Long Range Diffusion Reaction Model on Population Dynamics

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Abstract.

A model for long range diffusion reaction on population dynamics has been considered, and conditions for the existence and uniqueness of solutions to the model in $L^{p,q}$ norms has been obtained.

Keywords and Phrases: diffusion reaction, long range.

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1 Introduction

The dynamics of population has been described using mathematical models which have been very successful in giving good effect in the study of animal and human populations. Fisher [4] introduced a model for the spatial distribution of an advantageous gene as non-linear diffusion equations. Later, Hoppensteadt [6] p.50, derived an equation of age-dependent population growth which involves first order partial derivatives with respect to age and time, where Fife [3] considered reaction and diffusion systems which are distributed in 3-dimensional space or on a surface rather than on the line. In addition, Abual-rub studied diffusion in two dimensional spaces for which diffusion is more realistic and applicable in life. Most of these diffusion models deal with usual diffusion or short range diffusion. Such models have played a major role in the study of population dynamics. However, long range diffusion could also have a big influence on the dynamics of some populations with the form it takes depending on the nature of the populations themselves. Abual-rub talked about long range diffusion with population pressure in Plankton-Herbivore populations. He considered a model of the following form:

$$P_t - c\Delta^{(2)}P = aP + eP^2 - bPH + \frac{u}{\alpha+1}\Delta\left(P^{\alpha+1}\right) \tag{1}$$

$$P(x,0) = g(x), \qquad x \in R^2, \tag{2}$$

and

$$H_t - \ell\Delta^{(2)} H = kPH - dH^2 + \frac{u}{\alpha+1}\Delta\left(H^{\alpha+1}\right) \tag{3}$$

$$H(x,0) = h(x), \qquad x \in R^2, \tag{4}$$

where $P(x,t)$ and $H(x,t)$ represent the Plankton and Herbivore densities, respectively.

Here Δ represents the Laplacian operator and

$$\Delta^{(2)} = \sum_{i,j=1}^{2} \frac{\partial^4}{\partial x_i^2 \partial x_j^2}. \tag{5}$$

The existence and uniqueness of solutions to (1)-(4) have been proved by Abual-rub in the $L^{p,q}$ spaces. Okubo [8] p. 194, discussed the effect of density-dependent dispersal on population dynamics by considering the Gurtin and MacCamy [5] model which combines the flux with the population reaction term, $F(S)$, he considered diffusion-reaction problems in one dimension of the form:

$$\frac{\partial S}{\partial t} = K\frac{\partial^2 S^{m+1}}{\partial x^2} + F(s), \tag{6}$$

where

$$K = k\,(m+1) > 0. \tag{7}$$

Murray [7] p.245, which is one of the good books in mathematical biology, considered a long range diffusion model of population by taking the flux J to be:

$$J = -D_1 \nabla S + \nabla D_2\,(\Delta S) \tag{8}$$

where D_1 and D_2 are the constants which measure short range and long range effects, respectively. He obtained a long range diffusion approximation of the form:

$$\frac{\partial S}{\partial t} = \nabla \cdot D_1 \nabla S - \nabla \cdot \nabla (D_2 \Delta S). \tag{9}$$

For this model, Murray mentioned that the effect of short range diffusion is, usually, larger than that of long range diffusion, i.e. $D_1 > D_2$. In this paper we will see what happens if the effect of long range diffusion is larger. This assumption might not be realistic in general, but we think that it might be true in some rare cases of population dynamics such as for certain epidemics and Plankton-Herbivore systems.

2 Model

We will consider the two dimensional case in our model rather than the first dimensional case i.e., $x = (x_1, x_2)$, because it is more realistic that diffusion takes place in spaces and not along lines. Therefore, we will use ΔS instead of $\frac{\partial^2 S}{\partial x^2}$. As mentioned in the introduction we will assume that the effect of long range diffusion is larger than that of short range diffusion and investigate what will happen if at some stage D_1 is negligible compared with D_2. We believe that this might happen at some stages depending on the nature of the population and the nature of its dynamic. Its known that in short rang diffusion the flux J takes the following form

$$J = -D\nabla S. \tag{10}$$

Murray [7] p.245, derived the equation for flux J in (8). In our model, according to the above assumptions, we will consider the flux to be of the form

$$J = \nabla \left(D_2 \Delta S \right). \tag{11}$$

The conservation equation for S is given by

$$\frac{\partial S}{\partial t} = -\nabla \cdot J + F(S), \tag{12}$$

where $F(S)$ is the population reaction term. By substituting (11) into (12) we get the following model for long range diffusion reaction, namely

$$\frac{\partial S}{\partial t} = -D_2 \Delta^{(2)} S + F(S) \tag{13}$$

In this paper we will impose the initial condition on S , namely

$$S(x, 0) = g(x) \tag{14}$$

In addition, we will consider $F(S)$ to be directly proportional to S^n, i.e,

$$F(S) = aS^n \tag{15}$$

for some positive constant a and integer n which has to be determined later. The reason for writing S^n here is that in usual diffusion we have always S or S^2 but in long range diffusion things might differ and if it does we want to determine the right exponent, n, for S. Let $C = -D_2$, our model is thus

$$\frac{\partial S}{\partial t} - C\Delta^{(2)} S = aS^n, \tag{16}$$

$$S(x, 0) = g(x). \tag{17}$$

where the term $C\Delta^{(2)} S$ represents long range diffusion.

3 Existence and uniqueness of solutions:

We will look for solutions to model (16), (17) in the $L^{p,q}$ space, the function space consisting of Lebesgue measurable functions $S(x,t)$ such that $\|S\|_{p,q} < \infty$, where $\|(\cdot)\|_{p,q}$ is the norm in $L^{p,q}$ defined by :

$$\|S\|_{p,q} = \left[\int_0^T \left[\int_{R^2} |S|^p \, dx \right]^{\frac{q}{p}} dt \right]^{\frac{1}{q}} \tag{18}$$

We will now state and prove the main result in this paper.

3.1 LEMMA

The solution to model (16), (17), $S(x,t)$, exists and is unique in the space $L^{\frac{3}{2}(n-1),\frac{1}{2}(n-1)}$ for $n > 3$, whenever the initial data $g(x)$ is small enough in the norm of its space.

Proof. We begin by transforming equation (16) and the initial condition (17) into the following integral equation

$$S = a \int_0^t \int_{R^2} K(x-y, t-\tau) S^n(y,\tau) dy d\tau + \int_{R^2} K(x-y,t) g(y) dy \tag{19}$$

We will now rewrite (19) simply as

$$S = aK \odot S^n + K * g \tag{20}$$

where $\odot$ denotes the convolution in space and time and $*$ denotes the convolution in space only. Here the kernel K is the Fundamental solution to the homogeneous problem of (16), namely

$$K(x,t) = t^{-\frac{1}{2}} \phi \left(x t^{-\frac{1}{4}} \right) \text{ ,where K} \in \mathrm{C}^{\infty}(\mathrm{R}^2) \tag{21}$$

Using (21), K can be approximated by

$$|K(x,t)| \leq \frac{c}{\left(|x| + t^{\frac{1}{4}} \right)^2}, \quad t > 0 \tag{22}$$

Now, if $g \in L^q(R^2)$ we have

$$K * g \leq \int_{R^2} \frac{c g(y) \, dy}{\left(|x-y| + t^{1/4} \right)^2}.$$

We first take the p norm in t, namely

$$\|K * g\|_p \leq \left\| \int_{R^2} \frac{c g(y) \, dy}{\left(|x-y| + t^{1/4} \right)^2} \right\|_p .$$

Applying Minkowski's integral on the right hand side of the above inequality, we obtain

$$\begin{aligned} \|K * g\|_p &\leq c \int_{R^2} |g(y)| \left(\int_{R^+} \frac{dt}{\left(|x-y| + t^{1/4}\right)^{2p}} \right)^{\frac{1}{p}} dy \\ &\leq c\alpha \int_{R^2} |g(y)| \left(\frac{1}{\left(|x-y| + t^{1/4}\right)^{2p-4}} \right)^{\frac{1}{p}} dy \\ &= c\alpha \int_{R^2} \frac{|g(y)|\, dy}{\left(|x-y| + t^{1/4}\right)^{2-\frac{4}{p}}}, \end{aligned}$$

where α is a constant.
We now take the q norm in x of the above inequality to obtain

$$\|K * g\|_{p,q} \leq c\alpha \left\| \int_{R^2} \frac{|g(y)|\, dy}{\left(|x-y| + t^{1/4}\right)^{2-\frac{4}{p}}} \right\|_q .$$

The right hand side of the above inequality is less than or equal to constant$\cdot\|g\|_q$, if $\frac{1}{p} = \frac{1}{q} - \frac{4}{2p}$(using the Benedek-Panzone Potential Theorem [1], see Appendix). This implies that $p = 3q$ and hence

$$K * g \in L^{3q} \tag{23}$$

This concludes the proof for the initial data.

Now, for the first term in (20), note that we can rewrite (22) as

$$|K| \leq \frac{c}{\left(|x| + t^{\frac{1}{4}}\right)^2} = \frac{c}{\left(|x| + t^{\frac{1}{4}}\right)^{2+4-4}} \tag{24}$$

By doing the calculations to the first term in (20), using (24), similar to what has been done to the second term in (20) in the previous page 5, using (22), then applying the Benedek-Panzone Potential Theorem [1], see appendix, we conclude that

$$\frac{1}{r} = \frac{n}{p} - \frac{4}{2+4} = \frac{n}{p} - \frac{2}{3} \quad ; \quad 1 < \frac{p}{n} < \frac{3}{2} \tag{25}$$

Now, by setting $r = p$ in (25) we get :

$$p = \frac{3}{2}(n-1) \tag{26}$$

Using (25) and (26) we have :

$$n < \frac{3}{2}(n-1) < \frac{3}{2}n \tag{27}$$

Therefore, since $\frac{3}{2}(n-1) < \frac{3}{2}n$ is true always, we must have $n < \frac{3}{2}(n-1)$ which in turns gives :

$$n > 3 \tag{28}$$

To get a contraction mapping (see appendix) $L^p\left(R^2 \times R_+\right) \to L^p\left(R^2 \times R_+\right)$ in (20), the exponents in (23) and (26) must be equal, that is

$$\frac{3}{2}(n-1) = 3q \tag{29}$$

and thus

$$q = \frac{n-1}{2} \tag{30}$$

Hence

$$S(x,t) \in L^{\frac{3}{2}(n-1),\frac{1}{2}(n-1)} \tag{31}$$

Now, its enough to show the uniqueness of the solution.

Lets apply the mapping T to (20) to obtain :

$$T(S) = aK \odot S^n + K * g \tag{32}$$

Its easy to see that:

$$\|T(S)\|_{\frac{3}{2}(n-1)} \leq C(n)\,\|S\|^n_{\frac{3}{2}(n-1)} + \|h\|_{\frac{3}{2}(n-1)} \tag{33}$$

where h is an auxiliary function which represents the term $K * g$ in (32).

We are going now to compare equation (33) to the following mapping :

$$y = \alpha x^n + \beta \quad ; \quad (x \geq 0) \tag{34}$$

where both α and β are positive constants. Of course αx^n is convex and increases faster that a linear function.

Its obvious to see that if $\beta = 0$, there is only one non-zero root of (34) but if $0 < \beta < \delta$ (where δ is sufficiently small), we will have two roots, say $\widetilde{x_1}$ and $\widetilde{x_2}$.

Let $\widetilde{x_1}$ be the smallest root, then if $\widetilde{x_1}$ is small enough then the mapping T will be a contraction mapping which maps the ball of radius $\widetilde{x_1}$ into itself. This implies that the solution to the equation $S = T(S)$ in (32) exists and its unique in the ball of radius $\widetilde{x_1}$. This concludes the proof of Lemma 3.1.

Remark 1: The extension of the results in Lemma 3.1 to three or n dimensions is straight forward.

Remark 2: See [2] for a general method for studying long-time asymptotics of non-linear parabolic partial differential equations. In [2], p.898, Remark 1, the existence and uniqueness of solutions have been shown. Comparing our results with the results obtained in [2], we conclude that if we take $\beta = 4$, then equation (8) in [2], p. 898, is analogous to our equation (16) here and $u(x,t)$ used in [2] is the same as $K(x,t)$ used here in (21). This shows that our method coinsides with the method used in [2] and thus therorem 1 in [2] is applicable to our case.

4 CONCLUSION:

We conclude that solutions to our model (16), (17) can not exist in $L^{p,q}$ spaces unless $n > 3$. But this does not mean that there are no solutions for $n \leq 3$, because solution might exist for $n \leq 3$ but in other spaces different from $L^{p,q}$ spaces. Its very important to notice that under the assumption we have made at the beginning, namely the long range diffusion dominance, we have shown that $n > 3$. This means that we should have terms like S^4 or S^5 or of larger degree of S in the right hand side of (16) and this in turns says that we must have interaction between four Kinds of species or more in the population.

5 APPENDIX:

- BENEDEK-PANZONE POTENTIAL THEOREM :Let $X = E^n$ (the nth dimensional Euclidean space), and $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n-tuple of real numbers,$0 < \lambda_i < 1$. If P and Q are such that $\frac{1}{P} - \frac{1}{Q} = \Lambda$, $1 < P < \frac{1}{\Lambda}$, then $\left\| f * |x|^{\lambda - n} \right\|_Q \leq c \|f\|_P$ holds for every $f \in L^P$, where $\lambda = \sum_{i=1}^n \lambda_i$, and $c = c(\Lambda, P)$.

- CONTRACTION MAPPINGS :Let T be a mapping of a metric space X into itself. Then x is called a fixed point of T if $T(x) = x$. Suppose there exists a number $c < 1$ such that $\|T(x) - T(y)\| < c \|x - y\|$ for every pair of points $x, y \in X$. Then T is called a contraction mapping.

- FIXED POINT THEOREM:Every contraction mapping T defined on a complete metric space (or Banach space) has a unique fixed point.

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