T)$ we have $\text{ind}(\lambda I - T) \leq 0$, so $0 < \alpha(\lambda I - T) \leq \beta(\lambda I - T)$, and hence $\lambda \in \sigma_s(T)$. Now, if $\lambda \in \Delta_a(T)$ the SVEP for T^* entails by Theorem 2.2 that $\lambda \in \text{iso } \sigma_s(T)$, and hence $\lambda \in \text{iso } \sigma_s(T) \cap \text{iso } \sigma_a(T) = \text{iso } \sigma(T)$. ■

Corollary 4.7. Suppose that $T \in L(X)$ has SVEP and $\text{iso } \sigma_a(T) = \emptyset$. Then

$$\sigma_a(T) = \sigma_{wa}(T) = \sigma_k(T). \quad (10)$$

Analogously, if T^* has SVEP and $\text{iso } \sigma_s(T) = \emptyset$, then

$$\sigma_s(T) = \sigma_{ws}(T) = \sigma_k(T). \quad (11)$$

Proof. If T has SVEP then a -Browder's theorem holds for T . Since $\text{iso } \sigma_a(T) = \emptyset$, by Theorem 4.4 we have we have $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{wa}(T) = \emptyset$. Therefore $\sigma_a(T) = \sigma_{wa}(T)$ and this set coincides with the spectrum $\sigma_k(T)$, see [1, Chapter 2].

If T^* has SVEP and $\text{iso } \sigma_s(T) = \emptyset$, then $\text{iso } \sigma_a(T^*) = \text{iso } \sigma_s(T) = \emptyset$ and the first part implies that $\sigma_a(T^*) = \sigma_{wa}(T^*) = \sigma_k(T^*)$. By duality we then easily obtain that $\sigma_s(T) = \sigma_{ws}(T) = \sigma_k(T)$. ■

The first part of the previous corollary applies to a right weighted shift T on $\ell^p(\mathbb{N})$, where $1 \leq p < \infty$. In fact, if the spectral radius $r(T) > 0$ then $\text{iso } \sigma_a(T) = \emptyset$, since $\sigma_a(T)$ is a closed annulus (possibly degenerate), see Proposition 1.6.15 of [20], so (10) holds, while if $r(T) = 0$ then, trivially, $\sigma_a(T) = \sigma_{wa}(T) = \sigma_k(T) = \{0\}$. Of course, the equality (11) holds for any left weighted shift. Corollary 4.7 also applies to non-invertible isometry, since

for these operators we have $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, see [20].

As in Theorem 3.4, some characterizations of operators satisfying a -Browder's theorem may be given in terms of the quasi-nilpotent part $H_0(\lambda I - T)$.

Theorem 4.8. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

- (i) a -Browder's theorem holds for T .
- (ii) $H_0(\lambda I - T)$ is finite-dimensional for every $\lambda \in \Delta_a(T)$.
- (iii) $H_0(\lambda I - T)$ is closed for every $\lambda \in \Delta_a(T)$.

Note that in Theorem 4.8 does not appear a characterization of a -Browder's theorem in terms of the analytic core $K(\lambda I - T)$, analogous to that established in Theorem 3.4. The authors in [4] have proved only the following implication:

Theorem 4.9. *If $K(\lambda I - T)$ is finite-codimensional for all $\lambda \in \Delta_a(T)$ then a -Browder's theorem holds for T .*

It would be of interest to prove whenever the converse of the result of Theorem 4.9 holds.

Define

$$\sigma_2(T) := \sigma_{\text{wa}}(T) \cup \sigma_{\text{k}}(T).$$

Note that

$$\sigma_2(f(T)) \subseteq f(\sigma_2(T)) \quad \text{for all } f \in \mathcal{H}(\sigma(T)),$$

see Lemma 3.5 of [10]. A necessary and sufficient condition for the spectral mapping for $\sigma_2(T)$ is given in the next result.

Theorem 4.10. [10] *The spectral mapping theorem holds for $\sigma_2(T)$ precisely when $\text{ind}(\lambda I - T) \cdot \text{ind}(\mu I - T) \geq 0$ for each pair $\lambda, \mu \in \mathbb{C}$ such that $\lambda I - T \in \Phi_+(X)$ and $\mu I - T \in \Phi_-(X)$.*

Using the spectral mapping theorem for $\sigma_a(T)$, see Theorem 2.48 of [1], it is easy to derive the following result analogous to that established in Theorem 3.8

Theorem 4.11. [10] [12] *Suppose that both $T \in L(X)$ and $S \in L(X)$ satisfy a -Browder's theorem and $f \in \mathcal{H}(\sigma(T))$. Then we have:*

- (i) a -Browder's theorem holds for $f(T)$ if and only if $f(\sigma_2(T)) = \sigma_2(f(T))$.

(ii) *a*-Browder's theorem holds for the direct sum $T \oplus S$ if and only if $\sigma_2(T) \cup \sigma_2(S) = \sigma_2(T \oplus S)$.

(iii) *a*-Browder's theorem holds for the direct sum $T \oplus S$ if and only if $\sigma_{\text{wa}}(T) \cup \sigma_{\text{wa}}(S) = \sigma_{\text{wa}}(T \oplus S)$.

Also *a*-Browder's theorem survives under perturbation of Riesz operators K commuting with T , where T satisfies *a*-Browder's theorem. In fact, we have

$$\sigma_{\text{wa}}(T + K) = \sigma_{\text{wa}}(T), \quad \sigma_{\text{ub}}(T + K) = \sigma_{\text{ub}}(T),$$

see [23]. Similar equalities hold for quasi-nilpotent perturbations Q commuting with T , so that *a*-Browder's theorem holds for $T + Q$.

Note that *a*-Browder's theorem transfers successfully to $p(T)$, p a polynomial, if we assume that $p(\sigma_{\text{wa}}(T)) = \sigma_{\text{wa}}(p(T))$. In fact, we have:

Theorem 4.12. [12] *If the map $T \in L(X) \rightarrow \sigma_{\text{wa}}(T)$ is continuous at T_0 then *a*-Browder's theorem holds for T_0 . Furthermore, if *a*-Browder's theorem holds for T and p is a polynomial then *a*-Browder's theorem holds for $p(T)$ if and only if $p(\sigma_{\text{wa}}(T)) = \sigma_{\text{wa}}(p(T))$.*

We conclude by noting that, as Browder's theorem, *a*-Browder's theorem is equivalent to the discontinuity of some mappings.

Theorem 4.13. [4] *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

- (i) *T* satisfies *a*-Browder's theorem;
- (ii) the mapping $\lambda \rightarrow \ker(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_a(T)$ in the gap metric;
- (iii) the mapping $\lambda \rightarrow \gamma(\lambda I - T)$ is not continuous at every $\lambda \in \Delta_a(T)$;
- (iv) the mapping $\lambda \rightarrow (\lambda I - T)(X)$ is not continuous at every $\lambda \in \Delta_a(T)$ in the gap metric.

References

- [1] P. Aiena. *Fredholm and local spectral theory, with application to multipliers*. Kluwer Acad. Publishers, 2004.
- [2] P. Aiena, M. T. Biondi. *Some spectral mapping theorems for semi-Browder spectra through local spectral theory*. Rend. Circ. Mat. Palermo **53**(2004), 165-184.

- [3] P. Aiena, M. T. Biondi. *Browder's theorem and localized SVEP*. Mediterranean Journ. of Math. **2**(2005), 137-151.
- [4] P. Aiena, C. Carpintero, E. Rosas. *Some characterization of operators satisfying a -Browder theorem*. J. Math. Anal. Appl. **311**(2005), 530-544.
- [5] P. Aiena, M. L. Colasante, M. Gonzalez. *Operators which have a closed quasi-nilpotent part*. Proc. Amer. Math. Soc. **130**(9)(2002).
- [6] P. Aiena, B. P. Duggal. *Tensor products, multiplications and Weyl's theorem*. Rend. Circ. Mat. Palermo (2), **54**(2005), no. 3, 387-395.
- [7] P. Aiena, T. L. Miller, M. M. Neumann. *On a localized single valued extension property*. Math. Proc. Royal Irish Acad. **104A**(2004), no. 1, 17-34.
- [8] P. Aiena, O. Monsalve. *Operators which do not have the single valued extension property*. J. Math. Anal. Appl. **250**(2000), 435-448.
- [9] P. Aiena, E. Rosas. *The single valued extension property at the points of the approximate point spectrum*. J. Math. Anal. Appl. **279**(1)(2003), 180-188.
- [10] X. Cao, M. Guo, B. Meng. *A note on Weyl's theorem*. **133**(10)(2003), Proc. Amer. Math. Soc. 2977-2984.
- [11] R. E. Curto, Y. M. Han. *Weyl's theorem, a -Weyl's theorem, and local spectral theory*. J. London Math. Soc. (2) **67**(2003), 499-509.
- [12] S. V. Djordjević, Y. M. Han. *A Browder's theorem and spectral continuity*. Glasgow Math. J. **42**(2000), 479-486.
- [13] N. Dunford. *Spectral theory I. Resolution of the identity*. Pacific J. Math. **2** (1952), 559-614.
- [14] N. Dunford. *Spectral operators*. Pacific J. Math. **4** (1954), 321-354.
- [15] J. K. Finch. *The single valued extension property on a Banach space*. Pacific J. Math. **58** (1975), 61-69.
- [16] R. E. Harte, Woo Young Lee. *Another note on Weyl's theorem*. Trans. Amer. math. Soc. **349** (1997), 2115-2124.
- [17] Robin E. Harte, An-Hyun Kim. *Polaroid operators and Weyl's theorem II*. pre-print(2005).

- [18] H. Heuser. *Functional Analysis*. (1982), Marcel Dekker, New York.
- [19] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, New York. (1966).
- [20] K. B. Laursen, M. M. Neumann. *Introduction to local spectral theory*. Clarendon Press, Oxford 2000.
- [21] M. Mbekhta. *Sur la théorie spectrale locale et limite des nilpotents*. Proc. Amer. Math. Soc. **110** (1990), 621-631.
- [22] V. Müller. *Spectral theory of linears operators*. Operator Theory, Advances and Applications **139**, Birkäuser 2003.
- [23] V. Rakočević. *Semi-Browder operators and perturbations*. Studia Math. **122** (1996), 131-137.
- [24] P. Vrbová. *On local spectral properties of operators in Banach spaces*. Czechoslovak Math. J. **23**(98) (1973a), 483-92.