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A Research on Characterizations of Semi-$T_{1/2}$ Spaces

Un Levantamiento sobre Caracterizaciones de Espacios Semi-$T_{1/2}$

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Abstract

The goal of this article is to bring to your attention some of the salient features of recent research on characterizations of $Semi - T_{1/2}$ spaces.
Keywords and phrases: Topological spaces, generalized closed sets, semi-open sets, $Semi - T_{1/2}$ spaces, closure operator.

Resumen

El objetivo de este trabajo es presentar algunos aspectos resaltantes de la investigación reciente sobre caracterizaciones de los espacios $Semi - T_{1/2}$.
Palabras y frases clave: Espacios topológicos, conjuntos cerrados generalizados, conjuntos semi-abiertos, espacios $Semi - T_{1/2}$, operador clausura.

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1 Introduction

The concept of semi-open set in topological spaces was introduced in 1963 by N. Levine [11] i.e., if (X, τ) is a topological space and $A \subset X$, then A is semi-open ($A \in SO(X, \tau)$) if there exists $O \in \tau$ such that $O \subseteq A \subseteq Cl(O)$, where $Cl(O)$ denotes closure of O in (X, τ). The complement A^c of a semi-open set A is called semi-closed and the semi-closure of a set A denoted by $sCl(A)$ is the intersection of all semi-closed sets containing A.

After the work of N. Levine on semi-open sets, various mathematicians turned their attention to the generalizations of various concepts of topology by considering semi-open sets instead of open sets. While open sets are replaced by semi-open sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited.

In this direction, in 1975, S. N. Maheshwari and R. Prasad [12], used semi-open sets to define and investigate three new separation axiom called *Semi*-T_0, *Semi*-T_1 (if for $x, y \in X$ such that $x \neq y$ there exists a semi-open set containing x but not y or (resp. and) a semi-open set containing y but not x) and *Semi*-T_2 (if for $x, y \in X$ such that $x \neq y$, there exist semi-open sets O_1 and O_2 such that $x \in O_1, y \in O_2$ and $O_1 \cap O_2 = \emptyset$). Moreover, they have shown that the following implications hold.

$$\begin{array}{ccc} T_2 & \rightarrow & Semi-T_2 \\ \downarrow & & \downarrow \\ T_1 & \rightarrow & Semi-T_1 \\ \downarrow & & \downarrow \\ T_0 & \rightarrow & Semi-T_0 \end{array}$$

Later, in 1987, P. Bhattacharyya and B. K. Lahiri [1] generalized the concept of closed sets to semi-generalized closed sets with the help of semi-openness. By definition a subset of A of (X, τ) is said to be semi-generalized closed (written in short as sg-closed sets) in (X, τ), if $sCl(A) \subset O$ whenever $A \subset O$ and O is semi-open in (X, τ). This generalization of closed sets, introduced in [1], has no connection with the generalized closed sets as considered by N. Levine given in [10], although both generalize the concept of closed sets, this notions are in general independent. Moreover in [1], they defined the concept of a new class of topological spaces called *Semi*-$T_{1/2}$ (i.e., the spaces where the class of semi-closed sets and the sg-closed sets coincide). It is proved that every *Semi*-T_1 space is *Semi*-$T_{1/2}$ and every *Semi*-$T_{1/2}$ space is *Semi*-T_0, although none of these aplications is reversible.
The purpose of the present paper is to give some characterizations of *Semi*-$T_{1/2}$ spaces, incluing a characterization using a new topology that M. Caldas

and J. Dontchev [6] define as $\tau^{\wedge_s}$-topology. These characterizations are obtained mainly through the introduction of a concept of a generalized set or of a new class of maps.

2 Characterizations of Semi-$T_{1/2}$ Spaces

Recall ([15]) that for any subset E of (X,τ), $sCl^*(E) = \bigcap\{A : E \subset A(\in sD(X,\tau))\}$, where $sD(X,\tau) = \{A : A \subset X$ and A is sg-closed in $(X,\tau)\}$ and $SO(X,\tau)^* = \{B : sCl^*(B^c) = B^c\}$.

Similarly to W. Dunham [8], P. Sundaram, H. Maki and K. Balachandram in [15] characterized the *Semi-$T_{1/2}$* spaces as follows:

Theorem 2.1. *A topological space (X,τ) is a Semi-$T_{1/2}$ space if and only if $SO(X,\tau) = SO(X,\tau)^*$ holds.*

Proof. Necessity: Since the semi-closed sets and the sg-closed sets coincide by the assumption, $sCl(E) = sCl^*(E)$ holds for every subset E of (X,τ). Therefore, we have that $SO(X,\tau) = SO(X,\tau)^*$.
Sufficiency: Let A be a sg-closed set of (X,τ). Then, we have $A = sCl^*(A)$ and hence $A^c \in SO(X,\tau)$. Thus A is semi-closed. Therefore (X,τ) is *Semi-$T_{1/2}$*. □

Theorem 2.2. *A topological space (X,τ) is a Semi-$T_{1/2}$ space if and only if, for each $x \in X, \{x\}$ is semi-open or semi-closed.*

Proof. Necessity: Suppose that for some $x \in X$, $\{x\}$ is not semi-closed. Since X is the only semi-open set containing $\{x\}^c$, the set $\{x\}^c$ is sg-closed and so it is semi-closed in the *Semi-$T_{1/2}$* space (X,τ). Therefore $\{x\}$ is semi-open.
Sufficiency: Since $SO(X,\tau) \subseteq SO(X,\tau)^*$ holds, by Theorem 2.1, it is enough to prove that $SO(X,\tau)^* \subseteq SO(X,\tau)$. Let $E \in SO(X,\tau)^*$. Suppose that $E \notin SO(X,\tau)$. Then, $sCl^*(E^c) = E^c$ and $sCl(E^c) \neq E^c$ hold. There exists a point x of X such that $x \in sCl(E^c)$ and $x \notin E^c (= sCl^*(E^c))$. Since $x \notin sCl^*(E^c)$ there exists a sg-closed set A such that $x \notin A$ and $A \supset E^c$. By the hypothesis, the singleton $\{x\}$ is semi-open or semi-closed.
Case 1. $\{x\}$ is semi-open: Since $\{x\}^c$ is a semi-closed set with $E^c \subset \{x\}^c$, we have $sCl(E^c) \subset \{x\}^c$, i.e., $x \notin sCl(E^c)$. This contradicts the fact that $x \in sCl(E^c)$. Therefore $E \in SO(X,\tau)$.
Case 2. $\{x\}$ is semi-closed: Since $\{x\}^c$ is a semi-open set containing the sg-closed set $A(\supset E^c)$, we have $\{x\}^c \supset sCl(A) \supset sCl(E^c)$. Therefore $x \notin sCl(E^c)$. This is a contradiction. Therefore $E \in SO(X,\tau)$.
Hence in both cases, we have $E \in SO(X,\tau)$, i.e., $SO(X,\tau)^* \subseteq SO(X,\tau)$. □

As a consequence of Theorem 2.2, we have also the following characterization:

Theorem 2.3. *(X, τ) is Semi-$T_{1/2}$, if and only if, every subset of X is the intersection of all semi-open sets and all semi-closed sets containing it.*

Proof. Necessity: Let (X, τ) be a *Semi*-$T_{1/2}$ space with $B \subset X$ arbitrary. Then $B = \bigcap\{\{x\}^c; x \notin B\}$ is an intersection of semi-open sets and semi-closed sets by Theorem 2.2. The result follows.
Sufficiency: For each $x \in X$, $\{x\}^c$ is the intersection of all semi-open sets and all semi-closed sets containing it. Thus $\{x\}^c$ is either semi-open or semi-closed and hence X is *Semi*-$T_{1/2}$. □

Definition 1. A topological space (X, τ) is called a *semi-symmetric* space [3] if for x and y in X, $x \in sCl(\{y\})$ implies that $y \in sCl(\{x\})$.

Theorem 2.4. *Let (X, τ) be a semi-symmetric space. Then the following are equivalent.*
(i) (X, τ) is Semi-T_0.
(ii) (X, τ) is Semi-$T_{1/2}$.
(iii) (X, τ) is Semi-T_1.

Proof. It is enough to prove only the necessity of (i)↔(iii). Let $x \neq y$. Since (X, τ) is *Semi*-T_0, we may assume that $x \in O \subset \{y\}^c$ for some $O \in SO(X, \tau)$. Then $x \notin sCl(\{y\})$ and hence $y \notin sCl(\{x\})$. Therefore there exists $O_1 \in SO(X, \tau)$ such that $y \in O_1 \subset \{x\}^c$ and (X, τ) is a *Semi*-T_1 space. □

In 1995 J. Dontchev in [7] proved that a topological space is *Semi*-T_D if and only if it is *Semi*-$T_{1/2}$. We recall the following definitions, which will be useful in the sequel.

Definition 2. (i) A topological space (X, τ) is called a *Semi*-T_D space [9], if every singleton is either open or nowhere dense, or equivalently if the derived set $Cl(\{x\})\backslash\{x\}$ is semi-closed for each point $x \in X$.
(ii) A subset A of a topological space (X, τ) is called an α-open set [14], if $A \subset Int(Cl(Int(A)))$ and an α-closed set if $Cl(Int(Cl(A))) \subset A$.

Note that the family τ^α of all α-open sets in (X, τ) forms always a topology on X, finer than τ.

Theorem 2.5. *For a topological space (X, τ) the following are equivalent:*
(i) The space (X, τ) is a Semi-T_D space.
(ii) The space (X, τ) is a Semi-$T_{1/2}$ space.

Proof. (i)→(ii). Let $x \in X$. Then $\{x\}$ is either open or nowhere dense by (i). Hence it is α-open or α-closed and thus semi-open or semi-closed. Then X is a *Semi*-$T_{1/2}$ space by Theorem 2.2.
(ii)→(i). Let $x \in X$. We assume first that $\{x\}$ is not semi-closed. Then $X \backslash \{x\}$ is sg-closed. Then by (ii) it is semi-closed or equivalently $\{x\}$ is semi-open. Since every semi-open singleton is open, then $\{x\}$ is open. Next, if $\{x\}$ is semi-closed, then $Int(Cl(\{x\}) = Int(\{x\}) = \emptyset$ if $\{x\}$ is not open and hence $\{x\}$ is either open or nowhere dense. Thus (X, τ) is a *Semi*-T_D space. □

Again using semi-symmetric spaces we have the following theorem (see [3]).

Theorem 2.6. *For a semi−symmetric space (X, τ) the following are equivalent:*
(i) The space (X, τ) is a Semi-T_0 space.
(ii) The space (X, τ) is a Semi-D_1 space.
(iii) The space (X, τ) is a Semi-$T_{1/2}$ space.
(iv) The space (X, τ) is a Semi-T_1 space.

where a topological space (X, τ) is said to be a *Semi*-D_1 if for $x, y \in X$ such that $x \neq y$ there exists an sD-set of X (i.e., if there are two semi-open sets O_1, O_2 in X such that $O_1 \neq X$ and $S = O_1 \backslash O_2$) containing x but not y and an sD-set containing y but not x.

As an analogy of [13], M. Caldas and J. Dontchev in [5] introduced the $\wedge_s$-sets (resp. $\vee_s$-sets) which are intersection of semi-open (resp. union of semi-closed) sets. In this paper [5], they also define the concepts of $g.\wedge_s$-sets and $g.\vee_s$-sets.

Definition 3. In a topological space (X, τ), a subset B is called:
(i) $\wedge_s$-set (resp. $\vee_s$-set) if $B = B^{\wedge_s}$ (resp. $B = B^{\vee_s}$), where,
$B^{\wedge_s} = \bigcap\{O\colon O \supseteq B, O \in SO(X, \tau)\}$ and $B^{\vee_s} = \bigcup\{F\colon F \subseteq B, F^c \in SO(X, \tau)\}$.
(ii) Generalized $\wedge_s$-set (=$g.\wedge_s$-set) of (X, τ) if $B^{\wedge_s} \subseteq F$ whenever $B \subseteq F$ and $F^c \in SO(X, \tau)$.
(iii) Generalized $\vee_s$-set (=$g.\vee_s$-set) of (X, τ) if B^c is a $g.\wedge_s$-set of (X, τ).

By $D^{\wedge_s}$ (resp. $D^{\vee_s}$) we will denote the family of all $g.\wedge_s$-sets (resp. $g.\vee_s$-sets) of (X, τ).

In the following theorem ([5]), we have another characterization of the class of *Semi*-$T_{1/2}$ spaces by using $g.\vee_s$-sets.

Theorem 2.7. *For a topological space (X, τ) the following are equivalent:*
(i) (X, τ) is a Semi-$T_{1/2}$ space.
(ii) Every $g.\vee_s$-set is a $\vee_s$-set.

Proof. (i)→(ii). Suppose that there exists a $g.\vee_s$-set B which is not a $\vee_s$-set. Since $B^{\vee_s} \subseteq B$ ($B^{\vee_s} \neq B$), then there exists a point $x \in B$ such that $x \notin B^{\vee_s}$. Then the singleton $\{x\}$ is not semi-closed. Since $\{x\}^c$ is not semi-open , the space X itself is only semi-open set containing $\{x\}^c$. Therefore, $sCl(\{x\}^c) \subset X$ holds and so $\{x\}^c$ is a sg-closed set. On the other hand, we have that $\{x\}$ is not semi-open (since B is a $g.\vee_s$-set, and $x \notin B^{\vee_s}$). Therefore, we have that $\{x\}^c$ is not semi-closed but it is a sg-closed set. This contradicts the assumption that (X, τ) is a *Semi*-$T_{1/2}$ space.
(ii)→(i). Suppose that (X, τ) is not a *Semi*-$T_{1/2}$ space. Then, there exists a sg-closed set B which is not semi-closed. Since B is not semi-closed, there exists a point x such that $x \notin B$ and $x \in sCl(B)$. It is easily to see that the singleton $\{x\}$ is a semi-open set or it is a $g.\vee_s$-set. When $\{x\}$ is semi-open, we have $\{x\} \cap B \neq \emptyset$ because $x \in sCl(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is a $g.\vee_s$-set. If $\{x\}$ is not semi-closed, we have $\{x\}^{\vee_s} = \emptyset$ and hence $\{x\}$ is not a $\vee_s$-set. This contradicts (ii). Next, if $\{x\}$ is semi-closed, we have $\{x\}^c \supseteq sCl(B)$ (i.e., $x \notin sCl(B)$). In fact, the semi-open set $\{x\}^c$ contains the set B which is a sg-closed set. Then, this also contradicts the fact that $x \in sCl(B)$. Therefore (X, τ) is a *Semi*-$T_{1/2}$ space. □

M. Caldas in [4], introduces the concept of irresoluteness and the so called ap-irresolute maps and ap-semi-closed maps by using sg-closed sets. This definition enables us to obtain conditions under which maps and inverse maps preserve sg-closed sets. In [4] the author also characterizes the class of *Semi*-$T_{1/2}$ in terms of ap-irresolute and ap-semi-closed maps, where a map $f : (X, \tau) \to (Y, \sigma)$ is said to be: (i) Approximately irresolute (or ap-irresolute) if, $sCl(F) \subseteq f^{-1}(O)$ whenever O is a semi-open subset of (Y, σ), F is a sg-closed subset of (X, τ), and $F \subseteq f^{-1}(O)$; (ii) Approximately semi-closed (or ap-semi-closed) if $f(B) \subseteq sInt(A)$ whenever A is a sg-open subset of (Y, σ), B is a semi-closed subset of (X, τ), and $f(B) \subseteq A$.

Theorem 2.8. *For a topological space (X, τ) the following are equivalent:*
(i) (X, τ) is a Semi-$T_{1/2}$ space.
(ii) For every space (Y, σ) and every map $f : (X, \tau) \to (Y, \sigma)$, f is ap-irresolute.

Proof. (i)→(ii). Let F be a sg-closed subset of (X, τ) and suppose that $F \subseteq f^{-1}(O)$ where $O \in SO(Y, \sigma)$. Since (X, τ) is a *Semi*-$T_{1/2}$ space, F is semi-closed (i.e., $F = sCl(F)$). Therefore $sCl(F) \subseteq f^{-1}(O)$. Then f is ap-irresolute.
(ii)→(i). Let B be a sg-closed subset of (X, τ) and let Y be the set X with the topology $\sigma = \{\emptyset, B, Y\}$. Finally let $f : (X, \tau) \to (Y, \sigma)$ be the identity map.

By assumption f is ap-irresolute. Since B is sg-closed in (X, τ) and semi-open in (Y, σ) and $B \subseteq f^{-1}(B)$, it follows that $sCl(B) \subseteq f^{-1}(B) = B$. Hence B is semi-closed in (X, τ) and therefore (X, τ) is a *Semi-*$T_{1/2}$ space. □

Theorem 2.9. *For a topological space (X, τ) the following are equivalent:*
(i) (Y, σ) is a Semi-$T_{1/2}$ space.
(ii) For every space (X, τ) and every map $f:(X, \tau) \to (Y, \sigma)$, f is ap-semi-closed.

Proof. Analogous to Theorem 2.8 making the obvious changes. □

Recently, in [6], M. Caldas and J. Dontchev used the $g.\wedge_s$-sets to define a new closure operator $C^{\wedge_s}$ and a new topology $\tau^{\wedge_s}$ on a topological space (X, τ). By definition for any subset B of (X, τ), $C^{\wedge_s}(B) = \bigcap\{U : B \subseteq U, U \in D^{\wedge_s}\}$. Then, since $C^{\wedge_s}$ is a Kuratowski closure operator on (X, τ), the topology $\tau^{\wedge_s}$ on X is generated by $C^{\wedge_s}$ in the usual manner, i.e., $\tau^{\wedge_s} = \{B : B \subseteq X, C^{\wedge_s}(B^c) = B^c\}$.

We conclude the work mentioning a new characterization of *Semi-*$T_{1/2}$ spaces using the $\tau^{\wedge_s}$ topology.

Theorem 2.10. *For a topological space (X, τ) the following are equivalent:*
(i) (X, τ) is a Semi-$T_{1/2}$ space.
(ii) Every $\tau^{\wedge_s}$-open set is a $\vee_s$-set.

Proof. See [6]. □

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