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Fractional Integration and Fractional Differentiation for Jacobi Expansions.

Integración y Derivación Fraccionaria para desarrollos de Jacobi.

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Abstract

In this article we study the fractional Integral and the fractional Derivative for Jacobi expansion. In order to do that we obtain an analogous of P. A. Meyer's Multipliers Theorem for Jacobi expansions. We also obtain a version of Calderón's reproduction formula for the Jacobi measure. Finally, as an application of the fractional differentiation, we get a characterization for Potential Spaces associated to the Jacobi measure.

Key words and phrases: Fractional Integration, Fractional Differentiation, Jacobi expansions, Multipliers, Potential Spaces.

Resumen

En este trabajo estudiamos la integración y diferenciación fraccionaria para el caso de los desarrollos de Jacobi. Para ello obtenemos un teorema análogo al teorema de multiplicadores de P.A. Meyer para desarrollos de Jacobi. También obtenemos una versión de la fórmula de reproducción de Calderón para la medida de Jacobi. Finalmente, como una aplicacción de la diferenciación fraccionaria, obtenemos una caracterización de los Espacios Potenciales asociados a la medida de Jacobi.

Palabras y frases clave: Integración fraccionaria, Derivación fraccionaria, Desarrollos de Jacobi, Multiplicadores, Espacios Potenciales.

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1 Introduction.

Let us consider (normalized) Jacobi measure

$$\mu_{\alpha,\beta}(dx) = \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}dx, \tag{1}$$

for $x \in [-1,1]$, where $\alpha, \beta > -1$. This normalization gives a probability measure and it is not usually considered in classical orthogonal polynomial theory.

The one dimensional Jacobi operator is given by

$$\mathcal{L}^{\alpha,\beta} = (1-x^2)\frac{d^2}{dx^2} + (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx}. \tag{2}$$

It is easy to see that this is a symmetric operator on $L^2([-1,1],\mu_{\alpha,\beta})$.

Let $p_n^{\alpha,\beta}$ be the normalized Jacobi polynomials of degree $n \in \mathbb{N}$. Then the family $\{p_n^{\alpha,\beta}\}$ is an orthonormal Hilbert basis of $L^2([-1,1],\mu_{\alpha,\beta})$, that can be obtained by the Gram–Schmidt orthogonalization process with respect to the measure $\mu_{\alpha,\beta}$, applied to the monomials. It is well known that Jacobi polynomials are eigenfunctions of the Jacobi operator $\mathcal{L}^{\alpha,\beta}$ with eigenvalue $-\lambda_n = -n(n+\alpha+\beta+1)$, that is,

$$\mathcal{L}^{\alpha,\beta}p_n^{\alpha,\beta} = -n(n+\alpha+\beta+1)p_n^{\alpha,\beta}. \tag{3}$$

Since $\{p_n^{\alpha,\beta}\}$ is an orthonormal basis of $L^2([-1,1],\mu_{\alpha,\beta})$, we have the orthogonal decomposition

$$L^2([-1,1],\mu_{\alpha,\beta}) = \bigoplus_{n=0}^{\infty} C_n^{\alpha,\beta}, \tag{4}$$

where, for each n, $C_n^{\alpha,\beta}$ is the closed subspace generated by $p_n^{\alpha,\beta}$. This is called the Wiener–Jacobi chaos decomposition of $L^2([-1,1],\mu_{\alpha,\beta})$.

Let $J_n^{\alpha,\beta}$ be the orthogonal projection of $L^2([-1,1],\mu_{\alpha,\beta})$ onto $C_n^{\alpha,\beta}$. Then, for $f \in L^2([-1,1],\mu_{\alpha,\beta})$ we have

$$f = \sum_{n=0}^{\infty} J_n^{\alpha,\beta} f, \tag{5}$$

where $J_n^{\alpha,\beta} f = \hat{f}(n)p_n^{\alpha,\beta}$ with

$$\hat{f}(n) = \int_{-1}^{1} f(x)p_n^{\alpha,\beta}(x)\mu_{\alpha,\beta}(dx)$$

the nth-Jacobi–Fourier coefficient of f.

Let us now consider $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ the Jacobi semigroup. This is the Markov operator semigroup associated to the Markov probability kernel semigroup (see [2],[4])

$$P_t(x,dy) = \sum_{n=0}^{\infty} e^{-\lambda_n t} p_n^{\alpha,\beta}(x) p_n^{\alpha,\beta}(y) \mu_{\alpha,\beta}(dy) = p^{\alpha,\beta}(t,x,y)\mu_{\alpha,\beta}(dy),$$

that is

$$T_t^{\alpha,\beta} f(x) = \int_{-1}^{1} f(y) P_t(x,dy) = \int_{-1}^{1} f(y) p^{\alpha,\beta}(t,x,y)\mu_{\alpha,\beta}(dy).$$

Unfortunately, there is not a reasonable explicit representation for the kernel $p^{\alpha,\beta}(t,x,y)$.

The Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is a diffusion semigroup, conservative, symmetric, strongly continuous on $L^p([-1,1],\mu_{\alpha,\beta})$ of positive contractions on L^p, with infinitesimal generator $\mathcal{L}^{\alpha,\beta}$.

Moreover, for $\alpha,\beta > -\frac{1}{2}$ it can be proved that $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is also hypercontractive, that is to say that $T_t^{\alpha,\beta}$ is not only a contraction on $L^p([-1,1],\mu_{\alpha,\beta})$, but also for any initial condition $1 < q(0) < \infty$ there exists an increasing function $q : \mathbb{R}^+ \to [q(0),\infty)$, such that for every f and all $t \geq 0$,

$$\|T_t^{\alpha,\beta} f\|_{q(t)} \leq \|f\|_{q(0)}.$$

The proof that we know of this fact is an indirect one, obtained by D. Bakry in [3], that is based in proving that the Jacobi operator, for the parameters $\alpha,\beta > -1/2$, satisfies a Sobolev inequality, by checking that it satisfies a curvature-dimension inequality, and therefore a logarithmic Sobolev inequality and then use the well known equivalency due to L. Gross [8]. A detailed proof of this can be found in [2], see also [1]. From now on we will consider the Jacobi semigroup for the parameters $\alpha,\beta > -\frac{1}{2}$.

On the other hand, for $0 < \delta < 1$ we define the generalized Poisson–Jacobi semigroup of order δ, $\{P_t^{\alpha,\beta,\delta}\}$, as

$$P_t^{\alpha,\beta,\delta} f(x) = \int_0^{\infty} T_s^{\alpha,\beta} f(x) \mu_t^{\delta}(ds).$$

where $\{\mu_t^{\delta}\}$ are the stable measures on $[0,\infty)$ of order δ (*). The generalized

(*) The stable measures on $[0,\infty)$ of order δ are Borel measures on $[0,\infty)$ such that its Laplace transform verify $\int_0^{\infty} e^{-\lambda s}\mu_t^{\delta}(ds) = e^{-\lambda^{\delta} t}$. For δ fixed, $\{\mu_t^{\delta}\}$ form a semigroup with respect to the convolution operation in the parameter $\delta > 0$.

Poisson–Jacobi semigroup of order δ is a strongly continuous semigroup on $L^p([-1,1],\mu_{\alpha,\beta})$ with infinitesimal generator $(-\mathcal{L}^{\alpha,\beta})^\delta$.

In the case $\delta = 1/2$, we have the Poisson–Jacobi semigroup, that will be denoted as $P_t^{\alpha,\beta} = P_t^{\alpha,\beta,1/2}$. In this case we can explicitly compute $\mu_t^{1/2}$,

$$\mu_t^{1/2}(ds) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4s} s^{-3/2} ds$$

and by Bochner's formula we have

$$P_t^{\alpha,\beta} f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u}^{\alpha,\beta} f(x) du. \tag{6}$$

Then by (3),

$$T_t^{\alpha,\beta} p_n^{\alpha,\beta} = e^{-\lambda_n t} p_n^{\alpha,\beta},$$
$$P_t^{\alpha,\beta,\delta} p_n^{\alpha,\beta} = e^{-\lambda_n^\gamma t} p_n^{\alpha,\beta}.$$

Giving a function $\Phi : \mathbb{N} \to \mathbb{R}$ the multiplier operator associated to Φ is defined as

$$T_\Phi f = \sum_{k=0}^\infty \Phi(k) J_k^{\alpha,\beta} f,$$

for $f = \sum_{k=0}^\infty J_k^{\alpha,\beta} f$, a polynomial.

If Φ is a bounded function, then by Parseval's identity, T_Φ is bounded on $L^2([-1,1],\mu_{\alpha,\beta})$. In the case of Hermite expansions, the P.A. Meyer's Multiplier Theorem [10] gives conditions over Φ so that the multiplier T_Φ can be extended to a continuous operator on L^p for $p \neq 2$. In the next section we will prove an analogous result for the Jacobi expansions.

In section 3 we are going to define the Fractional Integration and Differentiation for Jacobi expansions, as well as Bessel Potentials associated to Jacobi measure. Using Meyer's multipliers Theorem we will see the L^p continuity of the Fractional Integration and of the Bessel Potentials and we give a characterization of the Potential Spaces. We also study the asymptotic behavior of the Poisson-Jacobi semigroup and we give a version of Calderon's reproducing formula.

2 P.A. Meyer's Multiplier Theorem for Jacobi expansions.

In order to establish the P.A. Meyer's Multiplier Theorem for Jacobi expansions we need some previous results. First, let us see that the orthogonal projections $J_n^{\alpha,\beta}$ can be extended to a continuous function on $L^p([-1,1],\mu_{\alpha,\beta})$.

Lemma 1. *If $1 < p < \infty$ then for every $n \in \mathbb{N}$, $J_n^{\alpha,\beta}$ can be extended to a continuous operator to $L^p([-1,1],\mu_{\alpha,\beta})$, that will also be denoted as $J_n^{\alpha,\beta}$, that is, there exists $C_{n,p} \in \mathbb{R}^+$ such that*

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p}\|f\|_p,$$

for $f \in L^p([-1,1],\mu_{\alpha,\beta})$.

Proof. Let us consider $p > 2$ and for the initial condition $q(0) = 2$, let t_0 be a positive number such that $q(t_0) = p$. Taking $g = J_n^{\alpha,\beta} f$, by the hypercontractive property, Parseval identity and Hölder inequality we obtain,

$$\|T_{t_0}^{\alpha,\beta} g\|_p = \|T_{t_0}^{\alpha,\beta} J_n^{\alpha,\beta} f\|_p \leq \|J_n^{\alpha,\beta} f\|_2 \leq \|f\|_2 \leq \|f\|_p.$$

Now, as $T_{t_0} J_n^{\alpha,\beta} f = e^{-t_0\lambda_n} J_n f$ we get

$$\|J_n^{\alpha,\beta} f\|_p \leq C_{n,p}\|f\|_p,$$

with $C_{n,p} = e^{t_0\lambda_n}$. For $1 < p < 2$ the result follows by duality. □

We also need the following technical result,

Lemma 2. *Let $1 < p < \infty$. Then, for each $m \in \mathbb{N}$ there exists a constant C_m such that*

$$\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \leq C_m e^{-tm}\|f\|_p.$$

Proof. Let $p > 2$ and for the initial condition $q(0) = 2$, let t_0 be a positive number such that $q(t_0) = p$.

If $t \leq t_0$, since $T_t^{\alpha,\beta}$ is a contraction, by the L^p- continuity of the projections

$$\begin{aligned} \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p &\leq \|(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \\ &\leq \|f\|_p + \sum_{k=0}^{m-1} \|J_k^{\alpha,\beta} f\|_p \\ &\leq (1 + \sum_{k=0}^{m-1} e^{t_0\lambda_k})\|f\|_p. \end{aligned}$$

But since $e^{t_0\lambda_k} \leq e^{t_0\lambda_m}$ for all $0 \leq k \leq m-1$ and $\lambda_m \geq m$ for all $m > 1$, therefore

$$\begin{aligned}\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p &\leq (1 + me^{t_0\lambda_m})\|f\|_p = C_m e^{-t_0\lambda_m}\|f\|_p \\ &\leq C_m e^{-tm}\|f\|_p,\end{aligned}$$

with $C_m = (1 + me^{t_0\lambda_m})e^{t_0 m}$.
Now suppose $t > t_0$. For $f = \sum_{k=0}^{\infty} J_k^{\alpha,\beta} f$, by the hypercontractive property,

$$\begin{aligned}\|T_{t_0}^{\alpha,\beta}T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots J_{m-1}^{\alpha,\beta})f\|_p^2 &\leq \|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots J_{m-1}^{\alpha,\beta})f\|_2^2 \\ &= \|T_t^{\alpha,\beta}(\sum_{k=m}^{\infty} J_k^{\alpha,\beta} f)\|_2^2 \\ &= \|\sum_{k=m}^{\infty} e^{-t\lambda_k} J_k^{\alpha,\beta} f\|_2^2 \\ &= \sum_{k=m}^{\infty} e^{-2t\lambda_k}\|J_k^{\alpha,\beta} f\|_2^2 \\ &\leq \sum_{k=m}^{\infty} e^{-2tk}\|J_k^{\alpha,\beta} f\|_2^2,\end{aligned}$$

as $\lambda_m \geq m$ for all $m \geq 1$. Then

$$\begin{aligned}\sum_{k=m}^{\infty} e^{-2tk}\|J_k^{\alpha,\beta} f\|_2^2 &\leq e^{-2tm}\sum_{k=0}^{\infty}\|J_{k+m}^{\alpha,\beta}\|_2^2 \leq e^{-2tm}\sum_{k=0}^{\infty}\|J_k^{\alpha,\beta}\|_2^2 \\ &= e^{-2tm}\|f\|_2^2 \leq e^{-2tm}\|f\|_p^2.\end{aligned}$$

Thus

$$\|T_{t_0}^{\alpha,\beta}T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \leq e^{-tm}\|f\|_p.$$

In particular,

$$\begin{aligned}\|T_t^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p &= \|T_{t_0}^{\alpha,\beta}T_{t-t_0}^{\alpha,\beta}(I - J_0^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \\ &\leq e^{-(t-t_0)m}\|f\|_p = C_m e^{-tm}\|f\|_p,\end{aligned}$$

with $C_m = e^{t_0 m}$. For $1 < p < 2$ the result follows by duality. □

Now, by the Minkowski integral inequality, we have an analogous result for the generalized Poisson–Jacobi semigroup.

Lemma 3. *Let $1 < p < \infty$. Then for each $m \in \mathbb{N}$, there exists C_m such that*

$$\|P_t^{\alpha,\beta,\gamma}(I - J_0^{\alpha,\beta} - J_1^{\alpha,\beta} - \cdots - J_{m-1}^{\alpha,\beta})f\|_p \leq C_m e^{-tm^{\gamma}}\|f\|_p.$$

From the generalized Poisson–Jacobi semigroup let us define a new family of operators $\{P^{\alpha,\beta}_{k,\gamma,m}\}_{k\in\mathbb{N}}$ by the formula

$$P^{\alpha,\beta}_{k,\gamma,m} f = \frac{1}{(k-1)!}\int_0^\infty t^{k-1} P^{\alpha,\beta,\gamma}_t (I - J^{\alpha,\beta}_0 - J^{\alpha,\beta}_1 - \cdots - J^{\alpha,\beta}_{m-1}) f dt.$$

By the preceding lemma and the Minkowski integral inequality we have,

Proposition 4. *If $1 < p < \infty$, then for every $m \in \mathbb{N}$ there is a constant C_m such that*

$$\|P^{\alpha,\beta}_{k,\gamma,m} f\|_p \le \frac{C_m}{m^{\gamma k}} \|f\|_p.$$

Observe that in particular if $f = p^{\alpha,\beta}_n$, $n \ge m$

$$P^{\alpha,\beta}_{k,\gamma,m} p^{\alpha,\beta}_n = \frac{1}{\lambda_n^{\gamma k}} p^{\alpha,\beta}_n. \tag{7}$$

Now we are ready to present P.A. Meyer's Multipliers Theorem for Jacobi expansions.

Theorem 5. *If for some $n_0 \in \mathbb{N}$ and $0 < \gamma < 1$*

$$\Phi(k) = h\left(\frac{1}{\lambda_k^\gamma}\right), \qquad k \ge n_0,$$

with h an analytic function in a neighborhood of zero, then T_Φ, the multiplier operator associated to Φ, admits a continuous extension to $L^p([-1,1], \mu_{\alpha,\beta})$.

Proof. Let

$$T_\Phi f = T^1_\phi f + T^2_\Phi f = \sum_{k=0}^{n_0-1} \Phi(k) J^{\alpha,\beta}_k f + \sum_{k=n_0}^{\infty} \Phi(k) J^{\alpha,\beta}_k f.$$

By the lemma 1 we have that

$$\|T^1_\Phi f\|_p \le \sum_{k=0}^{n_0-1} |\Phi(k)| \|J^{\alpha,\beta}_k f\|_p \le \left(\sum_{k=0}^{n_0-1} |\Phi(k)| C_k\right) \|f\|_p,$$

that is, T^1_Φ is L^p continuous. It remains to be seen that T^2_Φ is also L^p continuous.

By the hypothesis let us assume that h can be written as $h(x) = \sum_{n=0}^{\infty} a_n x^n$, for x in a neighborhood of zero, then

$$T_\Phi^2 f = \sum_{k=n_0}^{\infty} \Phi(k) J_k^{\alpha,\beta} f = \sum_{k=n_0}^{\infty} h\left(\frac{1}{\lambda_k^{\gamma}}\right) J_k^{\alpha,\beta} f = \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n \frac{1}{\lambda_k^{\gamma n}} J_k^{\alpha,\beta} f,$$

but since (7), for $k \geq n_0$, $\frac{1}{\lambda_k^{\gamma n}} J_k^{\alpha,\beta} f = P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f$, we have

$$\begin{aligned} T_\Phi^2 f &= \sum_{k=n_0}^{\infty} \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{\infty} P_{n,\gamma,n_0}^{\alpha,\beta} J_k^{\alpha,\beta} f \\ &= \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} \sum_{k=0}^{\infty} J_k^{\alpha,\beta} f = \sum_{n=0}^{\infty} a_n P_{n,\gamma,n_0}^{\alpha,\beta} f. \end{aligned}$$

Since $P_{n,\gamma,n_0}^{\alpha,\beta}$ is L^p continuous, by proposition 4, we obtain,

$$\begin{aligned} \|T_\Phi^2 f\|_p &\leq \sum_{n=0}^{\infty} |a_n| \|P_{n,\gamma,n_0}^{\alpha,\beta} f\|_p \\ &\leq \left(\sum_{n=0}^{\infty} |a_n| C_{n_0} \frac{1}{n_0^{\gamma n}}\right) \|f\|_p = C_{n_0} \left(\sum_{n=0}^{\infty} |a_n| \frac{1}{n_0^{\gamma n}}\right) \|f\|_p. \end{aligned}$$

Therefore, T_Φ is continuous in $L^p([-1,1], \mu_{\alpha,\beta})$. □

3 Fractional Integration and Differentiation.

As in the classical case, for $\gamma > 0$ we define the Fractional Integral of order γ, $I_\gamma^{\alpha,\beta}$, with respect to Jacobi measure, as

$$I_\gamma^{\alpha,\beta} = (-\mathcal{L}^{\alpha,\beta})^{-\gamma/2}. \tag{8}$$

$I_\gamma^{\alpha,\beta}$ is also called Riesz Potential of order γ.

Observe that, since zero is an eigenvalue of $\mathcal{L}^{\alpha,\beta}$, then $I_\gamma^{\alpha,\beta}$ is not defined over all $L^2([-1,1], \mu_{\alpha,\beta})$. Let $\Pi_0 = I - J_0^{\alpha,\beta}$ and denote also by $I_\gamma^{\alpha,\beta}$ the operator $(-\mathcal{L}^{\alpha,\beta})^{-\gamma/2}\Pi_0$. Then, this operator is well defined over all $L^2([-1,1], \mu_{\alpha,\beta})$. In particular, for Jacobi polynomials we have

$$I_\gamma^{\alpha,\beta} p_k^{\alpha,\beta} = \frac{1}{\lambda_k^{\gamma/2}} p_k^{\alpha,\beta}. \tag{9}$$

Thus, for f a polynomial in $L^2([-1,1],\mu_{\alpha,\beta})$ with Jacobi expansion $\sum_{k=0}^{\infty} J_k^{\alpha,\beta} f$, we have

$$I_\gamma^{\alpha,\beta} f = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\gamma/2}} J_k^{\alpha,\beta} f.$$

For the Fractional Integral of order $\gamma > 0$ we have the following integral representation,

$$I_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} P_t^{\alpha,\beta} f \, dt, \tag{10}$$

for f polynomial, where $P_t^{\alpha,\beta}$ is the Poisson–Jacobi semigroup. In order to prove that observe that for the Jacobi polynomials, we have, by the change of variables $s = \lambda_k^{1/2} t$,

$$\begin{aligned} \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} P_t^{\alpha,\beta} p_k^{\alpha,\beta} \, dt &= \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} e^{-\lambda_k^{1/2} t} \, dt \, p_k^{\alpha,\beta} \\ &= \frac{1}{\Gamma(\gamma)} \frac{1}{\lambda_k^{\gamma/2}} \int_0^\infty s^{\gamma-1} e^{-s} \, ds \, p_k^{\alpha,\beta} = \frac{1}{\lambda_k^{\gamma/2}} \, p_k^{\alpha,\beta}. \end{aligned}$$

The Meyer's multiplier theorem allows us to extend $I_\gamma^{\alpha,\beta}$ as a bounded operator on $L^p([-1,1],\mu_{\alpha,\beta})$, as next theorem shows.

Theorem 6. *The the Fractional Integral of order γ, $I_\gamma^{\alpha,\beta}$ admits a continuous extension, that it will also be denoted as denote $I_\gamma^{\alpha,\beta}$, to $L^p([-1,1],\mu_{\alpha,\beta})$.*

Proof. If $\gamma/2 < 1$, then $I_\gamma^{\alpha,\beta}$ is a multiplier with associated function

$$\Phi(k) = \frac{1}{\lambda_k^{\gamma/2}} = h\left(\frac{1}{\lambda_k^{\gamma/2}}\right)$$

where $h(z) = z$, which is analytic in a neighborhood of zero. Then the results follows immediately by Meyer's theorem.

Now, if $\gamma/2 \geq 1$, let us consider $\beta \in \mathbb{R}$, $0 < \beta < 1$ and $\delta = \frac{\gamma}{2\beta}$. Then $\delta\beta = \frac{\gamma}{2}$. Let $h(z) = z^\delta$, which is analytic in a neighborhood of zero. Then we have

$$h\left(\frac{1}{\lambda_k^{\beta}}\right) = \frac{1}{\lambda_k^{\delta\beta}} = \frac{1}{\lambda_k^{\gamma/2}} = \Phi(k).$$

Again the results follows applying Meyer's theorem. □

Now the Bessel Potential of order $\gamma > 0$, $\mathcal{J}_\gamma^{\alpha,\beta}$, associated to the Jacobi measure is defined as

$$\mathcal{J}_\gamma^{\alpha,\beta} = (I - \mathcal{L}^{\alpha,\beta})^{-\gamma/2}. \tag{11}$$

Observe that for the Jacobi polynomials we have

$$\mathcal{J}_\gamma^{\alpha,\beta} p_k^{\alpha,\beta} = \frac{1}{(1+\lambda_k)^{\gamma/2}} p_k^{\alpha,\beta},$$

and, therefore if $f \in L^2([-1,1], \mu_{\alpha,\beta})$ polynomial with expansion $\sum_{k=0}^{\infty} J_k^{\alpha,\beta} f$

$$\mathcal{J}_\gamma^{\alpha,\beta} f = \sum_{k=0}^{\infty} \frac{1}{(1+\lambda_k)^{\gamma/2}} J_k^{\alpha,\beta} f. \tag{12}$$

Again Meyer's theorem allows us to extend Bessel Potentials to a continuous operator on $L^p([-1,1], \mu_{\alpha,\beta})$,

Theorem 7. *The operator $\mathcal{J}_\gamma^{\alpha,\beta}$ admits a continuous extension, that it will also be denoted as $\mathcal{J}_\gamma^{\alpha,\beta}$, to $L^p([-1,1], \mu_{\alpha,\beta})$.*

Proof. Bessel Potential of order γ is a multiplier associated to the function $\Phi(k) = \left(\frac{1}{1+\lambda_k}\right)^{\gamma/2}$. Let $\beta \in \mathbb{R}$, $\beta > 1$ and $h(z) = \left(\frac{z^\beta}{z^\beta+1}\right)^{\gamma/2}$. Then h is an analytic function on a neighborhood of zero and

$$h\left(\frac{1}{\lambda_k^{1/\beta}}\right) = \left(\frac{1}{1+\lambda_k}\right)^{\gamma/2} = \Phi(k).$$

The results follows applying Meyer's theorem. □

Finally as in the classical case, we define the Fractional Derivative of order $\gamma > 0$, $D_\gamma^{\alpha,\beta}$, with respect to Jacobi measure as

$$D_\gamma^{\alpha,\beta} = (-\mathcal{L}^{\alpha,\beta})^{\gamma/2}. \tag{13}$$

Observe that, for the Jacobi polynomials we have,

$$D_\gamma^{\alpha,\beta} p_k^{\alpha,\beta} = \lambda_k^{\gamma/2} p_k^{\alpha,\beta}, \tag{14}$$

and therefore, by the density of the polynomials in $L^p([-1,1], \mu_{\alpha,\beta})$, $1 < p < \infty$, $D_\gamma^{\alpha,\beta}$ can be extended to $L^p([-1,1], \mu_{\alpha,\beta})$.

For the Fractional Derivative of order $0 < \gamma < 1$ we also have a integral representation,

$$D_{\gamma}^{\alpha,\beta} f = \frac{1}{c_\gamma} \int_0^\infty t^{-\gamma-1} (P_t^{\alpha,\beta} f - f) dt, \tag{15}$$

for f polynomial, where $c_\gamma = \int_0^\infty s^{-\gamma-1}(e^{-s} - 1) ds$, since, for the Jacobi polynomials, we have, by the change of variables $s = \lambda_k^{1/2} t$,

$$\begin{aligned} \int_0^\infty t^{-\gamma-1} (P_t^{\alpha,\beta} p_k^{\alpha,\beta} - p_k^{\alpha,\beta})\, dt &= \int_0^\infty t^{-\gamma-1} (e^{-\lambda_k^{1/2} t} - 1)\, dt\, p_k^{\alpha,\beta} \\ &= \lambda_k^{\gamma/2} \int_0^\infty s^{-\gamma-1} (e^{-s} - 1)\, ds\, p_k^{\alpha,\beta} \\ &= \lambda_k^{\gamma/2} c_\gamma\, p_k^{\alpha,\beta}. \end{aligned}$$

Now, if f is a polynomial, by (9) and (14) we have,

$$I_{\gamma}^{\alpha,\beta} (D_{\gamma}^{\alpha,\beta} f) = D_{\gamma}^{\alpha,\beta} (I_{\gamma}^{\alpha,\beta} f) = \Pi_0 f. \tag{16}$$

In [5] H. Bavinck has defined Fractional Integration and Differentiation for Jacobi expansions. Nevertheless the motivation, the methods and techniques use in his paper are totally different from ours.

Now we are going to give an alternative representation of $D_{\gamma}^{\alpha,\beta}$ and $I_{\gamma}^{\alpha,\beta}$ that are very useful in what follows. Before that, we need the following technical result of the asymptotic behavior of $\{P_t^{\alpha,\beta}\}_{t\geq 0}$ at infinity.

Lemma 8. *If $\int_{-1}^1 f(y) \mu_{\alpha,\beta}(dy) = 0$ and f has continuos derivatives up to the second order, then*

$$\left| \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) \right| \leq C_{f,\alpha,\beta} (1 + |x|) e^{-(\alpha+\beta+2)^{1/2} t}. \tag{17}$$

As a consequence we have that the Poisson-Jacobi semigroup $\{P_t^{\alpha,\beta}\}_{t\geq 0}$, has exponential decay on $(C_0^{\alpha,\beta})^{\perp} = \bigoplus_{n=1}^\infty C_n^{\alpha,\beta}$. More precisely, if we have $\int_{-1}^1 f(y) \mu_{\alpha,\beta}(dy) = 0$,

$$|P_t^{\alpha,\beta} f(x)| \leq C_{f,\alpha,\beta} (1 + |x|) e^{-(\alpha+\beta+2)^{1/2} t}. \tag{18}$$

Proof. First, let us prove that $\left|\frac{\partial}{\partial t}T_t^{\alpha,\beta}f(x)\right| \leq C_{f,\alpha,\beta}(1+|x|)e^{-(\alpha+\beta+2)t}$. Since $\frac{\partial}{\partial t}T_t^{\alpha,\beta}f = \mathcal{L}^{\alpha,\beta}T_t^{\alpha,\beta}f$ and

$$\begin{aligned}
\frac{\partial}{\partial x}T_t^{\alpha,\beta}f &= e^{-(\alpha+\beta+2)t}T_t^{\alpha+1,\beta+1}\left(\frac{\partial f}{\partial x}\right)\\
\frac{\partial^2}{\partial x^2}T_t^{\alpha,\beta}f &= e^{-2(\alpha+\beta+3)t}T_t^{\alpha+2,\beta+2}\left(\frac{\partial^2 f}{\partial x^2}\right)
\end{aligned}$$

we have,

$$\begin{aligned}
\left|\frac{\partial}{\partial t}T_t^{\alpha,\beta}f(x)\right| &\leq |1-x^2|e^{-2(\alpha+\beta+3)t}T_t^{\alpha+2,\beta+2}\left(\left|\frac{\partial^2 f}{\partial x^2}\right|\right)(x) +\\
&+ (|\beta-\alpha+1|+(\alpha+\beta+2)|x|)e^{-(\alpha+\beta+2)t}T_t^{\alpha+1,\beta+1}\left(\left|\frac{\partial f}{\partial x}\right|\right)(x).
\end{aligned}$$

Also

$e^{-2(\alpha+\beta+3)t} \leq e^{-(\alpha+\beta+2)t}$, $|1-x^2| \leq 1+|x|$, $|\beta-\alpha+1|+(\alpha+\beta+2)|x| \leq C_{\alpha,\beta}(1+|x|)$

and as f has continue derivatives up to the second order, there exist a constant C_f such that $\left|\frac{\partial f}{\partial x}\right| \leq C_f$ and $\left|\frac{\partial^2 f}{\partial x^2}\right| \leq C_f$, therefore,

$$\left|\mathcal{L}^{\alpha,\beta}T_t^{\alpha,\beta}f(x)\right| = \left|\frac{\partial}{\partial t}T_t^{\alpha,\beta}f(x)\right| \leq C_{f,\alpha,\beta}(1+|x|)e^{-(\alpha+\beta+2)t}.$$

Now,

$$\frac{\partial}{\partial t}P_t^{\alpha,\beta}f = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\frac{t}{2u}\mathcal{L}^{\alpha,\beta}T_{t^2/4u}^{\alpha,\beta}f du$$

hence, by the change of variables $u = (\alpha+\beta+2)s$ we have

$$\begin{aligned}
\left|\frac{\partial}{\partial t}P_t^{\alpha,\beta}f(x)\right| &\leq C_{f,\alpha,\beta}\frac{(1+|x|)}{2\sqrt{\pi}}\int_0^\infty e^{-u}u^{-3/2}te^{-(\alpha+\beta+2)t^2/4u}du\\
&= C_{f,\alpha,\beta}(1+|x|)\int_0^\infty e^{-(\alpha+\beta+2)s}\frac{t}{2\sqrt{\pi}}s^{-3/2}e^{-t^2/4s}ds\\
&= C_{f,\alpha,\beta}(1+|x|)\int_0^\infty e^{-(\alpha+\beta+2)s}\mu_t^{1/2}(ds)\\
&= C_{f,\alpha,\beta}(1+|x|)e^{-(\alpha+\beta+2)^{1/2}t}.
\end{aligned}$$

By hypothesis, since we are assuming that $\int_{-1}^1 f(y)\mu_{\alpha,\beta}(dy) = 0$,

$$\lim_{t\to\infty} P_t^{\alpha,\beta}f(x) = 0. \tag{19}$$

Then

$$\begin{aligned} \left|P_t^{\alpha,\beta} f(x)\right| &\leq \int_t^\infty \left|\frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x)\right| ds \leq C_{f,\alpha,\beta} \int_t^\infty (1+|x|) e^{-(\alpha+\beta+2)^{1/2} s} ds \\ &= C_{f,\alpha,\beta}(1+|x|) e^{-(\alpha+\beta+2)^{1/2} t}. \end{aligned}$$

□

Remember that, since $\{P_t^{\alpha,\beta}\}_{t\geq 0}$ is an strongly continuos semigroup, we have

$$\lim_{t\to 0^+} P_t^{\alpha,\beta} f(x) = f(x) \tag{20}$$

Now we are ready to give the alternate representation of $D_\gamma^{\alpha,\beta}$ and $I_\gamma^{\alpha,\beta}$,

Proposition 9. *Suppose $f \in C^2([-1,1])$ such that $\int_{-1}^1 f(y)\mu_{\alpha,\beta}(dy) = 0$, then*

$$D_\gamma^{\alpha,\beta} f = \frac{1}{\gamma c_\gamma} \int_0^\infty t^{-\gamma} \frac{\partial}{\partial t} P_t^{\alpha,\beta} f dt, \quad 0 < \gamma < 1, \tag{21}$$

$$I_\gamma^{\alpha,\beta} f = -\frac{1}{\gamma \Gamma(\gamma)} \int_0^\infty t^{\gamma} \frac{\partial}{\partial t} P_t^{\alpha,\beta} f dt, \quad \gamma > 0. \tag{22}$$

Proof. Let us start proving (21). Integrating by parts in (15) we get

$$\begin{aligned} D_\gamma^{\alpha,\beta} f(x) &= \frac{1}{c_\gamma} \lim_{\substack{a\to 0^+ \\ b\to\infty}} \int_a^b t^{-\gamma-1}\left(P_t^{\alpha,\beta} f(x) - f(x)\right) dt \\ &= \frac{1}{c_\gamma} \lim_{\substack{a\to 0^+ \\ b\to\infty}} \left\{ \frac{t^{-\gamma}}{-\gamma}\left(P_t^{\alpha,\beta} f(x) - f(x)\right)\Big|_a^b + \frac{1}{\gamma}\int_a^b t^{-\gamma} \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) dt \right\} \\ &= \frac{1}{\gamma c_\gamma} \int_0^\infty t^{-\gamma} \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) dt \end{aligned}$$

since, by (19), (20) and the previous lemma, we have

$$\lim_{b\to\infty} \left(\frac{P_b^{\alpha,\beta} f(x) - f(x)}{b^\gamma} \right) = 0$$

and

$$\begin{aligned} \lim_{a\to 0^+} \left| \frac{P_a^{\alpha,\beta} f(x) - f(x)}{a^\gamma} \right| &\leq \lim_{a\to 0^+} \frac{1}{a^\gamma} \int_0^a \left| \frac{\partial}{\partial s} P_s^{\alpha,\beta} f(x) \right| ds \\ &\leq C_{f,\alpha,\beta}(1+|x|) \lim_{a\to 0^+} \frac{1 - e^{-(\alpha+\beta+2)^{1/2} a}}{a^\gamma} = 0. \end{aligned}$$

Let us prove now (22). Again, by integrating by parts, we have

$$\begin{aligned}
I_\gamma^{\alpha,\beta} f(x) &= \frac{1}{\Gamma(\gamma)} \lim_{\substack{a\to 0^+\\ b\to\infty}} \int_a^b t^{\gamma-1} P_t^{\alpha,\beta} f(x) dt \\
&= \frac{1}{\Gamma(\gamma)} \lim_{\substack{a\to 0^+\\ b\to\infty}} \left\{ \frac{t^\gamma}{\gamma} P_t^{\alpha,\beta} f(x)\Big|_a^b - \frac{1}{\gamma}\int_a^b t^\gamma \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) dt \right\} \\
&= -\frac{1}{\gamma\Gamma(\gamma)} \int_0^\infty t^\gamma \frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) dt,
\end{aligned}$$

since, by the previous result

$$\lim_{b\to\infty} \left| P_b^{\alpha,\beta} f(x) b^\gamma \right| \leq C_{d,f}(1+|x|) \lim_{b\to\infty} b^\gamma e^{-(\alpha+\beta+2)^{-1/2} b} = 0$$

and

$$\lim_{a\to 0^+} \left| P_a^{\alpha,\beta} f(x) a^\gamma \right| = 0.$$

□

Observe that the previous proposition is also true for the Jacobi polynomials of order $n > 0$, and therefore is true for any nonconstant polynomial f such that $\int_{-1}^1 f(y)\mu_{\alpha,\beta}(dy) = 0$.

Now let us write

$$\begin{aligned}
P_t^{\alpha,\beta} f(x) &= \int_0^\infty T_s^{\alpha,\beta} f(x) \mu_t^{1/2}(ds) \\
&= \int_{-1}^1 [\int_0^\infty p^{\alpha,\beta}(s,x,y)\mu_t^{1/2}(ds)] f(y)\mu_{\alpha,\beta}(dy) \\
&= \int_{-1}^1 k^{\alpha,\beta}(t,x,y) f(y)\mu_{\alpha,\beta}(dy),
\end{aligned}$$

where

$$k^{\alpha,\beta}(t,x,y) = \int_0^\infty p^{\alpha,\beta}(s,x,y)\mu_t^{1/2}(ds) \tag{23}$$

and define the operator $Q_t^{\alpha,\beta}$ as

$$Q_t^{\alpha,\beta} f(x) = -t\frac{\partial}{\partial t} P_t^{\alpha,\beta} f(x) = \int_{-1}^1 q^{\alpha,\beta}(t,x,y) f(y)\mu_{\alpha,\beta}(dy), \tag{24}$$

with $q^{\alpha,\beta}(t,x,y) = -t\frac{\partial}{\partial t} k^{\alpha,\beta}(t,x,y)$.

Following [6] we get immediately from (21) and (22), the following formulas

Corollary 10. *Suppose f differentiable with continuos derivatives up to the second order such that $\int_{-1}^{1} f(y)\mu_{\alpha,\beta}(dy) = 0$, then we have*

$$-\gamma D_\gamma^{\alpha,\beta} f = \frac{1}{c_\gamma}\int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta} f dt,\ 0<\gamma<1, \tag{25}$$

$$\gamma I_\gamma^{\alpha,\beta} f = \frac{1}{\Gamma(\gamma)}\int_0^\infty t^{\gamma-1} Q_t^{\alpha,\beta} f dt,\ \gamma>0. \tag{26}$$

An interesting use of the family $\{Q_t^{\alpha,\beta}\}$ is that it allows to give a version of Calderón's reproduction formula for the Jacobi measure,

Theorem 11. *i) Suppose $f \in L^1([-1,1],\mu_{\alpha,\beta})$ such that $\int_{-1}^{1} f(y)\mu_{\alpha,\beta}(dy) = 0$, then we have*

$$f = \int_0^\infty Q_t^{\alpha,\beta} f \frac{dt}{t}. \tag{27}$$

ii) Suppose f a polynomial such that $\int_{-1}^{1} f(y)\mu_{\alpha,\beta}(dy) = 0$, then we have

$$f = C_\gamma \int_0^\infty \int_0^\infty t^{-\gamma} s^\gamma Q_t^{\alpha,\beta}\left(Q_s^{\alpha,\beta} f\right)\frac{ds}{s}\frac{dt}{t},\quad 0<\gamma<1. \tag{28}$$

Also,

$$\int_0^\infty \int_0^\infty t^{-\gamma} s^\gamma Q_t^{\alpha,\beta}\left(Q_s^{\alpha,\beta} f\right)\frac{ds}{s}\frac{dt}{t} = \int_0^\infty u\frac{\partial^2}{\partial u^2} P_u^{\alpha,\beta} f du. \tag{29}$$

Formula (28) is the version of Calderón's reproduction formula for the Jacobi measure.

Proof. i) Using (20) and (19) we have,

$$\int_0^\infty Q_t^{\alpha,\beta} f \frac{dt}{t} = \lim_{\substack{a\to 0^+\\ b\to\infty}} (-\int_a^b \frac{\partial}{\partial t} P_t^{\alpha,\beta} f dt) = \lim_{\substack{a\to 0^+\\ b\to\infty}} (-P_t^{\alpha,\beta} f)\Big|_a^b = f.$$

Let us prove (28), given f a polynomial such that $\int_{-1}^{1} f(y)\mu_{\alpha,\beta}(dy) = 0$, by Corollary 10, we have

$$D_\gamma^{\alpha,\beta}\left(I_\gamma^{\alpha,\beta} f\right) = -\frac{1}{\gamma c_\gamma}\int_0^\infty t^{-\gamma-1} Q_t^{\alpha,\beta}\left(I_\gamma^{\alpha,\beta} f\right) dt. \tag{30}$$

Now, by (26) and the linearity of $Q_t^{\alpha,\beta}$, we have

$$Q_t^{\alpha,\beta}\left(I_\gamma^{\alpha,\beta} f\right) = \frac{1}{\gamma\Gamma(\gamma)}\int_0^\infty s^{\gamma-1} Q_t^{\alpha,\beta}(Q_s^{\alpha,\beta} f)(y) ds.$$

Substituting in (30)

$$f = D_\gamma^{\alpha,\beta}\left(I_\gamma^{\alpha,\beta} f\right) = C_\gamma \int_0^\infty \int_0^\infty t^{-\gamma-1} s^{\gamma-1} Q_t^{\alpha,\beta}\left(Q_s^{\alpha,\beta} f\right) ds dt,$$

with $C_\gamma = -\frac{1}{\gamma^2 c_\gamma \Gamma(\gamma)}$.

To show (29), let us integrate by parts, and by Lemma 8 we have

$$\begin{aligned}\int_0^\infty u \frac{\partial^2}{\partial u^2} P_u^{\alpha,\beta} f(x) du &= u \frac{\partial}{\partial u} P_u^{\alpha,\beta} f(x)\Big|_0^\infty - \int_0^\infty \frac{\partial}{\partial u} P_u^{\alpha,\beta} f(x) du \\ &= -\int_0^\infty \frac{\partial}{\partial u} P_u^{\alpha,\beta} f(x) du = -P_u^{\alpha,\beta} f(x)\big|_0^\infty \\ &= P_0^{\alpha,\beta} f(x) = f(x).\end{aligned}$$

□

Now let us consider the Jacobi Potential Spaces. The Jacobi Potential space of order $\gamma > 0$, $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$, for $1 < p < \infty$, is defined as the completion of the polynomials with respect to the norm

$$\|f\|_{p,\gamma} := \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p.$$

That is to say $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ if, and only if, there is a sequence of polynomials $\{f_n\}$ such that $\lim_{n\to\infty} \|f_n - f\|_{p,\gamma} = 0$.

As in the classical case, the Jacobi Potential space $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ can also be defined as the image of $L^p([-1,1], \mu_{\alpha,\beta})$ under the Bessel Potential $\mathcal{J}_\gamma^{\alpha,\beta}$, that is,

$$L^p_\gamma([-1,1], \mu_{\alpha,\beta}) = \mathcal{J}_\gamma^{\alpha,\beta} L^p([-1,1], \mu_{\alpha,\beta}).$$

For the details of the proof of this equivalence, we refer to [2].

Now, let us consider some inclusion properties among Jacobi Potential Spaces,

Proposition 12. *i) If $p < q$, then $L^q_\gamma([-1,1], \mu_{\alpha,\beta}) \subseteq L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ for each $\gamma > 0$.*

ii) If $0 < \gamma < \delta$, then $L^p_\delta([-1,1], \mu_{\alpha,\beta}) \subseteq L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ for each $0 < p < \infty$.

Proof. *i)* For γ fixed, it follows immediately by Hölder's inequality.

ii) Let f be a polynomial and consider

$$\phi = (I - \mathcal{L}^{\alpha,\beta})^{\delta/2} f = \sum_{k=0}^{\infty} (1+\lambda_k)^{\delta/2} J_k^{\alpha,\beta} f,$$

which is also a polynomial. Then $\phi \in L^p_\delta([-1,1], \mu_{\alpha,\beta})$, $\|\phi\|_p = \|f\|_{p,\delta}$ and $\mathcal{J}^{\alpha,\beta}_{(\gamma-\delta)}\phi = (I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2}\phi = (I-\mathcal{L}^{\alpha,\beta})^{\gamma/2} f$. By the L^p continuity of Bessel Potentials,

$$\|f\|_{p,\gamma} = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f\|_p = \|\mathcal{J}_{(\gamma-\delta)}\phi\|_p \leq C_p \|f\|_{p,\delta}.$$

Now let $f \in L^p_\delta([-1,1], \mu_{\alpha,\beta})$. Then there exists $g \in L^p([-1,1], \mu_{\alpha,\beta})$ such that $f = \mathcal{J}^{\alpha,\beta}_\delta g$ and a sequence of polynomials $\{g_n\}$ in $L^p([-1,1], \mu_{\alpha,\beta})$ such that $\lim_{n\to\infty} \|g_n - g\|_p = 0$. Set $f_n = \mathcal{J}^{\alpha,\beta}_\delta g_n$. Then $\lim_{n\to\infty} \|f_n - f\|_{p,\delta} = 0$, and

$$\begin{aligned}\|f_n - f\|_{p,\gamma} &= \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2}(f_n - f)\|_p = \|(I - \mathcal{L}^{\alpha,\beta})^{\gamma/2}(I - \mathcal{L}^{\alpha,\beta})^{-\delta/2}(g_n - g)\|_p \\ &= \|(I - \mathcal{L}^{\alpha,\beta})^{(\gamma-\delta)/2}(g_n - g)\|_p = \|\mathcal{J}_{(\gamma-\delta)}(g_n - g)\|_p.\end{aligned}$$

By the L^p continuity of Bessel Potentials $\lim_{n\to\infty} \|f_n - f\|_{p,\gamma} = 0$.

Therefore,

$$\begin{aligned}\|f\|_{p,\gamma} &\leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\gamma} \\ &\leq \|f_n - f\|_{p,\gamma} + \|f_n\|_{p,\delta},\end{aligned}$$

taking limit as n goes to infinity, we obtain the result. □

Let us consider the space

$$L_\gamma([-1,1], \mu_{\alpha,\beta}) = \bigcup_{p>1} L^p_\gamma([-1,1], \mu_{\alpha,\beta}).$$

$L_\gamma([-1,1], \mu_{\alpha,\beta})$ is the natural domain of $D^{\alpha,\beta}_\gamma$. We define it in this space as follows.

Let $f \in L_\gamma([-1,1], \mu_{\alpha,\beta})$, then there is $p > 1$ such that $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ and a sequence $\{f_n\}$ polynomials such that $\lim_{n\to\infty} f_n = f$ in $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$. We define for $f \in L_\gamma([-1,1], \mu_{\alpha,\beta})$

$$D^{\alpha,\beta}_\gamma f = \lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n$$

in $L^p([-1,1], \mu_{\alpha,\beta})$. The next theorem shows that $D^{\alpha,\beta}_\gamma$ is well defined and also inequality (31) gives us a characterization of the Potential Spaces,

Theorem 13. *Let $\gamma > 0$ and $1 < p, q < \infty$.*

i) If $\{f_n\}$ is a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$ in $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$, then $\lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n \in L^p([-1,1], \mu_{\alpha,\beta})$ and the limit does not depend on the choice of the sequence $\{f_n\}$.

If $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta}) \bigcap L^q_\gamma([-1,1], \mu_{\alpha,\beta})$, then the limit does not depend on the choice of p or q.

Thus $D^{\alpha,\beta}_\gamma$ is well defined on $L_\gamma([-1,1], \mu_{\alpha,\beta})$.

ii) $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta})$ if, and only if, $D^{\alpha,\beta}_\gamma f \in L^p([-1,1], \mu_{\alpha,\beta})$. Moreover, there exists positive constants $A_{p,\gamma}$ and $B_{p,\gamma}$ such that

$$B_{p,\gamma}\|f\|_{p,\gamma} \leq \|D^{\alpha,\beta}_\gamma f\|_p \leq A_{p,\gamma}\|f\|_{p,\gamma}. \tag{31}$$

Proof.
ii) First, let us note that for $p = \sum_{n=0}^\infty J^{\alpha,\beta}_n p$ polynomial,

$$D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma p = \sum_{n=0}^\infty \left(\frac{\lambda_n}{1+\lambda_n}\right)^{\gamma/2} J^{\alpha,\beta}_n p,$$

that is, the operator $D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma$ is a multiplier with associated function $\Phi(n) = \left(\frac{\lambda_n}{1+\lambda_n}\right)^{\gamma/2} = h(\frac{1}{\lambda_n})$ where $h(z) = \left(\frac{1}{z+1}\right)^{\gamma/2}$, and therefore by Meyer's theorem it is L^p-continuos.

Let f be a polynomial and let ϕ be a polynomial such that $f = \mathcal{J}^{\alpha,\beta}_\gamma \phi$. We have that $\|f\|_{p,\gamma} = \|\phi\|_p$ and by the continuity of the operator $D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma$

$$\|D^{\alpha,\beta}_\gamma f\|_p = \|D^{\alpha,\beta}_\gamma \mathcal{J}^{\alpha,\beta}_\gamma \phi\|_p \leq A_{p,\gamma}\|\phi\|_p = A_{p,\gamma}\|f\|_{p,\gamma}.$$

To prove the converse, let us suppose that f polynomial, then $D^{\alpha,\beta}_\gamma f$ is also a polynomial, and therefore $D^{\alpha,\beta}_\gamma f \in L^p([-1,1], \mu_{\alpha,\beta})$. Consider

$$\phi = (I - \mathcal{L}^{\alpha,\beta})^{\gamma/2} f = \sum_{k=0}^\infty (1+\lambda_k)^{\gamma/2} J^{\alpha,\beta}_k f = \sum_{k=0}^\infty \left(\frac{1+\lambda_k}{\lambda_k}\right)^{\gamma/2} J^{\alpha,\beta}_k (D^{\alpha,\beta}_\gamma f).$$

The mapping

$$g = \sum_{k=0}^\infty J^{\alpha,\beta}_k g \mapsto \sum_{k=0}^\infty \left(\frac{1+\lambda_k}{\lambda_k}\right)^{\gamma/2} J^{\alpha,\beta}_k g$$

is a multiplier with associated function $\Phi(k) = \left(\frac{1+\lambda_k}{\lambda_k}\right)^{\gamma/2} = h(\frac{1}{\lambda_k})$ where $h(z) = (z+1)^{\gamma/2}$, so by Meyer's theorem, taking $g = D_\gamma^{\alpha,\beta} f$ we have

$$\|f\|_{p,\gamma} = \|\phi\|_p \leq B_{p,\gamma} \|D_\gamma^{\alpha,\beta} f\|_p.$$

Thus we get (31) for polynomials. For the general case consider $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta})$, then there exists $g \in L^p([-1,1], \mu_{\alpha,\beta})$ such that $f = \mathcal{J}_\gamma^{\alpha,\beta} g$ and a sequence $\{g_n\}$ of polynomials such that $\lim_{n\to\infty} \|g_n - g\|_p = 0$. Let $f_n = \mathcal{J}_\gamma^{\alpha,\beta} g_n$, then $\lim_{n\to\infty} \|f_n - f\|_{p,\gamma} = 0$. Then, by the continuity of the operator $D_\gamma^{\alpha,\beta} \mathcal{J}_\gamma^{\alpha,\beta}$ and as $\lim_{n\to\infty} \|g_n - g\|_p = 0$,

$$\lim_{n\to\infty} \|D_\gamma^{\alpha,\beta}(f_n - f)\|_p = \lim_{n\to\infty} \|D_\gamma^{\alpha,\beta} \mathcal{J}_\gamma^{\alpha,\beta}(g_n - g)\|_p = 0.$$

Then, as

$$B_{p,\gamma} \|f_n\|_{p,\gamma} \leq \|D_\gamma^{\alpha,\beta} f_n\|_p \leq A_{p,\gamma} \|f_n\|_{p,\gamma},$$

the results follows by taking the limit as n goes to infinity in this inequality.

i) Let $\{f_n\}$ be a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$ in $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$. Then, by (31)

$$\lim_{n\to\infty} \|D_\gamma^{\alpha,\beta} f_n\|_p \leq B_{p,\gamma} \lim_{n\to\infty} \|f_n\|_{p,\gamma} = B_{p,\gamma} \|f\|_{p,\gamma},$$

hence, $\lim_{n\to\infty} D_\gamma^{\alpha,\beta} f_n \in L^p([-1,1], \mu_{\alpha,\beta})$.

Now suppose that $\{q_n\}$ is another sequence of polynomials such that $\lim_{n\to\infty} q_n = f$ in $L^p_\gamma([-1,1], \mu_{\alpha,\beta})$. By (31)

$$B_{p,\gamma} \|f_n\|_{p,\gamma} \leq \|D_\gamma^{\alpha,\beta} f_n\|_p \leq A_{p,\gamma} \|f_n\|_{p,\gamma}$$

and

$$B_{p,\gamma} \|q_n\|_{p,\gamma} \leq \|D_\gamma^{\alpha,\beta} q_n\|_p \leq A_{p,\gamma} \|q_n\|_{p,\gamma}.$$

Taking the limit as n goes to infinity

$$B_{p,\gamma} \|f\|_{p,\gamma} \leq \lim_{n\to\infty} \|D_\gamma^{\alpha,\beta} f_n\|_p \leq A_{p,\gamma} \|f\|_{p,\gamma}$$

$$B_{p,\gamma} \|f\|_{p,\gamma} \leq \lim_{n\to\infty} \|D_\gamma^{\alpha,\beta} q_n\|_p \leq A_{p,\gamma} \|f\|_{p,\gamma},$$

therefore $\lim_{n\to\infty} D_\gamma^{\alpha,\beta} f_n = \lim_{n\to\infty} D_\gamma^{\alpha,\beta} q_n$ in $L^p([-1,1], \mu_{\alpha,\beta})$ and the limit does not depends on the choice of the sequence.

Finally, let us suppose that $f \in L^p_\gamma([-1,1], \mu_{\alpha,\beta}) \bigcap L^q_\gamma([-1,1], \mu_{\alpha,\beta})$ and, without loss of generality, that $p \leq q$. By Proposition 12, *i),*

$L^q_\gamma([-1,1],\mu_{\alpha,\beta}) \subseteq L^p_\gamma([-1,1],\mu_{\alpha,\beta})$ and therefore $f \in L^q_\gamma([-1,1],\mu_{\alpha,\beta})$. Then, if $\{f_n\}$ is a sequence of polynomials such that $\lim_{n\to\infty} f_n = f$ in $L^q_\gamma([-1,1],\mu_{\alpha,\beta})$ (hence in $L^p_\gamma([-1,1],\mu_{\alpha,\beta})$), we have

$$\lim_{n\to\infty} D^{\alpha,\beta}_\gamma f_n \in L^q([-1,1],\mu_{\alpha,\beta}) = L^p([-1,1],\mu_{\alpha,\beta}) \bigcap L^q([-1,1],\mu_{\alpha,\beta}).$$

Therefore the limit does not depends on the choice of p or q. □

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