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Locally Generated Semigroups

Semigrupos localmente generados

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Abstract

For a topological semigroup S, Lawson constructed a semigroup $\Gamma(S)$ with the property that any local homomorphism defined in a neighborhood of the identity of S to a topological semigroup T extends uniquely to a global homomorphism defined on $\Gamma(S)$. In this work we obtain conditions on S to topologize the semigroup $\Gamma(S)$ via an uniformity such that the extended homomorphism is continuous and such that $\Gamma(S)$ is a topological semigroup.
Key words and phrases: causal path, causal homotopy, topological semigroup, locally right divisible, locally causally path conected, locally causally simply connected.

Resumen

Dado un semigrupo topológico S Lawson construyó un semigrupo $\Gamma(S)$ con la propiedad de que cualquier homomorfismo local definido en un entorno de la identidad de S hacia un semigrupo topológico T se extiende de manera única a un homomorfismo global definido sobre $\Gamma(S)$. En este trabajo obtenemos condiciones sobre S para topologizar el semigrupo $\Gamma(S)$ via una uniformidad tal que el homomorfismo extendido es continuo y $\Gamma(S)$ es un semigrupo topológico.
Palabras y frases clave: camino causal, homotopía causal, semigrupo topológico, localmente divisible por la derecha, localmente causalmente conexo por arcos, localmente causalmente simplemente conexo.

1 Causal paths

We begin this section with the following definition:

Definition 1. *Let S be a topological semigroup with identity 1_S. A path $\alpha\colon [0,1] \to S$ such that $\alpha(0) = 1_S$ is called a* causal path *if the following property is satisfied: Given U a neighborhood of 1_S, there exists $\epsilon > 0$ such that whenever $s, t \in [0,1]$ with $s < t < s+\epsilon$, then $\alpha(t) \in \alpha(s)\cdot U$, i.e., $\alpha(t) = \alpha(s)u$ for some $u \in U$. Given causal paths $\alpha\colon [0,1] \to S$ and $\beta\colon [0,1] \to S$, we define the* concatenation *$\alpha * \beta\colon [0,1] \to S$ by*

$$\alpha * \beta = \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq 1/2, \\ \alpha(1)\beta(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Definition 2. *A subset W of a real topological vector space L is called a* cone *if it satifies the following conditions:*

(i) $W + W \subset W$,

(ii) $\mathbf{R}^+ \cdot \mathrm{W} \subset \mathrm{W}$,

(iii) $\overline{W} = W$, *that is, W is closed in L.*

Example 1. *In the additive semigroup $C = \{(x,y) \in \mathbf{R}^2 : \mathrm{x} \geq 0, \mathrm{y} \geq 0\}$ consider the path $\alpha(t) = tX$ where X is a unit vector in C. Given a neighborhood U of the identity of C pick a positive real number ϵ such that the intersection of C and the ball $B(0,\epsilon)$ is contained in U. If $s, t \in [0,1]$ with $s < t < s+\epsilon$, then $\alpha(t) = \alpha(s) + (t-s)X$. Clearly the vector $(t-s)X$ is in U. Observe that a similar argument shows that rays through the origin are causal paths in an arbitrary cone in $\mathbf{R}^{\mathrm{n}}$.*

The next example generalizes the preceding one.

Example 2. *Let S be a topological semigroup, and $\alpha\colon [0,\infty] \to S$ a one parameter subsemigroup. Then $\alpha|_{[0,1]}$ is a causal path of S. Indeed, given a neighborhood U of the identity, pick $\epsilon > 0$ such that if $0 < x < \epsilon$, then $\alpha(x) \in U$. That can be done by the continuity at zero of the path α. If $s, t \in [0,1]$ with $s < t < s+\epsilon$, then $\alpha(t) = \alpha(s)\alpha(t-s)$. Since $t - s < \epsilon$, then $\alpha(t-s) \in U$.*

Example 3. *Let $\alpha\colon I \to G$ be any path in a group G such that $\alpha(0) = 1$, where I is the closed unit interval. Given a neighborhood of the identity U of G we can choose a positive number ϵ such that if $s, t \in I$ with $s < t < s+\epsilon$, then $\alpha(s)^{-1}\alpha(t) \in U$. Therefore in a group any path is a causal path.*

Theorem 1. *Let $h\colon S \to T$ be a continuous homomorphism from the topological semigroup S to the topological semigroup T. If α is a causal path in S, then $h(\alpha)$ is a causal path in T.*

Proof. Let U be a neighborhood of 1_T. Then $h^{-1}(U)$ is a neighborhood of 1_S. Let $\epsilon > 0$ be such that if $s, t \in [0,1]$ with $s < t < s+\epsilon$, then $\alpha(t) = \alpha(s)u$ for some $u \in U$. Since h is a homomorphism we have that $(h \circ \alpha)(t) = (h \circ \alpha)(s)h(u)$ with $h(u) \in U$. □

Theorem 2. *The concatenation of two causal paths in a topological semigroup is again a causal path.*

Proof. Let α and β be causal paths in the topological semigroup S, and let U be a neighborhood of the identity of S. Pick a neighborhood W of the identity of S such that $W^2 \subset U$. Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be chosen corresponding to W, in the definition of causal paths for α and β respectively. Take $0 \leq \epsilon < \min(\epsilon_1/2, \epsilon_2/2)$. Suppose that $0 \leq s < t < s+\epsilon$, then we have the following three cases: 1) $0 \leq s < t \leq 1/2$, 2) $1/2 \leq s < t$ and 3) $s < 1/2 < t$.

In case 1) $(\alpha*\beta)(s) = \alpha(2s)$ and $(\alpha*\beta)(t) = \alpha(2t)$. Since $2s < 2t < 2s+\epsilon_1$, we have that $\alpha(2t) = \alpha(2s)w$ for some $w \in W$ and therefore $(\alpha*\beta)(t) \in (\alpha*\beta)(s)U$.

In case 2) $(\alpha*\beta)(t) = \alpha(1)\beta(2t-1)$ and $(\alpha*\beta)(s) = \alpha(1)\beta(2s-1)$, again since $2s < 2t < 2s+\epsilon_2$, subtracting 1 from this inequality we get that $2s-1 < 2t-1 < 2s-1+\epsilon_2$, hence $\beta(2t-1) = \beta(2s-1)w$ for some $w \in W$. Multiplying this last identity by $\alpha(1)$ we get $(\alpha*\beta)(t) = (\alpha*\beta)(s)w$, i.e., $(\alpha*\beta)(t) \in (\alpha*\beta)(s)U$.

In case 3) we have that $2s < 1 < 2t < 2s+2\epsilon < 1+2\epsilon$, which implies that $2s < 1 < 2s+\epsilon_1$. Hence, $\alpha(1) = \alpha(2s)w_1$ for some $w_1 \in W$. Also, since $1 < 2t < 1+2\epsilon$ which is equivalent to $0 < 2t-1 < \epsilon_2$ we have that $\beta(2t-1) = \beta(0)w_2 = w_2$ for some $w_2 \in W$. Now:

$$(\alpha*\beta)(t) = \alpha(1)\beta(2t-1) = \alpha(2s)w_1w_2 \in \alpha(2s)W^2 \subset \alpha(2s)U$$

which completes the proof of the proposition. □

We introduce now the notion of causal homotopy.

Definition 3. *Let α, $\beta\colon [0,1] \to S$ be causal paths in S with the same end point i.e., $\alpha(1) = \beta(1)$. A* causal homotopy *between α and β is a continuous function $H\colon [0,1]\times[0,1] \to S$ satisfying:*

a) $H(t,0) = \alpha(t)$ *for all* $t \in [0,1]$*,*

b) $H(t,1) = \beta(t)$ *for all* $t \in [0,1]$*,*

c) $H(0,s) = 1_S$*, and* $H(1,s) = \alpha(1) = \beta(1)$ *for all* $s \in [0,1]$*,*

d) the path $\gamma_s(t) = H(t,s)$ *is a causal path for all* $s \in [0,1]$*.*

Two paths are said to be causally homotopic *if there exists a causal homotopy between them. If* H *is a causal homotopy between the causal paths* α *and* β*, we write* $H\colon \alpha \sim \beta$*.*

Theorem 3. *The relation of causal homotopy is an equivalence relation on the set of causal paths, and the concatenation operation induces a well defined associative operation on the set* $\Gamma(S)$ *of causal homotopy classes of causal paths.*

Proof. Let S be a semigroup and let α be a causal path in S; then the map $H\colon [0,1]\times[0,1] \to S$ defined by $H(t,s) = \alpha(t)$ satisfies $H\colon \alpha \sim \alpha$. In other words, the relation of causal homotopy is reflexive. Suppose now that H is a causal homotopy map between α and β. Then the map $F(t,s) = H(t,1-s)$ is a causal homotopy map between β and α, which means that the relation of causal homotopy is symmetric. Suppose now that $F\colon \alpha \sim \beta$ and $G\colon \beta \sim \gamma$, then the map defined by

$$H(t,s) = \begin{cases} F(t,2s) & \text{for } 0 \le s \le 1/2, \\ G(t,2s-1) & \text{for } 1/2 \le s \le 1, \end{cases}$$

is a causal homotopy map between α and γ. So the relation of causal homotopy is transitive. We have proved that the relation of causal homotopy is an equivalence relation. We denote by $[\alpha]$ the causal homotopy class of the causal path α. We define a product in $\Gamma(S)$ by $[\alpha][\beta] = [\alpha * \beta]$. If $F\colon \alpha \sim \alpha'$ and $G\colon \beta \sim \beta'$ then the map defined by

$$H(t,s) = \begin{cases} F(2t,s) & \text{for } 0 \le t \le 1/2, \\ \alpha(1)G(2t-1,s) & \text{for } 1/2 \le t \le 1, \end{cases}$$

is a causal homotopy map between $\alpha * \beta$ and $\alpha' * \beta'$, i.e., $[\alpha][\beta] = [\alpha'][\beta']$, and therefore the concatenation induces a well defined product in $\Gamma(S)$.

For the last part of the proposition, consider the causal paths σ, τ, and ω. Define

$$F(s,t) = \begin{cases} \sigma\left(\frac{4s}{t+1}\right) & \text{for } 0 \le s \le 1/4(t+1), \\ \sigma(1)\tau(4s-t-1) & \text{for } 1/4(t+1) \le s \le 1/4(t+2), \\ \sigma(1)\tau(1)\omega\left(\frac{4s-t-2}{2-t}\right) & \text{for } 1/4(t+2) \le s \le 1, \end{cases}$$

to establish $(\sigma * \tau) * \omega \sim \sigma * (\tau * \omega)$ in $\Gamma(S)$.

So $\Gamma(S)$ has the structure of a semigroup with an identity. $\square$

2 Universal properties of $\Gamma(S)$

Let $\Gamma(S)$ denote the semigroup of causal homotopy classes of causal paths in the semigroup S, with the semigroup operation of concatenation.

Definition 4. *A* local homomorphism *on S is a function σ from a neighborhood of the identity U of S into a semigroup T endowed with a Hausdorff topology for which left translations are continuous satisfying:*

i) If $a, b, ab \in U$ then $\sigma(ab) = \sigma(a)\sigma(b)$,

ii) σ is continuous on U.

The next theorem is a major one. For a proof see [9].

Theorem 4. *Let S be a subsemigroup of a topological group G which contains the identity e in the closure of its interior, and let U be an open set of G containing e. Let $\sigma\colon S \cap U \to T$ be a local homomorphism. Then there exists a unique homomorphism $\hat{\sigma}\colon \Gamma(S) \to T$, such that $\hat{\sigma}([\alpha]) = \sigma(\alpha(1))$ whenever $\alpha\colon [0,1] \to G$ is a causal path such that $\alpha([0,1]) \subset U \cap S$.*

3 The uniform topology of $\Gamma(S)$

Our goal now is to define a suitable topology on $\Gamma(S)$ that makes it a topological semigroup. To do this, we define a uniformity on $\Gamma(S)$ and then we will consider the topology induced by that uniformity which we will call the uniform topology of $\Gamma(S)$.

Definition 5. *A topological semigroup is called* locally causally simply connected *if there exists a neighborhood U of the identity such that any two causal paths with the same end point and completely contained in U are causally homotopic.*

Example 4. *Let α be an arbitrary causal path in the semigroup C defined in the example 1. Consider the causal path defined by $\beta(t) = t\alpha(1)$. Then α and β are two causal paths in C with the same end point. We show that α and β*

are causally homotopic. Indeed, we define the map $H\colon I \times I \to C$ where I is the closed unit interval by:

$$H(s,t) = \begin{cases} \frac{s}{t}\alpha(t) & \textit{for } s < t, \\ \alpha(s) & \textit{for } t \leq s. \end{cases}$$

Clearly H is a causal homotopy map between α and β. So the semigroup C is causally simply connected.

Definition 6. *A topological semigroup is called* locally causally path connected *if there exist a basis of neighborhoods $\{U_\alpha : \alpha \in \Omega\}$ of the identity such that any point in U_α can be connected with the identity by a causal path completely contained in U_α for any $\alpha \in \Omega$.*

Definition 7. *A topological semigroup is said to be* locally right divisible *if given a neighborhood U of the identity, there exist a neighborhood V of the identity such that $\forall a, b \in V$ there exist $x, y \in U$ such that $ax = by$.*

Example 5. *Consider again the semigroup C given in the example 1. It is easy to see that the family $\{B(0,\epsilon) \cap C : \epsilon > 0\}$ is a basis of neighborhoods of the identity of C that satisfies the condition of the definition 6., therefore C is a locally causally path connected semigroup. Let U be a neighborhood of the identity of C. Set $V = U$. If $X, Y \in V$ take $A = Y$ and $B = X$, clearly $X + A = Y + B$, and $A, B \in U$. So C is a locally right divisible topological semigroup. The same argument shows that any commutative semigroup is locally right divisible.*

Notation. For a neighborhood U of the identity of a topological semigroup S we denote by $\tilde{U}$ and $[\tilde{U}]$ the following sets:

$$\tilde{U} = \{[\alpha] \in \Gamma(S) : \alpha([0,1]) \subset U\},$$

and

$$[\tilde{U}] = \{([\alpha],[\beta]) : [\alpha][\tilde{U}] \cap [\beta][\tilde{U}] \neq \emptyset\}.$$

Theorem 5. *Let S be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. Then the family*

$$\mathcal{A} = \{[\tilde{U}] : U \subset S \textit{ with } U \textit{ open and } 1_S \in U\}$$

is a base for a uniformity of $\Gamma(S)$.

Proof. We will show that the family $\mathcal{A}$ satisfies the conditions of Theorem 2. Chapter 6 of [7]. For $[\gamma] \in \Gamma(S)$, we have that $[\gamma]\tilde{U} \cap [\gamma]\tilde{U} = [\gamma]\tilde{U} \neq \emptyset$ since $[\gamma] \in [\gamma]\tilde{U}$, hence $([\gamma],[\gamma]) \in [\tilde{U}]$, i.e., the diagonal Δ is contained in $[\tilde{U}]$ for each open subset of S that contains the identity. Clearly $[\tilde{U}]^{-1} = [\tilde{U}]$. If U and V are neighborhoods of the identity of S, then $[U \tilde{\cap} V] \subset [\tilde{U}] \cap [\tilde{V}]$, so the intersection of two members of $\mathcal{A}$ contains a member of $\mathcal{A}$. So conditions a), b), and d) of Theorem 2. Chapter 6 of [7] are satisfied. Let's prove now that condition c) is also satisfied. Let U be a neighborhood of the identity of S. Since S is locally causally simply connected, we can pick a neighborhood of the identity $W \subset U$ such that any two causal paths in W with the same end point are causally homotopic. Pick a neighborhood V of the identity such that $V^2 \subset W$ and such that any point in V can be joined with the identity by means of a causal path completely contained in V. This is possible since S is locally causally path connected, and by the continuity of multiplication. Finally, since S is locally right divisible, we can pick $V' \subset V$ such that for all $a, b \in V'$ there exist $x, y \in V$ such that $ax = by$. We claim that $[\tilde{V}'] \circ [\tilde{V}'] \subset [\tilde{U}]$. To prove the claim, pick $([\alpha],[\beta]) \in [\tilde{V}'] \circ [\tilde{V}']$. Let $[\gamma] \in \Gamma(S)$ such that $([\alpha],[\gamma]) \in [\tilde{V}']$ and $([\gamma],[\beta]) \in [\tilde{V}']$. Therefore, there exist σ_i, ρ_i with $i = 1, 2$, and $\sigma_i([0,1]) \subset V'$, $\rho_i([0,1]) \subset V'$ such that

$$[\alpha][\rho_1] = [\gamma][\rho_2] \tag{1}$$

and

$$[\gamma][\sigma_1] = [\beta][\sigma_2]. \tag{2}$$

Now take $x, y \in V$ such that $\sigma_1(1)x = \sigma_2(1)y$. By the way V was chosen, there exist τ_1, τ_2 causal paths in V such that $\tau_1(1) = y$ and $\tau_2(1) = x$. Hence, $(\sigma_1 * \tau_2)(1) = (\sigma_2 * \tau_1)(1)$. Therefore, $(\sigma_1 * \tau_2)([0,1]) \subset V^2 \subset W$ and $(\sigma_2 * \tau_1)([0,1]) \subset V^2 \subset W$. Since any two causal paths in W with the same end point are causally homotopic, we conclude that

$$[\sigma_1 * \tau_2] = [\sigma_2 * \tau_1]. \tag{3}$$

Multiplying equations (1) and (2) on the right by $[\tau_1]$ and $[\tau_2]$ respectively, we get

$$[\gamma][\rho_2][\tau_1] = [\alpha][\rho_1][\tau_1] \tag{4}$$

and

$$[\gamma][\sigma_1][\tau_2] = [\beta][\sigma_2][\tau_2]. \tag{5}$$

Now combining equations (4) and (5) we get that $[\beta][\sigma_2 * \tau_2] = [\alpha][\rho_1 * \tau_1]$, but $(\sigma_2 * \tau_2)([0,1]) \subset V^2 \subset W \subset U$ and $(\rho_1 * \tau_1)([0,1]) \subset V^2 \subset W \subset U$, which means that $[\beta]\tilde{U} \cap [\alpha]\tilde{U} \neq \emptyset$, thus $([\alpha],[\beta]) \in [\tilde{U}]$, and therefore $[\tilde{V}'] \circ [\tilde{V}'] \subset [\tilde{U}]$. □

Theorem 6. *Let S be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. With the uniform topology, multiplication is continuous at the identity of $\Gamma(S)$.*

Proof. Let $[\tilde{U}]([e])$ be a neighborhood of $[e]$, the identity of the semigroup $\Gamma(S)$. We may assume that U is a neighborhood of the identity of S with the property that any two causal paths in U with the same end point are causally homotopic. Let V be a neighborhood of the identity of S with $V^2 \subset U$ and such that for any $x \in V$ there exists a causal path $\gamma\colon [0,1] \to V$ with $\gamma(1) = x$. Let W be a neighborhood of the identity of S such that $W \subset V$ and such that for all $a, b \in W$ there exist $x, y \in V$ with $ax = by$. Finally, pick W' with ${W'}^2 \subset W$. Consider $[\tilde{W}']([e])$; if $[\sigma], [\tau] \in [\tilde{W}']([e])$ then there exist $\sigma_1, \sigma_2, \tau_1, \tau_2\colon [0,1] \to W'$ such that

$$[\sigma * \sigma_1] = [\sigma_2], \tag{6}$$

and

$$[\tau * \tau_1] = [\tau_2]. \tag{7}$$

Now, observe that

$$(\sigma_1 * \tau * \tau_1)(1) = \sigma_1(1)(\tau * \tau_1)(1) = \sigma_1(1)\tau_2(1)$$

but $\sigma_1(1)\tau_2(1) \in {W'}^2 \subset W$. Also, $(\tau * \tau_1)(1) = \tau_2(1) \in W' \subset {W'}^2 \subset W$. So there exist $x, y \in V$ such that $(\sigma_1 * \tau * \tau_1)(1)x = (\tau * \tau_1)(1)y$. There exist $\alpha, \beta\colon [0,1] \to V$ causal paths such that $\alpha(1) = x$ and $\beta(1) = y$. Therefore $(\sigma_1 * \tau * \tau_1 * \alpha)(1) = (\tau * \tau_1 \; * \beta)(1)$, i.e., $(\sigma_1 * \tau_2 * \alpha)(1) = (\tau_2 * \beta)(1)$ and $(\tau_2 * \beta)([0,1]) \subset WV \subset V^2 \subset U$. Also, we have that $(\sigma_1 * \tau_2 * \alpha)([0,1]) \subset W'W'V \subset WV \subset V^2 \subset U$.

Since any two causal paths in U with the same end point are causally homotopic, we have that

$$[\tau_2 * \beta] = [\sigma_1 * \tau_2 * \alpha]. \tag{8}$$

Multiplying equations (6) and (7) we get: $[\sigma * \sigma_1 * \tau * \tau_1] = [\sigma_2 * \tau_2]$; therefore we conclude that

$$[\sigma * \sigma_1 * \tau * \tau_1 \; * \alpha] = [\sigma_2 * \tau_2 * \alpha]. \tag{9}$$

Combining equations (7), (8), and (9) we obtain $[\sigma*\tau*\tau_1*\beta] = [\sigma_2*\tau_2*\alpha]$.

Now, $(\tau_1*\beta)([0,1]) \subset W'V \subset V^2 \subset U$ and $(\sigma_2*\tau_2*\alpha)([0,1]) \subset {W'}^2V \subset V \subset U$, i.e., $[\sigma*\tau] \in [\tilde{U}]([e])$. This means that multiplication is continuous at the identity of $\Gamma(S)$. $\square$

Theorem 7. *Let S a be a locally causally simply connected, locally causally path connected, and locally right divisible topological semigroup. Then multiplication is continuous in the second variable.*

Proof. Let $[\alpha], [\beta] \in \Gamma(S)$, consider $[\tilde{U}]([\alpha*\beta])$, a neighborhood of $[\alpha*\beta]$ in $\Gamma(S)$. Consider now $[\tilde{U}]([\beta])$, a neighborhood of $[\beta]$ in $\Gamma(S)$. If $[\gamma] \in [\tilde{U}]([\beta])$, there exist $\gamma_1, \gamma_2 \colon [0,1] \to U$ such that $[\gamma*\gamma_1] = [\beta*\gamma_2]$, therefore $[\alpha*\gamma*\gamma_1] = [\alpha*\beta*\gamma_2]$, i.e., $[\alpha*\gamma] \in [\tilde{U}]([\alpha*\beta])$. $\square$

4 Uniform structure on groups

In a topological group G we can define a couple of uniformities, called the left and right uniformities respectively, such that the uniform topologies induced by them are compatible with the original topology of the group.

Let's denote by $\mathcal{U}$ the neighborhood system of the identity of the group G. For $V \in \mathcal{U}$ we write $V_r = \{(x,y) \in G \times G : xy^{-1} \in V\}$. Consider the set $\mathcal{A} = \{V_r : V \in \mathcal{U}\}$. It is clear that the diagonal Δ is contained in V_r for each $V \in \mathcal{U}$. Since the relations $yx^{-1} \in V$ and $xy^{-1} \in V^{-1}$ are equivalent, we have that $V_r^{-1} = (V^{-1})_r$. It is also clear that $(U \cap V)_r \subset U_r \cap V_r$. Finally, given $U \in \mathcal{U}$, pick $V \in \mathcal{U}$ such that $V^2 \subset U$; then it is easy to show that $V_r \circ V_r \subset U_r$. Therefore the set $\mathcal{A}$ is a base for a uniformity of the group G, called the right uniformity. Since $V_r[x] = V^{-1}x$ we have that the topology induced by this uniformity coincides with the original topology of the group G. The *left* uniformity is defined analogously, beginning with the sets $V_l = \{(x,y) \in G \times G : y^{-1}x \in V\}$. In general the right and left uniformities of a group G are different, but they define the same topology.

The following is a very useful result about topological groups.

Theorem 8. *Let G be a topological group. The uniformity generated by $\{\{(x,y) : xV \cap yV \neq \emptyset\} : V \in \mathcal{U}\}$ ($\{\{(x,y) : Vx \cap Vy \neq \emptyset\} : V \in \mathcal{U}\}$) is the left (right) uniformity, respectively.*

Proof. For $V \in \mathcal{U}$, set $V_{l'} = \{(x,y) : xV \cap yV \neq \emptyset\}$, $\mathcal{A} = \{V_l : V \in \mathcal{U}\}$ and $\mathcal{B} = \{V_{l'} : V \in \mathcal{U}\}$. We prove that $\mathcal{A}$ and $\mathcal{B}$ generate the same uniformity by showing that any element of $\mathcal{A}$ contains an element of $\mathcal{B}$ and that any element of $\mathcal{B}$ contains an element of $\mathcal{A}$. Indeed, pick $V \in \mathcal{U}$. If $(x, y) \in V_l$ then $y^{-1}x \in V$, this implies that $xV \cap yV \neq \emptyset$. Therefore, $V_l \subset V_{l'}$. On the other hand, if $V \in \mathcal{U}$, pick $W \in \mathcal{U}$ such that $WW^{-1} \subset V$. If $(x, y) \in W_{l'}$ then $xW \cap yW \neq \emptyset$. This implies that $y^{-1}x \in WW^{-1} \subset V$. Therefore $W_{l'} \subset V_l$. □

Theorem 9. *Let G be a topological group and let $S \subset G$ be a subsemigroup such that the identity of G is in $\overline{\mathrm{int}(S)}$. Then the uniformities defined by the following bases are equal:*

1) $\{V_a : V \in \mathcal{U}\}$, *where* $V_a = \{(x, y) \in S \times S : x^{-1}y \in V\}$,

2) $\{V_b : V \in \mathcal{U}\}$, *where* $V_b = \{(x, y) \in S \times S : xV \cap yV \neq \emptyset\}$,

3) $\{V_c : V \in \mathcal{U}\}$, *where* $V_c = \{(x, y) \in S \times S : x(V \cap S) \cap y(V \cap S) \neq \emptyset\}$.

Proof. That the uniformities defined by the bases 1) and 2) are equal follows from the previous theorem, since $V_a = (V^{-1})_l$. Let's prove that the uniformities defined by the bases 2) and 3) are equal. Indeed, pick $V \in \mathcal{U}$; if $x(V \cap S) \cap y(V \cap S) \neq \emptyset$, then $xV \cap yV \neq \emptyset$. Therefore $V_c \subset V_b$. This implies that the uniformity generated by the base 2) is contained in the uniformity generated by the base 3). To show the other inclusion, pick $V \in \mathcal{U}$. Set $Q = V \cap \mathrm{int}(S) \neq \emptyset$, since $1_G \in \overline{\mathrm{int}(S)}$. Take $s \in Q$ and set $W = Qs^{-1}$, then $W \in \mathcal{U}$. We show that $W_b \subset V_c$. If $(x, y) \in W_b$ then $xW \cap yW \neq \emptyset$, i.e., there exist $q_1, q_2 \in Q$ such that $xq_1s^{-1} = yq_2s^{-1}$. This implies that $xq_1 = yq_2$, hence $xQ \cap yQ \neq \emptyset$. But $Q = V \cap \int(S) \subset V \cap S$, therefore $x(V \cap S) \cap y(V \cap S) \neq \emptyset$, thus $(x, y) \in V_c$. Therefore the uniformity generated by the base 3) is contained in the uniformity generated by the base 2). □

5 Uniform structure on semigroups

It was shown above that the topology of a topological group can be described in terms of the right and left uniformities. The same technique does not work for topological semigroups due to the absence of inverses for the elements of the semigroup. However it is possible to define, for some semigroups, a uniformity closely related to the right and left uniformities of a group. We investigate in this section the kind of semigroups for which this is possible and study the relationship between the original topology of the semigroup and the topology induced by the uniformity.

Let S be a locally right divisible semigroup. For a neighborhood V of the identity of S we define V_r as the set of all pairs $(x, y) \in S \times S$ such that $xV \cap yV \neq \emptyset$. Consider now $\mathcal{A} = \{V_r : V \text{ is a neighborhood of } 1_S\}$.

Theorem 10. *Let S be a locally right divisible topological semigroup. The family $\mathcal{A}$ of subsets of $S \times S$ defined above is a uniformity for S.*

Proof. It is straightforward to see that $V_r^{-1} = V_r$ and $\Delta \subset V_r$ for any neighborhood V of the identity of the semigroup S. It is also clear that $(U \cap V)_r \subset U_r \cap V_r$ for U and V neighborhoods of the identity of S. Let U be an arbitrary neighborhood of the identity of the semigroup S. By the continuity of multiplication, we can choose a neighborhood W of the identity of S such that $W^2 \subset U$. Now, since S is a locally right divisible semigroup, we can pick a neighborhood $V \subset W$ of the identity such that $\forall a, b \in V$ there exist $x, y \in W$ such that $ax = by$. Finally pick a neighborhood V' of the identity of S such that ${V'}^2 \subset V$. Take $(x, y) \in V'_r \circ V'_r$, there exists $z \in S$ such that (x, z) and (z, y) are elements of V'_r. Therefore, there exist x', z', z'', y' elements of V' such that $xx' = zz'$ and $zz'' = yy'$. By the way V' and V were chosen, there exists $x_1, y_1 \in W$ such that $z'x_1 = z''y_1$. Combining this relation with the previous two relations, we get that $xx'x_1 = zz'x_1 = zz''y_1 = yy'y_1$. Now, $x'x_1 \in V'W \subset W^2 \subset U$ and $y'y_1 \in V'W \subset W^2 \subset U$ and, therefore $(x, y) \in U_r$. We have proved that $V' \circ V' \subset U$. We have shown that the family $\mathcal{A}$ satisfies the conditions of the definition of a uniformity for S (see [7], Chapter 6). □

The uniform topology of the semigroup S could be different from the original topology of S. In the rest of this chapter we are assuming that they are equal.

Definition 8. *A topological semigroup S is called* nice *if its topology is compatible with the uniform topology and it satisfies definitions 5, 6, and 7.*

Example 6. *For a subsemigroup S of a group G such that $1_G \in \overline{\mathrm{int}(S)}$ we saw in theorem 9 that the uniform topology of S is compatible with the relative topology of S. Therefore the class of these semigroups that satisfy the conditions given in the definitions 5, 6, and 7 are examples of nice semigroups. Particularly, cones in $\mathbf{R}^{\mathrm{n}}$ are nice semigroups.*

Theorem 11. *If S is a nice topological semigroup the map:*

$$[\alpha] \mapsto \alpha(1) \colon \Gamma(S) \to S$$

is continuous.

Proof. Let V be a neighborhood of $\alpha(1)$. Pick a neighborhood U of the identity such that $U_r(\alpha(1)) \subset V$. Consider $[\tilde{U}][\alpha]$, which is a neighborhood of α in $\Gamma(S)$. If $[\beta] \in [\tilde{U}]([\alpha])$, then there exist $\sigma_1, \sigma_2 \colon [0,1] \to U$ such that $[\beta * \sigma_1] = [\alpha * \sigma_2]$. Therefore, $\beta(1)\sigma_1(1) = \alpha(1)\sigma_2(1)$. Since both $\sigma_1(1)$ and $\sigma_2(1)$ are elements of U, we have that $\beta(1) \in U_r(\alpha(1))$. $\square$

6 Functorial properties of $\Gamma(S)$

Theorem 12. *Let S and T be nice topological semigroups and let $h \colon S \to T$ be a continuous homomorphism. Then the map $\overline{h} \colon \Gamma(S) \to \Gamma(T)$ defined by $\overline{h}([\alpha]) = [h(\alpha)]$ is a continuous homomorphism.*

Proof. We have that $\overline{h}([\alpha][\beta]) = \overline{h}([\alpha * \beta]) = [h(\alpha * \beta)]$, but

$$\alpha * \beta = \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq 1/2, \\ \alpha(1)\beta(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Then

$$h((\alpha * \beta(t)) = \begin{cases} h(\alpha(2t)) & \text{for } 0 \leq t \leq 1, \\ h(\alpha(1))h(\beta(2t-1)) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Therefore, $\overline{h}([\alpha][\beta]) = [(h \circ \alpha) * (h \circ \beta)] = [h \circ \alpha][h \circ \beta] = \overline{h}([\alpha])\overline{h}([\beta])$. This proves that $\overline{h}$ is a homomorphism.

Let's see now that $\overline{h}$ is continuous. Take $[\alpha] \in \Gamma(S)$, and consider $[\tilde{U}]([h \circ \alpha])$ which is a neighborhood of $\overline{h}([\alpha])$ in $\Gamma(T)$, where U is a neighborhood of the identity of T. By continuity of h, there exists a neighborhood V of the identity of S such that $h(V) \subset U$. Now, if $[\beta] \in [\tilde{V}]([\alpha])$, then there exist $\sigma_1, \sigma_2 \colon [0,1] \to V$ such that $[\beta][\sigma_1] = [\alpha][\sigma_2]$. Therefore,

$$\overline{h}([\beta])\overline{h}([\sigma_1]) = \overline{h}([\alpha])\overline{h}([\sigma_2])$$

But $\overline{h}([\sigma_1])$ and $\overline{h}([\sigma_2])$ maps the interval $[0,1]$ into U. In other words, $\overline{h}([\beta]) \in [\tilde{U}]([\overline{h}([\alpha]))$.

Clearly if $i \colon S \to S$ is the identity homomorphism, then $\overline{i} \colon \Gamma(S) \to \Gamma(S)$ is the identity homomorphism of the semigroup $\Gamma(S)$.

Also, if $h \colon S \to T$ and $g \colon T \to U$ are continuous homomorphisms, then $\overline{(h \circ g)}([\alpha]) = [(h \circ g)(\alpha)] = [h(g(\alpha))] = \overline{h}([g(\alpha)]) = (\overline{h} \circ \overline{g})([\alpha])$, therefore $\overline{(h \circ g)} = \overline{h} \circ \overline{g}$.

If $h \colon S \to S$ is an invertible homomorphism, then by the preceding results, we have that $\overline{i} = \overline{(h \circ h^{-1})} = \overline{h} \circ \overline{(h^{-1})}$. In other words, $(\overline{h})^{-1} = \overline{(h^{-1})}$. $\square$

7 Direct and semidirect products

Let S_1 and S_2 be topological semigroups. Suppose we have a homomorphism h from S_1 to the semigroup $\mathrm{Aut}(S_2)$ of automorphisms of S_2 such that the maps:

$$(s_1, s_2) \mapsto h(s_1)(s_2) \colon S_1 \times S_2 \to S_2$$

and

$$(s_1, s_2) \mapsto (h(s_1))^{-1}(s_2) \colon S_1 \times S_2 \to S_2$$

are continuous. Then in the cartesian product $S_1 \times S_2$ we define a new operation given by $(x_1, y_1)(x_2, y_2) = (x_1 h(y_1)(x_2), y_1 y_2)$. The set $S_1 \times S_2$ with this new product is called the semidirect product between S_1 and S_2 and it is denoted by $S_1 \ltimes S_2$. Then we can define a structure of a semidirect product between the semigroups $\Gamma(S_1)$ and $\Gamma(S_2)$ via the composition

$$\Gamma(S_2) \xrightarrow{\pi} S_2 \xrightarrow{h} \mathrm{Aut}(S_1) \xrightarrow{\phi} \mathrm{Aut}(\Gamma(S_1))$$

where π is the endpoint homomorphism and ϕ is defined by $(\phi(f))([\gamma]) = [f \circ \gamma]$. If we set $\psi = \phi \circ h \circ \pi$ then it is clear that $\Gamma(S_2) \xrightarrow{\psi} \mathrm{Aut}(\Gamma(S_1))$ is a homomorphism. This homomorphism is given by the formula $(\psi([\beta]))([\gamma]) = [h(\beta(1)) \circ \gamma]$. So therefore, we can define $\Gamma(S_1) \ltimes \Gamma(S_2)$ as it was done above.

In what follows we prove that the semigroups $\Gamma(S_1 \ltimes S_2)$ and $\Gamma(S_1) \ltimes \Gamma(S_2)$ are actually isomorphic. But first, let's prove the following:

Theorem 13. *Let S_1 and S_2 be topological semigroups. Suppose we have defined a semidirect product $S_1 \ltimes S_2$ via the homomorphism $h \colon S_2 \to Aut(S_1)$. Then $\alpha \times \beta$ is a causal path in $S_1 \times S_2$ if and only if α is a causal path in S_1 and β is a causal path in S_2.*

Proof. Suppose that $\alpha \times \beta$ is a causal path in $S_1 \ltimes S_2$. Since the projection map on the second coordinate $\pi_2 \colon S_1 \times S_2 \to S_2$ is a continuous homomorphism, by Theorem 1 we have that β is a causal path in S_2. Let's see that α is also a causal path in S_1. Let U be a neighborhood of e_1, the identity of S_1. The map $F \colon [0, 1] \times \{e_1\} \to S_1$ defined by $F(s, e_1) = h(\beta(s))(e_1) = e_1 \in U$ is continuous. Therefore, by Wallace's lemma, there exists V open in S_1 with $e_1 \in V$ such that $F([0, 1] \times V) \subset U$. In other words, $h(\beta(s))(v) \in U$ for all $v \in V$. Consider now the set $V \times S_2$, which is a neighborhood of (e_1, e_2). Then there exist $\epsilon > 0$ such that if $s < t < s + \epsilon$ then $(\alpha(t), \beta(t)) = (\alpha(s), \beta(s))(v_1, v_2)$ with $(v_1, v_2) \in V \times S_2$. Therefore, $\alpha(t) = \alpha(s)h(\beta(s))(v_1) \in \alpha(s)U$.

Suppose now that α and β are causal paths in S_1 and S_2 respectively. Let's prove that $\alpha \times \beta$ is a causal path in $S_1 \ltimes S_2$. Let $U_1 \times U_2$ be a neighborhood of (e_1, e_2) the identity element of $S_1 \ltimes S_2$. Consider the map $G\colon [0,1] \times \{e_1\} \to S_1$ defined by $G(s, e_1) = h(\beta(s))^{-1}(e_1) = e_1 \in U_1$. Since G is continuous, by Wallace's lemma, there exists a neighborhood V of e_1 in S_1 such that $G([0,1] \times V) \subset U_1$. In other words, $h(\beta(s))^{-1}(v) \in U_1$ for all $s \in [0,1]$ and for all $v \in V$. Now, pick $\epsilon > 0$ such that if $s < t < s + \epsilon$ then $\alpha(t) = \alpha(s)v$ and $\beta(t) = \beta(s)u$ for some $v \in V$ and some $u \in U$. Then $(\alpha(s), \beta(s)(h(\beta(s))^{-1}(v), u) = (\alpha(s)v, \beta(s)u) = (\alpha(t), \beta(t))$ and $(h(\beta(s))^{-1}(v), u) \in U_1 \times U_2$. □

Theorem 14. *Let $h\colon S_2 \to Aut(S_1)$ be a homomorphism, where S_1 and S_2 are topological semigroups, and suppose that the maps*

$$(s_1, s_2) \to h(s_1)(s_2)\colon S_1 \times S_2 \to S_2$$

and

$$(s_1, s_2) \to (h(s_1))^{-1}(s_2)\colon S_1 \times S_2 \to S_2$$

are continuous. Then the map $\Psi\colon \Gamma(S_1 \ltimes S_2) \to \Gamma(S_1) \ltimes \Gamma(S_2)$ defined by $\Psi([\alpha, \beta]) = ([\alpha], [\beta])$ is a semigroup isomorphism.

Proof. Take $([\alpha], [\beta]) \in \Gamma(S_1) \ltimes \Gamma(S_2)$. Then by the previous theorem, $[\alpha \times \beta]$ is an element of $\Gamma(S_1 \ltimes S_2)$. Therefore Ψ is clearly onto. Let's see now that Ψ is one–to–one. Suppose that $([\alpha], [\beta]) = ([\sigma], [\rho])$. Then there exists a continuous function $F\colon [0,1] \times [0,1] \to S_1$ such that $F(t, 0) = \alpha(t)$ for all $t \in [0,1]$, $F(t, 1) = \sigma(t)$ for all $t \in [0,1]$, $F(0, s) = e_1$ for all $s \in [0,1]$, $F(1, s) = \alpha(1) = \sigma(1)$ for all $s \in [0,1]$ and the path $F_s(t) = F(t, s)$ is a causal path for all fixed $s \in [0,1]$.

Also, there exists a continuous function $G\colon [0,1] \times [0,1] \to S_2$ such that $G(t, 0) = \beta(t)$ for all $t \in [0,1]$, $G(t, 1) = \rho(t)$ for all $t \in [0,1]$, $G(0, s) = e_2$ for all $s \in [0,1]$, $G(1, s) = \beta(1) = \rho(1)$ for all $s \in [0,1]$, and the path $G_s(t) = G(t, s)$ is a causal path for all s fixed in $[0,1]$.

Define now the map $F \times G\colon [0,1] \times [0,1] \to S_1 \times S_2$ by $F \times G(t, s) = (F(t, s), G(t, s))$ then clearly,

$$\begin{aligned}
F \times G(t, 0) &= (F(t, 0), G(t, 0)) = (\alpha(t), \beta(t)), \\
F \times G(t, 1) &= (F(t, 1), G(t, 1)) = (\sigma(t), \rho(t)), \\
F \times G(0, s) &= (F(0, s), G(0, s) = (e_1, e_2), \\
F \times G(1, s) &= (F(1, s), G(1, s)) = (\alpha(1), \beta(1)) = (\sigma(1), \rho(1)).
\end{aligned}$$

Also, according with the above theorem, $(F \times G)_s(t) = (F_s(t), G_s(t))$ is a causal path in $S_1 \ltimes S_2$. So we have proved that (α, β) is causally homotopic to (σ, ρ). In other words $[(\alpha, \beta)] = [(\sigma, \rho)]$, which proves that Ψ is one to one.

Let's see now that Ψ is a semigroup homomorphism. By the definition of the product of causal paths we have that $[(\alpha, \beta)][(\sigma, \rho)] = [(\alpha, \beta) * (\sigma, \rho)]$, but

$$\begin{aligned}
(\alpha, \beta) * (\sigma, \rho) &= \begin{cases} (\alpha, \beta)(2t) & \text{for } 0 \leq t \leq 1/2, \\ (\alpha, \beta)(1)(\sigma, \rho)(2t-1) & \text{for } 1/2 \leq t \leq 1/2. \end{cases} \\
&= \begin{cases} (\alpha, \beta)(2t) & \text{for } 0 \leq t \leq 1/2. \\ (\alpha(1)h(\beta(1)) \circ \sigma, \beta(1)\rho)(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases} \\
&= (\alpha, \beta) * (\sigma, \rho)(t) = (\alpha * h(\beta(1) \circ \sigma, \beta * \rho)(t),
\end{aligned}$$

so therefore,

$$\begin{aligned}
\Psi([(\alpha, \beta)][(\sigma, \rho)]) &= ([(\alpha * h(\beta(1)) \circ \sigma)], [(\beta * \rho)]) \\
&= ([\alpha][h(\beta(1)) \circ \sigma)], [\beta][\rho]) = ([\alpha], [\beta])([\sigma], [\rho]) \\
&= \Psi([(\alpha, \beta)])\Psi([(\sigma, \rho)]).
\end{aligned}$$

This proves that Ψ is a homomorphism. □

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