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Some Integral Inequalities

Algunas Desigualdades Integrales

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Abstract

In this note, we establish some integral inequalities by using elementary methods for certain classes of functions defined on finite intervals of the real line. Our results have some relationships with certain integral inequalities obtained by Feng Qi.
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Resumen

En esta nota se establecen algunas desigualdades integrales usando métodos elementales para ciertas clases de funciones definidas en intervalos finitos de la recta real. Nuestros resultados tienen alguna relación con ciertas desigualdades integrales obtenidas por Feng Qi.
Palabras y frases clave: Desigualdades integrales

1 Introduction and statement of the results

In this note, we establish some new integral inequalities by using analytic and elementary methods. These inequalities have some relationships with certain integral inequalities obtained by Feng Qi in [2]. We point out that one of Feng Qi's result (see [2]) is the following

Theorem A. *Let $n \geq 1$ be an integer and suppose that f has a continuous derivative of the $n-th$ order on $[a,b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ where $0 \leq i \leq n-1$. Then*

$$\int_a^b [f(x)]^{n+2}\,dx \geq \left[\int_a^b f(x)\,dx\right]^{n+1}. \tag{1}$$

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Our results will provide some similar inequalities for certain classes of functions f satisfying weaker conditions than required in the previous result. Before stating the results, we need the following definition.

Definition 1.1. Let $[a,b]$ be a finite interval of the real line $\mathbb{R}$. For each real number r, we denote $\mathbb{E}_r(a,b)$ the set of real continuous functions f on $[a,b]$ differentiable on $]a,b[$, such that $f(a) \geq 0$, and $f'(x) \geq r$ for all $x \in]a,b[$.

Our first result is the following.

Theorem B. *Let $\alpha > 0$, and let $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 1$, we have the following strict inequality:*

$$\int_a^b [f(x)]^{(\alpha+1)2^n-1}\,dx > \left[\int_a^b [f(x)]^\alpha\,dx\right]^{2^n}. \tag{2}$$

As a consequence of this result, we obtain the following

Corollary 1.1. *Let $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 0$, we have*

$$\left[\int_a^b [f(x)]^{(\alpha+1)2^{n+1}-1}\,dx\right]^{\frac{1}{2^{n+1}}} > \left[\int_a^b [f(x)]^{(\alpha+1)2^n-1}\,dx\right]^{\frac{1}{2^n}}. \tag{3}$$

A consequence of the previous corollary is the following

Corollary 1.2. *Let $f \in \mathbb{E}_2(a,b)$. Then for each integer $n \geq 2$, we have*

$$\int_a^b [f(x)]^{2^{n+1}-1}\,dx > \left[\int_a^b [f(x)]^3\,dx\right]^{2^{n-1}} > \left[\int_a^b f(x)\,dx\right]^{2^n}. \tag{4}$$

The strict inequality established in Theorem B and the second inequality in corollary 1.2 should be compared to those obtained in the paper [2]. But our results are obtained here under weaker assumptions than those used by Feng Qi (see [2]) to derive the inequality (1.1) (resp. (1.3)) stated in Proposition 1.1 (resp. Proposition 1.3). We point out that Proposition 1.3 of [2] is exactly Theorem A enunciated in this introduction. The proof of Theorem B will be given in the next section. The proof uses mathematical induction and the following auxillary proposition.

Proposition 1.1. *Let $f \in \mathbb{E}_2(a,b)$. Then for each real number $p \in]0,\infty[$, we have*

$$\int_a^b [f(x)]^{2p+1}\,dx > \left[\int_a^b [f(x)]^p\,dx\right]^2. \tag{5}$$

The paper is organized as follows. In the second section, we find the proofs of the results stated in the introduction. In the third section, we present a related inequality. The paper ends with some references. The references [1], [3], [4] and [5] are cited for the convenience of the reader who desires to have some acquaintance with the nice world of inequalities.

2 Proofs

2.1 We start by proving Proposition 1.1. Let $p > 0$. For every $t \in [a, b]$, we set

$$F(t) = \int_a^t [f(x)]^{2p+1}\, dx - \left[\int_a^t [f(x)]^p\, dx\right]^2.$$

A simple computation yields for all $t \in]a, b[$

$$\begin{aligned} F'(t) &= \left[[f(t)]^{p+1} - 2\int_a^t [f(x)]^p\, dx\right][f(t)]^p := G(t)[f(t)]^p. \\ G'(t) &= [(p+1)f'(t) - 2][f(t)]^p. \end{aligned}$$

Since $p > 0$ and $f \in \mathbb{E}_2(a, b)$ then $f'(t) \geq 2 > \frac{2}{p+1}$, thus f is strictly increasing on $[a, b]$. Therefore $f(t) > f(a) \geq 0$ for every $t \in]a, b]$. Consequently, G is strictly increasing on $[a, b]$. Since $G(a) = [f(a)]^{p+1} \geq 0$, we deduce that $G(t) > 0$ for all $t \in]a, b]$. So, $F'(t) > 0$ for all $t \in]a, b[$. We deduce that F is strictly increasing on $[a, b]$. In particular, we obtain $F(b) > F(a) = 0$. Therefore, the inequality (5) holds. □

2.2 Now, by induction we shall prove Theorem B. Let $\alpha \in]0, \infty[$. For every integer $n \geq 1$, we set $p_n(\alpha) = (\alpha + 1)2^n - 1$. Then we have $p_n(\alpha) > 0$ and $p_{n+1}(\alpha) = 2p_n(\alpha) + 1$, for each integer $n \geq 1$. We remark that $p_1(\alpha) = 2\alpha + 1$. Then a direct application of Proposition 1.1 shows that the inequalitiy (2) holds true for $n = 1$. Suppose that the inequality (2) holds for the integer n and let us prove it for $n + 1$. Since $p_{n+1}(\alpha) = 2p_n(\alpha) + 1$ and $p_n(\alpha) > 0$ then we may apply Proposition 1.1 and obtain the following strict inequality:

$$\int_a^b [f(x)]^{(\alpha+1)2^{n+1}-1}\, dx > \left[\int_a^b [f(x)]^{(\alpha+1)2^n-1}\, dx\right]^2. \tag{6}$$

By assumption we have

$$\int_a^b [f(x)]^{(\alpha+1)2^n-1}\, dx > \left[\int_a^b [f(x)]^{\alpha}\, dx\right]^{2^n}. \tag{7}$$

From (6) and (7) we deduce that the inequality (2) holds true for $n+1$. Thus our result is completely proved. □

Remark. From (6) we get all the statements contained in the corollaries 1.1 and 1.2.

3 A related inequality

We end this paper by giving a related integral inequality. More precisely, we have

Theorem C. *Let $[a,b]$ be a closed interval of $\mathbb{R}$, Let $p \geq 1$ be a real number and let $f \in \mathbb{E}_p(a,b)$. Then we have*

$$\int_a^b [f(x)]^{p+2}\,dx \geq \frac{1}{(b-a)^{p-1}} \left[\int_a^b f(x)\,dx\right]^{p+1}. \tag{8}$$

Proof. For every $t \in [a,b]$, we set

$$H(t) = \int_a^t [f(x)]^{p+2}\,dx - \frac{1}{(b-a)^{p-1}} \left[\int_a^t f(x)\,dx\right]^{p+1}.$$

Simple computations yield for all $t \in]a,b[$

$$\begin{aligned} H'(t) &= \left([f(t)]^{p+1} - \tfrac{p+1}{(b-a)^{p-1}} \left[\int_a^t f(x)\,dx\right]^p\right) f(t) := h(t)f(t), \\ h'(t) &= (p+1)\left([f(t)]^{p-1}f'(t) - \tfrac{p}{(b-a)^{p-1}} \left[\int_a^t f(x)\,dx\right]^{p-1}\right) f(t). \end{aligned}$$

Since f is increasing then $0 \leq \int_a^t f(x)\,dx \leq (b-a)f(t)$ for all $t \in [a,b]$. Therefore, we have

$$h'(t) \geq (p+1)(f'(t)-p)[f(t)]^p. \tag{9}$$

We deduce from (9) that h is increasing on $[a,b]$. Since $h(a) = [f(a)]^{p+1} \geq 0$, this shows that H is increasing on $[a,b]$. In particular, we have $H(b) \geq H(a) = 0$, which gives the desired inequality. □

As a consequence, by replacing $f(x)$ by $pf(x)$ in (8), we have the following

Corollary 3.1. *Let $p \geq 1$ be a real number and let $f \in \mathbb{E}_1(0,1)$. Then*

$$\int_0^1 [f(x)]^{p+2}\,dx \geq \frac{1}{p}\left[\int_0^1 f(x)\,dx\right]^{p+1} \tag{10}$$

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