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The Minimum Principle from a Hamiltonian Point of View

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Abstract. Let G be a complex Lie group and $G_{\mathbb{R}}$ a real form of G. For a $G_{\mathbb{R}}$-stable domain of holomorphy X in a complex G-manifold we consider the question under which conditions the extended domain $G \cdot X$ is a domain of holomorphy. We give an answer in term of $G_{\mathbb{R}}$-invariant strictly plurisubharmonic functions on X and the associate Marsden-Weinstein reduced space which is given by the Kaehler form and the moment map associated with the given strictly plurisubharmonic function. Our main application is a proof of the so called extended future tube conjecture which asserts that $G \cdot X$ is a domain of holomorphy in the case where X is the N-fold product of the tube domain in $\mathbb{C}^4$ over the positive light cone and G is the connected complex Lorentz group acting diagonally.

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Let $G_{\mathbb{R}}$ be a connected real form of a complex Lie group G and X a $G_{\mathbb{R}}$-stable domain in a complex G-manifold Z such that $G{\cdot}X = Z$. In this paper we consider the following question. Under which conditions on X is Z the natural domain of definition of the $G_{\mathbb{R}}$-invariant holomorphic functions on X? If Z is an open submanifold of a Stein manifold, then there is an envelope of holomorphy for Z. Consequently, every $G_{\mathbb{R}}$-invariant holomorphic function on X which extends to Z also extends to the envelope of holomorphy of Z. Thus one also has to ask under which additional requirements is Z a Stein manifold.

In order that an invariant holomorphic function extends to $Z = G \cdot X$ it is sufficient that X is orbit connected, i.e., for every $z \in Z$ the set $\{g \in G;\ g \cdot z \in X\}$ is connected (see [H]). Thus under this condition the main question is whether Z is a Stein manifold. Now if Z is a domain in a Stein manifold V, then Z itself is a Stein manifold if one can find a plurisubharmonic function Ψ on Z which goes to $+\infty$ at every boundary point of $\partial Z \subset V$. There is a natural way to construct

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G-invariant plurisubharmonic functions out of $G_{\mathbb{R}}$-invariant functions on X which was first proposed by Loeb in [L]. In this paper Loeb used an extended version of Kiselman's minimum principle ([K]) in order to construct invariant plurisubharmonic functions. The main idea is the following. Assume that there is a nice quotient $\pi : Z \to Z/G$ and let ϕ be a smooth $G_{\mathbb{R}}$-invariant plurisubharmonic function on X which is a strictly plurisubharmonic exhaustion on each fibre of $\pi|X$. Then the fibre wise minimum of ϕ defines a function ψ on Z/G which is a candidate for a plurisubharmonic function on Z/G. This procedure can be described in terms of Hamiltonian actions as follows.

Assume for simplicity that ϕ is strictly plurisubharmonic. Then $\omega := 2i\partial\bar{\partial}\phi$ defines an invariant Käher form on X and $\mu(x)(\xi) = d\phi(J\xi_X)$ is the associated moment map $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$. In this situation $\mu^{-1}(0)$ is the set of fibre wise critical points of ϕ which in good cases are exactly the points such that the restriction of ϕ to the fibre attains its minimum. Again under some additional assumption, it then follows from the principle of symplectic reduction that the reduced space $\mu^{-1}(0)/G_{\mathbb{R}}$ has a symplectic structure which in fact is Kählerian and moreover is given by the function ψ which is induced on $\mu^{-1}(0)/G_{\mathbb{R}}$ by $\phi|\mu^{-1}(0)$. It turns out that in the situation under consideration the procedures given by symplectic reduction and minimum principle are compatible. This is well known in the case where $G_{\mathbb{R}}$ is a compact Lie group (see e.g. [H-H-L], where a much more general result is proved) and we give here precise conditions such that it also works for a non compact group $G_{\mathbb{R}}$.

The application of Loeb's Minimum Principle is limited mainly to the case of free $G_{\mathbb{R}}$-actions. For the more general case of proper actions it seems that the Hamiltonian point of view is much more adequate. Moreover, for applications it is necessary to consider also domains X of G-spaces Z which do not admit a geometrical quotient Z/G. A typical example is given by the so called extended future tube which we will describe next.

Let $< \, , \, >$ denote the Lorentz product on $\mathbb{R}^4$ and also its $\mathbb{C}$-bilinear extension to $\mathbb{C}^4$. The future tube $\mathcal{T}$ is by definition the tube domain in $\mathbb{C}^4 = \mathbb{R}^4 + i\mathbb{R}^4$ over the positive light cone $C^+ = \{y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4;\ y_0 > 0,\ < y, y >= (y_0)^2 - (y_1)^2 - (y_2)^2 - (y_3)^2 > 0\}$, i.e.,

$$\mathcal{T} = \{z \in \mathbb{C}^4;\ \operatorname{Im} z \in C^+\}\,.$$

This domain is invariant under the action of the connected component $G_{\mathbb{R}}$ of the identity of the homogeneous Lorentz group $\mathrm{O}_{\mathbb{R}}(1,3)$. Now consider the N-fold product $\mathcal{T}^N$ with the diagonal action of $G_{\mathbb{R}}$. The extended future tube $(\mathcal{T}^N)^{\mathbb{C}}$ is by definition the orbit of $\mathcal{T}^N$ under the action of the complexified group G of $G_{\mathbb{R}}$. In other words

$$(\mathcal{T}^N)^{\mathbb{C}} = G \cdot \mathcal{T}^N = \{(g \cdot z_1, \ldots, g \cdot z_N);\ g \in G,\ z_j \in \mathcal{T}\}\,.$$

Note that G is the group $\mathrm{SO}_4(\mathbb{C})$ which is defined by the quadratic form $< \, , \, >$. Although there is no geometric quotient of Z, we have a quotient $\pi : (\mathbb{C}^4)^N \to (\mathbb{C}^4)^N /\!\!/ G$ which is given by the invariant holomorphic functions on $(\mathbb{C}^4)^N$ and it is a fundamental fact that the extended tube $(\mathcal{T}^N)^{\mathbb{C}}$ is saturated with respect to π ([H-W], see §3 for additional remarks). In this case it turns out that this invariant theoretical quotient has sufficiently many good properties in order to apply the main result of this paper which we formulate now.

Let V be a Stein G-manifold such that there exists almost a quotient $\pi : V \to V/\!\!/G$. More precisely we will assume that $V/\!\!/G$ is a complex space, $\pi : V \to V/\!\!/G$ is a G-invariant surjective holomorphic map and for an analytically Zariski open π-saturated subset V^0 of V the restriction map $\pi : V^0 \to V^0/\!\!/G$ is a holomorphic fibre bundle with typical fibre G/H. Thus $V^0/\!\!/G = V^0/G$ is a geometric quotient. Let X be a $G_{\mathbb{R}}$-stable domain in V such that $Z := G \cdot X$ is saturated with respect to $\pi : V \to V/\!\!/G$.

Theorem 1. *Let $\phi : X \to \mathbb{R}$ be a smooth non-negative $G_{\mathbb{R}}$-invariant plurisubharmonic function and assume that*

(i) *The fibres of π restricted to $X^0 := X \cap V^0$ are connected,*

(ii) *the restriction of $\phi^0 := \phi|X^0$ to the fibres of π restricted to X^0 is strictly plurisubharmonic,*

(iii) *ϕ^0 is proper mod $G_{\mathbb{R}}$ along $\pi|Z^0$ where $Z^0 := V^0 \cap Z$ and*

(iv) *ϕ is a weak exhaustion of X over $V/\!\!/G$,*

Then $Z = G \cdot X$ is a Stein manifold.

In the case where $G_{\mathbb{R}}$ acts properly on X^0 condition (iii) means that the map $\phi^0 \times \pi|X^0 : X^0 \to \mathbb{R} \times (Z^0/\!\!/G)$ induces a proper map $X^0/G_{\mathbb{R}} \to \mathbb{R} \times (Z^0/\!\!/G)$. By a weak exhaustion of X over $V/\!\!/G$ we mean a function which goes to $+\infty$ on a sequence if the corresponding sequence in $V/\!\!/G$ converges to a boundary point of $Z/\!\!/G$ in $V/\!\!/G$.

In the case where the G-action on Z^0 is assumed to be free, the theorem can be proved rather directly by applying Loeb's minimum principle. For a compact group it is a consequence of the methods presented in [H-H-K] (see also [H-H-L]).

In the last section we recall some previously known facts proved in [H-W] together with a more recent result in [Z] about the orbit geometry of the extended future tube in order to verify that the conditions of Theorem 1 are satisfied in the case of the extended future tube. This leads to a conceptual proof of the so called extended future tube conjecture in the last section.

Theorem 2. *The extended future tube is a domain of holomorphy.*

This result has conjecturally been known in constructive quantum field theory for more then thirty years. For its relevance and other publications concerning problems related to it we refer the reader to the literature ([B-L-T], [H-S], [J], [S-W], [S-V]).

There is a proof of Theorem 2 in [Z] which due to several mistakes and gaps is difficult to understand.

1. Hamiltonian actions on Kähler spaces.

Let $G_{\mathbb{R}}$ be a real connected Lie group and X a complex $G_{\mathbb{R}}$-space, i.e., $G_{\mathbb{R}}$ acts on X by holomorphic transformations such that the action $G_{\mathbb{R}} \times X \to X$, $(g, x) \to g \cdot x$, is real analytic. If ω is a smooth $G_{\mathbb{R}}$-invariant Kähler structure on X, then a $G_{\mathbb{R}}$-equivariant smooth map μ from X into the dual $\mathfrak{g}_{\mathbb{R}}^*$ of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is said to be an equivariant moment map if

$$d\mu_\xi = \imath_{\xi_X} \omega$$

holds on every $G_{\mathbb{R}}$-stable complex submanifold Y of X. Here ω denotes the Kähler form on Y induced by the Kählerian structure on X (see [H-H-L]), $\mu_\xi := < \mu, \xi >$ is the component of μ in the direction of $\xi \in \mathfrak{g}_{\mathbb{R}}$, ξ_X is the vector field on X induced by ξ and $\iota_{\xi_X}\omega$ denotes the one form given by contraction, i.e., $\eta \to \omega(\xi, \eta)$.

Example. If ω is given by a smooth strictly plurisubharmonic $G_{\mathbb{R}}$-invariant function ϕ, i.e., $\omega = 2i\partial\bar{\partial}\phi$ on every smooth part of X, then

$$\mu_\xi(x) := d\phi(J\xi_X) = (i(\partial - \bar{\partial})\phi)(\xi_X) = d^c\phi(\xi_X)$$

defines an equivariant moment map. This follows from invariance of ϕ, since in this case we have

$$d\mu_\xi = d\imath_{\xi_X} d^c\phi = -\imath_{\xi_X} dd^c\phi = \imath_{\xi_X} 2i\partial\bar{\partial}\phi\,.$$

Here we use the formula

$$\mathcal{L}_\xi\alpha = \imath_\xi d\alpha + d\imath_\xi\alpha$$

for all vector fields ξ and differential forms α where $\mathcal{L}_\xi$ denotes the Lie derivative in the direction of ξ.

Later we will need the following fact about the zero level set of μ.

LEMMA. *Assume that X is smooth and that $G_{\mathbb{R}}$ acts properly on X. If the dimension of the $G_{\mathbb{R}}$-orbits in $\mu^{-1}(0)$ is constant, then $\mu^{-1}(0)$ is a submanifold of X.*

Proof. Since the action is assumed to be proper, there is a local normal form for the moment map (see e.g. [A] or [H-L]). The statement is an easy consequence of this fact (see e.g. [A]. In [S-L] the argument is given for a compact group $G_{\mathbb{R}}$). □

Remark 1. It can be shown that the converse of the Lemma also holds. We will not use this fact here.

Remark 2. The properness assumption is very often satisfied. Since one may assume that $G_{\mathbb{R}}$ acts effectively, $G_{\mathbb{R}}$ is a Lie subgroup of the group I of isometries of the Riemannian manifold X. The group of isometries acts properly on X and consequently the $G_{\mathbb{R}}$-action on X is proper if and only if $G_{\mathbb{R}}$ is a closed subgroup of I. This is the case if and only if there is a point $x \in X$ such that $G_{\mathbb{R}} \cdot x$ is closed and the isotropy group $(G_{\mathbb{R}})_x := \{g \in G_{\mathbb{R}};\, g \cdot x = x\}$ is compact.

Remark 3. If $G_{\mathbb{R}}$ acts such that the isotropy groups are discrete, then μ has maximal rank. Thus in this case $\mu^{-1}(0)$ is obviously a submanifold of X. Moreover $T_x(\mu^{-1}(0)) = \ker d\mu(x)$ for all $x \in \mu^{-1}(0)$.

2. HAMILTONIAN ACTIONS ON INVARIANT DOMAINS

Let G be a connected complex Lie group and Z a holomorphic G-space, i.e., the action $G \times Z \to Z$ is assumed to be a holomorphic map. Let $G_{\mathbb{R}}$ be a connected real form of G. By an invariant domain in Z we mean in the following a $G_{\mathbb{R}}$-stable connected open subspace X of Z. In the homogeneous case we have the following

Lemma 1. *Let X be an invariant domain in Z and assume that Z is G-homogeneous. If the zero level set of $\mu : X \to \mathfrak{g}_{\mathbb{R}}$ is not empty, then $\mu^{-1}(0)$ is a Lagrangian submanifold of X and each connected component of $\mu^{-1}(0)$ is a $G_{\mathbb{R}}$-orbit.*

Proof. For $z_0 \in X$ let N be an open convex neighborhood of $0 \in \mathfrak{g}_{\mathbb{R}}$ such that $U := G_{\mathbb{R}} \cdot \exp iN \cdot z_0 \subset X$. Since $G_{\mathbb{R}} \cdot \exp iN$ is a neighborhood of $G_{\mathbb{R}}$ in G, the set U is a neighborhood of $G_{\mathbb{R}} \cdot z_0$ in X. The proof of Lemma 1 is a consequence of the following

Claim. $U \cap \mu^{-1}(0) = G_{\mathbb{R}} \cdot z_0$ for $z_0 \in \mu^{-1}(0)$.

In order to proof the claim, let $z \in U \cap \mu^{-1}(0)$ be given. Then there are $h \in G_{\mathbb{R}}$ and $\xi \in N$ such that $z = h \exp i\xi \cdot z_0 \in U \cap \mu^{-1}(0)$. Thus $\exp i\xi \cdot z_0 \in \mu^{-1}(0) \cap U$ and $z_t := \exp it\xi \cdot z_0 \in U$ for $t \in [0,1]$. Note that $J\xi_X(x) = \frac{d}{dt}|_{t=0} \exp it\xi \cdot x$ is the gradient flow of μ_ξ with respect to the Riemannian metric induced by ω. Thus, if z_t is not constant, then $t \to \mu_\xi(z_t)$ is strictly increasing. This contradicts $\mu_\xi(z_0) = 0 = \mu_\xi(z_1)$. Therefore $z_0 = \exp it\xi \cdot z_0$ for all $t \in \mathbb{R}$. This implies $z = h \cdot z_1 = h \cdot z_0 \in G_{\mathbb{R}} \cdot z_0$.

It is a consequence of the claim that every $G_{\mathbb{R}}$-orbit is closed in X. Therefore every component of $\mu^{-1}(0)$ is a $G_{\mathbb{R}}$-orbit. It remains to show that these orbits are Lagrangian. Since $\mu(G_{\mathbb{R}} \cdot z_0) = 0$ we have

$$0 = d\mu_\xi(\eta_X(z_0)) = \omega(\xi_X(z_0), \eta_X(z_0))$$

for all $\xi, \eta \in \mathfrak{g}_{\mathbb{R}}$. This means that $G_{\mathbb{R}} \cdot z_0$ is an isotropic submanifold of X. In particular, $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 \leq \dim_{\mathbb{C}} X$. In general the tangent space $T_{z_0}(G_{\mathbb{R}} \cdot z_0)$ spans $T_{z_0}X$ over $\mathbb{C}$. Thus $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 \geq \dim_{\mathbb{C}} G \cdot z_0 = \dim_{\mathbb{C}} X$. This shows that $\dim_{\mathbb{R}} G_{\mathbb{R}} \cdot z_0 = \frac{1}{2} \dim_{\mathbb{R}} X$. Hence $G_{\mathbb{R}} \cdot z_0$ is Lagrangian. □

Every Lagrangian submanifold of a Kähler manifold is totally real. Thus, if Z is G-homogeneous, then $\mu^{-1}(0)$ is a totally real submanifold of X. Note that the $G_{\mathbb{R}}$-orbits in $\mu^{-1}(0)$ are closed since they are connected components of the zero fibre of μ. Now if $G_{\mathbb{R}}$ is such that $0 \in \mathfrak{g}_{\mathbb{R}}^*$ is the only $G_{\mathbb{R}}$-fixed point, then $x \in \mu^{-1}(0)$ if and only if the orbit $G_{\mathbb{R}} \cdot x$ is isotropic. This condition holds for example for a semisimple Lie group.

It almost never happens that there is a $G_{\mathbb{R}}$-invariant Kähler form ω which is defined on Z. For example, if $G_{\mathbb{R}}$ is a simple non compact Lie group or more generally a semisimple Lie group without compact factors, then there does not exist a $G_{\mathbb{R}}$-invariant Kähler form on a non trivial holomorphic G-manifold Z. In order to see this, recall that since $G_{\mathbb{R}}$ is semisimple there is a moment map $\mu : Z \to \mathfrak{g}_{\mathbb{R}}^*$. Now let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup of $G_{\mathbb{R}}$. Then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is the Lie algebra of the maximal compact subgroup U of G. For $\xi \in i\mathfrak{p}$ the image of the one-parameter group $\gamma : t \to \exp it\xi$ lies in U and therefore there is a basis of $\mathfrak{p}$ consisting of ξ's such that the image of γ is compact, i.e., isomorphic to S^1. But γ is the flow of the gradient vector field of μ_ξ and therefore $t \to \mu_\xi(\gamma(t) \cdot z)$ is strictly increasing for every $z \in Z$. This implies that γ acts trivially on Z. Since G is semisimple and contains no compact factor, G itself is the smallest complex subgroup of G which contains $\exp \mathfrak{p}$. Thus G acts trivially on Z.

A geometric interpretation of the zero fibre $\mu^{-1}(0)$ of an equivariant moment map $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$ associated to a smooth $G_{\mathbb{R}}$-invariant strictly plurisubharmonic function $\phi : X \to \mathbb{R}$ (see Section 1, Example) can be given in the case where X is an invariant domain in Z as follows. For $x \in X$ let $\Omega(x) := \{g \cdot x;\ g \in G \text{ and } g \cdot x \in X\}$ be the local G-orbit of G through x in X where $(g, x) \to g \cdot x$ denotes the G-action on Z. Then by $G_{\mathbb{R}}$-invariance of ϕ we have

$$\mu^{-1}(0) = \{x \in X;\ \ x \text{ is a critical point of } \phi|\Omega(x)\}\,.$$

We consider now invariant domains X in G-homogeneous spaces Z such that there is a moment map associated to $\phi : X \to \mathbb{R}$ more closely. In order to do that we first introduce the notion of an exhaustion mod $G_{\mathbb{R}}$.

Let F be a complex space with a proper $G_{\mathbb{R}}$-action and let $F/G_{\mathbb{R}}$ be the space of $G_{\mathbb{R}}$-orbits endowed with the quotient topology. A $G_{\mathbb{R}}$-invariant function $f : F \to \mathbb{R}$ is said to be proper mod $G_{\mathbb{R}}$ if the induced map $\bar{f} : F/G_{\mathbb{R}} \to \mathbb{R}$ is proper. The map f is said to be an exhaustion mod $G_{\mathbb{R}}$ if $\bar{f}$ is an exhaustion, i.e., if for all $r \in \mathbb{R}$ the set $\{q \in F/G_{\mathbb{R}};\ \bar{f}(q) < r\}$ is relatively compact in F. Note that a $G_{\mathbb{R}}$-invariant continuous function which is bounded from below is proper mod $G_{\mathbb{R}}$ if and only if it is an exhaustion mod $G_{\mathbb{R}}$.

LEMMA 2. *Let Z be G-homogeneous and assume that the $G_{\mathbb{R}}$-action on X is proper. Let $\phi : X \to \mathbb{R}$ be a smooth strictly plurisubharmonic $G_{\mathbb{R}}$-invariant function which is an exhaustion mod $G_{\mathbb{R}}$. Then there is a $z_0 \in X$ such that*

$$G_{\mathbb{R}} \cdot z_0 = \mu^{-1}(0) = \{z \in X;\ \ \phi(z) \ \textit{is a minimal value of}\ \phi\}\,.$$

Proof. Since ϕ is plurisubharmonic and an exhaustion mod $G_{\mathbb{R}}$ there is a point $z_0 \in X$ which is a minimum for ϕ. In particular, $\mu^{-1}(0)$ is not empty where μ denotes the moment map associated with ϕ. We have to prove that $\mu^{-1}(0)$ is connected. By Lemma 1, every connected component of the set $M_\phi = \mu^{-1}(0)$ of critical points of ϕ is a $G_{\mathbb{R}}$-orbit. We claim that the $G_{\mathbb{R}}$-orbits are non degenerate in the sense that the Hessian of ϕ in normal directions is positive definite. This is seen as follows.

The vector fields $J\xi_X$, $\xi \in \mathfrak{g}_{\mathbb{R}}$ span the normal space at $x \in M_\phi$ and

$$(J\xi_X)(J\xi_X(\phi)) = \imath_{J\xi_X} d\mu_\xi = \omega(\xi_X, J\xi_X)\,.$$

Hence the Hessian at $x \in M_\phi$ is positive in the normal directions. Since ϕ is proper mod $G_{\mathbb{R}}$ and the gradient vector field of ϕ with respect to the $G_{\mathbb{R}}$-invariant Kähler metric given by $2i\partial\bar{\partial}\phi$ is $G_{\mathbb{R}}$-invariant, Lemma 2 follows from standard arguments in Morse Theory. □

In the situation of Lemma 2 every critical point of ϕ is a minimum and the set of these points is a $G_{\mathbb{R}}$-orbit and coincides with $\mu^{-1}(0)$.

We will now generalize the results in the homogeneous case to spaces Z which possess a geometric G-quotient and X is a weakly orbit connected invariant domain in Z. Here a $G_{\mathbb{R}}$-stable subset X of Z is said to be weakly orbit connected if for every $x \in X$ the local G-orbit $\Omega(x) := \{g \cdot x \in X;\, g \in G\}$ is connected.

Remark 1. A $G_{\mathbb{R}}$-invariant set X in Z is said to be orbit connected if for every $x \in X$ the set $\Omega_x := \{g \in G;\, g \cdot x \in X\}$ is connected. This is a stronger concept then weakly orbit connectedness.

Let Z be a holomorphic G-space such that there is a geometric quotient $\pi : Z \to Z/G$. By this we mean that the orbit space Z/G is a complex space such that the quotient map $\pi : Z \to Z/G$ is holomorphic. Moreover we assume that the structure sheaf of Z/G is the sheaf of invariants, i.e., for an open subset Q of Z/G a function $f : Q \to \mathbb{C}$ is holomorphic if and only if $f \circ \pi : \pi^{-1}(Q) \to \mathbb{C}$ is holomorphic.

Now let $X \subset Z$ be an invariant domain which lies surjectively over Z/G or equivalently such that $Z = G \cdot X$. Assume that $G_{\mathbb{R}}$ acts properly on X and that X is weakly orbit connected. Let $\phi : X \to \mathbb{R}$ be a smooth $G_{\mathbb{R}}$-invariant strictly plurisubharmonic function which is an exhaustion mod $G_{\mathbb{R}}$ along π, i.e., $\pi^{-1}(C) \cap \{x \in X;\, \phi(x) \le r\}/G_{\mathbb{R}} \subset X/G_{\mathbb{R}}$ is compact for every compact subset C in Z/G and $r \in \mathbb{R}$. We set $M_\phi = \mu^{-1}(0)$ where $\mu : X \to \mathfrak{g}_{\mathbb{R}}$ denotes the moment map associated with ϕ.

PROPOSITION 1. *The map $\bar\imath : M_\phi/G_{\mathbb{R}} \to Z/G$ induced by the inclusion $\imath : M_\phi \to Z$ is a homeomorphism. If X is a manifold, then M_ϕ is smooth and*

$$T_x M_\phi = \ker d\mu(x)$$

holds for all $x \in M_\phi$.

Proof. The map $\bar\imath$ is continuous and by Lemma 2 it is also a bijection. We claim that $\bar\imath$ is proper. Since the $G_{\mathbb{R}}$-action on M_ϕ is proper, $M_\phi/G_{\mathbb{R}}$ is a locally compact topological space. Thus properness of $\bar\imath$ implies that $\bar\imath$ is a homeomorphism.

Let (q_n) be a sequence in $M_\phi/G_{\mathbb{R}}$ and x_n a point in M_ϕ which lies over q_n. Assume that $(\pi(x_n)) = (\bar\imath(q_n))$ has a limit in Z/G and let $x_0 \in M_\phi$ be a point which lies over $\lim \pi(x_n)$. If some subsequence of $\phi(x_n)$ goes to infinity, then we may assume $\phi(x_n) > \phi(x_0) + 1$ for all n. Since $\pi : Z \to Z/G$ is an open map, there are $g_n \in G$ such that $\lim g_n \cdot x_n = x_0$ for some subsequence. This is a contradiction since $\phi(x_n) < \phi(g_n \cdot x_n)$ for all n such that $g_n \cdot x_n \in X$. Thus, since ϕ is assumed to be an exhaustion mod $G_{\mathbb{R}}$ along π, there are $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h_n \cdot x_n)$ converges to x_0. This implies that a subsequence of (q_n) converges in M_ϕ. So far we proved that $\bar\imath$ is a homeomorphism.

Assume now that X is smooth. The existence of a geometric quotient implies that the dimension of the G-orbits in Z is constant and therefore this is also true for the $G_{\mathbb{R}}$-orbits in M_ϕ (Lemma 1). Thus M_ϕ is a submanifold of X (Section 1, Lemma). Since $T_x M_\phi$ is a subspace of $\ker d\mu(x)$ and $\ker d\mu(x) = T_x(G_{\mathbb{R}} \cdot x) + T_x(G \cdot x)^\perp$, the claim follows from the obvious dimension count as follows. Let $d := \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot x$ for $x \in M_\phi$. Note that d is the complex dimension of the π-fibres. Thus $\dim_{\mathbb{R}} M_\phi = \dim_{\mathbb{R}} M_\phi/G_{\mathbb{R}} + d = \dim_{\mathbb{R}} Z/G + d = \dim_{\mathbb{R}} T_x(G \cdot x)^\perp + \dim_{\mathbb{R}} G_{\mathbb{R}} \cdot x$ implies that $T_x M_\phi = \ker d\mu(x)$ for all $x \in M_\phi$. $\square$

Remark 2. Without a reference to an embedding into a holomorphic G-space one can show that $\mu^{-1}(0)/G_{\mathbb{R}}$ is a complex space in a natural way (see [A-H-H] and [A]).

If $G_{\mathbb{R}}$ does not act properly on X, then let $\overline{G}_{\mathbb{R}}$ be the closure of $G_{\mathbb{R}}$ in the group I of isometries of the Kähler manifold X. Since the $G_{\mathbb{R}}$-orbits in $M_\phi = \mu^{-1}(0)$ are closed (Lemma 1), it follows that they coincide with the $\overline{G}_{\mathbb{R}}$-orbits. Moreover ϕ is $\overline{G}_{\mathbb{R}}$-invariant and $M_\phi = \bar{\mu}^{-1}(0) =: \overline{M}_\phi$, where $\bar{\mu}$ is the moment map associated with ϕ. Now if one redefines an exhaustion mod $G_{\mathbb{R}}$ along π in terms of sequences in X, then also in this case M_ϕ is smooth and $T_x M_\phi = T_x(G_{\mathbb{R}} \cdot x) \oplus T_x(G \cdot x)^\perp = \ker d\mu(x)$ holds for all $x \in M_\phi$.

Proposition 1 can be generalized to the case where $\phi : X \to \mathbb{R}$ is only assumed to be plurisubharmonic and strictly plurisubharmonic on the fibres. More precisely we have the following consequence which can be thought of as a version of Loeb's minimum principle (see [L]).

Corollary 1. *Let $X \subset Z$ be a weakly orbit connected invariant domain with $\pi(X) = Z$ and $\phi : X \to \mathbb{R}$ a smooth $G_{\mathbb{R}}$-invariant plurisubharmonic function which is an exhaustion mod $G_{\mathbb{R}}$ along π such that the restriction of ϕ to the local G-orbits in X is a strictly plurisubharmonic exhaustion mod $G_{\mathbb{R}}$. If $\pi : Z \to Z/G$ is a holomorphic bundle, then*

(i) $M_\phi = \mu^{-1}(0)$ *is smooth where* $\mu : X \to \mathfrak{g}_{\mathbb{R}}^*$, $\mu_\xi = d\phi(J\xi_X)$,

(ii) $T_x(M_\phi) = \ker d\mu(x)$ *for all* $x \in M_\phi$.

(iii) $M_\phi/\overline{G}_{\mathbb{R}}$ *is homeomorphic to* Z/G *and the function* $\psi : Z/G \to \mathbb{R}$ *which is induced by* $\phi|M_\phi$ *is a smooth plurisubharmonic function.*

Proof. We may assume that $G_{\mathbb{R}}$ acts properly on X and, since the statements are local over Z/G that Z/G is a Stein manifold. Let $\rho : Z \to \mathbb{R}$ be the the pull back of a strictly plurisubharmonic function on Z/G. Then $\phi + \rho$ is $G_{\mathbb{R}}$-invariant, strictly plurisubharmonic and an exhaustion mod $G_{\mathbb{R}}$ on the local G-orbits in X. Since $d\rho(J\xi_X) = 0$ for all $\xi \in \mathfrak{g}_{\mathbb{R}}$, the moment map associated with $\phi + \rho$ is the same as the moment map associated with ϕ. Thus Proposition 1 implies directly (i), (ii) and the first part of (iii). It remains to show that $\psi : Z/G \to \mathbb{R}$ is a smooth plurisubharmonic function.

For the plurisubharmonicity of ψ we recall the calculation in [H-H-L], §2. For $z \in M_\phi$ we have $T_z(M_\phi) = \ker d\mu(z) = T_z(G_{\mathbb{R}} \cdot z) \oplus T_z(G \cdot z)^\perp$. We may assume that $Z = G/H \times \Delta$ where Δ is an open neighborhood of 0 in $\mathbb{C}^d \cong T_z(G \cdot z)^\perp$, and $\pi(z) = 0$ where π is given by the projection on the second factor. Furthermore there is a section $\eta : \Delta \to M_\phi$, $\eta(w) = (\sigma(w), w)$ and therefore we have $\psi(w) = \phi(\eta(w))$. A direct calculation shows that

$$\partial\bar{\partial}\,\psi(0) = \partial\bar{\partial}\,\phi(\eta(0))\,.$$

Here one has to use that $d\phi(z) = 0$ and that $d\sigma(0) = 0$. Thus ψ is plurisubharmonic and smooth. □

If ϕ is strictly plurisubharmonic, then the proof shows that ψ is also strictly plurisubharmonic. For a proper $G_{\mathbb{R}}$-action the space Z/G is then given by symplectic reduction $M_\phi/G_{\mathbb{R}}$ and the induced Kählerian structure on Z/G is determined by the function $\psi(q) = \inf\limits_{x \in \pi^{-1}(q) \cap X} \phi(x)$ which is obtained by applying the minimum principle ([L]). Thus symplectic reduction and the minimum principle are compatible procedures.

For the remainder of this section we assume now that Z is a holomorphic G-manifold such that there is almost a quotient $Z/\!\!/G$. More precisely we will assume that $Z/\!\!/G$ is a complex space, $\pi : Z \to Z/\!\!/G$ is a surjective G-invariant holomorphic map and there is an analytically Zariski open π-saturated subset Z^0 of Z such that $\pi : Z^0 \to Z^0/\!\!/G$ is a geometric quotient, i.e., $Z^0/\!\!/G = Z^0/G$. Moreover, for the sake of simplicity we assume that $\pi : Z^0 \to Z^0/\!\!/G$ is a holomorphic fibre bundle.

Now let X be an invariant domain in Z with $\pi(X) = Z$ and assume that $X^0 := X \cap Z^0$ is weakly orbit connected. Let ϕ be a $G_{\mathbb{R}}$-invariant plurisubharmonic function such that $\phi^0 := \phi|X^0$ is smooth, strictly plurisubharmonic on the local G-orbits in X^0 and an exhaustion mod $G_{\mathbb{R}}$ along $\pi|Z^0$. Thus the restriction $\phi^0 := \phi|M^0_\phi$, $M^0_\phi := M_\phi \cap Z^0$ induces a plurisubharmonic function $\psi^0 : Z^0/\!\!/G \to \mathbb{R}$.

Lemma 3. *There is a unique G-invariant plurisubharmonic function $\Psi : Z \to [-\infty, +\infty)$ which extends $\Psi^0 := \psi^0 \circ \pi|Z^0$.*

Proof. The function $\Psi(z) = \inf_{g \in \Omega_z} \phi(g \cdot z)$ is upper semi-continuous on Z where $\Omega_z := \{g \in G;\, g \cdot z \in X\}$. Now $\Psi = \Psi^0$ on Z^0 (Lemma 2), and $Z \setminus Z^0$ is a proper analytic subset of Z. Thus Ψ is plurisubharmonic and by definition G-invariant. □

Remark 3. If $Z/\!\!/G$ is smooth and π is an open map, then ψ^0 extends uniquely to a plurisubharmonic function ψ on $Z/\!\!/G$. Of course in this case we have $\psi(q) = \inf_{x \in F_q} \phi(x)$, where $F_q := \pi^{-1}(q) \cap X$. If $\phi|F_q$ is an exhaustion mod $G_{\mathbb{R}}$, M_ϕ intersects every $G_{\mathbb{R}}$-stable closed analytic subset of F_q non trivially. But it might happen that $M_\phi \cap F_q$ is a union of several $G_{\mathbb{R}}$-orbits. On the other hand for $q \in Z^0/\!\!/G$ the intersection is exactly one $G_{\mathbb{R}}$-orbit.

Assume now in addition that Z is an open G-stable subspace of a holomorphic Stein G-manifold V which is saturated with respect to $\pi : V \to V/\!\!/G$. We say that $\phi : X \to \mathbb{R}$ is a weak exhaustion of X over $V/\!\!/G$ if $\limsup \phi(z_n) = +\infty$ for any sequence (z_n) in X such that $(\pi(z_n))$ converges to some q_0 in the boundary $\partial(Z/\!\!/G)$ in $V/\!\!/G$.

Theorem. *Let Z be a G-stable π-saturated open subspace of V, X an invariant domain in Z with $G \cdot X = Z$ and $\phi : X \to \mathbb{R}$ a $G_{\mathbb{R}}$-invariant plurisubharmonic function. Assume that*

(i) *X^0 is weakly orbit connected,*

(ii) *the restriction of $\phi^0 := \phi|X^0$ to the local G-orbits is strictly plurisubharmonic,*

(iii) *ϕ^0 is an exhaustion mod $G_{\mathbb{R}}$ along $\pi|Z^0$ and*

(iv) *ϕ is a weak exhaustion of X over $V/\!\!/G$,*

Then $Z = G \cdot X$ is a Stein manifold.

Proof. Let $z_0 \in \partial Z$ and $z_n \in Z$ be such that $z_0 = \lim z_n$. We have to show $\limsup \Psi(z_n) = +\infty$. Thus assume that $\Psi(z_n) < r$ for all n and some $r \in \mathbb{R}$.

There are $w_n \in G \cdot M_\phi^0 = Z^0$ such that $\Psi(w_n) < r$ and $z_0 = \lim w_n$. Let $w_n = g_n \cdot x_n$ where $g_n \in G$ and $x_n \in M_\phi^0$. Now $\Psi(w_n) = \Psi(x_n) = \phi(x_n) < r$ and, since $Z = G \cdot X$ is saturated, $\pi(x_n) = \pi(w_n) \to \pi(z_0) \in \partial(Z/\!\!/G)$. This contradicts the assumption that ϕ is a weak exhaustion. Thus Z is a domain in a Stein manifold with a plurisubharmonic weak exhaustion function and therefore Stein. $\square$

Remark 4. Elementary examples show that for a Stein $G_{\mathbb{R}}$-manifold some conditions are necessary in order that $G \cdot X$ is a Stein manifold. For example there is an $\mathrm{Sl}_2(\mathbb{R})$-invariant domain Ω of holomorphy in $\mathbb{C}^2$ such that $\mathrm{Sl}_2(\mathbb{C}) \cdot \Omega = \mathbb{C}^2 \setminus \{0\}$.

Now let G be complex reductive group and assume that the semistable quotient $\pi : Z \to Z/\!\!/G$ exists (see [H-M-P]). Thus $Z/\!\!/G$ is a complex space whose structure sheaf $\mathcal{O}_{Z/\!\!/G}(U) = \mathcal{O}_Z(\pi^{-1}(U)^G$ is the sheaf of invariants and every point in $Z/\!\!/G$ has an open Stein neighborhood such that the inverse image in Z is Stein. For example, if V is a holomorphic Stein G-manifold, then a semistable quotient $V/\!\!/G$ alway exists. Moreover it is shown in [H-M-P] that Z is a Stein space if and only if $Z/\!\!/G$ is a Stein space.

Assume that Z is connected and that some orbit of maximal dimension is closed. Then there exists a proper analytic subset A in $Z/\!\!/G$ such that $Z^o/\!\!/G = Z/\!\!/G \setminus A$ is a geometric quotient of $Z^o := \pi^{-1}(Z/\!\!/G \setminus A)$. In particular, every fibre of $\pi|Z^o$ is G-homogeneous or equivalently the dimension of the G-orbits in Z^o is constant. Every $x \in Z^o$ has a G-stable neighborhood U which is G-equivariantly biholomorphic to $G \times_H S$ where H is the isotropy group of G at x and S is a Stein space such that the connected component H^0 of the identity of H acts trivially on S. Here $G \times_H S$ denotes the bundle associated to the H-principal bundle $G \to G/H$. Thus locally $Z^o/\!\!/G$ is given by S/Γ where $\Gamma := H/H_0$ is a finite group. Moreover, there is an analytically Zariski open G-stable subset Z^{oo} of Z which is contained in Z^o such that the isotropy type is constant. This implies that Z^{oo} is a fibre bundle over $Z^{oo}/\!\!/G \subset Z/\!\!/G$.

3. Orbit geometry of the future tube.

In the following it will be convenient to introduce a linear coordinate change such that $< z, z >= (z_0)^2 - (z_1)^2 - (z_2)^2 - (z_3)^2$ has the form $z_0 z_1 - z_2 z_3$. Thus we set

$$Z := \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}$$

and obtain $\det Z =< z, z >$ and $\det \operatorname{Im} Z =< \operatorname{Im} z, \operatorname{Im} z >$ where $\operatorname{Im} Z := \frac{1}{2i}(Z - \bar{Z}^t)$.

Let $H := \{Z \in V;\ \operatorname{Im} Z > 0\}$ denote the generalized upper half plane where $V := \mathbb{C}^{2\times 2}$. Note that H is just the tube over the positive light cone in the new coordinates. Moreover H is stable with respect to the action of $G_{\mathbb{R}} := \mathrm{SL}_2(\mathbb{C})$ which is given by $G_{\mathbb{R}} \times H \to H,\ (g, Z) \to g * Z := gZ\bar{g}^t$. This action is not effective. The ineffectivity consists of $\Gamma = \{+I, -I\}$ and the quotient $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is the connected component of the identity of the homogeneous Lorentz group.

Let $H^N := H \times \cdots \times H \subset V \times \cdots \times V =: V^N$ denote the N-fold product of H and set $G := (G_{\mathbb{R}})^{\mathbb{C}} = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ where $G_{\mathbb{R}}$ is embedded in G via $g \to (g, \bar{g})$. The diagonal $G_{\mathbb{R}}$ action on V^N extends to a holomorphic G action $G \times V^N \to V^N,\ ((g, h), Z^1, \ldots, Z^N) \to (g, h) * (Z^1, \ldots, Z^N) := (gZ^1h^t, \ldots, gZ^Nh^t)$.

Theorem. *The extended future tube* $(H^N)^{\mathbb{C}} := G * H^N$ *is a domain of holomorphy.*

In the proof we will make an axiomatic use of the following statements

Fact 1 (see Streater Wightman [S-W], p. 66). The set H^N is orbit connected in V^N, i.e., $\{g \in G;\ g * Z \in H^N\}$ is connected for every $Z \in V^N$.

Fact 2. The extended future tube $G * H^N$ is saturated with respect to $\pi : V^N \to V^N /\!\!/ G$.

Fact 2 implies that the semistable quotient $G * H^N /\!\!/ G$ exists and is an open subset of $V^N /\!\!/ G$. The quotient map is given by restricting $\pi : V^N \to V^N /\!\!/ G$ to $G * H^N$.

There does not seem to be a proof in the literature of Fact 2 but there is a detailed proof for the whole complex orthogonal group in [H-W]. A slight modification of the proof there can be used for a proof of Fact 2. In order to be complete let us recall briefly the main steps. First we note that it is sufficient to show the following (see e.g. [H]).

Claim. If $Z \in H^N$, then the unique closed orbit $G * W$ in the closure of $\overline{G * Z}$ lies in $G * H^N$.

This can be seen as follows. Let $<\ ,\ >$ be the complex Lorenz product, i.e., the symmetric bilinear form on V which is associated to the quadratic form $\det : V \to V$. Thus V is just the standard representation of $\tilde{G} := \mathrm{O}_4(\mathbb{C})$. Note that $\tilde{G}$ has two connected components and the connected component of the identity is G. The functions $(Z^1, \ldots, Z^N) \to < Z^i, Z^j >$, form a set of generators for the algebra of the $\tilde{G}$-invariant polynomials on V^N. Thus the image of V^N in the set of symmetric $N \times N$-matrices of the map $\tilde{\pi}$ which sends $(Z^1, \ldots, Z^N)$ to the matrix $(< Z^i, Z^j >)$ is an affine variety which is isomorphic to $V^N /\!\!/ \tilde{G}$.

The matrices of rank 3 or 4 correspond to fibres of $\tilde{\pi}$ which are closed $\tilde{G}$-orbits. It follows that the G-orbit through every point $Z \in H^N$ such that the rank r of $\tilde{Z}$ is greater or equal to 3 is already closed. Now assume that $r \leq 2$. In this case the following is shown in [H-W]: There exists an $g \in \tilde{G}$, $\alpha_j \in \mathbb{C}$ and an $\omega \in V$ with $< \omega, \omega >= 0 =< \omega, W^j >$ such that

$$Z^j = g * W^j + \alpha_j \omega\ ,\ j = 1, \ldots, N\,.$$

The proof actually shows that one can choose $g \in G$, i.e., $\det g = 1$. Now an argument of Hall-Wightman ([H-W], p.21) implies that $g * W^j \in H$ for all j, i.e., $G * W \subset G * H^N$.

Fact 3. The function $\phi : H^N \to \mathbb{R}$, $\phi(Z^1, \ldots, Z^N) := \frac{1}{\det \operatorname{Im} Z^1} + \cdots + \frac{1}{\det \operatorname{Im} Z^N}$ is $G_{\mathbb{R}}$-invariant and strictly plurisubharmonic. Moreover, ϕ is a weak exhaustion of H^N.

The simplest way to see that ϕ is strictly plurisubharmonic is to note that $Z^j \to \frac{1}{\det \operatorname{Im} Z^j}$ it is given by the Bergmann kernel function on H. Since $\det \operatorname{Im} Z = 0$ for $Z \in \partial H$, ϕ is a weak exhaustion of H^N, i.e., $\phi(Z_k) \to +\infty$ if $\lim Z_k = Z_0 \in \partial(H^N) \subset V^N$.

Let $K_{\mathbb{R}} := \{(a, \bar{a});\ a \in \mathrm{SU}_2(\mathbb{C})\}$ be the maximal compact subgroup of $G_{\mathbb{R}}$. We set $V^0 := \{Z \in V;\ \det Z \neq 0\}$. Note that $V /\!\!/ G \cong \mathbb{C}$ and that after this identification the quotient map is given by $\det : V \to \mathbb{C}$. In particular, V^0 is saturated with respect to $V \to V/\!\!/G$.

LEMMA 1. *Let (W_n) be a sequence in H such that $(\pi(W_n))$ converges in $V/\!\!/G$. Then there exist $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h_n * W_n)$ converges in V.*

Proof. There exist $u_n \in K_{\mathbb{R}}$ such that

$$X_n := u_n * W_n =: \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}.$$

Since $(\pi(W_n)) = (\pi(X_n))$ converges, it follows that $|\det X_n| = |x_n y_n| \le R$ for some $R \ge 0$ and all n. Furthermore, $X_n \in H$ implies that $\frac{1}{4}|z_n|^2 < \operatorname{Im} x_n \operatorname{Im} y_n \le |x_n y_n| = |\det X_n|$. Therefore (z_n) is bounded. Now $0 < |x_n y_n| \le R$ implies that $|r_n^2 x_n| = |r_n^{-2} y_n|$ for some $r_n > 0$. In particular the sequence $(r_n^2 x_n, r_n^{-2} y_n)$ is bounded. Hence $h_n * W_n$ has a convergent subsequence where $h_n := r_n \cdot u_n \in G_{\mathbb{R}}$ and r_n is identified with $\left(\begin{pmatrix} r_n & 0 \\ 0 & \frac{1}{r_n} \end{pmatrix}, \begin{pmatrix} r_n & 0 \\ 0 & \frac{1}{r_n} \end{pmatrix}\right)$. □

Remark 1. Geometrically Lemma 1 asserts that H is relatively compact over $V/\!\!/G$ mod $G_{\mathbb{R}}$.

LEMMA 2. *Let (Z_n, W_n) be a sequence of points in $H \times H$ and assume that*

(i) *$\pi(Z_n, W_n)$ converges in $(V \times V)/\!\!/G$ and*

(ii) *$W_0 = \lim W_n$ exists in H.*

Then a subsequence of (Z_n) converges to a $Z_0 \in \overline{H}$.

Proof. Note that $V \times V^0$ is an open G-stable subset of $V \times V$ which is saturated with respect to $V \times V \to (V \times V)/\!\!/G$ and contains $H \times H$. The map $V \times V^0 \to V$, $(Z, W) \to ZW^{-1}$, is G-equivariant, where G acts on the image V by conjugation with the first component, i.e. by $\operatorname{int}(g, h) \cdot X = gXg^{-1}$. It is sufficient to show the following

Claim. A subsequence of (X_n) converges.

Since the image of $X_n := Z_n W_n^{-1}$ in $V/\!\!/\operatorname{int} G$ converges, the trace and the determinant of X_n and therefore the eigenvalues of X_n are bounded. Let $u_n = (a_n, \bar{a}_n) \in K_{\mathbb{R}}$ be such that $\operatorname{int} a_n \cdot X_n = (u_n * Z_n)(u_n * W_n)^{-1} = \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}$. Since $K_{\mathbb{R}}$ is compact, we may assume that $X_n = \begin{pmatrix} x_n & z_n \\ 0 & y_n \end{pmatrix}$.

Let $W_n =: \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ and $W_0 =: \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. By assumption we have $W_0 \in H$. Therefore $\operatorname{Im} d_0 \neq 0$. From

$$Z_n = X_n W_n = \begin{pmatrix} x_n a_n + z_n c_n & x_n b_n + z_n d_n \\ y_n c_n & y_n d_n \end{pmatrix} \in H$$

it follows that

$$\frac{1}{|z_n|^2}\big(\operatorname{Im}(x_n a_n + z_n c_n)\operatorname{Im}(y_n d_n) - \frac{1}{4}|x_n b_n + z_n d_n - \bar{y}_n \bar{c}_n|^2\big) > 0$$

for $z_n \neq 0$. Since the eigenvalues x_n, y_n and a_n, b_n, c_n, d_n are bounded, $d_0 \neq 0$ implies that $|z_n|$ is bounded. Thus (X_n) has a convergent subsequence. $\square$

Remark 2. The proofs of Lemma 1 and Lemma 2 use arguments which can be found at least implicitly in [Z] on p. 17.

In the above proof we used that $H \subset V^0$ which is implied by $\det \operatorname{Im} Z \leq |\det Z|$.

Corollary 1. *If $Z_n = (Z_n^1, \ldots, Z_n^N) \in H^N$ are such that $(\pi(Z_n))$ converges in $V^N/\!/G$ and (Z_n^N) converges in H, then (Z_n) has a convergent subsequence in $\overline{H}^N$.* $\square$

Lemma 3. *ϕ is a weak exhaustion of X over $V/\!/G$.*

Proof. Let $(Z_n) = ((Z_n^1, \ldots, Z_n^N))$ be a sequence in H^N such that $q := \lim \pi(Z_n) \in \partial(G{*}H^N/\!/G) \subset V^N/\!/G$ exists. There are $h_n \in G_{\mathbb{R}}$ such that a subsequence of $(h{*}Z_n^N)$ converges to $W^N \in \overline{H}$ (Lemma 1). Now, if $W^N \in \partial H$, then $\limsup \phi(Z_n) = +\infty$. Thus assume that $W^N \in H$. It follows that $(h_n * Z_n)$ has a subsequence which converges to $W \in \overline{H}^N$ (Corollary 1). But W is not in H^N, since $q = \pi(W) \in \partial(G * H^N/\!/G)$. Thus $W \in \partial H^N$ and therefore again $\limsup \phi(Z_n) = +\infty$ follows. $\square$

Lemma 4. *The function ϕ is an exhaustion mod $G_{\mathbb{R}}$ along π.*

Proof. For $r > 0$ let $Z_n \in H^N$, $Z_n =: (Z_n^1, \ldots, Z_n^N)$, be such that $\phi(Z_n) \leq r$ and assume that $\lim \pi(Z_n)$ exists in $G * H^N/\!/G$. Thus there are $h_n \in G_{\mathbb{R}}$ such that $(h_n * Z_n^N)$ has a subsequence which converges to some $W^N \in \overline{H}$. If $W^N \in \partial H$, then $\phi(Z_n)$ goes to infinity. This contradicts $\phi(h_n * Z_n) \leq r$. Thus $W^N \in H$ and therefore $(h_n * Z_n)$ has a subsequence with limit $W = (W^1, \ldots, W^N) \in \overline{H}^N$. The same argument as above implies that $W^j \in H$ for $j = 1, \ldots, N$. $\square$

Proof of the Theorem. From the invariant theoretical point of view the $G = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ action on V^N is the N-fold product of the standard representation of $\mathrm{SO}_4(\mathbb{C})$ on $\mathbb{C}^4$. It is well known that for any $N = 1, 2, \ldots$ the generic G-orbit in V^N is closed. Let $(V^N)^0$ denote the set of points in V^N which lie in a generic closed orbit, i.e., $(V^N)^0$ is a union of the fibres of the quotient $V \to V/\!/G$ which consist exactly of one G-orbit. Since the $G_{\mathbb{R}}$-action on H is proper, $G_{\mathbb{R}}$ acts properly on H^N. It follows from the results in §2 that there is a G-invariant plurisubharmonic function Ψ on $G * H^N$ which is a weak exhaustion. Thus $G * H^N$ is a domain of holomorphy. $\square$

Corollary 2. *The image $G * H^N/\!/G$ of H^N in $V^N/\!/G$ is an open Stein subspace.* $\square$

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Simple Models of Quasihomogeneous Projective 3-Folds

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Abstract. Let X be a projective complex 3-fold, quasihomogeneous with respect to an action of a linear algebraic group. We show that X is a compactification of SL_2/Γ, Γ a finite subgroup, or that X can be equivariantly transformed into $\mathbb{P}_3$, the quadric $\mathbb{Q}_3$, or into certain quasihomogeneous bundles with very simple structure.

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1 Introduction

Call a variety X quasihomogeneous if there is a connected algebraic group G acting algebraically on X with an open orbit. A rational map $X \dashrightarrow Y$ is said to be equivariant if G acts on Y and if the graph is stable under the induced action on $X \times Y$.

The class of varieties having an equivariant birational map to X is generally much smaller then the full birational equivalence class. The minimal rational surfaces are good examples: they are all quasihomogeneous with respect to an action of SL_2, but no two have an SL_2-equivariant birational map between them. On the other hand, if X is any rational SL_2-surface, then the map to a minimal model is always equivariant.

Generally, one may ask for a list of (minimal) varieties such that every quasihomogeneous X has an equivariant birational map to a variety in this list.

We give an answer for $\dim X = 3$ and G linear algebraic:

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Theorem 1.1. *Let X be a 3-dimensional projective complex variety. Let G be a connected linear algebraic group acting algebraically and almost transitively on X. Assume that the ineffectivity, i.e. the kernel of the map $G \to \mathrm{Aut}(X)$, is finite. Then either $G \cong SL_2$, and X is a compactification of SL_2/Γ, where Γ is finite and not cyclic, or there exists an equivariant birational map $X \dashrightarrow^{eq} Z$, where Z is one of the following:*

- *$\mathbb{P}_3$ or $\mathbb{Q}_3$, the 3-dimensional quadric*
- *a $\mathbb{P}_2$-bundle over $\mathbb{P}_1$ of the form $\mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$.*
- *a linear $\mathbb{P}_1$-bundle over a smooth quasihomogeneous surface Y, i.e. $Z \cong \mathbb{P}(E)$, where E is a rank-2 vector bundle over Y. If G is solvable, then E can be chosen to be split.*

If G is not solvable, then the map $X \dashrightarrow^{eq} Z$ factors into a sequence $X \leftarrow \tilde{X} \to Z$, where the arrows denote sequences of equivariant blow ups with smooth center.

A fine classification of the (relatively) minimal varieties involving SL_2 will be given in a forthcoming paper.

The result presented here is contained the author's thesis. The author would like to thank his advisor, Prof. Huckleberry, and Prof. Peternell for support and valuable discussions.

2 Existence of Extremal Contractions

The main tool we will use is Mori-theory. In order to utilize it, we show that in our context extremal contractions always exist.

Lemma 2.1. *Let X and G be as in 1.1, but allow for $\mathbb{Q}$-factorial terminal singularities. Then there exists a* Mori*-contraction.*

Proof. Let $\pi : \tilde{X} \to X$ be an equivariant resolution of the singularities of X, let $H < G$ be a (linear) algebraic subgroup and let $v_1 \in Lie(G)$ be the associated element of the Lie-algebra. Since $\tilde{X}$ is quasihomogeneous, we can find elements $v_2, v_3 \in Lie(G)$ such that the associated vector fields

$$\tilde{v_i}(x) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv_i)x \quad \in H^0(\tilde{X}, T\tilde{X})$$

are linearly independent at generic points of $\tilde{X}$. In other words,

$$\sigma := \tilde{v_1} \wedge \tilde{v_2} \wedge \tilde{v_3}$$

is a non-trivial holomorphic section of the anticanonical bundle $-K_{\tilde{X}}$. Because H is linear algebraic, the closure of a generic H-orbit is a rational curve, and H has a fixed point on this curve. Therefore $\tilde{v_1}$ has zeros, and the divisor given as the zero-set of σ is not trivial. In effect, we have shown that $-K_{\tilde{X}}$ is effective and not trivial.

If r is the index of X, then the line bundle $-rK_X$ is effective. We are finished if we exclude the possibility that $-rK_X$ is trivial. Assume that this is the case. The section σ not vanishing on the smooth points of X implies that $X \setminus \mathrm{Sing}(X)$ is G-homogeneous. But the terminal singularities are isolated. Thus, by [HO80, thm. 1 on p. 113], X is a cone over a rational homogeneous surface, a contradiction to $-rK_X$ trivial.

Consequently $-rK_X$ is effective and not trivial. So there is always a curve C intersecting an element of $|-rK_X|$ transversally. Hence $C.K_X < 0$ and there must be an extremal contraction. □

COROLLARY 2.2. *Let X and G be as in theorem 1.1 with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Let $\phi : X \to Y$ be an equivariant morphism with* $\dim Y < 3$. *Then there is a relative contraction over Y.*

Proof. If Y is a point, this follows directly from lemma 2.1. Otherwise, if $\eta \in Y$ generic, we know that the fiber X_η is smooth, does not intersect the singular set and is quasihomogeneous with respect to the isotropy group G_η. So there exists a curve $C \subset X_\eta$ with $C.K_{X_\eta} < 0$. Note that the adjunction formula holds, since X has isolated singularities and X_η does not intersect the singular set. Hence $K_{X_\eta} = K_X|_{X_\eta}$, and there must be an extremal ray $C \subset \overline{NE(X)}$ such that $\phi_*(C) = 0$. Thus, there exists a relative contraction. □

Recall that all the steps of the MORI minimal model program (i.e. extremal contractions and flips) can be performed in an equivariant way. For details, see [Keb96, chap. 3].

3 EQUIVARIANT RATIONAL FIBRATIONS

In this section we employ group-theoretical considerations in order to find equivariant rational maps from X to varieties of lower dimension. These will later be used to direct the minimal model program.

We start with the case that G is solvable.

LEMMA 3.1. *Let X and G be as in 1.1. Assume additionally that G is solvable. Then there exists an equivariant rational map $X \dashrightarrow^{eq} Y$ to a projective surface Y.*

Proof. Since G is solvable, there exists a one-dimensional algebraic normal subgroup N. Let H be the isotropy group of a generic point, so that $\Omega \cong G/H$, and consider the map

$$\Omega \cong G/H \to G/(N.H)$$

Recall that $N.H$ is algebraic. Since N is not contained in H (or else G acted with positive dimensional ineffectivity), the map has one-dimensional fibers. Now $\dim G/(N.H) > 0$ and $G/(N.H)$ can always be equivariantly compactified to a projective variety Y. This yields an equivariant rational map $X \dashrightarrow^{eq} Y$. □

Now consider the cases where G is not solvable.

LEMMA 3.2. *Let X and G be as above. Assume that G is neither reductive nor solvable. Then there exists an equivariant rational map $X \dashrightarrow^{eq} Y$ such that either*

1. *$Y \cong \mathbb{P}_3$, and $X \dashrightarrow^{eq} Y$ is birational, or* $\dim Y = 2$, *or*

2. $\dim Y = 1$*, and there exists a normal unipotent group* A *and a semisimple group* $S < G$*, acting trivially on* Y*. The unipotent part* A *acts almost transitively on generic fibers.*

Proof. Let $G = U \rtimes L$ be the LEVI decomposition of G, i.e. U is unipotent and L reductive and define A to be the center of U. Note that A is non-trivial. Since A is canonically defined, it is normalized by L, hence it is normal in G. Let H be the isotropy group of a generic point, Ω the open G-orbit, so that $\Omega \cong G/H$, and consider the map

$$\Omega \cong G/H \to G/(A.H)$$

There are two things to note. The first is that A is not contained in H (or else G acted with positive dimensional ineffectivity). So $\dim G/(A.H) < 3$. If $\dim G/(A.H) > 0$, it can always be equivariantly compactified $G/(A.H)$ to a variety Y yielding an equivariant rational map $X \dashrightarrow Y$. If $\dim G/(A.H) = 2$, we can stop here. If $\dim G/(A.H) = 1$, then note that A acts transitively on the fiber $A.H/H$. If $A.H$ does not contain a semi-simple group, we argue as in lemma 3.1 to find a subgroup H', $H < H' < A.H$ such that $\dim H'/H = 1$. Then $\dim G/H' = 2$, and again we are finished.

If $\dim G/(A.H) = 0$, then A acts transitively on Ω. In this case $A \cong \mathbb{C}^n$, and hence (because the G-action is algebraic) $\Omega \cong \mathbb{C}^3$. The theorem on MOSTOW fibration (see e.g. [Hei91, p. 641]) yields that L has to have a fixed point in Ω. Therefore, without loss of generality, $L < H$. As a next step, consider the group $B := (U \cap H)^0$. Since both U and H are normalized by L, B is as well. Elements in A commute with all elements of U, hence $A.B$ normalizes B as well. Then B is a normal subgroup of $U \rtimes L = G$. Note that $A.B = U$, because $A.B = A.(H \cap U) = (A.H) \cap U = G \cap U = U$. Consequently B acts trivially. Therefore $B = \{e\}$.

We are now in a position where we may write $G = A \rtimes_\rho L$, where ρ is the action of L on A (L acting by conjugation). Now $H = L$, hence $A \cong \Omega \cong \mathbb{C}^3$ and the L-action on $A \cong (\mathbb{C}^3, +)$ is linear. So G is a subgroup of the affine group and Ω can be equivariantly compactified to $\mathbb{P}_3$, yielding an equivariant rational map $X \dashrightarrow^{eq} \mathbb{P}_3$. □

We study case (1) of the preceding proposition in more detail.

LEMMA 3.3. *Let* X *be as above and assume that* G *is reductive. Assume furthermore that* G *is not semisimple. Then there is an equivariant rational map* $X \dashrightarrow^{eq} Z$*, where* $\dim Z = 2$.

Proof. As a first step, recall that $G = T.S$, where S is semisimple, T is a torus, and S and T commute and have only finite intersection. If η is a point in the open orbit and G_η the associated isotropy group, then $T \not\subset G_\eta$, or otherwise T would not act at all. For that reason we will be able to find a 1-parameter group $T_1 < T$, $T_1 \not\subset G_\eta$ and consider the map

$$\Omega := G/G_\eta \to G/(T_1.G_\eta).$$

Since T_1 has non-trivial orbits, $\dim G/(T_1.G_\eta) = 2$. If we compactify the latter in an equivariant way to a variety Z, we automatically obtain an equivariant rational map $X \dashrightarrow^{eq} Z$ as claimed. □

LEMMA 3.4. *Suppose G is semisimple. Then one of the following holds:*

1. *$G \cong SL_2$ and the open orbit Ω is isomorphic to SL_2/Γ, where Γ is finite and not contained in a* BOREL *subgroup.*
2. *$X \cong \mathbb{P}_3$*
3. *X is isomorphic to $F_{1,2}(3)$, the full flag variety*
4. *X is homogeneous and either $X \cong \mathbb{Q}_3$, the 3-dimensional quadric or X is a direct product involving only $\mathbb{P}_1$ and $\mathbb{P}_2$.*
5. *X admits an equivariant rational map $X \dashrightarrow^{eq} Y$ onto a surface.*

Proof. If $G \cong SL_2$, and Γ is embeddable into a BOREL group B, then Γ is in fact embeddable into a 1-dimensional torus T. Consider the map $G/\Gamma \to G/T$, and we are finished.

Assume for the rest of this proof that $G \not\cong SL_2$. Then the claim is already true in the complex analytic category: see [Win95, p. 3]. One must exclude torus bundles by the fact that they never allow an algebraic action of a *linear* algebraic group. □

We summarize a partial result:

COROLLARY 3.5. *Let X and G be as above. If there exists an equivariant map $X \dashrightarrow^{eq} \mathbb{P}_1$ and no such map to $\mathbb{P}_3$ or to a surface, then G is not solvable and there exist subgroups S and A as in lemma 3.2.*

4 THE CASE THAT Y IS A CURVE

In this section we investigate relatively minimal models over $\mathbb{P}_1$. The main proposition is:

PROPOSITION 4.1. *Let X and G be as in 1.1 with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Assume that $\phi : X \to \mathbb{P}_1$ is an extremal contraction. Assume additionally that there does not exist an equivariant rational map $X \dashrightarrow^{eq} Y$, where $\dim Y = 2$ or $Y \cong \mathbb{P}_3$. Then*

$$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}_1}(e) \oplus \mathcal{O}_{\mathbb{P}_1}(e) \oplus \mathcal{O}_{\mathbb{P}_1}),$$

with $e > 0$. In particular, X is smooth.

Proof. As a first step, we show that the generic fiber X_η is isomorphic to $\mathbb{P}_2$. As ϕ is a MORI-contraction, X_η is a smooth FANO surface. By corollary 3.5, the stabilizer $G_\eta < G$ of X_η contains a unipotent group A acting almost transitively on X_η and a semisimple part S. This already rules out all FANO surfaces other than $\mathbb{P}_2$. Furthermore, $S \cong SL_2$. Note that G_η stabilizes a unique line $L \subset X_\eta$ and that S acts transitively on L.

Set $D' := \overline{G.L}$ and remark that D' intersects the generic ϕ-fiber in the unique G_η-stable line: $D' \cap X_\eta = L$. We claim that D' is CARTIER. The desingularization $\tilde{D}'$ has a map to $\mathbb{P}_1$, the generic fiber is isomorphic to $\mathbb{P}_1$ and S acts non-trivially on all the fibers. Thus, $\tilde{D}'$ is isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$, and S does not have a fixed point on D'. Consequently, $\tilde{D}'$ does not intersect the singular set of X and is CARTIER.

Take D'' to be an ample divisor on Y. As ϕ is a Mori-contraction, the line bundle L associated to $D := D' + n\phi^*(D'')$, $n >> 0$, is ample on X. In this setting, a theorem of Fujita (cf. [BS95, Prop. 3.2.1]) yields that X is of the form $\mathbb{P}(E)$, where E is a vector bundle on $\mathbb{P}_1$.

The transition functions of E must commute with S, but the only matrices commuting with SL_2 are $Diag(\lambda, \lambda, \mu)$, hence $E = \mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O}(f)$ and $X \cong \mathbb{P}(\mathcal{O}(e-f) \oplus \mathcal{O}(e-f) \oplus \mathcal{O})$. □

For future use, we note

Lemma 4.2. *Let X and G be as in proposition 4.1. Then, by equivariantly blowing up and down, $X \dashrightarrow^{eq} \mathbb{P}(\mathcal{O}(e') \oplus \mathcal{O}(e') \oplus \mathcal{O})$ where the latter does not contain a G-fixed point.*

Proof. The semisimple group S fixes a unique point of each ϕ-fiber, so that there exists a curve C of S-fixed points. Suppose that G has a fixed point f. Then $f \in C$, and we can perform an elementary transformation $X \dashrightarrow^{eq} X'$ with center f, i.e. if X_μ is the ϕ-fiber containing f, then we blow up f and blow down the strict transform of the X_μ, again obtaining a linear $\mathbb{P}_2$-bundle of type $\mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$. This transformation exists, as has been shown in [Mar73]. Since all the centers of the blow-up and -down are G-stable, the transformation is equivariant.

We will use this transformation in order to remove G-fixed points. Let $g \in G$ be an element not stabilizing C. The curves gC and C meet in f. We know that after finitely many blow-ups of the intersection points of C and gC, the curves become disjoint, so that there no longer exists a G-fixed point! This, however, is exactly what we do when applying the elementary transformation. □

5 The case that Y is a surface

The cases that G is solvable or not solvable are in many respects quite different. Here we have to treat them separately.

5.1 The case G solvable

We will show that in this situation the open G-orbit can be compactified in a particularly simple way.

Proposition 5.1. *Let X and G be as in theorem 1.1. Assume additionally that G is solvable and $\phi : X \to Y$ is an equivariant map with connected fibers onto a smooth surface. Then there exists a splitting rank-2 vector bundle E on Y and an equivariant birational map $X \dashrightarrow^{eq} \mathbb{P}(E)$.*

We remark that if $y \in Y$ is contained in the open G-orbit, then it's preimage is quasihomogeneous with respect to the isotropy group G_y, hence isomorphic to $\mathbb{P}_1$. As a first step in the proof of proposition 5.1, we show the existence of very special divisors in X.

Notation 5.2. We call a divisor $D \subset X$ a "rational section" if it intersects the generic ϕ-fiber with multiplicity one.

In our context, such divisors always exist:

Lemma 5.3. *Let $\phi : X \to Y$ be as in lemma 5.1 and assume additionally that there exists a group $H^* \cong \mathbb{C}^*$ acting trivially on Y. Let D'_X be the fixed point set of the H^*-action. Then D'_X contains two rational sections as irreducible components.*

Proof. Let D_X be the union of those irreducible divisors in D'_X which are not preimages of curves or points by ϕ. The subvariety D_X intersects every generic ϕ-fiber at least once. Hence $D_X \neq 0$.

We claim that the set of branch points

$$M := \{y \in Y : \#(\phi^{-1}(y) \cap D_X) = 1\}$$

is discrete. Linearization of the H^*-action yields that for any point $f \in D_X \setminus \mathrm{Sing}(X)$, there is a unique H^*-stable curve intersecting D_X at f. Furthermore, the intersection is transversal. Assume $\dim M \geq 1$ and let y be a generic point in M. Then $\dim \phi^{-1}(y) = 1$ and $\phi^{-1}(y) = 1$ contains a smooth curve C as an irreducible component intersecting D_X. Now $C.D_X = 1$ and, because $C \cap D_X$ was the only intersection point by assumption, $\phi^{-1}(y).D_X = 1$. This is contrary to D_X intersecting the generic ϕ-fiber twice.

Set

$$N := \{\mu \in Y | \dim(X_\mu \cap D_X) > 0\} \cup M \cup \phi(\mathrm{Sing}(X)).$$

By definition N is finite and D_X is a 2-sheeted cover over $Y \setminus N$. Now Y is smooth and quasihomogeneous with respect to an algebraic action of the linear algebraic group G. Hence it is rational. This implies that $Y \setminus N$ is simply connected. Hence D_X has two connected components over $Y \setminus N$. Now the set $D_X \cap \phi^{-1}(N)$ is just a curve. Therefore D_X cannot be irreducible. □

Lemma 5.4. *Under the assumptions of lemma 5.1, there exists a G-stable rational section $E_1 \subset X$.*

Proof. If G is a torus, then there exists a subgroup T_1 acting trivially on Y. In this case we are finished by applying lemma 5.3. Thus we may assume that the unipotent part U of G is non-trivial. Let $\eta \in Y$ be a generic point and $x \in X_\eta \setminus \Omega$, where Ω denotes the open G-orbit in X. If x is unique, then the divisor $E_1 := \overline{G.x}$ has the required properties. Similarly, if U acts almost transitively on Y, then it's isotropy at η is connected and we may set $E_1 := \overline{U.x}$.

If neither holds, then necessarily $\dim U = 1$, and we can assume that U acts non-trivially on Y. Otherwise $X_\eta \setminus \Omega$ consists of a single point and we are finished as above. Let T_1 be a 1-dimensional subgroup of a maximal torus such that $I := U.T_1$ acts almost transitively on Y. If $\eta \in Y$ is generic, the isotropy group I_η is cyclic: I_η has two fixed points in X_η. Consequently, there exist at least two I-orbits whose closures D_i are rational sections.

Note that I is normal in G, i.e. all elements of G map I-orbits to I-orbits. If D_i are the only rational sections occurring as closures of I-orbits, they are automatically G-stable. Otherwise, all I-orbits are mapped injectively to Y, and at least one of these is G-stable. □

The existence of E_1 already yields a map to a $\mathbb{P}_1$-bundle.

Lemma 5.5. *Under the assumptions of lemma 5.1, there exists a rank-2 vector bundle E on Y (not necessarily split) and an equivariant birational map $X \dashrightarrow^{eq} \mathbb{P}(E)$.*

Proof. Set $E := (\phi_*(\mathcal{O}_X(E)))^{**}$. Since a reflexive sheaf on a smooth surface is locally free, E is a vector bundle. If $\Omega_Y \subset Y$ is the open orbit, $\phi^{-1}(\Omega_Y) \cong \mathbb{P}(E|_{\Omega_Y})$ (cf. [BS95, Prop. 3.2.1]), inducing a birational map $\psi : X \dashrightarrow \mathbb{P}(E)$. Note that $\phi_*(\mathcal{O}_X(E))$ is torsion free. In particular, $\phi_*(\mathcal{O}_X(E))$ is locally free over a G-stable cofinite set $Y_0 \subset Y$ so that, by the universal property of *Proj*, ψ is regular over Y_0. As $\psi|_{Y_0}$ is proper, it is equivariant. The automorphisms over Y_0 extend to the whole of $\mathbb{P}(E)$ by the RIEMANN extension theorem. Hence ψ is equivariant as claimed. $\square$

In order to show that E can be chosen to be split we need to find another rational section. We will frequently deal with the following situation, for which we fix some notation.

Notation 5.6. Let $\phi : X \to Y$ be as above and assume that there exists a map $\pi : Y \to Z \cong \mathbb{P}_1$, e.g. if Y is isomorphic to a (blown-up) HIRZEBRUCH surface Σ_n. Then, if $F \in Z$ is a generic point, set $F_Y := \pi^{-1}(F)$ and $F_X := \phi^{-1}(F_Y)$.

LEMMA 5.7. *In the setting of proposition 5.1, there exists a second rational section E_2. If E_1 is as constructed in lemma 5.4, then $E_1 \cap E_2$ is G-stable.*

Proof. If G is a torus, we are finished, as we have seen in the proof of lemma 5.4. Hence we may assume that $\dim U > 0$, where U is the unipotent part of G.

Suppose that U acts trivially on Y. Then we are able to choose a 2-dimensional torus $T < G$ such that T acts almost transitively on Y. If $\eta \in Y$ is generic, then the isotropy group T_η may not be cyclic, but since it has to fix the unique U-fixed point in X_η, its image $T_\eta \to \mathrm{Aut}(X_\eta)$ is contained in a BOREL group, hence cyclic. Consequently, T_η fixes another point x, and we may set $E_2 := \overline{T.x}$.

The other case is that U acts non-trivially on Y. We need to consider a mapping $\pi : Y \to Z \cong \mathbb{P}_1$. If $Y \cong \Sigma_n$, or a blow-up, there is no problem. If $Y \cong \mathbb{P}_2$, we note that, by G being solvable and BOREL's fixed point theorem (see [HO80, p. 32]), there exists a G-fixed point $y \in Y$. We can always blow up y and X_y in order to obtain a new $\mathbb{P}_1$-bundle over Σ_1. If we are able to construct our rational sections here, then we can simply take their images to be the desired rational sections in the variety we started with. So let us assume that $Y \not\cong \mathbb{P}_2$.

There exists a 1-dimensional normal unipotent subgroup $U_1 < G$. Assume first that U_1 acts non-trivially on Z. Using notation 5.6, F_Y is isomorphic to $\mathbb{P}_1$, F_X to a HIRZEBRUCH surface Σ_n. Choose a section $\sigma \subset F_X$ with the property that $\phi(\sigma \cap E_1)$ does not meet the open G-orbit in Y. As the stabilizer of F_X in G stabilizes E_1, so that $E_1 \cap F_X$ is either the infinity- or zero-section in $F_X \cong \Sigma_n$ or the diagonal in $F_X \cong \Sigma_0$, and G stabilizes a section of $Y \to \mathbb{P}_1$, this can always be accomplished. Set $E_1 := \overline{U_1.\sigma}$.

Secondly, we must consider the case that U_1 acts trivially on Z. We proceed similarly to the above. Choose a 1-dimensional group $G_1 < G$ such that the G_1-orbit in Z coincides with that of G. Now G_1 stabilizes at least one section $\sigma_Y \subset Y$ over Z which is not U_1-stable! Set $\sigma_X := \phi^{-1}(\sigma_Y)$ and consider a section $\sigma \subset \sigma_X$ over σ_Y such that $\phi(\sigma \cap E_1)$ is disjoint from the open G-orbit in Y. Then $E_1 := \overline{U_1.\sigma}$ is the divisor we were looking for.

We still have to show that the intersection $E_1 \cap E_2$ is G-stable. Note that by construction, $\phi(E_1 \cap E_2)$ does not meet the open G-orbit in Y. This, together with E_1 being G-stable, yields the claim. $\square$

We shall use the second rational section in order to transform E into a splitting bundle.

5.1.1 ELIMINATING VERTICAL CURVES

If $S \subset \phi(E_1 \cap E_2)$ is an irreducible curve which is a ϕ-fiber, then we say that E_1 and E_2 intersect vertically in S. We know that after blowing up S we obtain a $\mathbb{P}_1$-bundle over the blow-up of Y. Furthermore, the process is equivariant. The proper transforms of E_1 and E_2 are again rational sections. If they still intersect vertically, the blow-up procedure can be applied again. So we eventually obtain a sequence of blow-ups. The strict transforms of the E_1 and E_2 are again rational sections in X_i. We denote them by E_1^i or E_2^i, respectively. By the theorem on embedded resolution, we have:

LEMMA 5.8. *The sequence described above terminates, i.e. there exists a number $i \in \mathbb{N}$ such that the strict transforms E_1^i and E_2^i do not intersect vertically.*

5.1.2 ELIMINATING HORIZONTAL CURVES

We may now assume that E_1 and E_2 do not intersect vertically. Let $S \subset \phi(E_1 \cap E_2)$ be an irreducible curve. Then S gives rise to an elementary transformation as ensured by [Mar73]. Again, the transformation is equivariant and the strict transforms of E_1 and E_2 are rational sections. If they still intersect over S, we transform as before. Again one may use the embedded resolution to show (cf. [Keb96, thm. 5.30] for details):

LEMMA 5.9. *The sequence described above terminates after finitely many transformations, i.e. there exists a $j \in \mathbb{N}$ such that for all curves $C \subset E_1^{(j)} \cap E_2^{(j)}$ it follows that $\phi^{(j)}(C) \neq S$. Furthermore, if E_1 and E_2 do not intersect vertically, then $E_1^{(i)}$ and $E_2^{(i)}$ do not intersect vertically for all i.*

5.1.3 THE CONSTRUCTION OF INDEPENDENT SECTIONS

By lemma 5.8 the variety X can be transformed into a $\mathbb{P}_1$-bundle such that the strict transforms of E_1 and E_2 do not intersect in fibers. A second transformation will rid us of curves in $E_1 \cap E_2$ which are not contained in fibers. Since the latter transformation does not create new curves in the intersection, the strict transforms of E_1 and E_2 eventually become disjoint. The resulting space is the compactification of a line bundle.

LEMMA 5.10. *If E_1 and E_2 do not intersect, X is the compactification of a line bundle.*

Proof. Since E_1 and E_2 are disjoint, neither contains a fiber. Thus they are sections. $\square$

As a net result, we have shown proposition 5.1.

5.2 The case G not solvable

As first step, we show that X is again a linear $\mathbb{P}_1$-bundle. We do this under an additional hypothesis which will not impose problems in the course of the proof of theorem 1.1.

Lemma 5.11. *Let X and G be as in theorem 1.1, with the exception that X is allowed to have $\mathbb{Q}$-factorial terminal singularities. Let $\phi : X \to Y$ be a* Mori*-contraction to a surface and assume additionally that G is not solvable and that there exists an equivariant morphism $\psi : Y \to Y'$, where Y' is a smooth surface. Then X and Y are smooth and X is a linear $\mathbb{P}_1$-bundle over Y.*

Proof. First, we show that all ϕ-fibers are of dimension 1. If there exists a fiber X_μ which is not 1-dimensional, then $\dim X_\mu = 2$. Take a curve $C \subset Y$ so that $\mu \in C$. Set $D := \overline{\phi^{-1}(C \setminus \mu)}$. The divisor D intersects an irreducible component of X_μ. Now take a curve $R \subset X_\mu$ intersecting D in finitely many points. We have $R.D > 0$. However, all generic q-fibers X_η are homologous to R (up to positive multiples). So $X_\eta.D > 0$, contradicting the definition of D.

Secondly, we claim that X is smooth. Assume to the contrary and let $x \in X$ be a singular point, $\mu := \phi(x)$. Recall that terminal singularities in 3-dimensional varieties are isolated. Thus, if S is the semisimple part of G, then the fiber X_μ through x is pointwise S-fixed. Linearizing the S-action at a generic point $y \in X_\mu$, the complete reducibility of the S-representation yields an S-quasihomogeneous divisor D which intersects X_μ transversally at y and is Cartier in a neighborhood of y. The induced map $D \to Y'$ must be unbranched: Y' contains an S-fixed point and is therefore isomorphic to $\mathbb{P}_2$; but there is no equivariant cover of this other than the identity. So D is a rational section which is Cartier over a neighborhood of μ. If $H \in Pic(Y)$ is sufficiently ample, then $D + \phi^*(H)$ is ample, and [BS95, Prop. 3.2.1] applies, contradicting the assumption that X is singular.

Since X is smooth, the same theorem shows that in order to prove the lemma it is sufficient to show that there exists a rational section. If all the simple factors of S have orbits of dimension ≤ 2, then, after replacing the factors by their Borel groups, we obtain a solvable group G', acting almost transitively as well. In this case lemma 5.4 applies.

If $S' < S$ is a simple factor acting with 3-dimensional orbit on X, its action on Y is almost transitively. In particular, there exists a 2-dimensional group $B < S$, isomorphic to a Borel group in SL_2, which also acts almost transitively on Y. As in the proof of lemma 5.4, B has cyclic isotropy at a generic point of Y and so there exist two rational sections which are compactifications of B-orbits. $\square$

6 Proof of theorem 1.1

Prior to proving theorem 1.1, we still need to describe equivariant maps to $\mathbb{P}_3$ in more detail:

Lemma 6.1. *Let $X \overset{eq}{\dashrightarrow} \mathbb{P}_3$ be an equivariant birational map. Then either X has an equivariant rational fibration with 2-dimensional base variety or X and $\mathbb{P}_3$ are equivariantly linked by a sequence of blowing ups of X followed by a sequence of blow-downs.*

Proof. If the G-action on $\mathbb{P}_3$ has a fixed point, we can blow up this point and obtain a map from the blown-up $\mathbb{P}_3$ to $\mathbb{P}_2$. If there is no such G-fixed point in $\mathbb{P}_3$, then after replacing X by an equivariant blow-up, there is a regular equivariant map $\phi : X \to \mathbb{P}_3$. Recall that such a map factors through an extremal contraction. Since the base does not contain a fixed point, the classification of extremal contractions of smooth varieties yields the claim. □

Now we compiled all the results needed to finish the

Proof of theorem 1.1. Given X, we apply lemmata 3.1–3.4. Unless $X \cong Q_3$, $F_{1,2}(3)$ or a compactification of SL_2/Γ, Γ not cyclic, there exists an equivariant map $X \dashrightarrow^{eq} Y$, where Y is smooth and $Y \cong \mathbb{P}_3$, $\dim(Y) = 2$ or, if no other case applies, $\dim(Y) = 1$.

If $Y \cong \mathbb{P}_3$, then, by lemma 6.1, we may replace $\mathbb{P}_3$ by a surface, or else we are finished.

In the case of a map to Y with $\dim Y < 3$, we can blow up X equivariantly to obtains a morphism $\tilde{X} \to Y$. Recalling that all steps in the minimal model program (i.e. contractions and flips) are equivariant, we may perform a relative minimal model program over Y. In this situation corollary 2.2 shows that the program does not stop unless we encounter a contraction of fiber type $X' \to Y'$ and $\dim Y' < 3$. Note that $\dim Y' \geq \dim Y$.

In case that Y' is a surface, X' is the projectivization of a line bundle or can be equivariantly transformed into one (cf. lemma 5.5 and 5.11). If G is solvable, proposition 5.1 allows us to transform X into the projectivization of a splitting bundle over a surface.

If $\dim Y' = 1$ and there does not exist a map to one of the other cases, $X \cong \mathbb{P}(\mathcal{O}(e) \oplus \mathcal{O}(e) \oplus \mathcal{O})$ over $\mathbb{P}_1$, as was shown in proposition 4.1.

We still have to show that if G is not solvable, the map to one of the models in our list factors into equivariant monoidal transformations. Recall that it suffices to show that, after equivariantly blowing up, if necessary, the minimal models do not have a G-fixed point. We do a case-by-case checking:

$\mathbb{P}_2$-BUNDLES OVER $\mathbb{P}_1$: By lemma 4.2, these can be chosen not to contain a fixed point.

$\mathbb{P}_1$-BUNDLES OVER A SURFACE Y: If the semisimple part S of G acts trivially on Y, we can stop. Otherwise, if the S-action on Y has a fixed point f, we blow up f and the fiber over f and obtain a $\mathbb{P}_1$-bundle over Σ_1. Recall that actions of semisimple groups on Σ_n never have fixed points.

$\mathbb{P}_3$: This case has already been handled in lemma 6.1.

SL_2/Γ: After desingularizing and blowing up all fixed points, if any, the compactification of SL_2/Γ is fixed point free. Otherwise, linearization at a fixed point yields a contradiction to S acting almost transitively.

OTHER CASES: The remaining cases occur only when X is homogeneous (cf. lemma 3.4).

□

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Classical Motivic Polylogarithm According to Beilinson and Deligne

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Abstract. The main purpose of this paper is the construction in motivic cohomology of the *cyclotomic*, or *classical polylogarithm* on the projective line minus three points, and the identification of its image under the regulator to absolute (Deligne or l-adic) cohomology. By specialization to roots of unity, one obtains a compatibility statement on *cyclotomic elements* in motivic and absolute cohomology of abelian number fields. As shown in [BlK], this compatibility completes the proof of the Tamagawa number conjecture on special values of the Riemann zeta function.

The main constructions and ideas are contained in Beilinson's and Deligne's unpublished preprint "Motivic Polylogarithm and Zagier Conjecture" ([BD1]). We work out the details of the proof, setting up the foundational material which was missing from the original source: the paper contains an appendix on absolute Hodge cohomology with coefficients, and its interpretation in terms of Saito's Hodge modules. The second appendix treats K-theory and regulators for simplicial schemes.

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Introduction

The aim of this work is to present the construction of the class of the cyclotomic, or classical polylogarithm in motivic cohomology. It maps to the elements in Deligne and l–adic cohomology defined and studied in Beilinson's "Polylogarithm and cyclotomic elements" ([B4]). The latter elements can be seen as being represented by a pro–variation of Hodge structure, or a pro–l–adic sheaf on the projective line minus three points.

Our main interest lies in the specialization of these sheaves to roots of unity: they represent the "cyclotomic" one–extensions of Tate twists already studied by Soulé ([Sou5]), Deligne ([D5]) and Beilinson ([B2]).

Let us be more precise: denote by μ_d^0 the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, $d \geq 2$. We get an alternative proof of the following theorem of Beilinson's:

Corollary 9.6. Assume $n \geq 0$, and denote by $r_{\mathcal{D}}$ the regulator map

$$H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}\ .$$

There is a map of sets

$$\epsilon_{n+1} : \mu_d^0 \longrightarrow H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), \mathbb{Q}(n+1))$$

such that

$$r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu_d^0 \longrightarrow \bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}$$

maps a root of unity ω to $(-Li_{n+1}(\sigma\omega))_\sigma = \left(-\sum_{k\geq 1} \frac{\sigma\omega^k}{k^{n+1}}\right)_\sigma$.

Now fix a d–th primitive root of unity ζ in $\overline{\mathbb{Q}}$. This choice allows to identify continuous étale cohomology $H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$ with a $\mathbb{Q}_l$–subspace of

$$\left(\varprojlim_{r\geq 1} \left(\mathbb{Q}(\mu_{l^\infty}, \zeta)^* / (\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes n} \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\operatorname{Gal}(\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\zeta))} .$$

Note that there is a distinguished root of unity T in $\mathbb{Q}(\mu_d)$. As was observed already in [B4], the study of the cyclotomic polylogarithm gives a proof of [BlK], Conjecture 6.2 (cf. [Sou5], Théorème 1 for the case $n = 1$; [Gr], Théorème IV.2.4 for the local version if $(l, d) = 1$):

Corollary 9.7. Let ϵ_{n+1} be the map constructed in 9.6. Under the above inclusion, the l–adic regulator

$$r_l : H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), \mathbb{Q}(n+1)) \longrightarrow H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n} \cdot \frac{1}{n!} \cdot \left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r .$$

This result implies in particular that Soulé's cyclotomic elements in the group $K_{2n+1}(F) \otimes_{\mathbb{Z}} \mathbb{Z}_l$ (for an abelian number field F and a prime l) are induced by elements in K–theory itself (Corollary 9.8). Furthermore, the case $d = 2$ of 9.7 forms a central ingredient of the proof of the Tamagawa number conjecture modulo powers of 2 for odd Tate twists $\mathbb{Q}(n)$, $n \geq 2$ ([BlK], § 6). Finally, as shown in [KNF], Theorem 6.4, the general case of 9.7 implies the modified version of the Lichtenbaum conjecture for abelian number fields.

The main ideas necessary for both the construction of the motivic polylogarithm and the identification of the realization classes, together with a sketch of proof, are contained in the unpublished preprint "Motivic Polylogarithm and Zagier Conjecture" ([BD1]) and its predecessors [B4], [BD1p]. Our aim in this paper is to work out the details of the proofs. To do this we have to set up a lot of foundational material, which was missing from the original sources: K-theory of simplicial schemes, regulators to absolute (Hodge and l-adic) cohomology of simplicial schemes, and an interpretation of the latter as Ext groups of Hodge modules and l-adic sheaves respectively. This material is contained in the two appendices which we regard as our main contribution to the subject. We hope they prove to be useful in other contexts than that treated in the main text.

Other parts of [BD1] deal with (the weak version of) the Zagier conjecture. We do not treat this since a complete proof has been given by de Jeu ([Jeu]), although by somewhat different means from those used in [BD1].

We see two main groups of papers related to polylogarithms:

The first deals with mixed sheaves, i.e., variations of Hodge structure or l–adic mixed lisse sheaves. Maybe the nucleus of these papers is Deligne's observation that the analytic and topological properties of the dilogarithm Li_2, viewed as a multivalued holomorphic function on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, can be coded by saying that Li_2 is an entry of the period matrix of a certain rank three variation of $\mathbb{Q}$–Tate–Hodge structure on $\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}$.

We refer to [Rm], section 7.6 for a nice survey of the construction of a pro–variation on $\mathbb{P}^1_{\mathbb{C}}\backslash\{0,1,\infty\}$ containing all Li_k. The étale analogue is constructed in Beilinson's "Polylogarithm and Cyclotomic elements" ([B4]), where he defined pro–objects in the categories of l–adic sheaves on $\mathbb{P}^1\backslash\{0,1,\infty\}$. In both settings, the fibres at roots of unity different from 1 coincide with the cyclotomic extensions mentioned above.

The hope and indeed, the motivation underlying these papers is that once a satisfactory formalism of motivic sheaves is developed, the definition of polylogarithms should basically carry over. We would thus obtain polylogarithmic classes in Ext groups of motives, these groups being supposedly closely connected to K–theory, of which everything already defined on the level of realizations would turn out to be the respective regulator.

Nowhere is this hope documented more manifestly than in Beilinson's and Deligne's "Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs" ([BD2]): if there is such a motivic formalism, then the weak version of Zagier's conjecture necessarily holds: not only the values at roots of unity of higher logarithms, but also appropriate linear combinations of *arbitrary* values must lie in the image of the regulator.

For the time being, and in each case separately, honest work is needed to perform the K–theoretic constructions, and calculate their images under the regulators.

The second class of papers is concerned with precisely that task. In analogy with the above, one should first mention Bloch's "Application of the dilogarithm function in algebraic K–theory and algebraic geometry" ([Bl]).

Beilinson's "Higher regulators and values of L–functions" ([B2]) provided the K–theoretic construction of cyclotomic elements, together with the computation of their images in Deligne cohomology (loc. cit., Theorem 7.1.5, [Neu], [E]).

As for Zagier's conjecture, we mention Goncharov's "Polylogarithms and Motivic Galois Groups" ([Go]), where Zagier's conjecture, including the surjectivity statement is proved for K_5 of a number field, and de Jeu's "Zagier's Conjecture and Wedge Complexes in Algebraic K–theory" ([Jeu]), which contains the proof of the weak version of Zagier's conjecture, independently of motivic considerations, for K_{2n-1} of a number field, and arbitrary $n \geq 2$.

Typically, the objects of interest in this class of papers are complexes, cocycles, and symbols, i.e., objects which do not constantly afford a geometric, or sheaf–theoretic interpretation. It is by no means easy to see, say, how a concrete element in some Deligne cohomology group can be interpreted as an extension of variations of $\mathbb{R}$–Hodge structure. These and similar difficulties present themselves to the reader willing to translate from one class to the other.

The authors like to think of the present article as an attempt to bridge the gap between the two disciplines.

In a sense, the coarse structure of the article follows the above scheme: sections 1–6 are entirely sheaf–theoretic. Anything we say there is therefore a priori restricted to the level of realizations, i.e., non–motivic. In sections 7–9, K–theory enters. The appendices provide the foundations necessary to connect the two points of view.

Given that quite a lot has been said about the l–adic and Hodge theoretic incarnations of the classical polylogarithm ([B4], [BD2], [WiIV]), the reader may wonder why sheaf theoretic considerations still take up one third of this work.

Indeed, the construction of the motivic polylog could be achieved much more easily if a satisfactory formalism of mixed motivic sheaves were available. The necessity to replace a simple geometric situation by a rather complicated one, in order to replace complicated coefficients like $\mathcal{L}og$ by Tate twists, should be seen as the main source of difficulty in any attempt to the construction of motivic versions of polylogarithms.

We now turn to the description of the finer structure of the main text (sections 1–9):

In section 1, we normalize the sheaf theoretic notations used throughout the whole article.

Section 2 gives a quick axiomatic description of the logarithmic sheaf $\mathcal{L}og$, and the (small) polylogarithmic extension *pol*. The universal property (2.1) is needed only to connect the general definition of the logarithmic sheaf as a solution of a representability problem to the somewhat ad hoc, but much more geometric definition of section 4. A reader prepared to accept the results on the shape of the Hodge theoretic and l–adic incarnation of the polylogarithm (2.5, 2.6) may therefore take the constructions in sections 4 and 6 as a definition of both $\mathcal{L}og$ and *pol*, and view section 2 as an extended introduction providing background material.

In section 3, we establish the geometric situation used thereafter. As section 1, it is mainly intended for easier reference.

In section 4, we construct a pro-unipotent sheaf $\mathcal{G}$ on $\mathbb{U} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ as projective limit of relative cohomology objects of powers of $\mathbb{G}_m$ over $\mathbb{U}$ relative to certain singular subschemes. The transition maps are given by the boundary maps in the *relative residue sequence* (4.9). The universal property 2.1 then allows to identify $\mathcal{G}$ with the restriction of $\mathcal{L}og$ to $\mathbb{U}$ (4.11).

Section 5 contains a geometric proof of the *splitting principle* (5.2): the fibres of $\mathcal{L}og$ at roots of unity have split weight filtration. Since we need a proof which translates easily to the motivic situation, we return to Beilinson's original approach to the splitting principle ([B4], 4.2) which consists of an analysis of the action of the multiplication by natural numbers on our absolute cohomology groups.

The main objective of section 6 is the description of *pol* in terms of geometric data. The Leray spectral sequence suggests that one–extensions of $\mathbb{Q}(0)$ by $\mathcal{L}og$ should be described as elements of the projective limit of cohomology groups with Tate coefficients of powers of $\mathbb{G}_m$ relative to certain subschemes. The main result 6.6 allows to identify *pol* under this correspondence.

In Section 7 our main tool, the residue sequence is constructed in the setting of motivic cohomology (Proposition 7.2 and Lemma 7.3). The arguments are very much parallel to those used for absolute cohomology of realizations in section 4. However, we have to replace the singular schemes by explicit simplicial schemes with regular components. This is where the material of Appendix B enters.

Section 8 is the K–theoretic analogue of section 6. We consider a certain projective system of motivic cohomology groups. In order to identify its projective limit (Corollary 8.8) we use bijectivity or at least controlled injectivity of the regulator to Deligne cohomology, and the results of section 6. We are then able to define the universal motivic polylog (8.9).

In the final section 9 the motivic version of the splitting principle is shown (9.3). Again we strongly use the known behaviour of the regulator to show that the action of multiplication by natural numbers splits into eigenspaces. Applied to the universal motivic polylogarithm this induces the cyclotomic elements in motivic cohomology. In the light of section 5 it is clear from their very construction that they induce the right elements not only in Deligne but also in continuous étale cohomology. We conclude by drawing the corollaries which are the main results announced at the beginning (9.6–9.9).

The Appendices can be read independently of the main text and of each other. They are meant to be used as a reference, but a careful reader might actually want to read them first. We refer to the respective introductions for an account of their content.

The reader might find it useful to consult [HW] for an overview of the strategy of the proof of the main results.

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We thank C. Deninger for suggesting to us that the methods developed and results obtained in our respective PhD theses ([H1], [Wi]) might form a sound basis of a successful treatment of this theme.

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Contents

1 Mixed sheaves

We start by defining the sheaf categories which will be relevant for us. For our purposes, it will be necessary to work in the settings of mixed l–adic perverse sheaves ([H2]), and of algebraic mixed Hodge modules over $\mathbb{R}$ (A.2). Since the procedures are entirely analogous, we introduce, for economical reasons, the following rules: whenever an area of paper is divided by a vertical bar

the text on the left of it will concern the Hodge theoretic setting, while the text on the right will deal with the l–adic setting. Of course, we hope that before long, there will be a satisfactory formalism of mixed motivic sheaves providing a third setting to which our constructions can be applied. We let

$A := \mathbb{R}$, $F := \mathbb{Q}$,	$l :=$ a fixed prime number , $A := \mathbb{Z}\left[\frac{1}{l}\right]$, $F := \mathbb{Q}_l$

and set $B := \mathrm{Spec}(A)$.

For any reduced, separated and flat scheme X of finite type over B, we let

$X_{\mathrm{top}} := X(\mathbb{C})$ as a topol. space, $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q})$,	$X_{\mathrm{top}} := X \otimes_A \overline{\mathbb{Q}}$, $\mathrm{Sh}(X_{\mathrm{top}}) := \mathrm{Perv}(X_{\mathrm{top}}, \mathbb{Q}_l)$

the latter categories denoting the respective categories of perverse sheaves on X_{top} ([BBD], 2.2).

Next we define the category $\mathrm{Sh}(X)$: in the l–adic setting, we fix a pair $(\mathbf{S}, L)$ consisting of a horizontal stratification $\mathbf{S}$ of X ([H2], § 2) and a collection $L = \{L(S) \mid S \in \mathbf{S}\}$, where each $L(S)$ is a set of irreducible lisse l–adic sheaves on S. For all $S \in \mathbf{S}$ and $F \in L(S)$, we require that for the inclusion $j : S \hookrightarrow X$, all higher direct images $R^n j_* F$ are $(\mathbf{S}, L)$*–constructible*, i.e., have lisse restrictions to all $S \in \mathbf{S}$, which are extensions of objects of $L(S)$. We assume that all $F \in L(S)$ are pure.

We can make this more explicit: in our computations X will always be a locally closed subscheme of some $\mathbb{A}^n$; the stratification is by the number of vanishing coordinates in $\mathbb{A}^n$; $L(S)$ is the set of all Tate sheaves on S.

Following [H2], § 3, we define $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ as the full subcategory of $D^b_c(X, \mathbb{Q}_l)$ of complexes with $(\mathbf{S}, L)$–constructible cohomology objects. Note that all objects will be mixed. By [H2], § 3, $D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$ admits a perverse t–structure, whose heart we denote by $\mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

$\mathrm{Sh}(X) := \mathrm{MHM}_{\mathbb{Q}}(X/\mathbb{R})$ (see A.2.4)	$\mathrm{Sh}(X) := \mathrm{Perv}_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$.

Because of the horizontality requirement in the l–adic situation we have the full formalism of Grothendieck's functors only on the direct limit $D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$ of the $D^b_{(\mathbf{S},L)}(X_U, \mathbb{Q}_l)$, for U open in B, and $(\mathbf{S}, L)$ as above (see [H2], § 2). However, for a fixed morphism

$$\pi : X \longrightarrow Y ,$$

we have a notion of e.g. π_*–admissibility for a pair $(\mathbf{S}, L)$: this is the case if

$$D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l) \hookrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l) \xrightarrow{\pi_*} D^b_m(\mathfrak{U}_Y, \mathbb{Q}_l)$$

factors through some $D^b_{(\mathbf{T},K)}(Y, \mathbb{Q}_l)$. Our computations will show, at least a posteriori, that for our choice of $(\mathbf{S}, L)$ all functors which appear are admissible. We will not stress these technical problems and even suppress $(\mathbf{S}, L)$ from our notation.

As in [BBD], we denote by $\pi_*, \pi^*, \mathrm{Hom}$ etc. the respective functors on the categories

$D^b \operatorname{Sh}(X) := D^b \operatorname{MHM}_{\mathbb{Q}}(X/\mathbb{R})$,	$D^b \operatorname{Sh}(X) := D^b_{(\mathbf{S},L)}(X, \mathbb{Q}_l)$,

and $\mathcal{H}^q$ for the (perverse) cohomology functors.

We refer to objects of $\operatorname{Sh}(X)$ as sheaves, and to objects of $\operatorname{Sh}(X_{\mathrm{top}})$ as topological sheaves. Let us denote by

$$\mathbb{V} \mapsto \mathbb{V}_{\mathrm{top}}$$

the forgetful functor from $\operatorname{Sh}(X)$ to $\operatorname{Sh}(X_{\mathrm{top}})$. If we use the symbol $W.$, it will always refer to the weight filtration.

If X is smooth, we let

$\operatorname{Sh}^s(X) := \operatorname{Var}_{\mathbb{Q}}(X/\mathbb{R}) \subset \operatorname{Sh}(X)$ (see A.2.1) , $\operatorname{Sh}^s(X_{\mathrm{top}}) :=$ the category of $\mathbb{Q}$–local systems on X_{top}.	$\operatorname{Sh}^s(X) := \operatorname{Et}^{l,m}_{\mathbb{Q}_l}(X) \subset \operatorname{Sh}(X)$, the category of lisse mixed $\mathbb{Q}_l$–sheaves on X, $\operatorname{Sh}^s(X_{\mathrm{top}}) :=$ the category of lisse $\mathbb{Q}_l$–sheaves on X_{top}.

We refer to objects of $\operatorname{Sh}^s(X)$ as smooth sheaves, and to objects of $\operatorname{Sh}^s(X_{\mathrm{top}})$ as smooth topological sheaves. Denote by $U\operatorname{Sh}^s(X)$ the category of unipotent objects of $\operatorname{Sh}^s(X)$, i.e., those smooth sheaves admitting a filtration whose graded parts are pullbacks of smooth sheaves of $\operatorname{Sh}^s(B)$ via the structure morphism. Similarly, one defines $U\operatorname{Sh}^s(X_{\mathrm{top}})$.

REMARK: Note that in the l–adic situation, the existence of a weight filtration, i.e., an ascending filtration $W.$ by subsheaves indexed by the integers, such that Gr^W_m is of weight m, is not incorporated in the definition of Sh^s – compare the warnings in [H2], § 3. In the Hodge theoretic setting, the existence of a weight filtration is part of the data.

REMARK: We have to deal with a shift of the index when viewing e.g. a variation as a Hodge module, which occurs either in the normalization of the embedding

$$\operatorname{Var}_{\mathbb{Q}}(X/\mathbb{R}) \longrightarrow D^b \operatorname{MHM}_{\mathbb{Q}}(X/\mathbb{R})$$

or in the numbering of cohomology objects of functors induced by morphisms between schemes of different dimension. In order to conform with the conventions laid down in appendix A and [WiI], chapter 4, we chose the second possibility: a variation *is* a Hodge module, not just a shift of one such. Similarly, a lisse mixed $\mathbb{Q}_l$–sheaf *is* a perverse mixed sheaf. Therefore, if X is of pure relative dimension d over B, then the embedding

$$\operatorname{Et}^{l,m}_{\mathbb{Q}_l}(X) \longrightarrow D^b_m(\mathfrak{U}_X, \mathbb{Q}_l)$$

associates to $\mathbb{V}$ the complex concentrated in degree $-d$, whose only non–trivial cohomology object is $\mathbb{V}$.

As a consequence, the numbering of cohomology objects of the direct image (say) will differ from what the reader might be used to: e.g., the cohomology of a curve is concentrated in degrees -1, 0, and 1 instead of 0, 1, and 2. Similarly, one has to distinguish between the "naive" pullback $(\pi^s)^*$ of a smooth sheaf and the pullback π^* on the level of $D^b\,\mathrm{Sh}(X)$: $(\pi^s)^*$ lands in the category of smooth sheaves, while π^* of a smooth sheaf yields only a smooth sheaf up to a shift.

In the special situation of pullbacks, we allow ourselves one notational inconsistency: if there is no danger of confusion (e.g. in Theorem 2.1), we use the notation π^* also for the naive pullback of smooth sheaves. Similar remarks apply for smooth topological sheaves.

For a scheme $a : X \to B$, we define

$$F(n)_X := a^* F(n) \in D^b\,\mathrm{Sh}(X) ,$$

where $F(n)$ is the usual Tate twist on B.

If X is smooth, we also have the naive Tate twist

$$F(n) \in \mathrm{Sh}^s(X) \subset \mathrm{Sh}(X)$$

on X. If X is of pure dimension d, then we have the equality

$$F(n) = F(n)_X[d] .$$

In order to keep our notation transparent, we have the following

Definition 1.1. *For any morphism $\pi : X \longrightarrow S$ of reduced, separated and flat B–schemes we let*

$$\begin{aligned} \mathcal{R}_S(X,\ \cdot\ ) &:= \pi_* : D^b\,\mathrm{Sh}(X) \longrightarrow D^b\,\mathrm{Sh}(S) ,\\ \mathcal{H}^i_S(X,\ \cdot\ ) &:= \mathcal{H}^i\pi_* : D^b\,\mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(S) . \end{aligned}$$

Definition 1.2. *For a closed reduced subscheme Z of a separated, reduced, flat B–scheme X of finite type, with complement $j : U \hookrightarrow X$, and an object $M^\cdot$ of $D^b\,\mathrm{Sh}(X)$, define*

a)
$$\begin{aligned} R\Gamma_{\mathrm{abs}}(X, M^\cdot) &:= R\,\mathrm{Hom}_{D^b\,\mathrm{Sh}(X)}(F(0)_X, M^\cdot) ,\\ H^i_{\mathrm{abs}}(X, M^\cdot) &:= H^i R\Gamma_{\mathrm{abs}}(X, M^\cdot) , \end{aligned}$$

the absolute complex and absolute cohomology groups of X with coefficients in $M^\cdot$.

b)
$$\begin{aligned} R\Gamma_{\mathrm{abs}}(X, n) &:= R\Gamma_{\mathrm{abs}}(X, F(n)_X) ,\\ H^i_{\mathrm{abs}}(X, n) &:= H^i_{\mathrm{abs}}(X, F(n)_X) . \end{aligned}$$

c)
$$\begin{aligned} R\Gamma_{\mathrm{abs}}(X \ rel\ Z, n) &:= R\Gamma_{\mathrm{abs}}(X, j_! F(n)_U) ,\\ H^i_{\mathrm{abs}}(X \ rel\ Z, n) &:= H^i_{\mathrm{abs}}(X, j_! F(n)_U) , \end{aligned}$$

the relative absolute complex and relative absolute cohomology with Tate coefficients.

In the Hodge setting, absolute cohomology with Tate coefficients coincides with Beilinson's absolute Hodge cohomology over $\mathbb{R}$ (Theorem A.2.7). In the l-adic setting, it yields continuous étale cohomology (see the remark following Definition B.4.2).

Remark: If X is a scheme over S, then we have the formulae

$$R\Gamma_{\text{abs}}(X,\ \cdot\ ) = R\Gamma_{\text{abs}}\left(S, \mathcal{R}_S(X,\ \cdot\ )\right)\ ,$$
$$H^i_{\text{abs}}(X,\ \cdot\ ) = H^i_{\text{abs}}(S, \mathcal{R}_S(X,\ \cdot\ ))\ .$$

2 The Logarithmic Sheaf, and the Polylogarithmic Extension

We aim at a sheaf theoretic description of the (small) classical polylogarithm on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The first step is an axiomatic definition of the *logarithmic pro–sheaf*. We need the following result:

Theorem 2.1. *Let X be the complement in a smooth, proper B–scheme of an NC–divisor relative to B ([SGA1], Exp. XIII, 2.1), all of whose irreducible components are smooth over B. Let $x \in X(B)$, and write $a : X \to B$. The functor*

$$x^* : U\mathrm{Sh}^s(X) \longrightarrow \mathrm{Sh}^s(B)$$

is representable in the following sense:

a) *There is a pro–object*

$$\mathcal{G}en_x \in \text{pro–}U\mathrm{Sh}^s(X)\ ,$$

the generic pro–unipotent sheaf with basepoint x on X, which has a weight filtration satisfying

$$\mathcal{G}en_x / W_{-n}\mathcal{G}en_x \in U\mathrm{Sh}^s(X) \quad \textit{for all } n\ .$$

Note that this implies that the direct system

$$\left(R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x / W_{-n}\mathcal{G}en_x, \mathbb{V})\right)_{n \in \mathbb{N}}$$

of smooth sheaves on B becomes constant for any $\mathbb{V} \in U\mathrm{Sh}^s(X)$.

This constant value is denoted by

$$R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x, \mathbb{V})\ .$$

b) *There is a section*

$$1 \in \Gamma\left(B, x^*\mathcal{G}en_x\right)\ .$$

c) *The natural transformation of functors from $U\mathrm{Sh}^s(X)$ to $\mathrm{Sh}^s(B)$*

$$\begin{array}{rcl} ev : R^0 a_* \underline{\mathrm{Hom}}(\mathcal{G}en_x,\ \_\ ) & \longrightarrow & x^*\ , \\ \varphi & \longmapsto & (x^*\varphi)(1) \end{array}$$

is an isomorphism. Similarly for the transformation of functors from $U\mathrm{Sh}^s(X_{\mathrm{top}})$ *to* $\mathrm{Sh}^s(B_{\mathrm{top}})$

$$\begin{aligned} ev : R^0 a_* \underline{\mathrm{Hom}}\left((\mathcal{G}en_x)_{\mathrm{top}}, \_\right) &\longrightarrow x^* , \\ \varphi &\longmapsto (x^*\varphi)(1) . \end{aligned}$$

Consequently, the pairs $(\mathcal{G}en_x, 1)$ *and* $((\mathcal{G}en_x)_{\mathrm{top}}, 1)$ *are unique up to unique isomorphism.*

d) *The natural transformations of functors*

$$\begin{aligned} \mathrm{Hom}_{U\mathrm{Sh}^s(X)}(\mathcal{G}en_x, \_) &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}^s(B)}(F(0), x^*\_) \quad \textit{and} \\ \mathrm{Hom}_{U\mathrm{Sh}^s(X_{\mathrm{top}})}\left((\mathcal{G}en_x)_{\mathrm{top}}, \_\right) &\longrightarrow \Gamma(B_{\mathrm{top}}, x^*\_) \end{aligned}$$

from $U\mathrm{Sh}^s(X)$ *and* $U\mathrm{Sh}^s(X_{\mathrm{top}})$ *respectively are isomorphisms.*

Proof. For a)–c), we refer to

[WiI], Remark d) after Theorem 3.6,	[WiI], Theorem 3.5.i),

and loc. cit., Theorem 3.5.ii). Apply the functors $\mathrm{Hom}_{\mathrm{Sh}^s(B)}(F(0), \_)$ and $\Gamma(B_{\mathrm{top}}, \_)$ to the result in c) in order to obtain d). ☐

Remark: In the Hodge setting and for the constant base B, Theorem 2.1 is equivalent to the classification theorem for admissible unipotent variations of Hodge structure ([HZ], Theorem 1.6). In this case, $\mathcal{G}en_x$ is the *canonical variation with base point* x of loc. cit., section 1.

Now let

$$\begin{gathered} \mathbb{G}_m := \mathbb{G}_{m,B} , \quad \mathbb{U} := \mathbb{P}^1_B \setminus \{0, 1, \infty\}_B , \\ j : \mathbb{U} \hookrightarrow \mathbb{G}_m , \\ p : \mathbb{G}_m \longrightarrow B , \quad \widetilde{p} := p \circ j : \mathbb{U} \longrightarrow B . \end{gathered}$$

We may form the generic pro–unipotent sheaf with basepoint 1 on $\mathbb{G}_m$.

Definition 2.2. $\mathcal{L}og := \mathcal{G}en_1 \in$ *pro–*$U\mathrm{Sh}^s(\mathbb{G}_m)$ *is called the logarithmic pro–sheaf.*

As we shall see below, there is an isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot} \mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k) .$$

Assuming this for the moment, we now describe the higher direct images $\mathcal{H}^{\cdot}_B(\mathbb{U}, j^*\mathcal{L}og(1))$:

Theorem 2.3. a) $\mathcal{H}^q_B(\mathbb{U}, j^*\mathcal{L}og(1)) = 0$ *for* $q \neq 0$.

b) $\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))$ *has a weight filtration, and* $W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))\right)$ *is split. More precisely, any isomorphism* κ *as above induces an isomorphism*

$$W_{-1}\left(\mathcal{H}^0_B(\mathbb{U}, j^*\mathcal{L}og(1))\right) \xrightarrow{\sim} \prod_{k \geq 1} F(k) .$$

Remark: By these statements on the higher direct images of the pro–sheaf $j^*\mathcal{L}og(1)$, we mean the following:

a) For $q \neq 0$, the projective system

$$\mathcal{H}^q_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\right)_{n\geq 1}$$

is ML–zero.

b) κ induces a morphism of projective systems

$$\mathcal{H}^0_B\left(\mathbb{U}, j^*(\mathcal{L}og/W_{-2m}\mathcal{L}og)(1)\right)_{m\geq 1} \longrightarrow \left(\prod_{k=0}^m F(k)\right)_{m\geq 1}$$

of sheaves with a weight filtration, such that the weight ≤ -1–parts of the projective systems of kernels and co-kernels are ML–zero.

Proof. One uses the exact triangle

$$\begin{array}{ccc} 1_*1^! & \longrightarrow & \mathrm{id}_{\mathbb{G}_m} \\ {}_{[1]}\nwarrow & & \swarrow \\ & j_*j^* & \end{array}$$

or rather, $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \_)$ of it, and the fact that $\mathcal{H}^{\cdot}_B(\mathbb{G}_m, \mathcal{L}og)$ is easily computable. For the details, see [WiIII], Theorem 1.3. Or use 4.11 and 6.2, whose proof is independent of 2.3. $\square$

A fixed choice of

$$\kappa : \mathrm{Gr}^W_{\cdot}\, \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

induces in particular an isomorphism of $\mathrm{Gr}^W_{-2}\,\mathcal{L}og$ and $F(1)$. The theorem then enables one to define the *small polylogarithmic extension* as the extension

$$pol \in \mathrm{Ext}^1_{U\mathrm{Sh}^s(\mathbb{U})}\left(\mathrm{Gr}^W_{-2}\,\mathcal{L}og\,|_{\mathbb{U}}, \mathcal{L}og(1)\,|_{\mathbb{U}}\right)$$

mapping to the natural inclusion $F(1) \hookrightarrow \prod_{k\geq 1} F(k)$ under the isomorphism

$$\begin{aligned} \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right) &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}\left(F(1)_{\mathbb{U}}, \mathcal{L}og(1)\,|_{\mathbb{U}}\right) \\ &= \mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}\left(\widetilde{p}^*F(1), j^*\mathcal{L}og(1)\right) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(1), \prod_{k\geq 1} F(k)\right) \end{aligned}$$

induced by the projective limit of the edge homomorphisms in the Leray spectral sequence for $\widetilde{p}$, and the isomorphism of 2.3.b). Note that the definition of *pol* is independent of the choice of κ. For the details, we refer to [WiIII], Theorem 1.5 – as there, we define

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}\right) := \varprojlim_n \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}}\right) .$$

A description of $\mathcal{L}og$ and *pol*, in both incarnations, was given by Beilinson and Deligne; see [B4], 2.1, 3.1 and [BD1], § 1 for the Hodge version and [B4], 3.3 for the l–adic setting. The reader may find it useful to also consult [WiIV], chapters 3 and 4, setting $N = 1$ in the notation of loc. cit.

We recall the "values" of *pol* at spectra of cyclotomic fields: let $d \geq 2$, and $C := \operatorname{Spec}(R)$, where $R := A\left[\frac{1}{d}, T\right] / \Phi_d(T)$, where $\Phi_d(T)$ is the d-th cyclotomic polynomial.

C is canonically a closed, reduced subscheme of $\mathbb{G}_m \otimes_A A\left[\frac{1}{d}\right]$. For any integer b prime to d, there is an embedding

$$\begin{array}{rcl} i_b : C & \xrightarrow{\sim} & C \hookrightarrow \mathbb{G}_m \otimes_A A\left[\dfrac{1}{d}\right] \;, \\ \zeta & \longmapsto & \zeta^b \;. \end{array}$$

Since d is invertible on C, the image of i_b is actually contained in $\mathbb{U}$, and hence we may form the pullback of *pol* via i_b,

$$pol_b \in \operatorname{Ext}^1_{\operatorname{Sh}^s(C)} \left(F(1), \mathcal{L}og_b(1)\right) \;,$$

where $\mathcal{L}og_b$ denotes the pullback of $\mathcal{L}og$.

Now we have the following

Theorem 2.4 (Splitting Principle). *$\mathcal{L}og_b$ splits (uniquely) into a direct product*

$$\mathcal{L}og_b = \prod_{k \geq 0} \operatorname{Gr}^W_{-2k} \left(\mathcal{L}og_b\right) \;,$$

and $\operatorname{Gr}^W_{-2k} \left(\mathcal{L}og_b\right)$ is isomorphic to $F(k)$ for any $k \geq 0$.

Proof. [B4], 4, or [BD1], 3.6, or [WiIV], Lemma 3.10. Or use 4.11 and 5.2, whose proof is independent of 2.4. ☐

In order to identify pol_b with an element of

$$\prod_{k \geq 1} \operatorname{Ext}^1_{\operatorname{Sh}^s(C)} \left(F(1), F(k)\right) \;,$$

we need to fix an isomorphism

$$\kappa_b : \operatorname{Gr}^W_{\cdot} \mathcal{L}og_b \xrightarrow{\sim} \prod_{k \geq 0} F(k) \;.$$

By definition, κ_b is the pullback via i_b of the isomorphism

$$\kappa : \operatorname{Gr}^W_{\cdot} \mathcal{L}og \xrightarrow{\sim} \prod_{k \geq 0} F(k)$$

of pro–sheaves on $\mathbb{G}_m$ of [WiIV], chapters 3 and 4, which we briefly describe now:

By 2.1.d), there is a canonical projection

$$\varepsilon : \mathcal{L}og \longrightarrow F(0) \;.$$

Furthermore, there is a canonical isomorphism

$$\gamma : \mathrm{Gr}^W_{-2}\,\mathcal{L}og \xrightarrow{\sim} p^*\mathcal{H}^0_B\left(\mathbb{G}_m, F(0)\right)^\vee$$

given by the fact that both sides are equal to p^* of the mixed structure on the (abelianized) fundamental group $\pi_1(\mathbb{G}_{m,\mathrm{top}}, 1)$ (see [WiI], chapter 2).

Observe that there is an isomorphism

$$\mathrm{res} : \mathcal{H}^0_B\left(\mathbb{G}_m, F(0)\right) \xrightarrow{\sim} F(1)$$

given by the map "residue at 0".

Finally, both $\mathrm{Gr}^W_{\cdot}\,\mathcal{L}og$ and $\prod_{k\geq 0} F(k)$ carry a canonical multiplicative structure: for $\mathrm{Gr}^W_{\cdot}\,\mathcal{L}og$, this is a formal consequence of

[WiI], Corollary 3.4.ii)	[WiI], Corollary 3.2.ii)

(see Remark b) at the end of chapter 3 of loc. cit.).

Our isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot}\,\mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

is the unique isomorphism compatible with ε, $(\mathrm{res})^\vee\circ\gamma$, and the multiplicative structure of both sides.

Using the framing of $\mathcal{L}og_b$ given by κ_b, we may identify pol_b with an element of

$$\prod_{k\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(F(1), F(k)\right) \ ,$$

or, after twisting and forgetting the component "$k = 0$", as an element of

$$\prod_{k\geq 1} \mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(F(0), F(k)\right) \ .$$

Note that in the Hodge setting we do not lose any information by forgetting the component "$k = 0$" as there are no non–trivial extensions in $\mathrm{Sh}^s(C)$ of $F(0)$ by itself. This latter statement fails to hold in the l–adic context. It is however true that the zero–component of pol_b is trivial. One way to see this is via [WiIII], Corollary 2.2, where it is proved that there is in fact a mixed realization pol_b of which the above extensions are merely the Hodge and l–adic components. In the category of mixed realizations, there is a good concept of polarization, which ensures that there are no non–trivial extensions of pure realizations of the same weight. Alternatively, one uses Theorem 9.5, where it is proved that our pol_b lie in the image of the respective regulators. The claim then follows from the vanishing of $H^1_{\mathcal{M}}(C, 0)$.

Theorem 2.5 (Beilinson). *Under the isomorphism of A.2.12, we have in the Hodge setting:*

$$pol_b = \left((-1)^k \,\mathrm{Li}_k(\omega^b)\right)_{\omega,k} \in \prod_{k\geq 1}\left(\bigoplus_{\omega\in C(\mathbb{C})} \mathbb{C}/(2\pi i)^k\mathbb{Q}\right)^+ \ ,$$

where $\mathrm{Li}_k(z) := \sum_{n\geq 1}\frac{z^n}{n^k}$ *for* $|z| \leq 1$ *and* $z \neq 1$.

Proof. [B4], 4.1, or [BD1], 3.6.3.i), or [WiIV], Theorem 3.11. $\square$

Note that one may identify $C(\mathbb{C})$ with $\{\sigma : \mathbb{Q}(\mu_d) \hookrightarrow \mathbb{C}\}$ by associating to ω the unique embedding mapping $T \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$ to ω.

In the l–adic situation, choose a geometric point $\zeta \in C(\overline{\mathbb{Q}})$. It allows to identify C and

$$\operatorname{Spec}\left(\mathbb{Z}\left[\zeta, \frac{1}{ld}\right]\right) \; ,$$

and, furthermore, the category of continuous $\mathbb{Q}_l$–modules under the Galois group of $\mathbb{Q}(\zeta)$ that are mixed and unramified outside ld, and the category $\mathrm{Sh}^s(C) = \mathrm{Et}^{l,m}_{\mathbb{Q}_l}(C)$.

Given this, we think of $\mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(\mathbb{Q}_l(0), \mathbb{Q}_l(k)\right)$ as sitting inside

$$H^1_{cont}\left(\mathbb{Q}(\zeta), \mathbb{Q}_l(k)\right) \; .$$

Together with the natural map of Lemma B.4.9 we thus have an inclusion of $\mathrm{Ext}^1_{\mathrm{Sh}^s(C)}\left(\mathbb{Q}_l(0), \mathbb{Q}_l(k)\right)$ into

$$\left(\varprojlim_{r\geq 1}\left(\mathbb{Q}(\mu_{l^\infty}, \zeta)^* / (\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes(k-1)}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \; .$$

Theorem 2.6 (Beilinson). *Under the above inclusion, we have in the l–adic setting:*

$$pol_b = \left((-1)^{k-1} \cdot \frac{1}{d^{k-1}} \cdot \frac{1}{(k-1)!} \cdot \sum_{\alpha^{l^r} = \zeta^b} \left([1-\alpha] \otimes (\alpha^d)^{\otimes(k-1)}\right)\right)_{r,k\geq 1} \; .$$

Proof. [B4], 4.1, or [BD1], 3.6.3.ii), or [WiIV], Theorem 4.5. $\square$

Remarks: a) Using the defining property of *pol*, one can show (see [B4], 2.12 or [BD1], proof of 3.1.1) that it coincides with a specific subquotient of the generic pro-unipotent sheaf on $\mathbb{U}$. The specializations to spectra of cyclotomic fields of this subquotient were already studied in [D5], section 16. In particular, Theorems 2.5 and 2.6 are equivalent to the Hodge and l-adic versions of [D5], Théorème 16.24.
b) One of the main results of this work will be (Theorem 9.5) that the elements in 2.5 and 2.6, for fixed b and d, are the respective regulators of one and the same element in motivic cohomology. This implies that Soulé's construction of cyclotomic elements in the K–theory with $\mathbb{Z}_l$–coefficients of an abelian number field ([Sou2], Lemma 1, [Sou5]) actually factors over the image of K–theory proper (Corollary 9.8). As shown in [BlK], § 6, Theorem 9.5 also implies that the Tamagawa number conjecture modulo powers of 2 is also true for odd Tate twists (see our Corollary 9.9). Finally, 9.5 is used in [KNF], Theorem 6.4 to prove the modified version of the Lichtenbaum conjecture for abelian number fields.
c) There are relative versions of 2.1 and 2.3 for schemes over a base scheme S smooth over B. They allow to directly define the small polylogarithmic extension pol_S on $\mathbb{U} \times_B S$, which however turns out to be the base change to S of *pol*.

Remark: In our definition of *pol*, we chose not to follow [BD1], 3.1. The approach via the universal property of $\mathcal{L}og$ and the computation of its cohomology rather imitates that of Beilinson and Levin in the elliptic case ([BL], 1.2, 1.3). In fact, one of the predecessors of loc. cit. contains a unified definition of $\mathcal{L}og$ and *pol* for relative curves of arbitrary genus ([BLp], 1).

3 The Geometric Set-Up

For easier reference, we assemble the notation used in the next sections.
As before, we let

$A := \mathbb{R}$,	$l :=$ a fixed prime number, $A := \mathbb{Z}\left[\frac{1}{l}\right]$,

$$\begin{aligned} &B := \operatorname{Spec}(A) \ , \\ &\mathbb{G}_m := \mathbb{G}_{m,B} \ , \ \mathbb{U} := \mathbb{P}^1_B \setminus \{0,1,\infty\}_B \ . \end{aligned}$$

Furthermore, we let $\underline{S}$ denote a smooth separated scheme over B of pure relative dimension $d(\underline{S})$,

$$\underline{\alpha}, \underline{\beta} \in \mathbb{G}_m(\underline{S}) \ ,$$

$S \subset \underline{S}$ the open subscheme of $\underline{S}$ where $\underline{\alpha}$ and $\underline{\beta}$ are disjoint. We assume S to be dense in $\underline{S}$.

$$\begin{aligned} &j : S \hookrightarrow \underline{S} \ , \\ &i : \underline{S} \setminus S \hookrightarrow \underline{S} \ , \end{aligned}$$

where $\underline{S} \setminus S$ is equipped with the reduced scheme structure.

$$\underline{Z} := \underline{\alpha}(\underline{S}) \cup \underline{\beta}(\underline{S})$$

with the reduced scheme structure,

$$\underline{V} := \mathbb{G}_{m,\underline{S}} \setminus \underline{Z} \ .$$

For $n \geq 0$, define

$$\begin{aligned} &\underline{p}^n : \mathbb{G}^n_{m,\underline{S}} \to \underline{S} \ , \\ &\underline{v}^n : \underline{V}^n \hookrightarrow \mathbb{G}^n_{m,\underline{S}} \ , \\ &\underline{z}^{(n)} : \underline{Z}^{(n)} := \mathbb{G}^n_{m,\underline{S}} \setminus \underline{V}^n \hookrightarrow \mathbb{G}^n_{m,\underline{S}} \ , \end{aligned}$$

where $\underline{Z}^{(n)}$ carries the reduced scheme structure. (So $\underline{p}^0 = \underline{v}^0 = \mathrm{id}_{\underline{S}}$, and $\underline{Z}^{(0)} = \emptyset$.)

The base change of the above objects and morphisms to S is denoted by the same letters not underlined:

$$\begin{aligned}
&\alpha, \beta : S \to \mathbb{G}_{m,S} \ , \\
&Z := \alpha(S) \amalg \beta(S) \ , \\
&V := \mathbb{G}_{m,S} \setminus Z \ , \\
&p^n : \mathbb{G}^n_{m,S} \to S \ , \\
&v^n : V^n \hookrightarrow \mathbb{G}^n_{m,S} \ , \\
&z^{(n)} : Z^{(n)} \hookrightarrow \mathbb{G}^n_{m,S} \ .
\end{aligned}$$

Also, we define partial compactifications of p^n:

$$\begin{aligned}
&g^n : \mathbb{G}^n_{m,S} \hookrightarrow \mathbb{A}^n_S \ , \\
&h^{(n)} : H^{(n)} := \mathbb{A}^n_S \setminus \mathbb{G}^n_{m,S} \hookrightarrow \mathbb{A}^n_S \ ,
\end{aligned}$$

where again $H^{(n)}$ has the reduced structure,

$$\begin{aligned}
&\overline{p}^n : \mathbb{A}^n_S \to S \ , \\
&\overline{V} := \mathbb{A}^1_S \setminus Z \ , \\
&\overline{v}^n : \overline{V}^n \hookrightarrow \mathbb{A}^n_S \ , \\
&\overline{z}^{(n)} : \overline{Z}^{(n)} := \mathbb{A}^n_S \setminus \overline{V}^n \hookrightarrow \mathbb{A}^n_S \ ,
\end{aligned}$$

where $\overline{Z}^{(n)}$ is equipped with the reduced structure. (So $\overline{Z}^{(1)} = Z^{(1)} = Z$.)

Remarks: a) The $\underline{\text{under}}$lined objects should remind the reader that the partial compactification comes from the compactification j of the base S. The $\overline{\text{over}}$lined objects refer to compactification upstairs, induced from g^n.
b) For fixed n, we have a natural action of the symmetric group $\mathfrak{S}_n$ on our geometric situation.

For the purposes of K-theory in section 7 we will have to replace the singular scheme $Z^{(n)}$ by some smooth simplicial scheme. Put

$$Z^{(n)}_0 = Z \times_S \mathbb{G}^{n-1}_{m,S} \amalg \mathbb{G}_{m,S} \times Z \times_S \mathbb{G}^{n-2}_{m,S} \amalg \ldots \amalg \mathbb{G}^{n-1}_{m,S} \times_S Z$$

Note that $Z^{(n)}_0$ is a proper covering of $Z^{(n)}$. This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Z^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Z^{(n)}_{\cdot} = \operatorname{cosk}_0(Z^{(n)}_0 / \mathbb{G}^n_{m,S}) \ ,$$

i.e.,

$$Z^{(n)}_k = Z^{(n)}_0 \times_{\mathbb{G}^n_{m,S}} \cdots \times_{\mathbb{G}^n_{m,S}} Z^{(n)}_0 \qquad (k+1\text{-fold product}).$$

Put $Z_\cdot^{(0)} = \star$ (corresponding to the empty scheme). We will also use the simplicial scheme $\overline{Z}_\cdot^{(n)}$ which is attached to $\overline{Z}^{(n)}$ sitting in $\mathbb{A}^n_S$ in the same way. Finally let

$$\mathbb{G}^{\vee n}_{m,S} = \mathrm{Cone}(Z_\cdot^{(n)} \longrightarrow \mathbb{G}^n_{m,S})$$
$$\mathbb{A}^{\vee n}_S = \mathrm{Cone}(\overline{Z}_\cdot^{(n)} \longrightarrow \mathbb{A}^n_S)$$

where the cone is taken in the category of pointed simplicial sheaves on the big Zariski site (cf. the discussion in appendix B.1).

4 Geometric Origin of the Logarithmic Sheaf

In section 2, we defined a pro–sheaf

$$\mathcal{L}og \in \text{pro}-U\,\mathrm{Sh}^s(\mathbb{G}_m)$$

and an element

$$\begin{aligned} pol &\in \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), \mathcal{L}og(1)\,|_{\mathbb{U}}) \\ &= \varprojlim_n \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}(F(1), (\mathcal{L}og/W_{-n}\mathcal{L}og)(1)\,|_{\mathbb{U}})\,. \end{aligned}$$

The aim of this section is to identify $\mathcal{L}og\,|_{\mathbb{U}}$, or rather, its Noetherian quotients, as relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$ (Theorem 4.11).

Recall that according to our conventions, we have

$$F(0) = F(0)_{\mathbb{U}}[1] \quad ,$$

and hence we may view *pol* as an element of

$$\mathrm{Hom}_{D^b\,\mathrm{Sh}(\mathbb{U})}(F(0)_{\mathbb{U}}, \mathcal{L}og\,|_{\mathbb{U}}) = H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}})\ ,$$

where we have used the notation introduced in Definition 1.2.

For the schemes of section 3, we have the following

Definition 4.1. *For $n \geq 0$,*

$$\mathcal{G}^{(n)} := \mathcal{H}^0_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = \mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{sgn} \quad ,$$

where the superscript sgn refers to the sign–eigenspace under the natural action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{G}^n_{m,S}$ and V^n.
Observe in particular that $\mathcal{G}^{(0)} = F(0)$.

The following is an immediate consequence of the Künneth formula:

Lemma 4.2. *There is a canonical isomorphism*

$$\mathcal{G}^{(n)} \xrightarrow{\sim} \mathrm{Sym}^n\,\mathcal{G}^{(1)}\ .$$

We want to compute $\mathcal{G}^{(n)}$, and simultaneously construct, for each $n \geq 1$, a projection

$$\mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

via the "residue at 0", whose projective limit over n we shall then identify, for special α and β, and $S = \mathbb{U}$, with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic pro–sheaf to $\mathbb{U}$.

Let $H^{(n)}_{\mathrm{sing}}$ be the singular part of $H^{(n)}$ and $H^{(n)}_{\mathrm{reg}} := H^{(n)} \setminus H^{(n)}_{\mathrm{sing}}$ the smooth part. For any subscheme of $\mathbb{A}^n_S$, the subscript reg will mean the complement of $H^{(n)}_{\mathrm{sing}}$. We work with the following geometric arrangement:

$$\begin{array}{ccccc}
\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} & \longrightarrow & \overline{V}^n_{\mathrm{reg}} & \longleftarrow & V^n \\
{\scriptstyle \overline{v}^n_{H,\mathrm{reg}}}\big\downarrow & & {\scriptstyle \overline{v}^n_{\mathrm{reg}}}\big\downarrow & & {\scriptstyle v^n}\big\downarrow \\
H^{(n)}_{\mathrm{reg}} & \xrightarrow{h^{(n)}_{\mathrm{reg}}} & \mathbb{A}^n_{S,\mathrm{reg}} & \xleftarrow{g^n_{\mathrm{reg}}} & \mathbb{G}^n_{m,S}
\end{array}$$

Both squares are cartesian. All maps are either open or closed immersions, and each line gives in fact a smooth pair of S–schemes.

Lemma 4.3. *For any complex $M \in D^b\,\mathrm{Sh}(\mathbb{A}^n_{S,\mathrm{reg}})$ such that $(\overline{v}^n_{\mathrm{reg}})^*M$ is a shift of a smooth sheaf on $\overline{V}^n_{\mathrm{reg}}$, there is an exact triangle*

$$(*)\qquad \begin{array}{ccc}
(h^{(n)}_{\mathrm{reg}})_*\,(\overline{v}^n_{H,\mathrm{reg}})_!\left(\overline{v}^n_{H,\mathrm{reg}} \circ h^{(n)}_{\mathrm{reg}}\right)^* M(-1)[-2] & \longrightarrow & (\overline{v}^n_{\mathrm{reg}})_!\,(\overline{v}^n_{\mathrm{reg}})^*M \\
{}_{[1]}\nwarrow & & \swarrow \\
& (g^n_{\mathrm{reg}})_*\; v^n_!\left(v^n \circ g^n_{\mathrm{reg}}\right)^* M &
\end{array}$$

Proof. This is $(\overline{v}^n_{\mathrm{reg}})_!$ applied to the exact triangle obtained from purity for the closed immersion

$$\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \longrightarrow \overline{V}^n_{\mathrm{reg}}$$

of smooth schemes. ☐

We apply this lemma to $M = F(n)_{\mathbb{A}^n_{S,\mathrm{reg}}}$, and evaluate the cohomological functors $H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}}, \,\cdot\,)^{\mathrm{sgn}}$ on the triangle $(*)$. Following 1.2.c), we write everything as relative cohomology with Tate coefficients:

$$\begin{aligned}
\ldots \longrightarrow & H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}} \text{ rel } \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{\mathrm{sgn}} \longrightarrow H^i_{\mathrm{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\
& \longrightarrow H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{\mathrm{sgn}} \\
\longrightarrow & H^{i+1}_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}} \text{ rel } \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{\mathrm{sgn}} \longrightarrow \ldots
\end{aligned}$$

We refer to this as the *absolute residue sequence*.

Application of the cohomological functors $\mathcal{H}^i_S(\mathbb{A}^n_{S,\mathrm{reg}}, \,\cdot\,)^{\mathrm{sgn}}$ to the same exact triangle yields a long exact sequence of sheaves on S that we call the *relative residue* *sequence*:

$$\begin{aligned}\dots \longrightarrow &\mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \longrightarrow \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}} \\ &\longrightarrow \mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \\ \longrightarrow &\mathcal{H}^{i+1}_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{\mathrm{sgn}} \longrightarrow \dots\end{aligned}$$

Note that $\mathcal{G}^{(n)} = \mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}}$ occurs in this sequence.

We are now going to further analyse, and reshape these sequences. The final form will be achieved in Proposition 4.8 and Theorem 4.9.

First, we need to identify the terms

$$\begin{aligned} H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{\mathrm{sgn}}\ , &\quad n \geq 1\ , \\ \mathcal{H}^{i-1}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right)^{\mathrm{sgn}}\ , &\quad n \geq 1\ .\end{aligned}$$

The complement of $\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}$ in $H^{(n)}_{\mathrm{reg}}$ is given by

$$\overline{v}^n_{H,\mathrm{reg}} : \overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} \to H^{(n)}_{\mathrm{reg}}\ .$$

Since $\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n V^{n-1}$ under the identification

$$H^{(n)}_{\mathrm{reg}} = \coprod_{k=1}^n \mathbb{G}^{n-1}_{m,S}\ ,$$

and these components are permuted transitively by $\mathfrak{S}_n$, we conclude

Lemma 4.4.

a) $$(\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}} = \left(\coprod_{k=1}^n v^{n-1}\right)_! F(n-1)_{\coprod_{k=1}^n V^{n-1}}\ .$$

b) $$H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1) = \bigoplus_{k=1}^n H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)\ ,$$

and hence the sign–eigenspace $H^{i-1}_{\mathrm{abs}}(H^{(n)}_{\mathrm{reg}} \text{ rel } (\overline{Z}^{(n)} \cap H^{(n)}_{\mathrm{reg}}), n-1)^{sgn}$ *is isomorphic to*

$$H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{sgn}\ ,$$

where the last sgn refers to the action of $\mathfrak{S}_{n-1}$. *The isomorphism is given by projection onto the components unequal to* k, *for some* choice $k \in \{1, \dots, n\}$. *It is independent of the choice of* k.

c) $$\mathcal{R}_S\left(H^{(n)}_{\mathrm{reg}}, (\overline{v}^n_{H,\mathrm{reg}})_! F(n-1)_{\overline{V}^n_{\mathrm{reg}} \cap H^{(n)}_{\mathrm{reg}}}\right) = \bigoplus_{k=1}^n \mathcal{R}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right)\ .$$

As in b), the sign–eigenspace $\mathcal{H}_S^{i-1}\left(H_{\mathrm{reg}}^{(n)}, (\overline{v}_{H,\mathrm{reg}}^n)_! F(n-1)_{\overline{V}_{\mathrm{reg}}^n \cap H_{\mathrm{reg}}^{(n)}}\right)^{sgn}$ *is canonically isomorphic to*

$$\mathcal{H}_S^{i-1}\left(\mathbb{G}_{m,S}^{n-1}, v_!^{n-1} F(n-1)_{V^{n-1}}\right)^{sgn} .$$

For $i = n + d(\underline{S})$*, the latter equals* $\mathcal{G}^{(n-1)}$.

Proof. The only point that remains to be shown is the independence of the isomorphisms in b) and c) of the choice of k. Recall the identity

$$R\Gamma_{\mathrm{abs}}\left(\mathbb{G}_{m,S}^n \text{ rel } Z^{(n)}, n\right)^{\mathrm{sgn}} = R\Gamma_{\mathrm{abs}}\left(S, \mathcal{R}_S\left(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n}\right)\right)^{\mathrm{sgn}} .$$

We are going to prove in 4.6.d) that $\mathcal{H}_S^q(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n})^{\mathrm{sgn}} = 0$ for $q \neq n + d(\underline{S})$. So the associated spectral sequence degenerates, and shows that the independence of the map in b) follows from that of the map in c).
For c), we only need to consider $\mathcal{G}^{(n)} = \mathcal{H}_S^{n+d(\underline{S})}(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n})^{\mathrm{sgn}}$. There, our claim follows from Lemma 4.2, and the graded–compatibility of the cup product with boundary morphisms ([GH], Proposition 2.2 and Corollary 2.3). ☐

Remark: The arguments of this section would become simpler if we could use an object $\mathcal{R}_S^{\mathrm{sgn}}$ in c). However, we do not know whether it is possible to make a decomposition into eigenspaces in our triangulated categories.

By the identification of the lemma, the residue sequences define canonical *residue maps*

$$\mathrm{res} : H_{\mathrm{abs}}^i(\mathbb{G}_{m,S}^n \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \longrightarrow H_{\mathrm{abs}}^{i-1}(\mathbb{G}_{m,S}^{n-1} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} ,$$
$$\mathrm{res} : \mathcal{H}_S^i\left(\mathbb{G}_{m,S}^n, v_!^n F(n)_{V^n}\right)^{\mathrm{sgn}} \longrightarrow \mathcal{H}_S^{i-1}\left(\mathbb{G}_{m,S}^{n-1}, v_!^{n-1} F(n-1)_{V^{n-1}}\right)^{\mathrm{sgn}}$$

fitting into the relative and absolute residue sequences. In particular, observe that we have a residue map

$$\mathrm{res} : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n-1)} .$$

Now we concern ourselves with the identification of the remaining terms

$$H_{\mathrm{abs}}^i(\mathbb{A}_{S,\mathrm{reg}}^n \text{ rel } \overline{Z}_{\mathrm{reg}}^{(n)}, n)^{\mathrm{sgn}} \; , n \geq 0 \; ,$$
$$\mathcal{H}_S^i\left(\mathbb{A}_{S,\mathrm{reg}}^n, (\overline{v}_{\mathrm{reg}}^n)_! F(n)_{\overline{V}_{\mathrm{reg}}^n}\right)^{\mathrm{sgn}} \; , n \geq 0$$

of the residue sequences.

We use the following filtration of $\mathbb{A}_S^n$ by open subschemes:

$$F_k \mathbb{A}_S^n := \{(x_1, \ldots , x_n) \in \mathbb{A}_S^n \,|\, \text{at most } k \text{ coordinates vanish}\} .$$

So we have $F_n \mathbb{A}_S^n = \mathbb{A}_S^n$ and $F_0 \mathbb{A}_S^n = \mathbb{G}_{m,S}^n$.

The "graded pieces" of this filtration are

$$\begin{aligned} G_k \mathbb{A}_S^n &:= F_k \mathbb{A}_S^n \setminus F_{k-1} \mathbb{A}_S^n \\ &= \{(x_1, \ldots , x_n) \in \mathbb{A}_S^n \,|\, \text{precisely } k \text{ coordinates vanish}\} . \end{aligned}$$

$G_k\mathbb{A}^n_S$ is equipped with the reduced scheme structure. Note that it splits into several disjoint pieces. For $k \geq 2$ and any such piece, there is a transposition of $\mathfrak{S}_n$ acting trivially. By using triangles similar to $(*)$ for the inclusions

$$G_k\mathbb{A}^n_S \hookrightarrow F_k\mathbb{A}^n_S \hookleftarrow F_{k-1}\mathbb{A}^n_S \ ,$$

we conclude inductively that the sign–eigenpart of the cohomology of $H^{(n)}_{\mathrm{sing}}$ is trivial:

Lemma 4.5. *The adjunction morphism induces isomorphisms*

$$H^i_{\mathrm{abs}}(\mathbb{A}^n_S \ rel\ \overline{Z}^{(n)}, n)^{sgn} \xrightarrow{\sim} H^i_{\mathrm{abs}}(\mathbb{A}^n_{S,\mathrm{reg}} \ rel\ \overline{Z}^{(n)}_{\mathrm{reg}}, n)^{sgn} ,$$
$$\mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}\right)^{sgn} \xrightarrow{\sim} \mathcal{H}^i_S\left(\mathbb{A}^n_{S,\mathrm{reg}}, (\overline{v}^n_{\mathrm{reg}})_! F(n)_{\overline{V}^n_{\mathrm{reg}}}\right)^{sgn} .$$

By 4.4.b) and 4.5, the absolute residue sequence takes the form

$$\begin{aligned} \cdots \to & H^i_{\mathrm{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\mathrm{sgn}} \to H^i_{\mathrm{abs}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\ & \xrightarrow{\mathrm{res}} H^{i-1}_{\mathrm{abs}}(\mathbb{G}^{n-1}_{m,S} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} \\ \to & H^{i+1}_{\mathrm{abs}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}, n)^{\mathrm{sgn}} \to \dots \end{aligned}$$

Similarly, the relative residue sequence looks as follows:

$$\begin{aligned} \dots \longrightarrow & \mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} \longrightarrow \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{\mathrm{sgn}} \\ & \xrightarrow{\mathrm{res}} \mathcal{H}^{i-1}_S\left(\mathbb{G}^{n-1}_{m,S}, v^{n-1}_! F(n-1)_{V^{n-1}}\right)^{\mathrm{sgn}} \\ \longrightarrow & \mathcal{H}^{i+1}_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} \longrightarrow \dots \end{aligned}$$

For the computation of the term

$$\mathcal{H}^i_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}\right)^{\mathrm{sgn}} ,$$

we use the Künneth formula:

Lemma 4.6. a) $\mathcal{R}_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right)[0]$, *and the Künneth formula gives an isomorphism*

$$\mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right)^{sgn} \xrightarrow{\sim} \mathrm{Sym}^n \mathcal{H}^0_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) .$$

b) *The choice of an ordering of the sections α and β gives an isomorphism*

$$\mathcal{R}_S\left(\mathbb{A}^1_S, \overline{v}_!^1 F(1)\right) = \mathcal{H}^0_S\left(\mathbb{A}^1_S \overline{v}_!^1 F(1)\right)[0] \xrightarrow{\sim} F(1)[0] .$$

Up to sign, it is canonical.
c) *The isomorphisms of* a) *and* b) *induce an isomorphism*

$$\mathcal{H}^{n+d(\underline{S})}_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)_{\overline{V}^n}\right) = \mathcal{H}^0_S\left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right) \xrightarrow{\sim} F(n) .$$

It depends on the choice made in b) *only up to the sign $(-1)^n$. The group $\mathfrak{S}_n$ acts on these objects via the sign character.*
d) *For $i \neq 0$, we have*

$$\mathcal{H}^{i+n+d(\underline{S})}_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)_{V^n}\right)^{sgn} = \mathcal{H}^i_S\left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = 0 .$$

Proof. For b), consider the long exact cohomology sequence associated to the triangle

$$(**)\qquad \begin{array}{ccc} \overline{v}_!^1 F(1) & \longrightarrow & F(1) \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & \overline{z}_*^{(1)} F(1) & \end{array}$$

We have

$$\mathcal{H}_S^i\left(\mathbb{A}_S^1, F(1)\right) = \begin{cases} F(1)\,, & i=-1 \\ 0\,, & i \neq -1 \end{cases}$$

and

$$\mathcal{H}_S^i\left(\mathbb{A}_S^1, \overline{z}_*^{(1)} F(1)\right) = \begin{cases} \bigoplus\limits_{\alpha,\beta} F(1)\,, & i=0 \\ 0\,, & i \neq 0 \end{cases} .$$

The long exact cohomology sequence thus reads

$$0 \to \mathcal{H}_S^{-1}\left(\mathbb{A}_S^1, \overline{v}_!^1 F(1)\right) \to F(1) \xrightarrow{\Delta} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{H}_S^0\left(\mathbb{A}_S^1, \overline{v}_!^1 F(1)\right) \to 0\,.$$

If we let $\{\alpha,\beta\} = \{s_1, s_2\}$, then we identify the cokernel of

$$\Delta : F(1) \longrightarrow \bigoplus_{\alpha,\beta} F(1) = \bigoplus_{i=1}^{2} F(1)$$

with $F(1)$ by mapping $(f_{s_1}, f_{s_2}) \in \bigoplus_{i=1}^2 F(1)$ to $f_{s_2} - f_{s_1}$.
a) follows from b) since $\bigotimes^n F(1) = \mathrm{Sym}^n F(1)$.
c) is a consequence of a) and b).
d) follows from a) and the relative residue sequence by induction on n. ∎

On the level of absolute cohomology, the isomorphism of 4.6.c) induces an isomorphism

$$H_{\mathrm{abs}}^{i+n}\left(\mathbb{A}_S^n \text{ rel } \overline{Z}^{(n)}, n\right) = H_{\mathrm{abs}}^{i+n}\left(\mathbb{A}_S^n \text{ rel } \overline{Z}^{(n)}, n\right)^{\mathrm{sgn}} \xrightarrow{\sim} H_{\mathrm{abs}}^i(S,n)\,.$$

This gives the final shape of the absolute residue sequence:

$$\begin{aligned} \cdots \longrightarrow & H_{\mathrm{abs}}^i(S,n) \xrightarrow{\delta} H_{\mathrm{abs}}^{i+n}(\mathbb{G}_{m,S}^n \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \\ & \qquad\qquad \xrightarrow{\mathrm{res}} H_{\mathrm{abs}}^{i+n-1}(\mathbb{G}_{m,S}^{n-1} \text{ rel } Z^{(n-1)}, n-1)^{\mathrm{sgn}} \\ \longrightarrow & H_{\mathrm{abs}}^{i+1}(S,n) \xrightarrow{\delta} \cdots \end{aligned}$$

By 4.6.d), the relative residue sequence collapses into the short exact sequence of sheaves on S:

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \xrightarrow{\mathrm{res}} \mathcal{G}^{(n-1)} \longrightarrow 0\,.$$

In order to identify the long exact absolute cohomology sequence associated to this sequence with the absolute residue sequence, we need the following:

LEMMA 4.7. *Let $K \in D^b \operatorname{Sh}(X)$ be a complex of sheaves on a separated, reduced and flat B–scheme X. Suppose there is an action of a finite group G on K. Let χ be the character of an absolutely irreducible representation of G over F. For any object $\mathbb{V}$ with a G–action of an F-linear abelian category, denote by $\mathbb{V}(\chi)$ the χ–isotypical component of $\mathbb{V}$, i.e., the image under the projector*

$$e_\chi := \frac{1}{\#G} \sum_{g \in G} \chi(g^{-1}) \cdot g \ .$$

Suppose that $(\mathcal{H}^i K)(\chi)$ vanishes for all $i \neq 0$. Then

$$\operatorname{Hom}_{D^b}(F, K[i])(\chi) = \operatorname{Hom}_{D^b}(F, (\mathcal{H}^0 K)(\chi)[i])$$

Proof. By applying e_χ and $1 - e_\chi$, one checks the statement for a complex of the special form $K \cong \mathcal{H}^0 K$. For the general case, consider the spectral sequence for $\operatorname{Hom}_{D^b}(F, \,\cdot\, [i])$ induced by the truncation functors $\tau_{\leq n}$. It degenerates after applying e_χ. □

Now that we know that formation of absolute cohomology commutes with formation of sign eigenspaces, we have:

PROPOSITION 4.8. *The absolute residue sequence is the long exact sequence in absolute cohomology attached to the short exact sequence*

$$0 \longrightarrow F(n) \longrightarrow \mathcal{G}^{(n)} \xrightarrow{\text{res}} \mathcal{G}^{(n-1)} \longrightarrow 0 \, .$$

We conclude the computational part of this section by collecting our results:

THEOREM 4.9. a) *For $n \geq 0$, we have*

$$\mathcal{H}^0_S \left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = \mathcal{G}^{(n)} \ ,$$

and $\mathcal{H}^i_S \left(\mathbb{G}^n_{m,S}, v^n_! F(n)\right)^{sgn} = 0$ for $i \neq 0$.
b) *The residue at 0, i.e., the boundary map of $(*)$, gives an epimorphism*

$$\text{res} : \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

for $n \geq 1$.
c) *The Künneth formula gives an isomorphism*

$$\mathcal{H}^0_S \left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right) = \mathcal{H}^0_S \left(\mathbb{A}^n_S, \overline{v}_!^n F(n)\right)^{sgn} \xrightarrow{\sim} \ker(\text{res})$$

for $n \geq 1$. A choice of an ordering of the sections α and β induces an isomorphism

$$F(n) \xrightarrow{\sim} \ker(\text{res}) \ ,$$

which depends on this choice only up to the sign $(-1)^n$.
d) *Let $\mathcal{G}^{(n)} \xrightarrow{\sim} \operatorname{Sym}^n \mathcal{G}^{(1)}$ be the canonical isomorphism of 4.2, and*

$$\operatorname{Sym}^n F(0) \xrightarrow{\sim} F(0) \ ,$$
$$\operatorname{Sym}^n F(1) \xrightarrow{\sim} F(n)$$

the isomorphisms given by multiplication. Then the diagrams

$$\begin{array}{ccc} \mathcal{G}^{(n)} & \longrightarrow & F(0) \\ \downarrow \wr & & \uparrow \wr \\ \mathrm{Sym}^n \mathcal{G}^{(1)} & \longrightarrow & \mathrm{Sym}^n F(0) \end{array}$$

and

$$\begin{array}{ccc} F(n) & \longrightarrow & \mathcal{G}^{(n)} \\ \uparrow \wr & & \downarrow \wr \\ \mathrm{Sym}^n F(1) & \longrightarrow & \mathrm{Sym}^n \mathcal{G}^{(1)} \end{array}$$

commute. Here, the horizontal maps are given by the successive residue maps, and by c) respectively.
e) *Let* $W_{-2n-1}\,\mathcal{G}^{(n)} := 0$,

$$W_{-2k}\,\mathcal{G}^{(n)} := W_{-2k+1}\,\mathcal{G}^{(n)} := \ker(\mathcal{G}^{(n)} \to \mathcal{G}^{(k-1)}) \quad \textit{for} \quad 1 \le k \le n\,,$$

and $W_0\,\mathcal{G}^{(n)} := \mathcal{G}^{(n)}$. *The choice in c) induces isomorphisms*

$$\mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^n F(i)\ ,$$

which by their construction fit into commutative diagrams

$$\begin{array}{ccc} \mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n)} & \xrightarrow{\sim} & \bigoplus_{i=0}^n F(i) \\ \mathrm{Gr}^W_\cdot\,\mathrm{res}\downarrow & & \downarrow \mathrm{can} \\ \mathrm{Gr}^W_\cdot\,\mathcal{G}^{(n-1)} & \xrightarrow{\sim} & \bigoplus_{i=0}^{n-1} F(i) \end{array}$$

The filtration $W_\cdot$ *is therefore the weight filtration of* $\mathcal{G}^{(n)}$.

Proof. a), b) and c) follow from the previous results. The commutativity of the first diagram in d) follows from the definition of the residue map. For the second diagram, we use the fact that the Künneth formula of 4.2 is compatible with the Künneth formula of the proof of 4.6.a). For e), apply induction on n. □

Recall that S is the open subscheme of $\underline{S}$ where the sections $\underline{\alpha}$ and $\underline{\beta}$ of $\mathbb{G}_{m,\underline{S}}$ are disjoint. For special S, α and β, the following is the main step towards the identification of the projective limit of the $\mathcal{G}^{(n)}$ with the restriction $\mathcal{L}og\,|_{\mathbb{U}}$ of the logarithmic sheaf:

Lemma 4.10. a) *There is a unique smooth sheaf* $\underline{\mathcal{G}}^{(n)}$ *on* $\underline{S}$ *extending* $\mathcal{G}^{(n)}$. *It has a weight filtration.*
b) *There is a canonical isomorphism*

$$\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Sym}^n \underline{\mathcal{G}}^{(1)}\ ,$$

and a unique isomorphism

$$\eta^{(n)} : \mathrm{Gr}^W_\cdot\,\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^n F(i)\ ,$$

which is compatible with the isomorphism of 4.9.e).
c) *The weight filtration of $i^*\underline{\mathcal{G}}^{(n)}$ is split: there is a canonical isomorphism*

$$i^*\underline{\mathcal{G}}^{(n)} \xrightarrow{\sim} \mathrm{Gr}^W_{\cdot}\, i^*\underline{\mathcal{G}}^{(n)} \xrightarrow[b)]{\sim} \bigoplus_{i=0}^{n} F(i)\ .$$

Here, i denotes the inclusion of $\underline{S} \setminus S$ into $\underline{S}$.
d) *There is an exact sequence*

$$0 \longrightarrow i_*F(1) \longrightarrow \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

of sheaves on $\underline{S}$.

Proof. If there is any smooth sheaf as in a), then it will automatically be unique, and hence b) follows from a), and 4.9.d), e). Also, it will suffice, because of 4.9.d), to show the lemma for the case $n = 1$.
There we have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & i_*F(1) \arrow[d] & & & \\
\mathcal{H}^{-1} \arrow[r] & F(1) \arrow[r, "\Delta"] & K \arrow[r] \arrow[d] & \mathcal{H}^0 \arrow[r] & F(0) \arrow[r] & 0 \\
 & & \bigoplus_{\alpha,\beta} F(1) \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

where

$$K = \mathcal{H}^{-1}\,\mathrm{Cone}(\delta : \bigoplus_{\alpha,\beta} F(1)_{\underline{S}}[d(\underline{S})] \to i_*F(1)_{\underline{S}\setminus S}[d(\underline{S})])$$

with $\delta(v_1, v_2) := v_1 - v_2$ (in terms of constructible sheaves this is just $\mathrm{Ker}\,\delta$ shifted in the appropriate degree to define a perverse sheaf). The horizontal sequence is, as in the proof of 4.6.b), the long exact cohomology sequence on $\underline{S}$ associated to the short exact sequence on $\mathbb{G}_{m,\underline{S}}$

$$(**) \qquad 0 \to \underline{z}_*^{(1)} F(1) \to \underline{v}_!^1 F(1) \to F(1) \to 0\ ,$$

where we have set

$$\mathcal{H}^i := \mathcal{H}^i_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right)\ .$$

We thus get the equality

$$\mathcal{R}_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right) = \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}_!^1 F(1)\right)[0]\ ,$$

and an exact sequence of sheaves on $\underline{S}$

$$0 \to K/\Delta\,(F(1)) \to \mathcal{H}^0_{\underline{S}}\left(\mathbb{G}_{m,\underline{S}}, \underline{v}^1_! F(1)\right) \to F(0) \to 0 \ ,$$

whose restriction to S is isomorphic, via the choice of an ordering of α and β, to

$$0 \to F(1) \to \mathcal{G}^{(1)} \to F(0) \to 0 \ .$$

Push out of the above via the morphism

$$K/\Delta(F(1)) \to \left(\bigoplus_{\alpha,\beta} F(1)\right)/\Delta(F(1)) \ ,$$

whose kernel is $i_*F(1)$ (recall again that we use perverse indices), gives the desired extension $\underline{\mathcal{G}}^{(1)}$. By construction b) and d) hold. Applying i^* to the pushout diagram and taking cohomology, we see that the sheaf $i^*\underline{\mathcal{G}}^{(1)}[-1]$ is the pushout of $F(0)$ via

$$0 \hookrightarrow F(1) \ ,$$

and we get c). □

We now specialize our geometric situation: we let

$$\begin{aligned} &\underline{S} := \mathbb{G}_{m,B} \ , \\ &\underline{\alpha} := 1 : \mathbb{G}_{m,B} \twoheadrightarrow B \hookrightarrow \mathbb{G}_{m,B} \ , \\ &\underline{\beta} := \mathrm{id} : \mathbb{G}_{m,B} \to \mathbb{G}_{m,B} \ . \end{aligned}$$

So we have $S = \mathbb{U}$ and $\underline{S} \setminus S = 1_B$, the closed subscheme of $\mathbb{G}_{m,B}$ given by the immersion 1 of B into $\mathbb{G}_{m,B}$.

After having made precise which choice of normalization we have and in how far it affects our identifications, we now fix it: we let

$$s_1 := \alpha = 1 \text{ and } s_2 := \beta = \mathrm{id} \text{ in 4.9.c).}$$

We thus get a projective system $(\underline{\mathcal{G}}^{(n)})_{n\geq 0}$ of smooth Tate sheaves on $\mathbb{G}_{m,B}$ with

$$\underline{\mathcal{G}}^{(n)}\,|_{1_B} = \bigoplus_{i=0}^{n} F(i) \ .$$

By the universal property of $\mathcal{L}og$ (Theorem 2.1.d)), there is a unique morphism

$$\varphi : \mathcal{L}og \to \underline{\mathcal{G}} := \varprojlim_n \underline{\mathcal{G}}^{(n)}$$

such that $\varphi\,|_{1(B)}$ sends $1 \in \Gamma(B, \mathcal{L}og\,|_{1_B})$ to

$$1 : F(0) \hookrightarrow \prod_{i=0}^{\infty} F(i) = \underline{\mathcal{G}}\,|_{1(B)} \ .$$

Theorem 4.11. *φ is an isomorphism.*

Proof. The claim can be shown on the level of the underlying topological sheaves. The l–adic statement follows from the statement for the topological spaces of $\mathbb{C}$–valued points by comparison – recall that we are dealing with locally constant sheaves.
Over $\mathbb{C}$, the fibre at 1 of the pro–local system $\mathcal{L}og_{\mathrm{top}}$ equals the completion of the group ring $\mathbb{Q}[\pi_1]$ of $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1) \cong \mathbb{Z}$ with respect to the augmentation ideal a. The representation of π_1 is given by multiplication; compare the general construction in [WiI], 2.5–2.7. In particular, we have

$$\mathcal{L}og_{\mathrm{top}} = \varprojlim_n \mathrm{Sym}^n(\mathcal{L}og_{\mathrm{top},\geq -2}) \; ,$$

where $\mathcal{L}og_{\mathrm{top},\geq -2} := \mathcal{L}og_{\mathrm{top}}/a^2$ is of dimension two. Now in the category of unipotent local systems on $\mathbb{G}_m(\mathbb{C})$, the pro–sheaf $\mathcal{L}og_{\mathrm{top}}$ has the universal property of Theorem 2.1.d).
We apply this universal property to $\underline{\mathcal{G}}_{\mathrm{top},\geq -2} := \underline{\mathcal{G}}^{(1)}_{\mathrm{top}}$. The resulting map factors over φ_{top}. Since $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ is two-dimensional, the representation of $\mathbb{Q}[\pi_1]$ is necessarily trivial on a^2, and we get a morphism of local systems

$$\varphi_{\mathrm{top},\geq -2} : \mathcal{L}og_{\mathrm{top},\geq -2} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top},\geq -2}$$

giving rise to a morphism

$$\varprojlim_n \mathrm{Sym}^n(\varphi_{\mathrm{top},\geq -2}) : \mathcal{L}og_{\mathrm{top}} \longrightarrow \underline{\mathcal{G}}_{\mathrm{top}} \; .$$

Again because of the universal property of $\mathcal{L}og_{\mathrm{top}}$, this morphism is identical to φ_{top}. It therefore suffices to show that $\varphi_{\mathrm{top},\geq -2}$ is bijective, which amounts to saying that the coinvariants of $\underline{\mathcal{G}}_{\mathrm{top},\geq -2}$ under the action of π_1 are one–dimensional. But taking coinvariants under π_1 of a unipotent variation $\mathbb{V}$ amounts to computing singular cohomology

$$H^1(\mathbb{G}_m(\mathbb{C}), \mathbb{V}) = \mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \mathbb{V}) \; .$$

Firstly, we claim that

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \begin{cases} F(-1)\,, & i = 0 \\ 0\,, & i \neq 0 \end{cases} \quad :$$

e.g., identify the left hand side with

$$\begin{aligned} & H^{i+2}\left(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), \Delta(\mathbb{G}_m(\mathbb{C})) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)\right) \\ \cong\; & H^{i+2}\left(\mathbb{G}_m(\mathbb{C}) \times \mathbb{G}_m(\mathbb{C}), (\mathbb{G}_m(\mathbb{C}) \times \{1\}) \cup (\{1\} \times \mathbb{G}_m(\mathbb{C})), F(1)\right) \; , \end{aligned}$$

and apply the Künneth formula. From the proof of 4.10, we recall – remember that we have $\underline{S} = \mathbb{G}_m$:

$$\mathcal{R}_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right) = \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)[0] \; ,$$

from which we conclude:

$$\mathcal{H}^i_{\mathrm{Spec}(\mathbb{R})}\left(\mathbb{G}_{m,\mathbb{R}}, \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\left(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\right)\right) = \begin{cases} F(-1)\,, & i = 0 \\ 0\,, & i \neq 0 \end{cases} \quad .$$

The long exact sequence obtained by applying $\mathcal{R}_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \_)$ to the exact sequence of 4.10.d)

$$0 \longrightarrow 1_* F(1) \longrightarrow \mathcal{H}^0_{\mathbb{G}_{m,\mathbb{R}}}\big(\mathbb{G}_{m,\mathbb{R}} \times \mathbb{G}_{m,\mathbb{R}}, \underline{v}^1_! F(1)\big) \longrightarrow \underline{\mathcal{G}}^{(1)} \longrightarrow 0$$

then shows that

$$\mathcal{H}^0_{\mathrm{Spec}(\mathbb{R})}(\mathbb{G}_{m,\mathbb{R}}, \underline{\mathcal{G}}^{(1)}) = F(-1)\ .$$

$\square$

Remark: The geometric situation used in this section is identical to the one of [BD1], 4.1–4.3 (see in particular loc. cit., 4.1.9). The comparison statement of our Proposition 4.8 is implicit in loc. cit., 4.3.3. We mention that basically the same geometric arrangement was used in [Jeu]. More precisely, writing down the iterated cone construction of loc. cit., one arrives at a simplicial object which is homotopy equivalent to Beilinson's and Deligne's construction used here.

5 The Splitting Principle Revisited

In order to be able to translate easily to the motivic context, we recall Beilinson's original proof ([B4], 4) of the splitting of the logarithmic pro-sheaf over spectra of cyclotomic fields (Theorem 2.4).

First, we return to the general situation considered at the beginning of section 4. For $N \geq 1$, we have the morphism of S-schemes

$$\begin{array}{rcl} \phi : \mathbb{G}_{m,S} & \longrightarrow & \mathbb{G}_{m,S}\ , \\ x & \longmapsto & x^N\ , \end{array}$$

and for each $n \geq 0$, the induced morphism

$$\phi^n : \mathbb{G}^n_{m,S} \longrightarrow \mathbb{G}^n_{m,S}\ .$$

We work under the additional assumption

(A) $$\phi\circ\alpha = \alpha\ , \quad \phi\circ\beta = \beta\ .$$

If this is the case, we have $(\phi^n)^{-1}(V^n) \subset V^n$, and hence get a morphism

$$(\phi^n)^* v^n_! F(n) \longrightarrow v^n_! F(n)\,,$$

and hence a morphism

$$(\phi^n)^\sharp : v^n_! F(n) \longrightarrow \phi^n_* v_! F(n)\,,$$

which after application of p^n_* and projection onto the sign-eigenpart induces

$$(\phi^n)^\sharp : \mathcal{G}^{(n)} \longrightarrow \mathcal{G}^{(n)}\ .$$

We need to understand the action of $(\phi^n)^\sharp$ on $\mathcal{G}^{(n)}$, and on absolute cohomology. First, we establish in how far $(\phi^n)^\sharp$ is compatible with the residue at 0:

Lemma 5.1. a) *Under any isomorphism*

$$\mathrm{Gr}^W_{\cdot}\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\;,$$

the map $\mathrm{Gr}^W_{\cdot}(\phi^n)^\sharp$ *is multiplication by* N^{n-i} *on* $F(i)$.
b) *For any* $n \geq 1$, *the diagram*

$$\begin{tikzcd}
\mathcal{G}^{(n)} \arrow[r, "(\phi^n)^\sharp"] \arrow[d, "\mathrm{res}_n"'] & \mathcal{G}^{(n)} \arrow[d, "\mathrm{res}_n"] \\
\mathcal{G}^{(n-1)} \arrow[r, "N\cdot(\phi^{n-1})^\sharp"] & \mathcal{G}^{(n-1)}
\end{tikzcd}$$

commutes.

Proof. Since the morphisms in b) are strict with respect to the weight filtration, it suffices to check that

$$\mathrm{Gr}^W_{\cdot}(\mathrm{res}_n)\circ \mathrm{Gr}^W_{\cdot}(\phi^n)^\sharp = N\cdot \mathrm{Gr}^W_{\cdot}(\phi^{n-1})^\sharp \circ \mathrm{Gr}^W_{\cdot}(\mathrm{res}_n)\;.$$

But if we choose the isomorphism of 4.9.e), then $\mathrm{Gr}^W_{\cdot}(\mathrm{res}_n)$ is simply the canonical projection

$$\bigoplus_{i=0}^{n} F(i) \twoheadrightarrow \bigoplus_{i=0}^{n-1} F(i)\;,$$

and therefore b) follows from a). For a), we note first that it suffices to show the statement for one choice of isomorphism

$$\mathrm{Gr}^W_{\cdot}\,\mathcal{G}^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i)\;.$$

This time, we use the isomorphism on graded objects induced by 4.2, thereby reducing ourselves to the case $n = 1$. There, we consider the long exact cohomology sequence associated to the exact sequence

$$0 \to z_*^{(1)} F(1) \to v_!^1 F(1) \to F(1) \to 0\;,$$

and the cohomological functors $\mathcal{H}^i_S(\mathbb{G}_{m,S},\,\cdot\,)$. We know the cohomology of $\mathbb{G}_m$:

$$\mathcal{H}^i_S\left(\mathbb{G}_{m,S}, F(1)\right) = \left\{ \begin{array}{ccl} F(1) & , & i=-1 \\ F(0) & , & i=0 \\ 0 & , & i\notin\{-1,0\} \end{array} \right. .$$

Of course, we know the cohomology of two points:

$$\mathcal{H}^i_S\left(\mathbb{G}_{m,S}, z_*^{(1)} F(1)\right) = \left\{ \begin{array}{ccl} \bigoplus_{\alpha,\beta} F(1) & , & i=0 \\ 0 & , & i\neq 0 \end{array} \right. .$$

We get an exact sequence

$$0 \to F(1) \stackrel{\Delta}{\to} \bigoplus_{\alpha,\beta} F(1) \to \mathcal{G}^{(1)} = \mathcal{H}^0_S\left(\mathbb{G}_{m,S}, v^1_! F(1)\right) \to F(0) \to 0 \ .$$

and because of assumption (A), it carries an action of $(\phi^n)^\sharp$. But this action can be identified on $\mathcal{H}^i_S(\mathbb{G}_{m,S}, F(1))$ and $\mathcal{H}^i_S\left(\mathbb{G}_{m,S}, z_*^{(1)} F(1)\right)$: it is trivial on the $F(1)$, and multiplication by N on $F(0)$. ☐

Certainly (A) is only satisfied in very special situations, namely if α and β are supported in the schemes of $(N-1)$-torsion of $\mathbb{G}_{m,S}$.

Let again $d \geq 2$, $C := \mathrm{Spec}(R)$, where $R := A[\frac{1}{d}, T]/\Phi_d(T)$ as in section 2. For b prime to d, consider

$$\begin{array}{rcl} i_b : C & \xrightarrow{\sim} & C \hookrightarrow \mathbb{G}_m \ , \\ \zeta & \longmapsto & \zeta^b \ . \end{array}$$

The pullback $\mathcal{L}og_b$ of the pro-sheaf $\mathcal{L}og\,|_{\mathbb{U}}$ on $\mathbb{U}$ via i_b is identical to the projective limit of the sheaves $\mathcal{G}_b^{(n)}$ obtained by setting

$$\begin{aligned} &\underline{S} := C \ , \\ &\underline{\alpha} := 1 : C \to B \hookrightarrow \mathbb{G}_m \ , \\ &\underline{\beta} := i_b \ . \end{aligned}$$

Since (A) is satisfied with $N = d+1$, we may apply 5.1, and conclude:

Corollary 5.2. *$\mathcal{G}_b^{(n)}$ splits into a direct sum*

$$\mathcal{G}_b^{(n)} = \bigoplus_{i=0}^{n} \mathrm{Gr}^W_{-2i}\, \mathcal{G}_b^{(n)} \ .$$

Therefore, there is a unique isomorphism

$$\eta_b^{(n)} : \mathcal{G}_b^{(n)} \xrightarrow{\sim} \bigoplus_{i=0}^{n} F(i) \ ,$$

which is compatible with the isomorphism $\eta^{(n)}$ of 4.10.b).

Proof. $F(i) \subset \mathcal{G}_b^{(n)}$ is the eigenspace of $(d+1)^{n-i}$ under the morphism $(\phi^n)^\sharp$. ☐

We conclude with the implications of 5.1 and 5.2 for absolute cohomology with coefficients. For this, recall the absolute residue sequence for $n \geq 1$

$$\ldots \to H^{\cdot}_{\mathrm{abs}}(C, n) \to H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} \stackrel{\mathrm{res}}{\to} H^{\cdot+n-1}_{\mathrm{abs}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{\mathrm{sgn}} \to \ldots$$

introduced after 4.6, where we have set

$$H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} := H^{\cdot+n}_{\mathrm{abs}}(\mathbb{G}^{n}_{m,C} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} \ ,$$

thus saving enough space to get the above sequence into a single line.

Corollary 5.3. a) *For $n \geq 1$, the absolute residue sequence splits into short exact sequences*

$$0 \to H^{\cdot}_{\text{abs}}(C,n) \to H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \to H^{\cdot+n-1}_{\text{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \to 0\ .$$

b) *For $N = d+1$, the map $(\phi^n)^*$ acts on the short exact sequences of a): there is a commutative diagram*

$$\begin{array}{ccccc}
H^{\cdot}_{\text{abs}}(C,n) & \to & H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \to & H^{\cdot+n-1}_{\text{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \\
\text{id}\downarrow & & (\phi^n)^*\downarrow & & (d+1)\cdot(\phi^{n-1})^*\downarrow \\
H^{\cdot}_{\text{abs}}(C,n) & \to & H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \to & H^{\cdot+n-1}_{\text{abs}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn}
\end{array}$$

Proof. By 4.8, the absolute residue sequence is the absolute cohomology sequence for the exact sequence of sheaves on C

$$0 \to F(n) \to \mathcal{G}^{(n)}_b \overset{\text{res}_b}{\to} \mathcal{G}^{(n-1)}_b \to 0\ .$$

Therefore, a) follows from 5.2, while b) follows from 5.1.b) and the fact that under the identification of 4.9.a)

$$H^{\cdot}_{\text{abs}}(C,\mathcal{G}^{(n)}) \xrightarrow{\sim} H^{\cdot+n}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\text{sgn}}\ ,$$

the map induced by

$$(\phi^n)^\sharp : \mathcal{G}^{(n)}_b \to \mathcal{G}^{(n)}_b$$

is the map $(\phi^n)^*$ of the absolute cohomology groups. ☐

It follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\text{sgn}}$ are $1, d+1, \ldots, (d+1)^n$. The eigenspace decomposition yields

$$\eta^{(n)}_b : H^{n+1}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\text{sgn}} = H^{n+1}_{\text{abs}}(\mathbb{G}^{n}_{m,C} \text{ rel } Z^{(n)},n)^{\text{sgn}} \xrightarrow{\sim} \bigoplus_{i=0}^{n} H^1_{\text{abs}}(C,i)\ ,$$

which in sheaf theoretic terms corresponds to the decomposition

$$\eta^{(n)}_b : \text{Ext}^1_{\text{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) \xrightarrow{\sim} \bigoplus_{i=0}^{n} \text{Ext}^1_{\text{Sh}(C)}(F(0),F(i))$$

given by Corollary 5.2.

The pullback pol_b of the small polylogarithmic extension pol on $\mathbb{U}$ is an element of

$$\begin{aligned}
\varprojlim_{n\geq 1} \text{Ext}^1_{\text{Sh}(C)}(F(0),\mathcal{G}^{(n)}_b) &= \varprojlim_{n\geq 1} H^{n+1}_{\text{abs}}(\mathbb{G}^{n}_{m,C} \text{ rel } Z^{(n)},n)^{\text{sgn}} \\
&= \varprojlim_{n\geq 1} H^{n+1}_{\text{abs}}(\mathbb{G}^{\vee n}_{m,C},n)^{\text{sgn}}\ .
\end{aligned}$$

We have shown that, using the eigenspace decomposition for the action of the $(\phi^n)^\sharp$, these groups are isomorphic to

$$\prod_{k\geq 0} \mathrm{Ext}^1_{\mathrm{Sh}(C)}(F(0), F(k)) = \prod_{k\geq 0} H^1_{\mathrm{abs}}(C,k) \ .$$

2.5 and 2.6 describe pol_b as an element in this group.

Actually, in order to relate the above decomposition to the one used for 2.5 and 2.6, we shall need to compare the isomorphism

$$\eta := \varprojlim_{n\geq 1} \eta^{(n)} : \mathrm{Gr}^W_{\cdot} \underline{\mathcal{G}} \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of 4.10.b) to the isomorphism

$$\kappa : \mathrm{Gr}^W_{\cdot} \underline{\mathcal{G}} = \mathrm{Gr}^W_{\cdot} \mathcal{L}og \xrightarrow{\sim} \prod_{k\geq 0} F(k)$$

of section 2.

A priori, we know that the isomorphisms

$$\eta_{-2k} \ , \ \kappa_{-2k} : \mathrm{Gr}^W_{-2k} \underline{\mathcal{G}} \xrightarrow{\sim} F(k)$$

satisfy an identity of the type

$$\eta_{-2k} = q_{-2k} \cdot \kappa_{-2k} \ ,$$

for a constant $q_{-2k} \in F^*$.

We remark that in order to prove the main results announced in the introduction, all one needs to know is that q_{-2k} is a rational number, which is independent of whether we work in the Hodge or the l–adic setting.

In order to exhibit the precise relation of the motivic analogue of *pol* (see section 8) to the cyclotomic elements in K–theory (see Corollary 9.6.b)), we need to identify q_{-2k}.

Proposition 5.4. *We have the equality*

$$\eta_{-2k} = k! \cdot \kappa_{-2k} \ .$$

Proof. Because of the compatibility of κ_0 with the canonical projection

$$\varepsilon : \underline{\mathcal{G}} \longrightarrow F(0) \ ,$$

we have $\eta_0 = \kappa_0$. In order to show $\eta_{-2} = \kappa_{-2}$ we compare the classes of $\underline{\mathcal{G}}^{(1)}$ in

$$\mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{G}_m)}(F(0), F(1))$$

induced by η_{-2} and κ_{-2} respectively. Let

$K := \mathbb{C}$,	$K := \mathbb{Q}$,

and choose any K–valued point t of $\mathbb{U}$. Of course, the value of q_{-2} can still be detected from the extensions of

mixed $\mathbb{Q}$–Hodge structures	Galois modules

given by the pullback $t^*\underline{\mathcal{G}}^{(1)}$ of $\underline{\mathcal{G}}^{(1)}$ via t. In both settings, there is a natural morphism of $K^* \otimes_{\mathbb{Z}} F$ into the respective $\mathrm{Ext}^1\,(F(0), F(1))$ (see e.g.

[WiIV], Theorem 3.7).	[WiIV], Theorem 4.6).

By

[WiIV], Proposition 3.13.a),	[WiIV], Proposition 4.7.a),

the class of $t^*\underline{\mathcal{G}}^{(1)}$, calculated in the framing given by κ_{-2}, equals the image of $t \in K^*$ under this morphism. By [Sch], 2.7, the same holds for the framing given by η_{-2} – note that here it is vital to choose the ordering of the sections $\underline{\alpha}$ and $\underline{\beta}$ in the way we did before 4.11. For $k \geq 2$, let

$$\varphi_0^{(k)} : \underline{\mathcal{G}}^{(k)} \xrightarrow{\sim} \mathrm{Sym}^k\, \underline{\mathcal{G}}^{(1)}$$

be the isomorphism of 4.10.b). By 4.9.d), the diagram

$$\begin{array}{ccc} \underline{\mathcal{G}}^{(k)} & \longrightarrow & F(0) \\ \varphi_0^{(k)} \downarrow \wr & & \uparrow \wr \\ \mathrm{Sym}^k \underline{\mathcal{G}}^{(1)} & \longrightarrow & \mathrm{Sym}^k F(0) \end{array}$$

commutes. By [WiIV], Theorem 3.12.a), the commutativity of this diagram characterizes $\varphi_0^{(k)}$ uniquely. From loc. cit., Theorem 3.12.b) and c), we know that the diagram

$$\begin{array}{ccc} F(k) & \xrightarrow{\frac{1}{k!}\cdot\kappa^{-1}} & \underline{\mathcal{G}}^{(k)} \\ \uparrow \wr & & \wr \downarrow \varphi_0^{(k)} \\ \mathrm{Sym}^k F(1) & \xrightarrow{\mathrm{Sym}^k\,\kappa^{-1}} & \mathrm{Sym}^k \underline{\mathcal{G}}^{(1)} \end{array}$$

commutes. So our identity

$$\eta_{-2k} = k! \cdot \kappa_{-2k}$$

follows from 4.9.d). □

6 Polylogs in Absolute Cohomology Theories

In section 4, we showed that the logarithmic pro–sheaf is the projective limit of relative cohomology objects with coefficients in Tate twists of certain schemes over $\mathbb{U}$. The Leray spectral sequence suggests that is should be possible to recover *pol* as a projective limit of elements in absolute cohomology with Tate coefficients of these schemes, and indeed this is what we do in Theorem 6.6. That the coefficients are Tate is of course the central point: it allows us, in section 7, to imitate the construction of this section, and thus to define a motivic version of *pol*. This detour is necessary because we know, up to date, of no satisfactory formalism of mixed motivic sheaves, whose absolute cohomology with Tate coefficients would give back motivic cohomology defined via K-theory.

We return to the geometric situation set up before 4.11, and start by computing the higher direct images of the restriction of $\mathcal{L}og$ to $\mathbb{U}$:

LEMMA 6.1. a) *The inclusion* $F(1) \hookrightarrow \underline{\mathcal{G}}^{(1)}$ *and the projection* $\underline{\mathcal{G}}^{(1)} \twoheadrightarrow F(0)$ *induce natural isomorphisms*

$$\begin{aligned} F(1)_B &\stackrel{\sim}{\longrightarrow} \mathcal{H}_B^{-1}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) , \\ \mathcal{H}_B^{0}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(1)}\right) &\stackrel{\sim}{\longrightarrow} \mathcal{H}_B^{0}\left(\mathbb{G}_m, F(0)\right) , \end{aligned}$$

and the latter group is isomorphic to $F(-1)_B$ *via the map "residue at* 0*".*
b) *The inclusion* $F(n) \hookrightarrow \underline{\mathcal{G}}^{(n)}$ *and the projection* $\underline{\mathcal{G}}^{(n)} \twoheadrightarrow F(0)$ *induce natural identifications*

$$\mathcal{H}_B^{i}\left(\mathbb{G}_m, \underline{\mathcal{G}}^{(n)}\right) = \begin{cases} F(n)_B \, , & i=-1 \\ F(-1)_B \, , & i=0 \\ 0 \, , & i \notin \{-1,0\} \end{cases} .$$

Proof. The statements need only be checked on the level of local systems. Part a) is shown in the proof of 4.11. From there, we also recall that we have to compute the invariants and coinvariants under the action of the group $\pi_1 := \pi_1(\mathbb{G}_m(\mathbb{C}), 1)$, or equivalently, of a generator of π_1. Using 4.10.b), we may deduce b) from a). ☐

COROLLARY 6.2.

$$\mathcal{H}_B^{i}\left(\mathbb{U}, \mathcal{G}^{(n)}\right) = \begin{cases} F(n)_B & , \ i=-1 \\ 0 & , \ i \notin \{-1,0\} \end{cases} .$$

For $i=0$*, the sheaf* $\mathcal{H}_B^{0}\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$ *is the direct sum of* $\bigoplus_{k=1}^{n} F(k-1)_B$ *and an object which is an extension of* $F(-1)_B$ *by itself.*

Proof. By [WiI], Theorem 4.3, there is a weight filtration on $\mathcal{H}_B^{i}\left(\mathbb{U}, \mathcal{G}^{(n)}\right)$. Now use the exact triangle

$$\begin{array}{ccc} 1_*1^! & \longrightarrow & \mathrm{id}_{\mathbb{G}_{m,B}} \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & j_*j^* & \end{array}$$

purity, and 4.10.c). ☐

REMARK: In the setting of Hodge modules, where a concept of polarization is available, any extension of pure objects of the same weight is necessarily split.

The map $\mathcal{H}_B^{0}\left(\mathbb{U}, \mathcal{G}^{(n)}\right) \to F(0)$ of the corollary yields in particular a map "residue at 1_B", for $n \geq 1$,

$$\mathrm{res} : H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^0_{\mathrm{abs}}\left(B, \mathcal{R}_B(\mathbb{U}, \mathcal{G}^{(n)})\right) \to H^0_{\mathrm{abs}}(B, 0) .$$

Definition 6.3. *Let $n \geq 1$. The map*

$$\mathrm{res} : H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) = H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \; rel \; Z^{(n)}, n)^{sgn} \to H^0_{\mathrm{abs}}(B, 0)$$

is called the total residue map.

For later reference, we note

Corollary 6.4. $H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}} \; rel \; Z^{(1)}, 1) = 0$.

Proof. We have

$$H^1_{\mathrm{abs}}(\mathbb{G}^1_{m,\mathbb{U}} \text{ rel } Z^{(1)}, 1) = H^{-1}_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) \; ,$$

which because of 6.2 equals $H^0_{\mathrm{abs}}(B, F(1)) = 0$. □

Next we have

Lemma 6.5. i) *The transition morphism*

$$\mathrm{res} : \mathcal{G}^{(n)} \twoheadrightarrow \mathcal{G}^{(n-1)}$$

satisfies

$$\begin{aligned} &\mathcal{H}^{-1}_B(\mathbb{U}, \mathrm{res}) = 0 : F(n)_B \to F(n-1)_B \; , \\ &\mathcal{H}^0_B(\mathbb{U}, \mathrm{res}) : \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n)}) \to \mathcal{H}^0_B(\mathbb{U}, \mathcal{G}^{(n-1)}) \end{aligned}$$

is surjective with kernel $F(n-1)_B$.

In particular, the total residue for $n \geq 2$ factors over the total residue for $n-1$: there is a commutative diagram

$$\begin{tikzcd} H^0_{\mathrm{abs}}(B, 0) & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \arrow[l, "\mathrm{res}"'] \arrow[d, "\mathrm{res}"] \\ & H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n-1)}) \arrow[ul, "\mathrm{res}"] \end{tikzcd}$$

ii) *The Leray spectral sequences, for $n \geq 0$, give exact sequences*

$$0 \longrightarrow H^1_{\mathrm{abs}}(B, n) \xrightarrow{\delta} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) \xrightarrow{\mathrm{res}} H^0_{\mathrm{abs}}(B, 0) \longrightarrow 0 \; .$$

The map

$$\delta : H^1_{\mathrm{abs}}(B, n) \to H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

is the composition of $H^1_{\mathrm{abs}}(B, n) \to H^1_{\mathrm{abs}}(\mathbb{U}, n) = H^0_{\mathrm{abs}}(\mathbb{U}, F(n))$ and the map induced by the inclusion of $F(n)$ into $\mathcal{G}^{(n)}$, in other words, the same noted map of the residue sequence.

The projective limit of the above sequences identifies

$$H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}}) := \varprojlim_n H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)})$$

and $H^0_{\mathrm{abs}}(B, 0)$.

iii) *There are unique splittings*

$$s_n : H^0_{\text{abs}}(B,0) \hookrightarrow H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)})$$

of the sequences in ii), for any $n \geq 0$*, such that for any* $n \geq 1$ *we have a commutative diagram*

$$\begin{tikzcd} H^0_{\text{abs}}(B,0) \arrow[r, "s_n"] \arrow[dr, "s_{n-1}"'] & H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \arrow[d, "\text{res}"] \\ & H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n-1)}) \end{tikzcd}$$

Proof. i) The first statement is clear. For the second, either go through the construction or observe that the direct image of the morphism $\mathbb{U}_{\text{top}} \to B_{\text{top}}$ has cohomological dimension one, hence $\mathcal{H}^0_B(\mathbb{U}, \,\cdot\,)$ is right exact on smooth sheaves.
ii) We have the Leray spectral sequence

$$E_2^{p,q} = H^p_{\text{abs}}\left(B, \mathcal{H}^q_B(\mathbb{U},\mathcal{G}^{(n)})\right) \Rightarrow H^{p+q}_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \; ,$$

whose low-term sequence reads

$$0 \to H^1_{\text{abs}}(B,n) \to H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \to H^0_{\text{abs}}(B,0) \overset{d_2^{(n)}}{\to} H^2_{\text{abs}}(B,n) \; .$$

By i), the Mittag–Leffler condition is satisfied for the projective system $(H^1_{\text{abs}}(B,n))_{n\geq 0}$, and therefore,

$$H^0_{\text{abs}}(\mathbb{U},\mathcal{L}og|_{\mathbb{U}}) = \varprojlim_n \ker(d_2^{(n)}) = H^0_{\text{abs}}(B,0)$$

since the projective system $(\text{im}(d_2^{(n)}))_{n\geq 0} \subset (H^2_{\text{abs}}(B,n))_{n\geq 0}$ is ML-zero.
But then any of the

$$H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \to H^0_{\text{abs}}(B,0)$$

must be surjective as well.
iii) Apply ii). ∎

Denote by *pol*$^{(n)}$ the image of the small polylogarithmic extension *pol* under

$$H^0_{\text{abs}}(\mathbb{U},\mathcal{L}og\,|_{\mathbb{U}}) \to H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)}) \; .$$

Theorem 6.6. a) *Under the isomorphism*

$$H^0_{\text{abs}}(\mathbb{U},\mathcal{L}og\,|_{\mathbb{U}}) \xrightarrow{\sim} H^0_{\text{abs}}(B,0)$$

of 6.5 ii), the small polylogarithmic extension pol is mapped to 1.
b) *For each* $n \geq 0$*, the map*

$$s_n : H^0_{\text{abs}}(B,0) \to H^0_{\text{abs}}(\mathbb{U},\mathcal{G}^{(n)})$$

maps 1 *to pol*$^{(n)}$.

Proof. This is the definition of *pol* and the s_n. $\square$

Recall (4.9.a)) that we may identify

$$\begin{aligned} H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(n)}) &= H^0_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n))^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}}, v^n_! F(n)_{V^n})^{\mathrm{sgn}} \\ &= H^{n+1}_{\mathrm{abs}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}} . \end{aligned}$$

In section 8, we are going to prove a motivic analogue of 6.5.ii), and then *define pol* as the element in

$$\varprojlim_n H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,\mathbb{U}} \text{ rel } Z^{(n)}, n)^{\mathrm{sgn}}$$

mapping to 1 under the isomorphism to $H^0_{\mathcal{M}}(B, 0)$.

In order to prove a motivic version of 6.5.ii), we shall frequently use injectivity of the Beilinson regulator on certain motivic cohomology groups, and two technical results on $H^{\cdot}_{\mathrm{abs}}$, that will occupy the rest of this section.

While this may appear artificial at first sight, we remind the reader that in the motivic setting, we cannot make use of any sheaf theoretic means like Leray spectral sequences.

An important means will be the *localization sequence* associated to the geometric situation

$$\{0,1\}_B \hookrightarrow \mathbb{A}^1_B \hookleftarrow \mathbb{U} .$$

It is the result of the degeneration of the Leray spectral sequence and reads

$$\begin{aligned} \cdots \to & H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U}, p) \to H^{\cdot -1}_{\mathrm{abs}}(\{0,1\}_B, p-1) \\ \to & H^{\cdot +1}_{\mathrm{abs}}(\mathbb{A}^1_B, p) \to \ldots \end{aligned}$$

LEMMA 6.7. a) *The structure morphism is an isomorphism*

$$H^{\cdot}_{\mathrm{abs}}(B, p) \xrightarrow{\sim} H^{\cdot}_{\mathrm{abs}}(\mathbb{A}^1_B, p) .$$

b) *The boundary map is trivial, i.e., we have short exact sequences*

$$0 \to H^{\cdot}_{\mathrm{abs}}(B, p) \to H^{\cdot}_{\mathrm{abs}}(\mathbb{U}, p) \to \bigoplus_{i=0}^{1} H^{\cdot -1}_{\mathrm{abs}}(B, p-1) \to 0 .$$

Proof. For a), note that $\mathcal{R}_B \left(\mathbb{A}^1_B, F(p)_{\mathbb{A}^1_B} \right) = F(p)_B[0]$.

b) follows from the fact that there are B-valued points of $\mathbb{U}$. $\square$

In particular, for $p = 1$, we have the exact sequence

$$0 \longrightarrow H^1_{\mathrm{abs}}(B, 1) \longrightarrow H^1_{\mathrm{abs}}(\mathbb{U}, 1) \xrightarrow{\partial} \bigoplus_{i=0}^{1} H^0_{\mathrm{abs}}(B, 0) \longrightarrow 0 .$$

The last map equals the map of Ext groups

$$\partial : \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})} \left(F(0), F(1) \right) \longrightarrow \mathrm{Hom}_{\mathrm{Sh}(B)} \left(F(0), \mathcal{H}^0_B \left(\mathbb{U}, F(1) \right) \right)$$

obtained from the Leray spectral sequence; observe that the residues at 0_B and 1_B provide an isomorphism

$$\mathcal{H}^0_B\left(\mathbb{U}, F(1)\right) \xrightarrow{\sim} \bigoplus_{i=0}^{1} F(0) \; .$$

We have a natural map

$$\mathcal{O}(\mathbb{U})^* \to H^1_{\mathrm{abs}}(\mathbb{U}, 1) \; .$$

Its composition with

$$\partial : H^1_{\mathrm{abs}}(\mathbb{U}, 1) \longrightarrow \bigoplus_{i=0}^{1} H^0_{\mathrm{abs}}(B, 0)$$

associates to a function on $\mathbb{U}$ its orders at 0 and 1 respectively.

We need to understand the composition

$$\begin{aligned} \mathrm{res} \circ \partial : H^1_{\mathrm{abs}}(\mathbb{U}, 1) &= \mathrm{Ext}^1_{\mathrm{Sh}(\mathbb{U})}\left(F(0), F(1)\right) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{Sh}(B)}\left(F(0), \mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(1)}\right)\right) \; . \end{aligned}$$

Observe that due to 6.2, the last group is equal to $H^0_{\mathrm{abs}}(B, 0)$. Furthermore, we recall from the proof of 6.2 and the definition of res that the composition

$$\bigoplus_{i=0}^{1} F(0) = \mathcal{H}^0_B\left(\mathbb{U}, F(1)\right) \longrightarrow \mathcal{H}^0_B\left(\mathbb{U}, \mathcal{G}^{(1)}\right) \xrightarrow{\mathrm{res}} F(0)$$

is given by projection onto the "1"–component of $\bigoplus_{i=0}^{1} F(0)$. We have thus proved:

Lemma 6.8. *Consider the non–vanishing functions t and $1-t$ on $\mathbb{U}$. We have*

$$\mathrm{res} \circ \partial(t) = 0 \; , \quad \mathrm{res} \circ \partial(1-t) = 1 \; .$$

In particular, the map

$$\delta : H^1_{\mathrm{abs}}(\mathbb{U}, 1) \longrightarrow H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{G}^{(1)}) = H^2_{\mathrm{abs}}\left(\mathbb{G}^1_{m,\mathbb{U}} \; rel \; Z^{(1)}, 1\right)$$

does not map $1-t \in \mathcal{O}(\mathbb{U})^*$ to zero.

Proof. Observe that $\mathrm{res} \circ \partial$ factorizes through δ. ◻

Remark: The main technical result of this section, 6.5.ii) corresponds to [BD1], 3.1.6.ii). Observe that *pol* and the *polylogarithmic class* Π_ϕ of loc. cit. do not quite agree: in our notation,

$$\Pi_\phi \in H^0_{\mathrm{abs}}\left(\mathbb{U}, \mathcal{L}og(1)|_{\mathbb{U}}\right) \; ,$$

while $pol \in H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og\,|_{\mathbb{U}})$. The connection is as follows: there is a canonical monomorphism

$$\iota : \mathcal{L}og(1) \longrightarrow \mathcal{L}og$$

(identifying $\mathcal{L}og(1)$ with $W_{-2}\mathcal{L}og$), and *pol* is the push out of Π_ϕ via ι. The present definition of the polylog seems more natural since it is an element of an $H^0_{\mathrm{abs}}(B, 0)$-module of rank one, which is canonically trivialized. By contrast, $H^0_{\mathrm{abs}}(\mathbb{U}, \mathcal{L}og(1)|_{\mathbb{U}})$ is of rank two.

7 Calculations in K-theory

The next step is to do the constructions of section 4 with K-groups, or more precisely, with relative K-cohomology as introduced in appendix B.2. For technical reasons we will have to use simplicial schemes to replace the singular schemes that appeared before. All constructions will be compatible with the regulator maps to absolute Hodge cohomology (appendix A and B.5.8) and to continuous étale cohomology (appendix B.4.6).

A priori these regulators have values in absolute cohomology groups for the same simplicial object (cf. B.4.2 and B.5.2). Using B.4.5 and B.5.7 these absolute cohomology groups are then identified with (relative) cohomology of singular schemes. This identification is made tacitly.

Let $B = \mathrm{Spec}(\mathbb{Z})$ and S a smooth affine B-scheme. We will work in the category of smooth S-schemes. K-cohomology is taken on the Zariski site over B.

Before returning to the geometric situation introduced in section 3, we have to check a technical lemma. Let us consider the following general construction: Let X be a smooth quasi-projective S-scheme and Y a closed subscheme of X which is itself also smooth over S. Put

$$Y_0^{(n)} = Y \times_S X^{n-1} \amalg X \times_S Y \times X^{n-2} \amalg \ldots \amalg X^{n-1} \times_S Y \ .$$

Note that $Y_0^{(n)}$ is a proper covering of the singular scheme

$$Y^{(n)} = X^n \smallsetminus (X \smallsetminus Y)^n \ .$$

This is the easiest case of a morphism of schemes with cohomological descent, meaning that for any reasonable cohomology theory the cohomology of $Y^{(n)}$ will agree with the cohomology of the smooth simplicial scheme

$$Y_\cdot^{(n)} = \mathrm{cosk}_0(Y_0^{(n)}/X^n) \ ,$$

i.e.,

$$Y_k^{(n)} = Y_0^{(n)} \times_{X^n} \cdots \times_{X^n} Y_0^{(n)} \qquad (k+1\text{-fold product}).$$

For étale cohomology and absolute Hodge cohomology, the corresponding results are B.4.5 and B.5.6 respectively.

We will work in the setting of spaces, i.e., pointed simplicial sheaves of sets on the Zariski site of smooth B-schemes. We refer to appendix B.1 for details and terminology. We use the notation

$$X_\cdot^{\vee n} = \mathrm{Cone}(Y_\cdot^{(n)} \longrightarrow X^n)$$

for the space that computes relative cohomology for the closed embedding (cf. B.1.5).

The space $Y_\cdot^{(n)}$ does not become degenerate above any simplicial degree. However, we have:

Lemma 7.1. a) *$Y_\cdot^{(n)}$ is isomorphic in $\mathrm{Ho}\, s\mathbf{T}$ to a simplicial scheme which is degenerate above degree $n-1$.*

b) *In particular,. $Y_\cdot^{(n)}$ and $X_\cdot^{\vee n}$ are K-coherent.*

c) *$X_\cdot^{\vee n}$ is a space constructed from schemes in a finite diagram over X^n in the sense of B.2.13.*

d) *If T is another closed subscheme of X which is smooth over S and disjoint of Y, then the inclusions*

$$T^i \times_S X^{n-i} \longrightarrow X^n$$

are tor-independent of all morphisms in the diagram in c).

Proof. By definition

$$Y_0^{(n)} = Y_1 \amalg \cdots \amalg Y_n$$

where Y_i is the reduced closed subscheme of X^n of those points, whose i-th coordinate lies in Y. This induces a decomposition of $Y_k^{(n)}$ into disjoint subschemes of the form $Y_{i_1} \times_{X^n} \cdots \times_{X^n} Y_{i_k}$. Actually this subscheme is canonically isomorphic to

$$Y_{i_1} \cap \cdots \cap Y_{i_k} = \{(x_1, \ldots, x_n) \in X^n \mid x_{i_j} \in Y \text{ for } 1 \le j \le k\} .$$

We get the following more familiar form of the simplicial scheme

$$Y_k^{(n)} = \coprod_{I \in \{1,\ldots,n\}^k} \bigcap_{i \in I} Y_i .$$

Let $\Delta(n)$ be the simplicial set with

$$\Delta(n)_k = \{(i_0, \ldots, i_k) \mid 1 \le i_0 \le \cdots \le \ldots i_k \le n\} .$$

We define the simplicial scheme $Y_\cdot^{\Delta(n)}$ by

$$Y_k^{\Delta(n)} = \coprod_{I \in \Delta(n)_k} \bigcap_{i \in I} Y_i$$

It is degenerate above the simplicial degree $n-1$ and from our previous considerations we see that it is a natural subspace of $Y_\cdot^{(n)}$. We consider these simplicial schemes as spaces in the sense of appendix B.1 by adding a disjoint base point $\star$.
For a scheme U in the big Zariski site over B we consider the morphism of simplicial sets

$$Y_\cdot^{\Delta(n)}(U) \longrightarrow Y_\cdot^{(n)}(U) \quad .$$

By the combinatorial Lemma B.6.2 it induces an isomorphism of homotopy sets. Hence the inclusion is a weak homotopy equivalence of spaces.
b) is an immediate consequence of a) and B.2.3.b). Recall that Y and X were assumed smooth over B. We already have seen that all components of $X^{\vee n}$ are disjoint unions of X^n-schemes of the form $Y_{i_1} \cap \cdots \cap Y_{i_k}$ and a disjoint base point. All morphisms between the scheme components are given by the natural closed immersions between

them. The condition on the tor-dimension required in B.2.13 follows because are schemes are regular. T, Y and X are all flat over S, hence the maps in the diagram

$$\begin{tikzcd} & X \times_S Y \arrow[d] \\ T \times_S X \arrow[r] & X \times_S X \end{tikzcd}$$

are easily seen to be tor-independent. The inclusions of T and Y into X are trivially tor-independent because this is a local condition. $\square$

Basically this lemma tells us that all conditions hold that are needed to apply the machinery of appendix B.2. We have a well-behaved relative motivic cohomology theory (cf. B.2.11).

Now we return to the geometric situation set up in section 3. We consider

$$\begin{tikzcd} Z_\cdot^{(n)} \arrow[r] \arrow[d] & \mathbb{G}_{m,S}^n \arrow[d] \\ \overline{Z}_\cdot^{(n)} \arrow[r] & \mathbb{A}_S^n \end{tikzcd}$$

where $Z = \overline{Z} = \alpha(S) \amalg \beta(S)$ with disjoint S-rational points α and β of $\mathbb{G}_{m,S}$. There is a simplicial operation of $\mathfrak{S}^n$ on the situation which induces an operation on relative K-cohomology and on motivic cohomology.

Proposition 7.2. *There is a natural residue map*

$$H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \; rel \; Z_\cdot^{(n)}, j)^{sgn} \xrightarrow{\mathrm{res}_n} H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \; rel \; Z_\cdot^{(n-1)}, j-1)^{sgn}$$

where sgn means the sign eigen-space under the operation of the respective symmetric group.

Moreover, there is a long exact sequence

$$\begin{aligned} \cdots \longrightarrow & H^{i-2}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \; rel \; Z_\cdot^{(n-1)}, j-1)^{sgn} \longrightarrow H^i_{\mathcal{M}}(\mathbb{A}^n_S \; rel \; \overline{Z}_\cdot^{(n)}, j)^{sgn} \\ & \longrightarrow H^i_{\mathcal{M}}(\mathbb{G}^n_{m,S} \; rel \; Z_\cdot^{(n)}, j)^{sgn} \\ \longrightarrow & H^{i-1}_{\mathcal{M}}(\mathbb{G}^{n-1}_{m,S} \; rel \; Z_\cdot^{(n-1)}, j-1)^{sgn} \longrightarrow \cdots \quad . \end{aligned}$$

Under the regulators, the long exact sequences are compatible with the ones in absolute cohomology (after 4.5).

Remark: Recall that $Z_\cdot^{(0)} = \star$ and hence $H^k_{\mathcal{M}}(\mathbb{G}^0_{m,S} \text{ rel } Z_\cdot^{(0)}, j) = H^k_{\mathcal{M}}(S, j)$ by definition.

Proof. We filter $\mathbb{A}^n_S$ by the open subschemes $F_k\mathbb{A}^n_S$ defined just before Lemma 4.5. In particular, $F_0\mathbb{A}^n_S = \mathbb{G}^n_{m,S}$. Again $G_k\mathbb{A}^n_S = F_k\mathbb{A}^n_S \smallsetminus F_{k-1}\mathbb{A}^n_S$. We use the notation $F_k\mathbb{A}_\cdot^{\vee n}$ and $G_k\mathbb{A}_\cdot^{\vee n}$ for the induced open respectively locally closed subspaces of $\mathbb{A}_\cdot^{\vee n}$. Note that the situation is still symmetric under permutation of coordinates. Hence there is a compatible operation of the symmetric group on the space constructed from schemes $F_k\mathbb{A}_\cdot^{\vee n}$.

The closed immersion $G_k\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot}$ satisfies the first condition in (TC) in B.2.13. The maps we have to consider for the rest of (TC) are locally of the form considered in 7.1.d). Hence B.2.19 applies, i.e., we can use the localization sequences for motivic cohomology induced by the triples $F_{k-1}\mathbb{A}^{\vee n}_{\cdot} \to F_k\mathbb{A}^{\vee n}_{\cdot} \leftarrow G_k\mathbb{A}^{\vee n}_{\cdot}$. We get

$$\begin{aligned}\dots \longrightarrow & H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \longrightarrow H^{i+2}_{\mathcal{M}}(F_k\mathbb{A}^{\vee n}_{\cdot}, j+1) \longrightarrow H^{i+2}_{\mathcal{M}}(F_{k-1}\mathbb{A}^{\vee n}_{\cdot}, j+1)\\ \longrightarrow & H^{i+1}_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) \longrightarrow \dots\end{aligned}$$

The sequence remains exact when we take sign–eigenspaces. Now let us compute one of the groups involved.

$$H^i_{\mathcal{M}}(G_k\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{\{1\le a_1<a_2<\dots<a_k\le n\}} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \dots, a_k), j)$$

where

$$\mathbb{G}_{m,S}(a_1, \dots, a_k) = \{(x_1, \dots, x_n) \mid x_i = 0 \text{ if } i = a_j \text{ for some } j;\ x_i \neq 0 \text{ else } \}\ .$$

The decomposition corresponds to the decomposition of $G_k\mathbb{A}^n$ into its connected components. The notation $\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} \mathbb{G}_{m,S}(a_1, \dots, a_k)$ means the open subspace lying over the locally closed scheme. Now consider the operation of the symmetric group. If $k > 1$, then there is for each component some transposition which acts trivially, namely one that interchanges two vanishing coordinates. Hence the sign–eigenspace vanishes altogether. For $k = 1$, the decomposition has the form

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j) = \bigoplus_{a=1,\cdots,n} H^i_{\mathcal{M}}(\mathbb{A}^{\vee n}_{\cdot} \times_{\mathbb{A}^n} (\mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}), j)\ .$$

The operation of the symmetric group permutes the factors transitively. The stabilizer of one summand is the symmetric group $\mathfrak{S}^{n-1}$. We get

$$H^i_{\mathcal{M}}(G_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}} \cong H^i_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,S}, j)^{\text{sgn}}$$

where the sign eigenspace on the right hand side is taken with respect to the smaller symmetric group $\mathfrak{S}^{n-1}$. We have a choice of isomorphism here and use the one that identifies $\mathbb{G}^{n-1}_{m,S}$ with $\mathbb{G}^{n-1}_{m,S} \times \{0\}$. Putting these results in the long exact sequences we get iteratively

$$H^i_{\mathcal{M}}(\mathbb{A}^n \text{ rel } \overline{Z}^{(n)}_{\cdot}, j)^{\text{sgn}} = H^i_{\mathcal{M}}(F_n\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}} \xrightarrow{\cong} \cdots H^i_{\mathcal{M}}(F_1\mathbb{A}^{\vee n}_{\cdot}, j)^{\text{sgn}}\ .$$

So the above sequence, for $k = 1$, gives the desired residue sequence. We can do the same construction for absolute cohomology (Hodge or l-adic) considered as generalized cohomology theories. By B.4.6, B.5.8 and B.3.7, the long exact sequences for motivic cohomology will be compatible via the regulator with the ones in generalized cohomology. The next step is to pass from generalized cohomology to cohomology of abelian sheaves. By B.4.5 and B.5.7 this can be done. In fact we get precisely the residue sequence for absolute cohomology constructed in section 4. $\square$

REMARK: a) By B.2.19, we have the same maps and long exact sequences for the K-cohomology groups themselves. However, note that there is a Riemann-Roch hidden in the compatibility of the localization sequence in K-cohomology and absolute cohomology.
b) We shall show injectivity of the Beilinson regulator on

$$H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, n)^{\text{sgn}}$$

in Proposition 8.7. Together with Lemma 4.4.b), it shows that the residue map on

$$H^{i}_{\mathcal{M}}(\mathbb{G}^n_{m,S} \text{ rel } Z^{(n)}_{\cdot}, j)^{\text{sgn}}$$

does not depend on the choice of embedding of $\mathbb{G}^{n-1}_{m,S}$ in

$$\bigcup_{a=1,\cdots,n} \mathbb{G}^{a-1}_{m,S} \times \{0\} \times \mathbb{G}^{n-a}_{m,S}$$

of the above proof, if $(i,j) = (n+1,n)$. Since we are only interested in these special indices, we chose to exclude from the statement of 7.2 the dependence of res_n in the general case from the above choice.

LEMMA 7.3. *Let $2j \geq k$. Then*

$$H^k_{\mathcal{M}}(\mathbb{A}^n_S \text{ rel } \overline{Z}^{(n)}_{\cdot}, j) \cong H^{k-n}_{\mathcal{M}}(S, j)$$

where the isomorphism is induced by a choice of ordering of the sections α and β. It is compatible with the identification in 4.6 under the regulator map. $\mathfrak{S}_n$ operates by sign on the left hand side.

REMARK: Here and in the sequel we put $H^i_{\mathcal{M}}(S, j) = 0$ if $j < 2i$. This makes sense as S is regular and the corresponding K–group vanishes (see B.2.3).

Proof. Fix j. We consider the skeletal spectral sequence B.2.12. We have

$$E_1^{p,q} = H^q_{\mathcal{M}}((\mathbb{A}^{\vee n}_S)_p, j) \ .$$

We will show that the only non-trivial E_2-terms are concentrated in one vertical line

$$E_2^{n,q} = H^q_{\mathcal{M}}(S, j) \ .$$

This means that the spectral sequence converges in the strongest possible way. This yields isomorphisms as stated. Before we can check this we need some preparation. If $X_{\cdot}$ is a space constructed from schemes, we denote by $Cp(X_{\cdot})$ the simplicial set of its connected components. $Cp(\overline{Z}^{(n)}_{\cdot})$ has the same singular cohomology as $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ (cf. proof of 7.1) which is the simplicial set attached to a CW-complex dual to the boundary of the n-dimensional hypercube (note that $\overline{Z}$ has two disjoint components). This means that $Cp(\overline{Z}^{\Delta(n)}_{\cdot})$ has a 1-vertex for every $(n-1)$-cell of the cube etc. In particular we see that it has the homotopy type of an $(n-1)$-sphere. $Cp(\mathbb{A}^n_S)$ is of course contractible. It follows that $Cp(\mathbb{A}^{\vee n}_{\cdot})$ has singular cohomology concentrated in degree n where it is one-dimensional.

Let us make this more explicit:
In order to compute the cohomology of a cosimplicial group it suffices to consider the sub-complex corresponding to nondegenerate simplices. $Cp(\overline{Z}_\cdot^{\Delta(n)})$ is completely degenerate from cosimplicial degree n on. In degree $n-1$, there is one nondegenerate simplex for each vertex of the hypercube. They are indexed by $\{\alpha,\beta\}^n$. Hence any element of $H^nCp(\mathbb{A}_\cdot^{\vee n}) = H^{n-1}(Cp(\overline{Z}_\cdot^{\Delta(n)}))$ is represented by an element of

$$K^{n-1} = \bigoplus_{\{\alpha,\beta\}^n} \mathbb{Q}\,.$$

Let g be a generator of the cohomology group. $Cp(\overline{Z}_\cdot^{(n)})$ does not become degenerate. The nondegenerate part in degree $n-1$ is given by one copy of $\{\alpha,\beta\}^n$ for each possible permutation of the numbers $0,\dots,n-1$. It is easy to see that $((-1)^{\mathrm{sgn}(\sigma)}g)_\sigma$ is in the kernel of the differential. It represents the generator of cohomology of $Cp(\overline{Z}_\cdot^{(n)})$. We see that $\mathfrak{S}_n$ operates by the sign of the permutation.

We choose the generator g of cohomology given by the tuple

$$(-1)^{s(i_1)+\cdots+s(i_n)} \in \mathbb{Q}_{i_1\times\cdots\times i_n}$$

where $i_k \in \{\alpha,\beta\}$ and $s(\alpha)=1$, $s(\beta)=0$. This choice of generator amounts to picking the ordering $\alpha<\beta$ and extending it by the Künneth-formula. Now let us analyze our E_1-term: For fixed q we have the complex attached to the cosimplicial abelian group $H^q_{\mathcal{M}}((\mathbb{A}_S^{\vee n})_p, j)_{p\in\mathbb{N}_0}$. All connected components of $\mathbb{A}_\cdot^{\vee n}$ are isomorphic to a copy of some power of $\mathbb{A}^1_S$. By the homotopy property of K-theory we have

$$H^q_{\mathcal{M}}((\mathbb{A}_S)_p^{\vee n}, j)_{p\in\mathbb{N}_0} = H^q_{\mathcal{M}}(S,j)\otimes_{\mathbb{Q}} C_\cdot^{\vee n}$$

where $C_\cdot^{\vee n}$ is the cosimplicial vector space computing singular cohomology of $Cp(\mathbb{A}_\cdot^{\vee n})$. By the previous considerations we already know its cohomology. It also follows that the operation of $\mathfrak{S}_n$ on our motivic cohomology is by the sign.
Now compare our isomorphism to the one constructed in the realization. We have the same spectral sequence there (attached to the weight filtration). The identification of the E_2-term also uses Künneth-formula and choice of an ordering of the sections. □

Using this identification we obtain the *motivic residue sequence*:

$$\begin{aligned}\dots \longrightarrow & H^{k-n}_{\mathcal{M}}(S,j) \longrightarrow H^k_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n}, j)^{\mathrm{sgn}} \longrightarrow H^{k-1}_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n-1}, j-1)^{\mathrm{sgn}}\\ \longrightarrow & H^{k-n+1}_{\mathcal{M}}(S,j) \longrightarrow \dots\end{aligned}$$

for $2j\ge k$. By construction, we have the following:

Theorem 7.4. *Under the regulator, the motivic residue sequence maps to the absolute residue sequence of section 4.*

Note that the residue sequences for all indices k and n organize into a spectral sequence connecting the relative motivic cohomology of $\mathbb{A}_\cdot^{\vee ?}$ and the relative motivic cohomology of $\mathbb{G}_m^{\vee ?}$. In particular for each n there is the converging cohomological spectral sequence

$$E_1^{pq} = H^{p+q-n}_{\mathcal{M}}(S,p) \Rightarrow H^{p+q}_{\mathcal{M}}(\mathbb{G}_{m,S}^{\vee n}, n) = H^{p+q}_{\mathcal{M}}(\mathbb{G}_{m,S}^n \text{ rel } Z_\cdot^{(n)}, n)\,.$$

This is the motivic version of the weight spectral sequence in absolute cohomology. We refer to it as the *motivic residue spectral sequence*.
Remark: As in section 6, the residue sequence, or equivalently, the residue spectral sequence turns out to be the central technical tool in the construction of the motivic polylog (see Definition 8.9). The spectral sequence is identical to the one constructed in [BD1], 4.2.6. The definition and basic properties of motivic cohomology of simplicial schemes (B.1, B.2) allow to justify the construction.

At this point, we should stress that the proof of the innocent looking Theorem 7.4 requires the whole of the theory covered in the appendices.

8 Universal Motivic Polylogarithm

We now return to the special situation used in section 6. Let $B = \mathrm{Spec}(\mathbb{Z})$. We consider now the case $S = \mathbb{U}$. Let $\alpha = 1$, and β the diagonal section of $\mathbb{U} \times_B \mathbb{G}_{m,B}$.

First we compute the motivic cohomology of $\mathbb{U}$. We use the embedding of $\mathbb{U}$ into $\mathbb{A}^1$ to do so. The long exact localization sequence B.2.18 reads

$$\begin{aligned} \ldots \longrightarrow & H^{n-2}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \longrightarrow H^n_{\mathcal{M}}(\mathbb{A}^1_B, j) \longrightarrow H^n_{\mathcal{M}}(\mathbb{U}, j) \\ \longrightarrow & H^{n-1}_{\mathcal{M}}(0(B) \amalg 1(B), j-1) \longrightarrow \ldots \end{aligned}$$

By the homotopy property of K-theory we get

$$\begin{aligned} \ldots \longrightarrow & H^n_{\mathcal{M}}(B, j) \longrightarrow H^n_{\mathcal{M}}(\mathbb{U}, j) \longrightarrow H^{n-1}_{\mathcal{M}}(B, j-1) \oplus H^{n-1}_{\mathcal{M}}(B, j-1) \\ \longrightarrow & H^{n+1}_{\mathcal{M}}(B, j) \longrightarrow \ldots \end{aligned}$$

The Gysin map for the inclusion of a point in the affine line vanishes by [Q2] Thm 8 ii. Hence we are actually dealing with a system of short exact sequences. As all motivic cohomology groups of B vanish for $n > 1$ this sequence only gives non-trivial cohomology of $\mathbb{U}$ for $n = 0, 1, 2$.

Lemma 8.1. *For $B = \mathrm{Spec}(\mathbb{Z})$ we have*

$$\begin{aligned} H^0_{\mathcal{M}}(\mathbb{U}, i) &= \begin{cases} \mathbb{Q} & \text{if } i = 0, \\ 0 & \text{else,} \end{cases} \\ H^1_{\mathcal{M}}(\mathbb{U}, j) &= \begin{cases} 0 & \text{for } j < 1, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{for } j = 1, \\ H^1_{\mathcal{M}}(B, j) & \text{for } j > 1, \end{cases} \\ H^2_{\mathcal{M}}(\mathbb{U}, j) &= H^1_{\mathcal{M}}(B, j-1) \oplus H^1_{\mathcal{M}}(B, j-1)\ , \\ H^n_{\mathcal{M}}(\mathbb{U}, j) &= 0 \text{ if } n > 2. \end{aligned}$$

Proof. Clear from the above using B.2.20 □

By Borel's Theorem (B.5.9) the Beilinson regulator

$$H^i_{\mathcal{M}}(X, j) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^i_{\mathfrak{H}^p}(X_{\mathbb{R}}/\mathbb{R}, j)$$

is injective for $X = \mathrm{Spec}(\mathbb{Z})$, even an isomorphism but in the one case $H^1_{\mathcal{M}}(B, 1)$ where the codimension is one. (We call Beilinson regulator what strictly speaking is its tensor product with $\mathbb{R}$.) This implies that it is also an isomorphism for $H^i_{\mathcal{M}}(\mathbb{U},k)$ with the exception of the indices $(1,1)$ and $(2,2)$ where the codimension is 1 resp. 2.

This means that many of the residue maps are actually isomorphisms. The following computations are carried out in the case $B=\operatorname{Spec}(\mathbb{Z})$. With a little more effort they generalize to the case of the ring of integers of a number field.

Consider the residue sequence for $n=j=1$ and $S=\mathbb{U}$.

$$0=H^0_{\mathcal{M}}(\mathbb{U},1)\longrightarrow H^1_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)\longrightarrow H^0_{\mathcal{M}}(\mathbb{U},0)$$
$$\longrightarrow H^1_{\mathcal{M}}(\mathbb{U},1)\xrightarrow{\delta} H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)\longrightarrow H^1_{\mathcal{M}}(\mathbb{U},0)=0\ .$$

The Beilinson regulator induces a map between the above sequence and the residue sequence in section 4. On $H^0_{\mathcal{M}}(\mathbb{U},0)\otimes\mathbb{R}$, the regulator is an isomorphism, and on $H^1_{\mathcal{M}}(\mathbb{U},1)\otimes\mathbb{R}$ it is injective of codimension one. By 6.4, the absolute Hodge cohomology group $H^1_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_\mathbb{R}}/\mathbb{R},1)$ vanishes. Hence the map from the first to the second line is injective and the regulator is injective of codimension one on $H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. Furthermore, this last group is one dimensional.

The image of δ under the Beilinson regulator is the map occurring in 6.8 for $n=1$.

Definition 8.2. *Let s_1 be the composition of the maps*

$$\mathbb{Q}\ =\ H^0_{\mathcal{M}}(B,0)\ \xrightarrow{i_1}\ \bigoplus_{i=0,1}H^0_{\mathcal{M}}(B,0)\ =\ H^1_{\mathcal{M}}(\mathbb{U},1)\ \xrightarrow{\delta}\ H^2_{\mathcal{M}}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$$

where i_1 is the inclusion of the 1*-summand and δ is the map of the residue sequence.*

Lemma 8.3. *s_1 is an isomorphism.*

Proof. Because of dimension reasons we only have to check that δ does not vanish on the image of i_1. This follows from 6.8. □

Definition 8.4. *Let* res_1 *be the inverse of s_1. We define the total residue map*

$$\operatorname{res}: H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}\longrightarrow\mathbb{Q}\ \ .$$

by composition of the residue maps in our long exact sequence 7.2 with res_1.

We now have to check that the total residue map deserves its name. By definition and 6.5.i) it suffices to consider res_1.

Lemma 8.5. *The regulators map the motivic* res_1 *to* res_1 *in absolute cohomology.*

Proof. Let us consider the situation of 6.8. The morphism

$$\mathcal{O}(\mathbb{U})^*\longrightarrow H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1)$$

factors through $H^1_{\mathcal{M}}(\mathbb{U},1)=K_1(\mathbb{U})_\mathbb{Q}$. There is a commutative diagram

$$\begin{tikzcd}
 & H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1) \arrow[r] & \bigoplus_{i=0,1}H^0_{H_{\mathfrak{H}^p}}(B_\mathbb{R}/\mathbb{R},0) \\
\mathcal{O}(\mathbb{U})^* \arrow[r] & H^1_{\mathcal{M}}(\mathbb{U},1) \arrow[u] \arrow[r, "\cong"] & \bigoplus_{i=0,1}H^0_{\mathcal{M}}(B,0) \arrow[u, "\cong"]
\end{tikzcd},$$

hence the functions t and $1-t$ on $\mathbb{U}$ correspond to the canonical generators of the two summands. We consider the commutative diagram for absolute Hodge cohomology

$$\begin{array}{ccccc} H^1_{\mathfrak{H}^p}(\mathbb{U}_\mathbb{R}/\mathbb{R},1) & \xrightarrow{\delta_\mathbb{R}} & H^2_{\mathfrak{H}^p}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}_\mathbb{R}}/\mathbb{R},1) & \xrightarrow{\mathrm{res}_\mathbb{R}} & H^0_{\mathfrak{H}^p}(B_\mathbb{R}/\mathbb{R},0) \\ \uparrow & & \uparrow r & & \\ \bigoplus_{i=0,1} H^0_\mathcal{M}(B,0) & \xrightarrow{\delta} & H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1) & & \end{array},$$

By 6.8 the composition from the bottom left to the top right corner is given by the projection to the 1-component tensored by $\mathbb{R}$. It follows that $(\mathrm{res}_R \circ r) \otimes \mathbb{R}$ is an isomorphism. In turn δ vanishes on the 0-component and is an isomorphism on the 1-component. But then by definition $\mathrm{res}_1 \circ \delta$ is also the projection to the 1-summand. As δ is surjective, this suffices. The same argument works in the étale situation. □

LEMMA 8.6. *There is a short exact sequence*

$$0 \longrightarrow H^1_\mathcal{M}(B,2) \longrightarrow H^3_\mathcal{M}(\mathbb{G}^{\vee 2}_{m,\mathbb{U}},2)^{sgn} \xrightarrow{res} \mathbb{Q} \longrightarrow 0$$

and the Beilinson regulator is an isomorphism on the middle term.

Proof. This is nothing but the residue sequence using our computation of $H^2_\mathcal{M}(\mathbb{G}^{\vee 1}_{m,\mathbb{U}},1)$. The zeroes on both sides come from vanishing cohomology groups. Comparison with the short exact sequence 6.5.ii) shows that the regulator is an isomorphism. □

PROPOSITION 8.7. *There are short exact sequences*

$$0 \longrightarrow H^1_\mathcal{M}(B,n) \xrightarrow{\delta_n} H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \xrightarrow{\mathrm{res}} \mathbb{Q} \longrightarrow 0\,.$$

The Beilinson regulator is injective on all $H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$. It is even an isomorphism for $n>1$.

Proof. The $n=1$ and $n=2$ cases are the previous lemmas. By induction, one checks that all $H^n_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}}$ vanish for $n \geq 1$. Hence the residue sequence reads

$$0 \to H^1_\mathcal{M}(B,n) \xrightarrow{\delta_n} H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{\mathrm{sgn}} \to H^n_\mathcal{M}(\mathbb{G}^{\vee n-1}_{m,\mathbb{U}},n-1)^{\mathrm{sgn}} \to H^2_\mathcal{M}(\mathbb{U},n)\,.$$

By the five lemma and inductive hypothesis we see that the regulator is an isomorphism on the middle term for n. We need the previous lemma to get started.
Now consider the sequences of the proposition. All maps are well-defined. It follows from 6.5.ii) that the sequence is exact. □

COROLLARY 8.8. *There are canonical splittings $s_n : \mathbb{Q} \to H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$ such that the diagram*

$$\begin{array}{ccc} H^0_\mathcal{M}(B,0) & \xrightarrow{s_n} & H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn} \\ & s_{n-1}\searrow & \downarrow \mathrm{res} \\ & & H^n_\mathcal{M}(\mathbb{G}^{\vee n-1}_{m,\mathbb{U}},n-1)^{sgn} \end{array}$$

commutes. They are compatible with the ones in 6.5.iii). Furthermore, the group $\varprojlim H^{n+1}_\mathcal{M}(\mathbb{G}^{\vee n}_{m,\mathbb{U}},n)^{sgn}$ is canonically isomorphic to $\mathbb{Q}$.

Proof. $\mathrm{Im}(res_n)$ is isomorphic to $\mathbb{Q}$ by the total residue on $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,\mathbb{U}}, n)^{\mathrm{sgn}}$. This induces the same splitting as in 6.5. □

Definition 8.9. *For $n \in \mathbb{N}$ the system $pol_n = s_n(1)$ defines the universal motivic polylogarithm.*

By construction pol_n is mapped to the polylogarithmic system in absolute Hodge cohomology and continuous étale cohomology.

Remark: The main result of this section, 8.7 is identical to [BD1], 4.3.4. Although part of the argument involves only constructions within K-theory, the proof of 8.7 relies heavily on a detailed analysis of the behaviour of the regulator between the motivic and absolute residue sequences.

9 The Cyclotomic Case

Let $d \geq 2$. As before let $R = A[1/d, T]/\Phi_d(T)$ the ring of d-integers of the cyclotomic field of d-th roots of unity. Put $C = \operatorname{Spec} R$. Let ζ be a primitive d-th root of unity in $\overline{\mathbb{Q}}$, and b an integer prime to d. We work in the situation $S = C$, $\alpha = 1 \in \mathbb{G}_m(C)$, and $\beta = i_b \in \mathbb{G}_m(C)$ as in section 5.

Lemma 9.1. a) *For $n \geq 0$ we have*

$$H^n_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} = H^n_{\mathcal{M}}(\mathbb{G}^n_{m,C} \text{ rel } Z^{(n)}_{\cdot}, n)^{sgn} = \mathbb{Q} \quad .$$

The Beilinson and the l-adic regulators are isomorphisms.

b) *For $n \geq 1$, the residue sequence induces short exact sequences*

$$0 \longrightarrow H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{sgn} \longrightarrow 0 \quad .$$

The l-adic regulator is injective on the group $H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn}$ for $n \geq 1$.

Proof. For $n = 0$ we have $H^0_{\mathcal{M}}(\mathbb{G}^{\vee 0}_{m,C}, 0) = H^0_{\mathcal{M}}(C, 0)$, which is canonically isomorphic to $\mathbb{Q}$ by B.2.20. In particular both regulator are isomorphisms.
$H^1_{\mathcal{M}}(\mathbb{G}^{\vee 0}_{m,C}, 0)$ and its counterpart in absolute cohomology vanish.
Consider the following bit of the residue sequence for $n \geq 1$:

$$H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n+1}_{m,C}, n+1)^{\mathrm{sgn}} \to H^n_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} \to H^1_{\mathcal{M}}(C, n+1)$$

The first map is injective since $H^0_{\mathcal{M}}(C, n+1) = 0$. The l-adic regulator is always injective on the last term by B.4.8. By inductive hypothesis it is an isomorphism on the middle term. By Cor. 5.3, the last map vanishes in absolute cohomology. This implies a) for $n+1$. In the next bit of the long exact sequence

$$H^1_{\mathcal{M}}(C, n) \longrightarrow H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C}, n)^{\mathrm{sgn}} \longrightarrow H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1)^{\mathrm{sgn}} \longrightarrow (*) \quad ,$$

the first map is injective by a). For $n \geq 2$ we have $(*) = H^2_{\mathcal{M}}(C, n) = 0$, while for $n = 1$ the term

$$H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C}, n-1) = H^1_{\mathcal{M}}(C, 0)$$

vanishes. Hence in any case we end up with the short exact sequence in b). The regulator maps it to the short exact sequence 5.3. By induction and B.4.8 we can control the injectivity of the l–adic regulator. □

Remark: The Beilinson regulator is not injective on $H^1_{\mathcal{M}}(C,1)$ because d is inverted in C.

Consider the morphism $\phi : \mathbb{G}_{m,C} \to \mathbb{G}_{m,C}$ that raises points to the $d+1$-th power. As in section 5 it induces a morphism of spaces $\phi^n : \mathbb{A}^{\vee n}_C \to \mathbb{A}^{\vee n}_C$. By contravariance it induces an operation on motivic cohomology.

Lemma 9.2 ([BD1], Remark (ii) on page 78). *$(\phi^n)^*$ operates on the short exact sequence of the previous lemma as follows:*

$$\begin{array}{ccccc} H^1_{\mathcal{M}}(C,n) & \longrightarrow & H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \longrightarrow & H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \\ {\scriptstyle id}\downarrow & & {\scriptstyle (\phi^n)^*}\downarrow & & \downarrow{\scriptstyle (d+1)(\phi^{n-1})^*} \\ H^1_{\mathcal{M}}(C,n) & \longrightarrow & H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} & \longrightarrow & H^n_{\mathcal{M}}(\mathbb{G}^{\vee n-1}_{m,C},n-1)^{sgn} \end{array}.$$

Proof. This description follows immediately from the injectivity of the l-adic regulator and Cor. 5.3.b). □

Remark: The operation $(\phi^n)^*$ on $H^1_{\mathcal{M}}(C,n)$ is given by the operation on $H^{n+1}_{\mathcal{M}}(\mathbb{A}^{\vee n}_C,n)$. It is easy to check that it is trivial by considering the operation on the starting terms of the degenerating skeletal spectral sequence. To understand the compatibility with the residue map in terms of K-theory is a lot harder. The factor $d+1$ is induced by a push-forward from a non-reduced scheme to its reduction. The theory in Appendix B is not even set up to handle such schemes.

As in the case of absolute cohomology it follows that the eigenvalues of $(\phi^n)^*$ on $H^{n+1}_{\mathcal{M}}(\mathbb{G}_{m,C} \text{ rel } Z^{(n)},n)^{\text{sgn}}$ are $1,d+1,\ldots,(d+1)^{n-1}$.

Lemma 9.3. *The eigenspace decomposition yields a splitting*

$$\eta_b^{(n)} : H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \xrightarrow{\sim} \bigoplus_{1\le i\le n} H^1_{\mathcal{M}}(C,i)\ ,$$

which is compatible with the splitting $\eta_b^{(n)}$ after Cor. 5.3. There is a canonical isomorphism

$$\eta_b : \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} \xrightarrow{\sim} \prod_{i\ge 1} H^1_{\mathcal{M}}(C,i)\ .$$

Proof. The first assertion is clear by construction. The second follows because the eigenspace decomposition is compatible with the residue map. □

Definition 9.4. *Let $i_b : C \to \mathbb{U}$ be as before. Let pol_b be the pullback of the universal polylogarithm system pol defined in 8.9 to the inverse limit $\varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} = \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^n_{m,C}$ rel $Z^{(n)}_{\cdot},n)^{sgn}$. Via the isomorphism η_b of 9.3, we have constructed an element in $\prod_{i\ge 1} H^1_{\mathcal{M}}(C,i)$.*

Theorem 9.5. *Under the regulators, the element*

$$pol_b \in \varprojlim H^{n+1}_{\mathcal{M}}(\mathbb{G}^{\vee n}_{m,C},n)^{sgn} = \prod_{i\ge 1} H^1_{\mathcal{M}}(C,i)$$

is mapped to the elements

$$pol_b \in \varprojlim H^{n+1}_{\mathrm{abs}}(\mathbb{G}^{\vee n}_{m,C}, n)^{sgn} = \prod_{i\geq 1} H^1_{\mathrm{abs}}(C, i)$$

constructed at the end of section 5.

Proof. This follows from the construction. □

We list the consequences of this result: denote by μ_d^0 the set of primitive d–th roots of unity in $\mathbb{Q}(\mu_d)$.

Firstly, the description of the regulator to absolute Hodge cohomology yields an alternative proof of the following:

Corollary 9.6. *Assume $n \geq 0$.*

a) ([B2], 7.1.5, [Neu], II.1.1, [E], 3.9.)
There is a map of sets

$$\begin{aligned}\epsilon_{n+1} : \mu_d^0 &\longrightarrow H^1_{\mathcal{M}}(C, n+1)\\ &\qquad (= H^1_{\mathcal{M}}(\operatorname{Spec}\mathbb{Q}(\mu_d), n+1) \text{ for } n \geq 1)\end{aligned}$$

such that

$$\begin{aligned} r_{\mathcal{D}} \circ \epsilon_{n+1} : \mu_d^0 &\longrightarrow H^1_{\mathfrak{H}^p}(\operatorname{Spec}\mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1)\\ &\underset{A.2.12}{\xrightarrow{\sim}} \left(\bigoplus_{\sigma:\mathbb{Q}(\mu_d)\hookrightarrow\mathbb{C}} \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}\right)^+\end{aligned}$$

maps a root of unity ω to $(-Li_{n+1}(\sigma\omega))_\sigma$. For $n \geq 1$, this property characterizes the map ϵ_{n+1} uniquely.
b) For a root of unity $T^b \in \mathbb{Q}(\mu_d) = \mathbb{Q}[T]/\Phi_d(T)$, the element

$$\epsilon_{n+1}(T^b) \in H^1_{\mathcal{M}}(C, n+1)$$

is given by

$$\epsilon_{n+1}(T^b) := (-1)^n \cdot \frac{1}{(n+1)!} \cdot (\ (n+1)\text{–component of } pol_b)\ .$$

Proof. Note that a) really *is* Beilinson's formulation of the result: his normalization of the isomorphism

$$H^1_{\mathfrak{H}^p}(\operatorname{Spec}\mathbb{Q}(\mu_d)_{\mathbb{R}}/\mathbb{R}, n+1) \xrightarrow{\sim} \left(\bigoplus_\sigma \mathbb{C}/(2\pi i)^{n+1}\mathbb{R}\right)^+$$

differs from ours by the factor -1. The unicity assertion is a direct consequence of the injectivity of the regulator. So our claim follows from 2.5, and from 5.4. □

In [B2], the above compatibility statement is used to prove Gross's conjecture about special values of Dirichlet L-functions. An alternative proof of this conjecture, using an entirely different geometric construction, is given in section 3 of [Den].

Recall that the l–adic regulator r_l factorizes as follows:

$$\begin{aligned} K_{2n+1}(C) \otimes_{\mathbb{Z}} \mathbb{Q} = H^1_{\mathcal{M}}(C, n+1) &\hookrightarrow H^1_{\mathcal{M}}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(C_{(l)}, n+1) \\ &\hookrightarrow H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \ , \end{aligned}$$

where we let $C_{(l)} := C \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{l}]$.

For the rest of this section, we fix $\zeta \in C(\overline{\mathbb{Q}})$. As was observed already in [B4], the study of the cyclotomic polylog yields a proof of the following result:

COROLLARY 9.7. *Assume $n \geq 0$.*
a) ([Sou5], Théorème 1 for the case $n = 1$; [Gr], Théorème IV.2.4 for the local version if $(l, d) = 1$.)
Let d and ϵ_{n+1} be as in 9.6. Let l be a prime. Under the embedding of 2.6, the l–adic regulator

$$r_l : H^1_{\mathcal{M}}(C, n+1) \longrightarrow H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

maps $\epsilon_{n+1}(T^b)$ to

$$\frac{1}{d^n} \cdot \frac{1}{n!} \cdot \left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r \ .$$

b) Conjecture 6.2 of [BlK] holds.

Proof. a) is 2.6 and 5.4. As for b), it remains to check the comparison statement of [BlK], Conjecture 6.2 for the root of unity 1. For this, observe the relations

$$\begin{aligned} c_{n+1}(1) &= \frac{2^n}{1-2^n} c_{n+1}(-1) \ , \\ c_{n+1,2}(1) &= \frac{2^n}{1-2^n} c_{n+1,2}(-1) \end{aligned}$$

in the notation of loc. cit., if $n \geq 1$ ([D5], Proposition 3.13.1.i)). $\square$

Soulé has constructed maps

$$\varphi_l : \mu_d^0 \to K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l$$

for any prime l (see end of Appendix B.4 for more details).

The l–adic regulator

$$r_l : K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^1_{cont}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1) \qquad \text{(Prop. B.4.10)}$$

takes $\varphi_l(T^b)$ to the cyclotomic element in continuous Galois cohomology

$$\left(\sum_{\alpha^{l^r} = \zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r$$

defined by Soulé and Deligne (cf. [Sou2], page 384, [D5], 3.1, 3.3).

Corollary 9.8. *For each d and n, there is a unique map*

$$\varphi : \mu_d^0 \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d))$$

such that for each prime number l, the map

$$\begin{aligned} \varphi_l : \mu_d^0 \to & K_{2n+1}(C_{(l)}) \otimes_{\mathbb{Z}} \mathbb{Z}_l \\ & \hookrightarrow K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \end{aligned}$$

equals the composition of φ and the natural map

$$K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \to K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d)) \otimes_{\mathbb{Z}} \mathbb{Z}_l \ .$$

Furthermore, the map $\varphi \otimes_{\mathbb{Z}} \mathbb{Q}$ agrees with

$$\epsilon'_{n+1} : \mu_d^0 \to H^1_{\mathcal{M}}(\operatorname{Spec} \mathbb{Q}(\mu_d), n+1)$$

given by $d^n \cdot n! \cdot \epsilon_{n+1}$.

Proof. The uniqueness assertion is a formal consequence of the finite generation of $K_{2n+1}(\operatorname{Spec} \mathbb{Q}(\mu_d))$: to give an element in a finitely generated abelian group M is the same as giving elements in $M \otimes_{\mathbb{Z}} \mathbb{Q}$ and all $M \otimes_{\mathbb{Z}} \mathbb{Z}_l$, which coincide in $M \otimes_{\mathbb{Z}} \mathbb{Q}_l$. By 9.7, the maps $r_l \circ \varphi_l$ and $r_l \circ \epsilon'_{n+1}$ agree for all l. From Theorem B.4.8, we conclude that φ_l and ϵ'_{n+1} agree as maps to $K_{2n+1} \otimes_{\mathbb{Z}} \mathbb{Q}_l$. $\square$

As shown by Bloch and Kato, Corollary 9.7 implies the validity of the following also for even n:

Corollary 9.9. *Let $n \geq 1$.*
Then the Tamagawa number conjecture ([BlK], Conjecture 5.15) is true modulo a power of 2 for the motif $\mathbb{Q}(n+1)$.

Proof. [BlK], Theorem 6.1.i) gives the complete proof for odd n, which is independent of anything said in the present article. In loc.cit., Theorem 6.1.ii), it is shown that the conjecture holds for even n if [BlK], 6.2 holds. But the latter is the content of 9.7. $\square$

Finally, the compatibility statement of 9.7 forms a central ingredient in the proof of the modified version of the Lichtenbaum conjecture for abelian number fields ([KNF], Theorem 6.4).

A Absolute Hodge Cohomology with Coefficients

The aim of this appendix is to provide a natural interpretation of absolute Hodge cohomology as extension groups in the category of algebraic Hodge modules over $\mathbb{R}$ (A.2.7). That such a sheaf–theoretic interpretation should be possible was already anticipated by Beilinson ([B1], 0.3), long before Hodge modules were defined.

The appendix is divided into two subsections. The first (A.1) starts with a summary of those parts of Saito's theory relevant to us. The central result is A.1.8, where we prove that for a smooth scheme $a : U \to \operatorname{Spec}(\mathbb{C})$, the polarizable Hodge complex $\underline{R\Gamma}(U,F)$ of [D3], (8.1.12) and [B1], § 4 is a representative for $a_*F(0)_U$, the object in the derived category of polarizable F–Hodge structures defined via Saito's formalism ([S2], 4.3). As a consequence, we are able (A.1.10) to identify absolute Hodge cohomology of a smooth scheme U over $\mathbb{C}$, as defined in [B1], § 5: it equals the Ext groups of Tate twists in the category of algebraic Hodge modules on U. The compatibility between the approaches of Deligne–Beilinson and of Saito will come as no surprise to the experts (see e.g. [S3], (2.8)). However, we were unable to find a quotable reference.

In A.2, we turn to the variant of the theory we really need: algebraic Hodge modules over $\mathbb{R}$. These live on the complexification of separated, reduced schemes of finite type over $\mathbb{R}$, and are basically the objects fixed by the natural involution on the category of mixed Hodge modules given by complex conjugation. The comparison statement for absolute Hodge cohomology over $\mathbb{R}$ (Theorem A.2.7) then follows formally from A.1.10.

A.1 Algebraic Mixed Hodge Modules

In [S2], § 4, the category $\mathrm{MHM}_A(X)$ of *algebraic mixed A–Hodge modules* is defined, where A is a field contained in $\mathbb{R}$, and X a separated reduced scheme of finite type over $\mathbb{C}$.

Saito's construction admits the full formalism of Grothendieck's functors $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ on the level of bounded derived categories $D^b\,\mathrm{MHM}_A$ ([S2], 4.3, 4.4) and a forgetful functor

$$\mathrm{rat} : \mathrm{MHM}_A(X) \longrightarrow \mathrm{Perv}_A(\overline{X})$$

to the category of perverse sheaves on the topological space $\overline{X}$ underlying $X(\mathbb{C})$, which have algebraic stratifications such that the restrictions of their cohomology sheaves to the strata are local systems. By the definition of MHM_A, which we shall partly sketch in a moment, rat is faithful and exact. The functor rat on the level of derived categories is compatible with Grothendieck's functors ([S2], 4.3, 4.4).

For smooth X, one constructs $\mathrm{MHM}_A(X)$ as an abelian subcategory ([S1], Proposition 5.1.14) of the category $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$, whose objects are

$$((M, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha) \ ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$, i.e., a regular holonomic algebraic $\mathcal{D}_X$–module M together with a good filtration $F^{\cdot}$, and $K \in \mathrm{Perv}_A(\overline{X})$. $W_{\cdot}$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \xrightarrow{\sim} K \otimes_A \mathbb{C}$$

respecting $W_{\cdot}$. Here, DR denotes the de Rham functor from the category of $\mathcal{D}_X$–modules to the category of perverse sheaves.

We note that by definition, the weight graded objects of all algebraic Hodge modules satisfy a certain polarizability condition (see [S1], 5.2.10).

Call an algebraic Hodge module on a smooth variety *smooth* if the underlying perverse sheaf is a local system up to a shift.

Theorem A.1.1 (Saito). *Let X be smooth and separated. Then there is an equivalence*

$$\mathrm{Var}_A(X) \xrightarrow{\sim} \mathrm{MHM}_A(X)^s$$

between the category of admissible variations of mixed A–Hodge structure ([Ks]) and the category of smooth algebraic A–Hodge modules on X.

Proof. This is the remark following [S2], Theorem 3.27. $\square$

In particular, we see that $\mathrm{MHM}_A(\mathrm{Spec}(\mathbb{C}))$ is the category MHS_A of polarizable mixed A–Hodge structures.

If $\mathbb{V}$ is a variation on X with underlying local system $\mathrm{For}(\mathbb{V})$, then the perverse sheaf underlying the Hodge module $\mathbb{V}$ under the correspondence of A.1.1 is

$$\mathrm{For}(\mathbb{V})[d]$$

if X is of pure dimension d.

It turns out that the definition of Tate twists in $\mathrm{MHM}_A(X)$ is compatible with the above equivalence only up to shift:

Definition A.1.2 ([S2], (4.5.5)). *Let $n \in \mathbb{Z}$, and $A(n) \in \mathrm{MHS}_A$ the usual Tate twist. For a separated reduced scheme $a : X \to \mathrm{Spec}(\mathbb{C})$, define*

$$A(n)_X := a^* A(n) \in D^b \,\mathrm{MHM}_A(X) \; .$$

If X is smooth and of pure dimension d, then $A(n)_X[d]$ is the variation of Hodge structure, which one denotes $A(n)$.

For arbitrary X, the complex $A(n)_X$ will not even be the shift of a Hodge module, but a proper element of $D^b \,\mathrm{MHM}_A(X)$, whose cohomology objects $\mathcal{H}^p A(n)_X$ are a priori trivial only for $p > \dim X$ ([S2], (4.5.6)).

We note again that we follow Saito's convention and write e.g. π_* for the functor on derived categories

$$D^b \,\mathrm{MHM}_A(X) \longrightarrow D^b \,\mathrm{MHM}_A(Y)$$

induced by a morphism $\pi : X \to Y$.

In order to compare the Hodge structures on Betti cohomology given by Saito's and Deligne's constructions, we need to go into the details of [S2]:

Theorem A.1.3 (Saito). *Let $j : U \hookrightarrow X$ be an open immersion of smooth separated schemes over $\mathbb{C}$, with $Y := X \setminus U$ a divisor with normal crossings. If X is of pure dimension d, then*

$$j_* A(0)_U[d] = \mathcal{H}^d j_* A(0)_U \in \mathrm{MHM}_A(X) \subset \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

equals the object

$$(w_X(*Y), (j_{\mathrm{top}})_* A_U[d], \alpha) \;,$$

where $w_X(*Y)$ denotes the $\mathcal{D}_X$–module $\Omega^d_X(\log Y)$, and $(j_{\mathrm{top}})_*$ the direct image for the derived category of perverse sheaves.

The de Rham complex with logarithmic singularities is quasi-isomorphic to $w_X(*Y) \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X[-d] = DR(w_X(*Y))[-d]$*, hence*

$$DR(w_X(*Y)) = \Omega^{\cdot}_X(\log Y)[d]$$

(compare [Bo3], VIII, 13.1), and

$$\alpha : \Omega^{\cdot}_X(\log Y)[d] \xrightarrow{\sim} (j_{\mathrm{top}})_* \mathbb{C}\,[d]$$

is the usual quasi-isomorphism

$$\Omega^{\cdot}_X(\log Y) \xrightarrow{\sim} (j_{\mathrm{top}})_* \Omega^{\cdot}_U \xleftarrow{\sim} (j_{\mathrm{top}})_* \mathbb{C}$$

(compare [D2], 3.1), shifted by d.

The Hodge filtration $F^{\cdot}$ *on* $w_X(*Y)$ *is induced from the stupid filtration, while the weight filtrations* $W.$ *on* $w_X(*Y)$ *and* $(j_{\mathrm{top}})_*\mathbb{C}\,[d]$ *are those induced from the canonical filtration on* $(j_{\mathrm{top}})_*\Omega_U$*, shifted by d.*

Proof. The equation $j_*A(0)_U[d] = \mathcal{H}^d j_* A(0)_U$ follows from the faithfulness of rat and the fact that the corresponding statement for $(j_{\mathrm{top}})_*$ is true since j is affine. In our geometric situation, the explicit construction of j_* of *any* admissible variation of A–Hodge structure is carried out in the proof of [S2], Theorem 3.27. For $A(0)_U$, it specializes to our claim. $\square$

In [B1], 3.9, Beilinson extends Deligne's notion of Hodge complexes ([D3], 8.1) to the polarizable situation:

Definition A.1.4 (Beilinson). *A mixed A–Hodge complex*

$$K = ((K_{\mathbb{C}}, F^{\cdot}, W.), (K, W.), \alpha)$$

is called polarizable if the cohomology objects of the weight n *Hodge complexes* $\mathrm{Gr}^W_n(K)$ *are polarizable A–Hodge structures.*

Remark: The weight filtration $W.$ of a mixed Hodge complex K induces mixed Hodge structures on its cohomology. Observe however that $\mathrm{Gr}^W_n(H^iK)$ is of weight $n+i$.

As in the non–polarizable situation, Beilinson proves:

Theorem A.1.5 ([B1], Lemma 3.11). *There is an equivalence of categories between* $D^b\,\mathrm{MHS}_A$ *and the derived category of polarizable A–Hodge complexes.*

Let X be smooth and separated over $\mathbb{C}$. Forgetting part of the structure of a Hodge module yields a functor

$$\mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(X)\;.$$

Here, $T(X)$ is the category of triples

$$M^{\cdot} = ((M^{\cdot}, F^{\cdot\cdot}, W^{\cdot}_{.}), (K^{\cdot}, W^{\cdot}_{.}), \alpha^{\cdot})\;,$$

where $(M^{\cdot}, F^{\cdot\cdot}, W_{\cdot}^{\cdot})$ is a class in the filtered derived category $D^bW(\mathrm{MF}_h(\mathcal{D}_X))$ of $\mathrm{MF}_h(\mathcal{D}_X)$, and $(K^{\cdot}, W_{\cdot}^{\cdot})$ a class in the filtered derived category of sheaves of A–vector spaces on $X(\mathbb{C})$, denoted by $D^bW(X(\mathbb{C}), A)$. Furthermore, the map $\alpha^{\cdot}$ is an isomorphism

$$DR(M^{\cdot}) \xrightarrow{\sim} K^{\cdot} \otimes_A \mathbb{C}$$

respecting $W_{\cdot}^{\cdot}$.

Recall that in order to obtain a class in $D^bW(X(\mathbb{C}), A)$ from a complex of perverse sheaves, one applies the realization functor of [BBD], 3.1.9.

The global section functor Γ can be derived on $D^bW(X(\mathbb{C}), A)$. By [S1], 2.3, we have a functor $R\Gamma$ on $D^bW(\mathrm{MF}_h(\mathcal{D}_X))$ if X is proper, and the two constructions are compatible with the comparison isomorphism $\alpha^{\cdot}$ of any object in $T(X)$ ([S1], 2.3.7). We end up with an object

$$R\Gamma M^{\cdot} = (R\Gamma(M^{\cdot}, F^{\cdot\cdot}, W_{\cdot}^{\cdot}), R\Gamma(K^{\cdot}, W_{\cdot}^{\cdot}), R\Gamma\alpha^{\cdot})$$

of $T(\mathrm{Spec}(\mathbb{C}))$. The functor

$$\underline{R\Gamma} := R\Gamma \circ \mathrm{For} : C^b\,\mathrm{MHM}_A(X) \longrightarrow T(\mathrm{Spec}(\mathbb{C}))$$

factorizes through $D^b\,\mathrm{MHM}_A(X)$.

Our second comparison result is the following:

Theorem A.1.6. *Let $a : X \to \mathrm{Spec}(\mathbb{C})$ be smooth and proper, and $M^{\cdot}$ an object of $D^b\,\mathrm{MHM}_A(X)$. Write*

$$\mathrm{For}\, M^{\cdot} = ((M^{\cdot}, F^{\cdot\cdot}, W_{\cdot}^{\cdot}), (K^{\cdot}, W_{\cdot}^{\cdot}), \alpha^{\cdot}) \in T(X)\ .$$

a)

$$\underline{R\Gamma} M^{\cdot} = (R\Gamma(M^{\cdot}, F^{\cdot\cdot}, W_{\cdot}^{\cdot}), R\Gamma(K^{\cdot}, W_{\cdot}^{\cdot}), R\Gamma\alpha^{\cdot})$$

is a mixed polarizable A–Hodge complex.

b) *The class of $\underline{R\Gamma} M^{\cdot}$ in the derived category of polarizable Hodge complexes is canonically isomorphic, under the identification of A.1.5, to*

$$a_* M^{\cdot} \in D^b\,\mathrm{MHS}_A\ .$$

c) *Let $f : Y \to X$ be a (proper) morphism of smooth and proper schemes over $\mathbb{C}$, and let b denote the structure morphism of Y, such that*

$$b = a \circ f\ .$$

For any $N^{\cdot} \in D^b\,\mathrm{MHM}_A(Y)$ together with a morphism $\eta : M^{\cdot} \to f_* N^{\cdot}$ in $D^b\,\mathrm{MHM}_A(X)$, the morphism

$$a_*\eta : a_* M^{\cdot} = \underline{R\Gamma} M^{\cdot} \longrightarrow \underline{R\Gamma} N^{\cdot} = b_* N^{\cdot} = a_* f_* N^{\cdot}$$

equals, under the isomorphism of a), the morphism

$$(R\Gamma\eta, R\Gamma\eta, R\Gamma\eta)$$

of A–Hodge complexes.

Proof. a) We may assume that $M^\cdot$ is pure of some weight. Using [S2], (4.5.4), we are reduced to the case where $M^\cdot = M$ is a Hodge module of weight n, and we have to show that $\underline{R\Gamma} M$ is a polarizable Hodge complex of the same weight. Axiom (CH 1) of [D3], (8.1.1) follows from [S2], Proposition 2.16, in particular (2.16.5), applied to $\mathrm{pr}^* M$, where

$$\mathrm{pr} : X \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \longrightarrow X \; .$$

Furthermore, by the remark following [S2], (4.2.9), and by loc. cit., 2.15, we have isomorphisms in $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_{\mathrm{Spec}(\mathbb{C})}, A)$

$$\underline{R^i\Gamma} M := \big(R^i\Gamma(M, F^\cdot, W_\cdot[i]), R^i\Gamma(K, W_\cdot[i]), R^i\Gamma\alpha\big) \xrightarrow{\sim} \mathcal{H}^i a_* M \; .$$

Since the right hand side is a polarizable Hodge structure of weight $i+n$ ([S2], (4.5.2)), we have (CH 2), and in addition, polarizability.
b) In the proof of a), we constructed a functor

$$a_*^\sim := \underline{R\Gamma} : D^b \,\mathrm{MHM}_A(X) \longrightarrow D^b \,\mathrm{MHS}_A \; ,$$

such that

$$\mathcal{H}^i a_*^\sim = \mathcal{H}^i a_* : \mathrm{MHM}_A(X) \hookrightarrow D^b \,\mathrm{MHM}_A(X) \longrightarrow \mathrm{MHS}_A$$

for all i. Composition with $j_* : D^b \,\mathrm{MHM}_A(U) \to D^b \,\mathrm{MHM}_A(X)$ for open immersions $j : U \hookrightarrow X$ defines

$$(a \circ j)_*^\sim := a_*^\sim \circ j_* : D^b \,\mathrm{MHM}_A(U) \longrightarrow D^b \,\mathrm{MHS}_A \; .$$

But for affine U, $(a \circ j)_*$ is the left derived functor of

$$\mathcal{H}^0 (a \circ j)_* : \mathrm{MHM}_A(U) \longrightarrow \mathrm{MHS}_A$$

([S2], proof of Theorem 4.3.). If U is affine, then so is $j : U \hookrightarrow X$, and hence j_* is exact. Therefore,

$$\mathcal{H}^0 (a \circ j)_* = \mathcal{H}^0 a_* \circ j_* : \mathrm{MHM}_A(U) \xrightarrow{j_*} \mathrm{MHM}_A(X) \xrightarrow{\mathcal{H}^0 a_*} \mathrm{MHS}_A$$

coincides with $\mathcal{H}^0 (a \circ j)_*^\sim$, and we get a natural transformation

$$(a \circ j)_* \longrightarrow (a \circ j)_*^\sim \; ,$$

which is an isomorphism, since this is true on the level of cohomology objects, as one checks on the level of vector spaces. Observe that this natural transformation is compatible with restriction to smaller affine subschemes of X. Now recall ([S2], proof of 4.3) that the functor a_* is constructed using the Čech complex associated to an affine covering of X (for details, see [B3], 3.4). In the same way, the functor $a_*^\sim$ is recoverable from the $(a \circ j)_*^\sim$. We end up with an isomorphism of a_* and $a_*^\sim$, which is independent of the covering.
c) In the proof of b), we constructed a natural isomorphism

$$\kappa : a_* \xrightarrow{\sim} a_*^\sim$$

of functors from $D^b\,\mathrm{MHM}_A(X)$ to $D^b\,\mathrm{MHS}_A$. For $f = \mathrm{id}$, our claim is therefore proved. For the general situation, we use the same techniques as in the proof of b) to first construct a natural isomorphism

$$b_* \xrightarrow{\sim} a_*^\sim \circ f_*$$

of functors from $D^b\,\mathrm{MHM}_A(Y)$ to $D^b\,\mathrm{MHS}_A$, and then to see that the triangle

$$\begin{array}{ccc} b_* & \longrightarrow & a_* \circ f_* \\ & \searrow & \downarrow \kappa \\ & & a_*^\sim \circ f_* \end{array}$$

commutes. □

Corollary A.1.7 (cf. [S3], (2.8)). *Let $j : U \hookrightarrow X$ be a smooth compactification of a smooth and separated scheme $a : U \to \mathrm{Spec}(\mathbb{C})$, such that $Y := X \setminus U$ is a divisor with normal crossings.*

a) *$a_* A(0)_U \in D^b\,\mathrm{MHS}_A$ is isomorphic, under the identification of A.1.5, to the class of the mixed polarizable A–Hodge complex*

$$\underline{R\Gamma}(U, A) := \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U, \alpha)$$

of [D3], (8.1.12) and [B1], § 4 (with the same notation).

b) *If $f : X \to X'$ is a morphism of compactifications $j : U \hookrightarrow X$ and $j' : U \hookrightarrow X'$ of U as in a), then f induces an isomorphism*

$$\underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_{X'}(\log Y'), (j'_{\mathrm{top}})_* A_U) \xrightarrow{\sim} \underline{R\Gamma}(DR^{-1}\Omega^{\cdot}_X(\log Y), (j_{\mathrm{top}})_* A_U)$$

([D3], remark preceding (8.1.17)), so $\underline{R\Gamma}(U, A)$ depends only on U.

The isomorphism in a) also depends only on U.

c) *In particular, the Hodge structures on*

$$\mathrm{rat}(\mathcal{H}^n a_* A(n)_U) = H^n_B(U(\mathbb{C}), (2\pi i)^n A)$$

given by Deligne's and Saito's constructions coincide.

Proof. a) Combine A.1.3 and A.1.6.b).
b) Use A.1.6.c).
c) follows from a) and b). □

Actually, the statement A.1.6.c) implies the functoriality property we were after: we have two functors

$$(Sm/\mathbb{C})^0 \longrightarrow D^b\,\mathrm{MHS}_A\ ,$$

where $(Sm/\mathbb{C})$ denotes the category of smooth separated schemes over $\mathbb{C}$:

$$\begin{aligned} \underline{R\Gamma}(\_\,, A) &: U \longmapsto \underline{R\Gamma}(U, A)\,, \\ {}_*(A) &: (a : U \longrightarrow \mathrm{Spec}(\mathbb{C})) \longmapsto a_*\,(A(0)_U)\ . \end{aligned}$$

Corollary A.1.8. *The isomorphism of A.1.7.a) is functorial in $U \in Sm/\mathbb{C}$. In other words, there is a natural isomorphism*

$$_*(A) \xrightarrow{\sim} \underline{R\Gamma}(\_, A)$$

of functors from $(Sm/\mathbb{C})^0$ to $D^b \operatorname{MHS}_A$.

Proof. Let

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "f"'] & X' \arrow[d, "f"] \\ U \arrow[r, hook, "j"] & X \end{tikzcd}$$

be a commutative diagram of smooth and separated schemes over $\mathbb{C}$, where X' and X are proper, and $Y' := X' \setminus U'$ and $Y := X \setminus U$ are divisors with normal crossings. We have a morphism

$$(*) \qquad j_* A(0)_U \longrightarrow f_*(j'_* A(0)_{U'}) \ .$$

Application of $(a_X)_*$ gives the morphism

$$(a_U)_* A(0)_U \longrightarrow (a_{U'})_* A(0)_{U'}$$

belonging to the functoriality requirement for $_*(A)$. Our claim follows from A.1.6.c), applied to a shift of the morphism $(*)$. $\square$

Definition A.1.9. *Let $X/\mathbb{C}$ be separated, reduced and of finite type, and $M^\cdot$ an object of $D^b \operatorname{MHM}_A(X)$.*

a) *The absolute Hodge complex of X with coefficients in $M^\cdot$ is*

$$R\Gamma_{\mathfrak{H}^p}(X, M^\cdot) := R\operatorname{Hom}_{D^b \operatorname{MHM}_A(X)}(A(0)_X, M^\cdot) \ .$$

b) *Its cohomology groups*

$$H^i_{\mathfrak{H}^p}(X, M^\cdot) := H^i R\Gamma_{\mathfrak{H}^p}(X, M^\cdot)$$

are called absolute Hodge cohomology groups of X with coefficients in $M^\cdot$.

c) *We denote absolute Hodge cohomology with coefficients in Tate twists by*

$$H^i_{\mathfrak{H}^p}(X, n) := H^i_{\mathfrak{H}^p}(X, A(n)_X) \ .$$

d) *For a closed reduced subscheme Z of X with complement $j : U \hookrightarrow X$, we define relative absolute Hodge cohomology with coefficients in Tate twists as*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z, n) := H_{\mathfrak{H}^p}(X, j_! A(n)_U) \ .$$

Note that if X is smooth and of pure dimension d, and if

$$M^{\cdot} = M \in \mathrm{MHM}_A(X) ,$$

then the right hand side of A.1.9.b), being equal to

$$\mathrm{Hom}_{D^b \, \mathrm{MHM}_A(X)}(A(0)_X[d], M[d+i]) ,$$

admits an interpretation as the group of $(d+i)$–extensions of Hodge modules modulo Yoneda equivalence.

COROLLARY A.1.10. *If X is smooth and separated over $\mathbb{C}$, and $n \in \mathbb{Z}$, then*

$$R\Gamma_{\mathfrak{H}^p}(X,n) = R\Gamma_{\mathfrak{H}^p}(X,A(n)_X) \quad \textit{and} \quad H^{\cdot}_{\mathfrak{H}^p}(X,n) = H^{\cdot}_{\mathfrak{H}^p}(X,A(n)_X)$$

coincide functorially with the same noted objects of [B1], § 5.

Proof. This follows from A.1.8 and the adjunction formula

$$R\,\mathrm{Hom}_{D^b \, \mathrm{MHM}_A(X)}(A(0)_X, M^{\cdot}) = R\,\mathrm{Hom}_{D^b \, \mathrm{MHS}_A}(A(0), a_* M^{\cdot}) .$$

$\square$

REMARK: The Leray spectral sequence for $a : X \to \mathrm{Spec}(\mathbb{C})$ yields exact sequences

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}_A}\left(A(0), H^{i-1}\right) \to H^i_{\mathfrak{H}^p}(X, A(n)_X) \to \mathrm{Hom}_{\mathrm{MHS}_A}\left(A(0), H^i\right) \to 0$$

(with $H^k := H^k_B(X(\mathbb{C}), (2\pi i)^n A))$ since MHS_A has cohomological dimension one ([B1], Corollary 1.10). Comparing them with the analogous sequences for $H^i_{\mathfrak{H}}$, we see that

$$H^i_{\mathfrak{H}^p}(X, A(n)_X) = H^i_{\mathfrak{H}}(X, A(n)_X)$$

(in the notation of [B1], § 5) if $H^{i-1}_B(X(\mathbb{C}), (2\pi i)^n A)$ has weights smaller than zero, which is the case if $i \le n$ ($i \le 2n$ if X is proper).

Observe that this is the same range of indices where Deligne cohomology coincides with $H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X)$ ([N], (7.1)): we have natural morphisms

$$H^i_{\mathfrak{H}^p}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X, \mathbb{R}(n)_X) ,$$

both of which are isomorphisms if $i \le n$ ($i \le 2n$ if X is proper).

A.2 ALGEBRAIC MIXED HODGE MODULES OVER $\mathbb{R}$

Algebraic Hodge modules over $\mathbb{R}$ are defined as the category of Hodge modules fixed under a certain involution given by complex conjugation. We start by constructing this involution:

Let $X/\mathbb{C}$ be smooth, and let ${}^{\iota}X$ denote the complex conjugate scheme. We have an equivalence

$$\iota^* : \mathrm{Var}_A({}^{\iota}X) \xrightarrow{\sim} \mathrm{Var}_A(X)$$

of the categories of admissible variations, induced by complex conjugation

$$\iota : X(\mathbb{C}) \longrightarrow {}^{\iota}X(\mathbb{C}) ,$$

and defined as follows:

The local system and the weight filtration on $X(\mathbb{C})$ are the pullbacks via ι of the local system and the weight filtration on ${}^{\iota}X(\mathbb{C})$, and the Hodge filtration on $X(\mathbb{C})$ is the pullback of the conjugate of the Hodge filtration on ${}^{\iota}X(\mathbb{C})$.

ι^* preserves admissibility, and behaves, in an obvious sense, involutively. In particular, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$.

Definition A.2.1. *Let $X/\mathbb{R}$ be smooth and separated.*

a) *The category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ consists of pairs $(\mathbb{V}, F_\infty)$, where $\mathbb{V}$ is an object of $\mathrm{Var}_A(X \otimes_{\mathbb{R}} \mathbb{C})$, and F_∞ is an isomorphism*

$$\mathbb{V} \xrightarrow{\sim} \iota^*\mathbb{V}$$

of variations such that $\iota^* F_\infty = F_\infty^{-1}$.

In the category $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$, we may define Tate twists $A(n)$: F_∞ acts via multiplication by $(-1)^n$.

b) *$\mathrm{Var}_A(X/\mathbb{R})$, the category of admissible variations of mixed A–Hodge structure over $\mathbb{R}$, is the full subcategory of $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$ of pairs $(\mathbb{V}, F_\infty)$ which are graded–polarizable: for $n \in \mathbb{Z}$, there is a morphism*

$$\mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \otimes_A \mathrm{Gr}_n^W(\mathbb{V}, F_\infty) \longrightarrow A(-n)$$

in $\mathrm{Var}_A^{\sim}(X/\mathbb{R})$, such that the induced morphism

$$\mathrm{Gr}_n^W \mathbb{V} \otimes_A \mathrm{Gr}_n^W \mathbb{V} \longrightarrow A(-n)$$

is a polarization in the usual sense.

Remark: We note that implicit in our definition is a descent datum over $\mathbb{R}$ of the bifiltered flat vector bundle on $X \otimes_{\mathbb{R}} \mathbb{C}$ underlying any admissible variation $(\mathbb{V}, F_\infty)$ of mixed A–Hodge structure over $\mathbb{R}$:

For this claim to make sense, recall first ([D1], II, Théorème 5.9) that any flat analytic vector bundle on $X(\mathbb{C})$ carries a canonical algebraic structure. If the vector bundle underlies an admissible variation, then the Hodge filtration is a filtration by algebraic subbundles ([Ks], Proposition 1.11.3).

Now the descent datum is given by the anti-linear isomorphism

$$c_{DR} := F_{\mathrm{diff}}(F_\infty) \circ c_\infty = c_\infty \circ F_{\mathrm{diff}}(F_\infty) : F_{\mathrm{diff}}(\mathbb{V}) \xrightarrow{\sim} F_{\mathrm{diff}}(\iota^*\mathbb{V})$$

of the C^∞–bundles underlying $\mathbb{V}$ and $\iota^*\mathbb{V}$. Here, c_∞ denotes the anti-linear involutions given by complex conjugation of coefficients, and F_{diff} is the forgetful functor to C^∞–bundles.

Lemma A.2.2. *The category $\mathrm{Var}_A(\mathrm{Spec}(\mathbb{R})/\mathbb{R})$ equals the category MHS_A^+ of mixed polarizable A–Hodge structures over $\mathbb{R}$ ([B1], § 7).*

Proof. Straightforward. □

Our aim is to generalize our definition of sheaves over $\mathbb{R}$ to algebraic Hodge modules.

For smooth and separated $X/\mathbb{C}$, recall that $\mathrm{MHM}_A(X)$ is an abelian subcategory of $\mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$. Objects of the latter are

$$((M, F^{\cdot}, W_{\cdot}), (K, W_{\cdot}), \alpha) \ ,$$

where $(M, F^{\cdot})$ is an object of the category $\mathrm{MF}_h(\mathcal{D}_X)$ of regular holonomic algebraic $\mathcal{D}_X$–modules with a good filtration, and $K \in \mathrm{Perv}_A(\overline{X})$. $W_{\cdot}$ is a locally finite ascending filtration, and α is an isomorphism

$$DR(M) \stackrel{\sim}{\longrightarrow} K \otimes_A \mathbb{C}$$

respecting W.

The equivalence

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^{\iota}X}, A) \stackrel{\sim}{\longrightarrow} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A)$$

is constructed componentwise:

The perverse sheaf and the weight filtration on $X(\mathbb{C})$ are the pullbacks via $\iota : X(\mathbb{C}) \to {}^{\iota}X(\mathbb{C})$ of the perverse sheaf and the weight filtration on ${}^{\iota}X(\mathbb{C})$.

The equivalence

$$\iota^* : \mathrm{Mod}_{\mathcal{D}_{{}^{\iota}X}} \stackrel{\sim}{\longrightarrow} \mathrm{Mod}_{\mathcal{D}_X} \ ,$$

which by construction will respect holonomicity, comes about as follows:

Given a $\mathcal{D}_{{}^{\iota}X}$–module N, we may form the inverse image (in the sense of sheaves of abelian groups) $\iota^{-1}N$, which is a $\iota^{-1}\mathcal{D}_{{}^{\iota}X}$–module. All we therefore need is an isomorphism $c_\infty : \iota^{-1}\mathcal{D}_{{}^{\iota}X} \stackrel{\sim}{\longrightarrow} \mathcal{D}_X$ of sheaves of rings extending the isomorphism $c_\infty : \iota^{-1}\mathcal{O}_{{}^{\iota}X} \stackrel{\sim}{\longrightarrow} \mathcal{O}_X$ given by complex conjugation of coefficients – we then define

$$\iota^* N := \iota^{-1}N \otimes_{\iota^{-1}\mathcal{D}_{{}^{\iota}X}} \mathcal{D}_X \ .$$

Of course, the map c_∞ is itself given by conjugation of coefficients: in local coordinates $x_1, \ldots, x_n$, we have

$$c_\infty \left(\sum_\alpha f_\alpha \partial_{\overline{x}}^{\overline{\alpha}} \right) = \sum_\alpha (c_\infty \circ f_\alpha \circ \iota) \partial_x^\alpha \ .$$

Altogether, we get

$$\iota^* : \mathrm{MF}_h\mathrm{W}(\mathcal{D}_{{}^{\iota}X}, A) \stackrel{\sim}{\longrightarrow} \mathrm{MF}_h\mathrm{W}(\mathcal{D}_X, A) \,,$$

which again behaves involutively.

Going through the definition, one checks that ι^* induces

$$\iota^* : \mathrm{MHM}_A({}^{\iota}X) \stackrel{\sim}{\longrightarrow} \mathrm{MHM}_A(X) \ .$$

Using local embeddings as in [S2], 2.1, we can define ι^* for any scheme X, which is separated, reduced and of finite type over $\mathbb{C}$. Furthermore, if X is defined over $\mathbb{R}$, we get an involution ι^* on $\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})$.

Theorem A.2.3. *Let X and Y be separated and reduced schemes of finite type over $\mathbb{C}$.*

a) *ι^* is compatible with $\underline{\mathrm{Hom}}$, $\otimes$, and $\mathbb{D}$: e.g., for $M^\cdot, N^\cdot \in D^b \mathrm{MHM}_A({}^\iota X)$, we have*

$$\underline{\mathrm{Hom}}_X(\iota^* M^\cdot, \iota^* N^\cdot) = \iota^* \underline{\mathrm{Hom}}_{{}^\iota X}(M^\cdot, N^\cdot) .$$

b) *If $\pi : X \to Y$ is a morphism, then ι^* is compatible with $\pi_!$, $\pi^!$, π^*, π_*: e.g., for $M^\cdot \in D^b \mathrm{MHM}_A({}^\iota X)$, we have*

$$\iota^* ({}^\iota \pi)_* M^\cdot = \pi_* (\iota^* M^\cdot) \in D^b \mathrm{MHM}_A(Y) .$$

Proof. This follows from the definitions. □

Definition A.2.4. a) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. The category $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ consists of pairs (M, F_∞), where M is an object of $\mathrm{MHM}_A(X \otimes_{\mathbb{R}} \mathbb{C})$, and F_∞ is an isomorphism*

$$M \xrightarrow{\sim} \iota^* M$$

such that $\iota^* F_\infty = F_\infty^{-1}$.

By A.2.3.b), we have $a^! A(n) \in \mathrm{MHM}_A^{\sim}(X/\mathbb{R})$.

b) *Let $a : X \to \mathrm{Spec}(\mathbb{R})$ be smooth and separated. $\mathrm{MHM}_A(X/\mathbb{R})$, the category of algebraic mixed A–Hodge modules over $\mathbb{R}$ on X, is the full subcategory of $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$ of pairs (M, F_∞) which are graded–polarizable: for any $n \in \mathbb{Z}$, there is a morphism*

$$\mathrm{Gr}_n^W (M, F_\infty) \otimes_A \mathrm{Gr}_n^W (M, F_\infty) \longrightarrow a^! A(-n)$$

in $\mathrm{MHM}_A^{\sim}(X/\mathbb{R})$, such that the induced morphism

$$\mathrm{Gr}_n^W M \otimes_A \mathrm{Gr}_n^W M \longrightarrow a^! A(-n)$$

is a polarization in the sense of [S1], 5.2.10.

As in A.1.1, we identify the category of smooth objects in $\mathrm{MHM}_A(X/\mathbb{R})$ with $\mathrm{Var}_A(X/\mathbb{R})$.

c) *For an arbitrary separated and reduced scheme X of finite type over $\mathbb{R}$, one defines the category $\mathrm{MHM}_A(X/\mathbb{R})$ using local embeddings as in [S2], 2.1.*

Remark: a) As in the case of variations over $\mathbb{R}$, we get a descent datum over $\mathbb{R}$ for the bifiltered $\mathcal{D}_{X \otimes_{\mathbb{R}} \mathbb{C}}$–module underlying any Hodge module over $\mathbb{R}$ on a smooth and separated scheme X over $\mathbb{R}$.
b) As in [S2], (4.2.7), the category $\mathrm{MHM}_A(Z/\mathbb{R})$, for any closed reduced subscheme Z of X, is equivalent to the category of Hodge modules over $\mathbb{R}$ on X with support in Z.

Theorem A.2.5. *There is a formalism of Grothendieck's functors $\pi_!$, $\pi^!$, π^*, π_*, $\underline{\mathrm{Hom}}$, $\otimes$, $\mathbb{D}$ on $D^b\,\mathrm{MHM}_A(\cdot/\mathbb{R})$. It is compatible with the forgetful functor*

$$D^b\,\mathrm{MHM}_A(\cdot/\mathbb{R}) \longrightarrow D^b\,\mathrm{MHM}_A(\cdot \otimes_{\mathbb{R}} \mathbb{C})\ .$$

Proof. By A.2.3, we may e.g. define

$$\pi_!(M^\cdot, F^\cdot_\infty) := (\pi_! M^\cdot, \pi_! F^\cdot_\infty)\ .$$

□

Definition A.2.6. *Let $X/\mathbb{R}$ be separated, reduced and of finite type, and $M^\cdot$ an object of $D^b\,\mathrm{MHM}_A(X/\mathbb{R})$.*

a) *The absolute Hodge complex of $X/\mathbb{R}$ with coefficients in $M^\cdot$ is*

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^\cdot) := R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X, M^\cdot)\ .$$

b) *Its cohomology groups*

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M^\cdot) := H^i R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^\cdot)$$

are called absolute Hodge cohomology groups of $X/\mathbb{R}$ with coefficients in $M^\cdot$.

c) *We denote absolute Hodge cohomology with coefficients in Tate twists by*

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, n) := H^i_{\mathfrak{H}^p}(X/\mathbb{R}, A(n)_X)\ .$$

d) *For a closed reduced subscheme Z of X with complement $j : U \hookrightarrow X$, we define relative absolute Hodge cohomology with coefficients in Tate twists as*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/\mathbb{R}, n) := H_{\mathfrak{H}^p}(X/\mathbb{R}, j_! A(n)_U)\ .$$

Again, if X is smooth and of pure dimension d, and $M^\cdot = M \in \mathrm{MHM}_A(X)$, we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, M) = \mathrm{Ext}^{d+i}_{\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X[d], M)\ .$$

We have statements analogous to A.1.1–A.1.10 for the situation over $\mathbb{R}$. For reference, we note explicitly:

Theorem A.2.7. *If X is smooth and separated over $\mathbb{R}$, and $n \in \mathbb{Z}$, then*

$$R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, n) \quad \text{and} \quad H^\cdot_{\mathfrak{H}^p}(X/\mathbb{R}, n)$$

coincide functorially with the absolute Hodge complex and cohomology groups of [B1], § 7.

Next, we have

LEMMA A.2.8. *Let $X/\mathbb{R}$ be separated, reduced and of finite type, and $M^{\cdot}$ an object of $D^b\,\mathrm{MHM}_A(X/\mathbb{R})$. Then the forgetful functor*

$$D^b\,\mathrm{MHM}_A(X/\mathbb{R}) \longrightarrow D^b\,\mathrm{MHM}_A(X\otimes_{\mathbb{R}}\mathbb{C})$$

induces functorial isomorphisms

$$\begin{array}{rcl} R\Gamma_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) & \stackrel{\sim}{\longrightarrow} & R\Gamma_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C}, M^{\cdot})^+ \ , \\ H^{\cdot}_{\mathfrak{H}^p}(X/\mathbb{R}, M^{\cdot}) & \stackrel{\sim}{\longrightarrow} & H^{\cdot}_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C}, M^{\cdot})^+ \ . \end{array}$$

Here, the superscript $+$ denotes the fixed part of the action of the involution ι^* on

$$R\,\mathrm{Hom}_{D^b\,\mathrm{MHM}_A(X\otimes_{\mathbb{R}}\mathbb{C})}(A(0)_{X\otimes_{\mathbb{R}}\mathbb{C}}, M^{\cdot}) \ .$$

In particular, the category MHS_A^+ has cohomological dimension one since this is true for MHS_A. Furthermore, observe that the above action of $\mathbb{Z}/2\mathbb{Z}$ on $R\Gamma_{\mathfrak{H}^p}(X\otimes_{\mathbb{R}}\mathbb{C}, A(n)_{X\otimes_{\mathbb{R}}\mathbb{C}})$ is precisely that of [B1], § 7.

COROLLARY A.2.9. *Let $X/\mathbb{R}$ be separated, reduced and of finite type. The forgetful functor*

$$\mathrm{rat} : \mathrm{MHM}_A(X/\mathbb{R}) \longrightarrow \mathrm{Perv}_A(\overline{X\otimes_{\mathbb{R}}\mathbb{C}})$$

is faithful and exact.

REMARK: Again we have

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, A(n)_X) = H^i_{\mathfrak{H}}(X/\mathbb{R}, A(n)_X)$$

if $i \le n$ ($i \le 2n$ if X is proper). We have natural morphisms

$$H^i_{\mathfrak{H}^p}(X/\mathbb{R}, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathfrak{H}}(X/\mathbb{R}, \mathbb{R}(n)_X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)_X)\,,$$

which are isomorphisms in the same range of indices.

We conclude with an explicit formula for Ext^1 in $\mathrm{MHM}_A(X/\mathbb{R})$ of a finite scheme $X/\mathbb{R}$.

THEOREM A.2.10. *For any $H \in \mathrm{MHS}_A^+$, there is a canonical isomorphism*

$$\begin{array}{rcl} \big(W_0H_{\mathbb{C}}/(W_0H_A + W_0F^0H_{\mathbb{C}})\big)^+ & \stackrel{\sim}{\longrightarrow} & \mathrm{Ext}^1_{\mathrm{MHS}_A^+}(A(0), H) \\ & = & H^1_{\mathfrak{H}^p}(\mathrm{Spec}(\mathbb{R})/\mathbb{R}, H)\,, \end{array}$$

where the superscript $+$ on the left hand side denotes the fixed part of the de Rham–conjugation

$$\begin{array}{rcl} W_0H_{\mathbb{C}}/(W_0H_A + W_0F^0H_{\mathbb{C}}) & \stackrel{c_\infty}{\longrightarrow} & W_0H_{\mathbb{C}}/(W_0H_A + W_0\overline{F}^0H_{\mathbb{C}}) \\ & = & W_0\,\iota^*H_{\mathbb{C}}/(W_0\,\iota^*H_A + W_0F^0\iota^*H_{\mathbb{C}}) \\ & \stackrel{F_\infty}{\longrightarrow} & W_0H_{\mathbb{C}}/(W_0H_A + W_0F^0H_{\mathbb{C}})\,. \end{array}$$

The isomorphism is given by sending the class of $h \in W_0 H_{\mathbb{C}}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id}_H \end{pmatrix} .$$

This means that we equip $\mathbb{C} \oplus H_{\mathbb{C}}$ with the diagonal weight and Hodge filtrations, and the A–rational structure extending the A–rational structure H_A of $H_{\mathbb{C}}$ by the vector

$$1 - h \in \mathbb{C} \oplus H_{\mathbb{C}} ,$$

thereby obtaining an extension E of $A(0)$ by H in the category MHS_A.

The conjugate extension $\iota^* E \in \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), \iota^* H)$ is given, with the same notation, by the matrix

$$\begin{pmatrix} 1 & 0 \\ -F_\infty(h) & \mathrm{id}_{\iota^* H} \end{pmatrix} ,$$

and the extension of F_∞ to an isomorphism

$$F_\infty : E \xrightarrow{\sim} \iota^* E$$

sends $1 - h$ to $1 - F_\infty(h)$. Thus

$$(F_\infty)_{\mathbb{C}} = \mathrm{id} \oplus (F_\infty)_{\mathbb{C}} : \mathbb{C} \oplus H_{\mathbb{C}} \longrightarrow \mathbb{C} \oplus \iota^* H_{\mathbb{C}} .$$

Proof. Using [B1], § 1 or [Jn3], Lemma 9.2 and Remark 9.3.a), we see that there is an isomorphism

$$W_0 H_{\mathbb{C}} / (W_0 H_A + W_0 F^0 H_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H) .$$

Note that our normalization follows that of Jannsen, and therefore differs from that of Beilinson by the factor -1.
In general, if $h \in W_0 H_{\mathbb{C}}$ corresponds to an extension E in MHS_A, then $c_\infty h \in W_0 \iota^* H_{\mathbb{C}}$ corresponds to $\iota^* E$, and its pullback via

$$F_\infty : \iota^* H \longrightarrow H ,$$

is described by $F_\infty c_\infty h$. The action of the involution on $\mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)$ therefore corresponds to $F_\infty c_\infty$ on the left hand side of the above isomorphism. ☐

Corollary A.2.11. *Let $X/\mathbb{R}$ be finite and reduced, and $M \in \mathrm{MHM}_A(X/\mathbb{R})$. Then there is a canonical isomorphism*

$$\begin{aligned} & \left(\bigoplus_{x \in X(\mathbb{C})} W_0 M_{x,\mathbb{C}} / (W_0 M_{x,A} + W_0 F^0 M_{x,\mathbb{C}}) \right)^+ \\ \underset{A.2.10}{\xrightarrow{\sim}} \; & \mathrm{Ext}^1_{\mathrm{MHS}_A^+} \left(A(0), \bigoplus_{x \in X(\mathbb{C})} M_x \right) \\ = \; & H^1_{\mathfrak{H}^p}(X/\mathbb{R}, M) . \end{aligned}$$

Proof. The last isomorphism is given by the observation that we have

$$\mathrm{MHM}_A(X) = \bigoplus_{x \in X(\mathbb{C})} \mathrm{MHS}_A \ .$$

□

Corollary A.2.12. *For $X/\mathbb{R}$ finite and reduced, and $n \geq 1$, we have*

$$\begin{aligned} \left(\bigoplus_{x \in X(\mathbb{C})} \mathbb{C}/(2\pi i)^n A \right)^+ &\xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHM}_A(X/\mathbb{R})}(A(0)_X, A(n)_X) \\ &= H^1_{\mathfrak{H}^p}(X/\mathbb{R}, n) \ . \end{aligned}$$

Here, the superscript $+$ denotes the fixed part with respect to the conjugation on both $X(\mathbb{C})$ and $\mathbb{C}/(2\pi i)^n A$, and the isomorphism associates to $(z_x)_{x \in X(\mathbb{C})}$ the extension, whose stalk at $x \in X(\mathbb{C})$ is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{(2\pi i)^n} \cdot z_x & 1 \end{pmatrix} :$$

if e_0 and e_n are the base vectors $1 \in F \subset \mathbb{C}$ and $(2\pi i)^n \in (2\pi i)^n A \subset \mathbb{C}$, then the Hodge structure is specified by

$$F^0 := \langle e_0 \rangle_{\mathbb{C}} \ , \quad W_{-2n} \otimes_A \mathbb{C} = \langle e_n \rangle_{\mathbb{C}} \ ,$$

and the A–rational structure is generated by e_n and

$$e_0 - \frac{1}{(2\pi i)^n} \cdot z_x e_n \ .$$

Proof. This is A.2.11 and A.2.10, using the basis (e_n) of $A(n)$. □

B K-Theory of Simplicial Schemes and Regulators

We start with a presentation of K-theory (B.2.1) for simplicial schemes in terms of generalized cohomology. Applied to a regular scheme, we get back its K-groups (cf. B.2.3.a)). Next we define λ-operations on K-cohomology (cf. B.2.10). Motivic cohomology of simplicial schemes, in particular relative motivic cohomology (B.2.11) is introduced as graded pieces of the γ-filtration with respect to these λ-operations. This discussion is based on the extremely useful (unfortunately unpublished) paper [GSo1] by Gillet and Soulé. More often than not the results in B.1 and B.2 will be due to them. The wish for a complete published reference made us go over the material again. Meanwhile an alternative approach to K-theory of simplicial schemes and λ-operations was also worked out by Levine [Le]. De Jeu was the first to use the setting of [GSo1] to define motivic cohomology of simplicial objects. In his article [Jeu] he proves Riemann-Roch in this setting. We give a more general version in B.2.18.

We then construct regulators (i.e., Chern classes) from K-cohomology to continuous étale cohomology (B.4) and to absolute Hodge cohomology (B.5) in this situation. Our main interest is the construction of a long exact sequence for relative

K-cohomology of simplicial schemes as well as for their motivic cohomology which is mapped to the corresponding long exact sequences in sheaf cohomology (B.3.8).

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B.1 Generalized Cohomology Theories

We need a framework which is general enough to treat K-theory and the usual cohomology theories in parallel. It turns out such a framework is given by homotopical algebra as axiomatized by Quillen in [Q1].

We define cohomology of spaces (=simplicial sheaves of sets) with coefficients in another space (B.1.4). We then construct a long exact sequence for relative cohomology in this context (B.1.6). Finally we deduce the spectral sequence relating generalized cohomology of a space to generalized cohomology of its components (B.1.7).

A systematic investigation of generalized cohomology for Grothendieck topologies was carried out by Jardine, in particular [Jr2]. We recapitulate the definitions for the convenience of the reader. A first introduction to the necessary simplicial methods is [M].

We fix a regular affine irreducible base scheme B of finite Krull dimension. In our applications B is either a field or an open subscheme of the ring of integers of a number field. We fix a small category of noetherian finite dimensional B–schemes which is closed under finite disjoint unions and contains all open subschemes of all its objects. We turn it into a site using the Zariski topology. Typically this will be a subcategory of all smooth schemes over the base B.

Let $\mathbf{T}$ be the topos of sheaves of sets on our Zariski site over B. Let $s\mathbf{T}$ be the category of pointed simplicial $\mathbf{T}$-objects. Its objects will be called *spaces* in the sequel. We denote the final and initial object of $s\mathbf{T}$ by $\star$.
Remark: A space is given by a simplicial sheaf of sets $X_.$ and a simplicial map ι from $\star$ (the constant simplicial sheaf all of whose components are given by the constant sheaf $\tilde{\star}$ attached to the set with one element) to $X_.$. Equivalently we can consider it as a simplicial object in the category of sheaves pointed by $\tilde{\star}$.

Let X be a scheme. We can also see it as an object of $\mathbf{T}$. The corresponding constant simplicial object pointed by a disjoint base point,

$$U \mapsto \mathrm{Mor}_B(U, X) \cup \{\star\} \qquad \text{for connected } U \in \mathbf{T},$$

will also be denoted X.

Definition B.1.1. *A space is said to be constructed from schemes if all components are representable by a scheme in the site plus a disjoint base point.*

Note that any simplicial scheme (whose components are schemes in the site) gives rise to a space constructed from schemes but there are many spaces constructed from schemes which do not come from simplicial schemes. The main example is the mapping cone of a map of schemes taken in $s\mathbf{T}$ (cf. B.1.5 below).

If P is a property of schemes and if the space X is constructed from schemes, we say X has P if the scheme parts of the components have P.

The easiest way to define the homotopy sets $\pi_n(X, x)$ of a simplicial set X with basepoint $x \in X_0$ is to take the homotopy sets of its geometric realization. $\pi_n(X, x)$

is a group for $n \geq 1$, even abelian for $n \geq 2$. If X is a space and K a finite simplicial set (i.e., all K_n are finite), then we define the space $X \otimes K$ componentwise as the sum of pointed sheaves

$$n \mapsto \bigvee_{\sigma \in K_n} X_n \; .$$

Definition B.1.2 (Brown, Gersten, Gillet, Soulé). *Let X be a space and $f : X \to Y$ be a map of spaces.*

a) *f is called a weak equivalence if all stalks $f_P : X_P \to Y_P$ are weak equivalences of simplicial sets, i.e., if f_P induces an isomorphism on all homotopy sets for all choices of base point.*

b) *f is called a cofibration if for all schemes U in $\mathbf{T}$ the induced map $f(U) : X(U) \to Y(U)$ is injective.*

c) *f is called a fibration if it has the following lifting property: given a commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration and a weak equivalence, there exists a map $B \to X$ that makes the diagram commute.

d) *For two spaces X and Y, let $\mathrm{Hom}_.(X, Y)$ be the pointed simplicial set*

$$n \mapsto \mathrm{Hom}_{s\mathbf{T}}(X \otimes \Delta(n), Y)$$

where $\Delta(n)$ is the standard simplicial n-simplex (e.g. [M] 5.4) pointed by zero.

This is the pointed version of the global theory discussed in [Jr2] §2.

Quillen's notion of a closed model category axiomatizes the properties which are needed in order to pass to a homotopy category which behaves similar to the homotopy category of CW-spaces.

Proposition B.1.3 (Brown, Gersten, Joyal). *$s\mathbf{T}$ is a pointed closed simplicial model category in the sense of Quillen [Q1].*

Proof. For a model category we need fibrations, cofibrations and weak equivalences satisfying a set of axioms ([Q1] I Def. 1). This is [GSo1] Theorem 1. Gillet and Soulé attribute this theorem to Joyal (letter to Grothendieck). For simplicial sheaves a published proof of all properties can be found in [Jr2] Cor. 2.7. It is an abstract non-sense fact that with the category of simplicial sheaves the category of pointed simplicial sheaves is also a model category. It is pointed by $\star$. The simplicial structure ([Q1] II Def. 1) is given by B.1.2.d). $\square$

Technical Remark: Note that the unique map $\star \to X$ is always a cofibration, i.e., all spaces are cofibrant. A space will be called fibrant if the unique map $X \to \star$ is a fibration. If a space is fibrant, than its sections $X(U)$ over a scheme U form a simplicial set satisfying Kan's extension condition (cf. [M] 1.3). However, this property does not suffice to make X fibrant. Part of the proof of the proposition is the existence of fibrant resolutions. In fact, the construction in [Jr2] Lemma 2.5 is even functorial.

Let $\mathrm{Ho}(s\mathbf{T})$ be the homotopy category associated to the model category $s\mathbf{T}$ by localizing at the class of weak equivalences. As usual we will write $[X, Y]$ for the morphisms from X to Y in the homotopy category. If Y is fibrant, then this set is given by the set of morphisms from X to Y in $s\mathbf{T}$ up to simplicial homotopy. For general Y, we compute $[X, Y]$ by $[X, \tilde{Y}]$ where $\tilde{Y}$ is a fibrant resolution of Y.
Remark: The category of pointed presheaves with the same notions as in B.1.2 is also a pointed model category. By [Jr2], Lemma 2.6 the map from a presheaf to its sheafification is a weak equivalence and we get the same homotopy category from presheaves or sheaves.

If X is a space, then its suspension SX is given by $X \otimes \Delta(1)/\sim$ where $\sim$ is the usual equivalence relation generated by $(x, 0) \sim (x, 1)$. By [Q1] Ch. I 2, the loop space functor Ω is right adjoint to S on the homotopy category.

There are two natural ways of thinking about $\mathrm{Ho}(s\mathbf{T})$. From the point of view of algebraic topology it corresponds to the category of CW-complexes with morphisms up to homotopy. From the point of view of homology theory it corresponds to the category of homological complexes which are concentrated in positive degrees with morphisms up to homotopy. S and Ω shift the complexes. This second point of view is not quite precise - note that in general morphisms in $\mathrm{Ho}(s\mathbf{T})$ form pointed sets rather than groups.

Definition B.1.4. *For any space A we define cohomology of spaces with coefficients in A by setting*

$$H^{-m}_{s\mathbf{T}}(X, A) = [S^m X, A] \quad \textit{for } m \geq 0 \quad .$$

This is a pointed set for $m = 0$, a group for $m > 0$ and even an abelian group for $m > 1$. If A belongs to an infinite loop spectrum, i.e., if there are spaces A_i for $i \geq 0$ with $A_0 = A$ and weak equivalences $A_i \to \Omega A_{i+1}$, then we also define cohomology groups with positive indices by setting

$$H^{n-m}_{s\mathbf{T}}(X, A) = [S^m X, A_n] \quad \textit{for } m, n \geq 0 \quad .$$

Note that the set only depends on $n - m$ because the suspension S and the loop functor Ω are adjoint.

Definition B.1.5. *Let $f : X \to Y$ be a map of spaces. Then the mapping cone of f is the space*

$$C(f) = X \otimes \Delta(1) \amalg Y/\sim$$

where $\sim$ is the usual equivalence relation of the mapping cone (i.e., $(x, 1) \sim f(x)$, $(x, 0) \sim \star$). For any map of spaces $f : X \to Y$, we define relative cohomology by

$$H^{-m}_{s\mathbf{T}}(Y \textit{ rel } X, A) = H^{-m}_{s\mathbf{T}}(C(f), A) \quad .$$

$C(f)$ is the standard construction of the homotopy cofibre of a map.

PROPOSITION B.1.6. *For any morphism $f : X \to Y$ of spaces there is a long exact cohomology sequence:*

$$\to H^{-m}_{s\mathbf{T}}(Y,A) \to H^{-m}_{s\mathbf{T}}(X,A) \to H^{-m+1}_{s\mathbf{T}}(Y \text{ rel } X, A) \to H^{-m+1}_{s\mathbf{T}}(Y,A) \; .$$

Proof. By [Q1] Ch. I 3 we have the above long exact sequence attached to the triple of spaces

$$X \xrightarrow{i} Y' \longrightarrow Y' \vee_X \star$$

if i is a cofibration. The mapping cylinder of f is defined as $X \otimes \Delta(1) \vee_X Y$. It is weakly equivalent to Y, and the induced mapping $X \to X \otimes \Delta(1) \vee_X Y$ is a cofibration. The mapping cone of f is nothing but the cofibre of this inclusion. Hence the long exact sequence of the lemma is a special case of Quillen's with $Y' = X \otimes \Delta(1) \vee_X Y$. $\square$

If A is only a space, then the sequence will end at the index zero. There is no reason for the last arrow to be right exact. The $H^0_{s\mathbf{T}}$ are only pointed sets. The $H^{-1}_{s\mathbf{T}}$ are groups, all others are even abelian groups. However, if A is an infinite loop spectrum, then all cohomology groups will be abelian groups and the sequence is unbounded in both directions.

We will consider a couple of spectral sequences which are constructed by means of homotopical algebra. Their differentials are

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q+r-1} \quad .$$

We refer to this behaviour as homological spectral sequence as opposed to a cohomological spectral sequences with differentials

$$d_r : E_r^{p,q} \longrightarrow E_r^{p-r,q-r+1} \quad .$$

In the same way as with the long exact sequences which involve pointed sets we also have to be careful about our spectral sequences. They will be constructed by the method of Bousfield-Kan (cf. [BouK] Ch. IX §§4-5). We refer to them as spectral sequences of Bousfield-Kan type. We give an overview over their properties. They look like this:

$$E_r^{p,q} \Rightarrow L^{q-p} \qquad q \geq p \geq 0, \; r \geq 1$$

with homological differentials.

$$L^{q-p}, E_r^{p,q} = \begin{cases} \text{are abelian groups} & \text{if } q-p \geq 2; \\ \text{are groups} & \text{if } q-p = 1; \\ \text{are pointed sets} & \text{if } q-p = 0. \end{cases}$$

We have $E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{im} d^{p-r,q-r+1}$. (Treat non-existing $E_r^{p,q}$ as zero for this formation.) By [BouK] IX 4.2.iv) this makes also sense for $p = q$. Let

$$E_\infty^{p,q} = \varprojlim\nolimits_r E_r^{p,q} = \bigcap_{r>p} E_r^{p,q} \; .$$

There is a descending cofiltration Q_* on the limit term L^n (i.e., Q_iL^n is a quotient of L^n). Let

$$e_\infty^{p,q} = \mathrm{Ker}\left(Q_pL^{q-p} \longrightarrow Q_{p-1}L^{q-p}\right) \ .$$

In general, there will be an injection $e_\infty^{p,q} \longrightarrow E_\infty^{p,q}$. Convergence is a more complicated question. The spectral sequence stabilizes if all projective systems $(E_r^{p,q})_{r>p}$ become eventually stable. Then we have complete convergence ([BouK] IX 5.3). Hence the cofiltration on the limit term is exhaustive ($\varprojlim Q_sL^n = L^n$), and we have isomorphisms

$$e_\infty^{p,q} \xrightarrow{\cong} E_\infty^{p,q} \qquad \text{for } p-q>0.$$

Note that even then the case $p = q$ has to be discussed separately. We refer to this problem and more generally the fact that pointed sets rather then groups appear as the fringe effect.

Proposition B.1.7. a) *Let X and A be spaces. The filtration of X by its skeletons sq_nX induces a spectral sequence of Bousfield-Kan type for its A-cohomology*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p, A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X, A) \qquad \text{for } q \geq p \geq 0 \ .$$

It converges completely if X is degenerate above some degree (i.e., if there is N such that for $n \geq N$, X_n is covered by the image of the degeneracy maps.).

b) *If A is an infinite loop spectrum and X as in a), then we have a converging homological spectral sequence*

$$E_1^{p,q} = H_{s\mathbf{T}}^{-q}(X_p, A) \Rightarrow H_{s\mathbf{T}}^{-(q-p)}(X, A) \qquad \text{for } p \geq 0 \ .$$

Proof. This is the hypercohomology spectral sequence of [GSo1] 1.2.3. We sketch their proof: We can assume A to be fibrant. We can construct a weak equivalence $X' \to X$ such that $sk_pX'/sk_{p-1}X' \cong S^pX_p$. The $\mathrm{Hom}_.(sk_pX', A)$ form a tower of fibrations of simplicial sets converging to $\mathrm{Hom}_.(X, A)$. The attached Bousfield-Kan spectral sequence ([BouK] §4 -§5) has starting terms

$$\begin{aligned} E_1^{p,q} &= \pi_{q-n}\,\mathrm{Hom}_.(sk_pX'/sk_{p-1}X', A) \\ &= \pi_{q-p}\,\mathrm{Hom}_.(S^pX_p, A) = H_{s\mathbf{T}}^{-q}(X_p, A) \ . \end{aligned}$$

This finishes the construction of the spectral sequence. In order to discuss convergence we consider the same spectral sequence attached to X itself. It stabilizes by the assumption on degeneracy (see [BouK] §5). Both spectral sequences agree from $r = 2$ on.

For b) we consider the spectral sequence in a) for each space in the spectrum. By shifting q accordingly we get a direct system of spectral sequences whose limit is the one we are interested in. □

Remark: It would be much nicer to work with spectra and their homotopy category throughout. It would be a triangulated category. It would help to get rid of the fringe effects. However, the question of convergence of the spectral sequences does not get easier, the reason behind this being that all these spectral sequences are constructed for some kind of homotopy limit, and projective limits are not exact. However, the literature we want to use is in the setting of spaces. The reason is that we want to use the λ-ring structure in order to define motivic cohomology and the λ-operators do not deloop.

B.2 K-theory

We now introduce higher algebraic K–theory of spaces as a generalized cohomology theory. It gives back usual K-theory in the case of regular schemes (B.2.3). We then define λ-operators on these K-cohomology groups (B.2.10). This allows definition of motivic cohomology of spaces as graded parts of the γ-filtration (B.2.11). We then prove a Grothendieck-Riemann-Roch type theorem (B.2.18). As a consequence we get a long exact localization sequence for motivic cohomology (B.2.19).

Recall that all schemes in the site underlying $\mathbf{T}$ are assumed to be noetherian and finite dimensional.

Let $\mathbf{K}$ be the space $\mathbb{Z} \times \mathbb{Z}_\infty BGl$ where $\mathbb{Z}_\infty BGl$ is the simplicial sheaf associated to the simplicial presheaf $U \mapsto \mathbb{Z}_\infty BGl(U) = \varinjlim \mathbb{Z}_\infty BGl_n(U)$. $\mathbf{K}$ is pointed by $0 \times \varinjlim BGl_n(E_n)$. It is in fact part of an infinite loop spectrum. We also need the "unstable" spaces $\mathbf{K}^N = \mathbb{Z} \times \mathbb{Z}_\infty BGl_N$. There are natural transition maps $\mathbf{K}^N \to \mathbf{K}^{N+1} \to \mathbf{K}$. As K-groups commute with direct limits, the stalk of $\mathbf{K}$ in a point P on $U \in \mathbf{T}$ is weakly equivalent to

$$\mathbf{K}_P \cong \mathbb{Z} \times \mathbb{Z}_\infty BGl(\mathcal{O}_P) \ .$$

where $\mathcal{O}_P$ is the stalk of the structural sheaf.

Remark: Even though it is well-known that K-theory is defined by a spectrum, it is not completely trivial to define it as a functor from schemes to spectra (rather than just a functor up to homotopy). We refer to [GSo2], 5.1.2 for the details of this construction. For a different account of K-theory as a presheaf and its properties (including the product structure) we also refer to Jardine's book [Jr4].

Definition B.2.1 (Gillet, Soulé). *For any space X in $s\mathbf{T}$ we define its K-cohomology*

$$H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = [S^m X, \mathbf{K}] \qquad \text{for } m \in \mathbb{Z}$$

and the unstable K-groups $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}^N)$ for $m \geq 0$. Following [GSo1] we call a space K-coherent if $\varinjlim H^{-m}_{s\mathbf{T}}(X, \mathbf{K}^N) \to H^{-m}_{s\mathbf{T}}(X, \mathbf{K})$ for $m \geq 0$ is an isomorphism.

Proposition B.2.2 (Brown). *Let $\mathcal{K}_q$ be the sheafification of the presheaf $Y \mapsto H^{-q}_{s\mathbf{T}}(Y, \mathbf{K})$. Let X be a scheme in $\mathbf{T}$. There is a homological spectral sequence*

$$E_2^{pq} \Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X, \mathbf{K})$$

with

$$E_2^{pq} = H^p_{\mathrm{ZAR}}(X, \mathcal{K}_q)$$

It converges completely.

Proof. For $q-p \geq 0$ this is the spectral sequence [GSo1] Prop. 2. The basic version for the small Zariski site was constructed in [BrG] Theorem 3. Our generalization follows from the proof of [Jr2] 3.4 and 3.5, which deals with the étale topology. The key is to construct a Postnikov-tower for $\mathbf{K}$. This is done as in in the proof of [BrG] Thm 3. We then have to check that the homotopy sheaves of $\mathbf{K}$ are isomorphic to the homotopy sheaves of the limit of its Postnikov-tower. It suffices to check this for the small Zariski site Zar/Y for all schemes Y in $\mathbf{T}$. Hence we are reduced to the situation considered in loc. cit. Note that Y was assumed to be noetherian and finite dimensional. We extend to arbitrary p, q using the full K-theory spectrum. Convergence follows because X has finite cohomological dimension. □

Remark: We could generalize the spectral sequence to arbitrary spaces X. $H^p_{\mathrm{ZAR}}(X, \mathcal{K}_q)$ would have to be understood as in B.3. Convergence would not be guaranteed anymore.

The most important application of this proposition is that it allows to transport properties which are well-known for cohomology with coefficients in an abelian sheaf to cohomology with coefficients in a space. One such property is the comparison between different Zariski sites.

Proposition B.2.3 (Gillet, Soulé, de Jeu). a) *Let X be a noetherian regular finite dimensional scheme in the site. Then one has the equality $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = K_m(X)$, where the right hand side means Quillen K-theory of the scheme X. In particular, $H^{-m}_{s\mathbf{T}}(X, \mathbf{K}) = 0$ for $m < 0$.*

b) *Let X be a space constructed from schemes. Assume that all components are regular Noetherian finite dimensional schemes and that X is degenerate above some simplicial degree. Then X is K-coherent.*

Proof. The constant case is proved in [GSo1] 2.2.2 Prop. 5. We sketch a slightly different argument: We use the converging Brown spectral sequence and comparison theorems for sheaf cohomology to show that it suffices to prove the proposition in the case of $\mathbf{T} = \mathrm{Zar}/X$. (Note that the existence of the whole spectrum means we do not have to worry about fringe effects.) In this case we have a Mayer-Vietoris sequence for K-theory ([Q2] Rem. 3.5) and hence the presheaf defining K-cohomology is pseudo-flasque in the sense of Brown and Gersten ([BrG] p. 285). By loc. cit. Thm. 4 this implies a) for the site Zar/X.
The vanishing follows because the K-theory spectrum is connective. The generalization to spaces constructed from schemes using the skeletal spectral sequence was carried out in [Jeu] 2.1 (1) and Lemma 2.1. □

Corollary B.2.4. *If X is a space meeting the conditions of part b) of the proposition, then its K-cohomology does not depend on the category of schemes underlying the topos.*

Proof. If X is constant, then we always get its K-theory. For more general X we have to use the converging skeletal spectral sequence. There are no fringe problems because $\mathbf{K}$ is an infinite loop spectrum. □

The direct sum of matrices (cf. [Lo] 1.2.4) together with addition on $\mathbb{Z}$ induces a compatible system of maps

$$\mathbf{K}^N \times \mathbf{K}^N \to \mathbf{K}\ .$$

Our aim is to show that its direct limit defines an H-group structure on $\mathbf{K}$. It will be used to define addition on K-cohomology.

Lemma B.2.5. *Let G, G' be algebraic groups over $\mathbb{Z}$, E a subgroup of G with $E = [E, E]$. Let $f_1, f_2 : G' \to G$ be homomorphisms which differ by conjugation by a global section of E. Then the induced maps*

$$\mathbb{Z}_\infty BG' \xrightarrow{f_1, f_2} \mathbb{Z}_\infty BG$$

agree in the homotopy category of spaces.

Proof. The construction in [Lo] A.3. is functorial. Hence it yields a free homotopy η between Bf_1 and Bf_2. By construction we get a commutative diagram

$$\begin{tikzcd} \Delta(1) \times \star \arrow[r, "\eta|_{\Delta(1)\times\star}"] \arrow[d] & \mathbb{Z}_\infty BE \arrow[d, "i"'] \\ \Delta(1) \times \mathbb{Z}_\infty BG' \arrow[r, "\eta"] & \mathbb{Z}_\infty BG\ . \end{tikzcd}$$

The composition of η with $d : \mathbb{Z}_\infty BG \to C(i)$ is a homotopy between df_1 and df_2. Now it suffices to show that d is a weak equivalence, i.e., that $\mathbb{Z}_\infty BE$ is contractible. This can be checked on stalks. As homotopy groups commute with direct limits it is enough to show that $\mathbb{Z}_\infty BE(U)$ is contractible for all affine schemes U. We consider the diagram

$$\begin{tikzcd} BE(U) \arrow[r, "\phi"] \arrow[d] & BE(U)^+ \arrow[d] \\ \mathbb{Z}_\infty BE(U) \arrow[r, "\mathbb{Z}_\infty(\phi)"] & \mathbb{Z}_\infty BE(U)^+\ . \end{tikzcd}$$

By definition of Quillen's $+$-construction (see [Lo] ch. 1.1) ϕ induces an isomorphism on homology. Hence $\mathbb{Z}_\infty(\phi)$ is a weak equivalence ([BlK] Ch. I, 5.5). $BE(U)^+$ is contractible because $[E(U), E(U)] = E(U)$ ([Lo] Proposition 1.1.7). Hence $\mathbb{Z}_\infty BE(U)^+$ is also contractible. □

The standard application of this lemma is with $G' = Gl_n$, $G = Gl$ and E the subgroup generated by elementary matrices (which contains all even permutation matrices), see [Lo] 1.1.10.

Proposition B.2.6. *The direct sum of matrices induces an H-group structure on K.*

Proof. The same proof as in [Lo] Theorem 1.2.6 allows to check the identities of an H-space. On finite level, they hold up to conjugation with a permutation matrix.

By the previous lemma this implies that they hold in the homotopy category. We use that the transition maps $K^N \to K^{N+1}$ are cofibrations in order to show that the maps on finite level define one on $\mathbf{K}$. For the existence of a homotopy inverse we argue differently. An H-space is an H-group if and only if the shear map

$$\mathbf{K} \times \mathbf{K} \to \mathbf{K} \times \mathbf{K} \ , (k_1, k_2) \mapsto (k_1, k_1 + k_2)$$

is a weak equivalence. This can be checked on stalks. But the stalks of $\mathbf{K}$ are the simplicial sets computing K-theory of local rings. They are H-groups with the same addition by the affine case [Lo] 1.2.6. □

Remark: We now have two H-group structures on $\mathbf{K}$: the explicit one we just have constructed and one because $\mathbf{K}$ is a loop space as part of a spectrum. We expect them to be equal but have not been able to prove it. They certainly induce the same addition on higher K-cohomology groups. On $H^0_{s\mathbf{T}}(X, \mathbf{K})$ they agree at least if X is represented by a scheme because they do for K-theory of schemes. This is enough for our needs. In the sequel the addition on K-cohomology is the one of the proposition.

The next aim is the definition of a multiplicative structure on $\mathbf{K}$. We start with the operation of $\mathbb{Z}$ on $\mathbf{K}$. The H-group structure on $\mathbf{K}$ allows to define a map of spaces

$$\mu_{\mathbb{Z}} : \mathbb{Z} \times \mathbf{K} \to \mathbf{K} \quad .$$

It vanishes on $\mathbb{Z} \times \star \vee \star \times \mathbf{K}$ and hence factors over $\mathbb{Z} \wedge \mathbf{K}$.

The construction of the Loday product [Lo] 2.1.5

$$\mathbb{Z}_\infty BGl_N(U) \wedge \mathbb{Z}_\infty BGl_N(U) \to \mathbb{Z}_\infty BGl(U)$$

is functorial in U. Together with the product $\mu_{\mathbb{Z}}$ on the factor $\mathbb{Z}$ it defines a system of maps

$$\mu_{\mathbf{K}} : \mathbf{K}^N \wedge \mathbf{K}^N \to \mathbf{K}$$

(compatible up to homotopy), which defines a product

$$[Y, \mathbf{K}] \times [Y, \mathbf{K}] \to [Y, \mathbf{K}]$$

for all K–coherent spaces Y. It turns all $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n \geq 0$ into a ring, possibly without unity.

Remark: Note that this product on $[Y, \mathbf{K}]$ is zero on $H^{-n}_{s\mathbf{T}}(Y, \mathbf{K})$ for $n > 0$ (cf. [Kr] Ex. 1 p. 243). The same map $\mu_{\mathbf{K}}$ of spaces also induces a non-trivial product

$$[S^n Y, \mathbf{K}] \times [S^m Y, \mathbf{K}] \to [S^{n+m} Y, \mathbf{K}] \quad .$$

This is the one which is usually called Loday product. We do not need it in the sequel.

Let S^0 be the simplicial version of the 0-sphere, i.e., the constant simplicial sheaf associated to $\{0, 1\}$ pointed by 0. We will use the notation $K_0(s\mathbf{T})$ for $H^0_{s\mathbf{T}}(S^0, \mathbf{K})$. It is a ring with unity where the ring structure is induced by the ring structure on $\mathbb{Z}$.

Lemma B.2.7. *If the site underlying* $\mathbf{T}$ *has a final object* X*, then*

$$K_0(X) \cong K_0(s\mathbf{T}) .$$

Proof. If X is the final object of the site, then the space we denote by X is equal to S^0. □

The following lemma generalizes an operation of $K_0(X)$ which was explained to us by de Jeu in the case where Y is constructed from X–schemes.

Lemma B.2.8. *Let* Y *be a space in* $s\mathbf{T}$*. Then the ring* $K_0(s\mathbf{T})$ *operates on* $H^{-n}_{s\mathbf{T}}(Y,\mathbf{K})$ *for* $n \geq 0$ *and makes it into an* $K_0(s\mathbf{T})$*-algebra.*

Proof. If Y is a space in $s\mathbf{T}$, then there is canonical isomorphism $Y \cong S^0 \wedge Y$. The product $\alpha \in K_0(s\mathbf{T})$ with $\beta \in H^{-n}_{s\mathbf{T}}(Y,\mathbf{K})$ is defined by the composition

$$Y \longrightarrow S^0 \wedge Y \xrightarrow{\alpha\wedge\beta} \mathbf{K} \wedge \mathbf{K} \xrightarrow{\mu_{\mathbf{K}}} \mathbf{K} \ .$$

□

Lemma B.2.9 (Gillet, Soulé). *Let* G *be a group over* $\mathbb{Z}$*. Let* $R_{\mathbb{Z}}(G)$ *be the Grothendieck group of representations of* G *on free* $\mathbb{Z}$*–modules of finite type.*

a) *Let* A *be an* N*-dimensional representation of* G*. There is a canonical class in* $[\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}]$ *which depends only on the equivalence class of* A*. The direct sum of representations is mapped to the sum of classes.*

b) *The map in a) induces an algebra homomorphism*

$$r : R_{\mathbb{Z}}(G) \to [\mathbb{Z} \times \mathbb{Z}_\infty BG, \mathbf{K}] .$$

Proof. We follow [GSo1] 3.2 or the affine case [Kr] 3. By choice of a basis of an N-dimensional representation A induces a map of sheaves

$$A : G \to Gl_N$$

and hence by functoriality a map

$$r'(A) : \mathbb{Z}_\infty BG \to \{N\} \times \mathbb{Z}_\infty BGl_N \to \mathbf{K}^N \ .$$

For different choices of basis the maps differ by conjugation with an element of $\alpha \in Gl_N$. The matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ is in the perfect subgroup $E = [Gl, Gl]$ hence by Lemma B.2.5 the image of $r'(A)$ in $[\mathbb{Z}_\infty BG, \mathbf{K}^{2N}]$ does not depend on the choice of matrix. Viewed as map to $\mathbf{K}$, this $r'(A)$ extends to the factor $\mathbb{Z}$ using the above product $\mu_{\mathbb{Z}}$. The last statement of a) follows by definition of the H-group structure on $\mathbf{K}$.
For b) we have to check that the relations of the Grothendieck-group are mapped to zero and that the multiplicative structure is well-behaved. We first prove the analogue of [Kr] Theorem 3.1: The canonical maps

$$Gl\begin{pmatrix} ? & 0 \\ 0 & ? \end{pmatrix} \overset{\longleftarrow}{\longrightarrow} Gl\begin{pmatrix} ? & ? \\ 0 & ? \end{pmatrix}$$

induce weak equivalences of simplicial sheaves after applying $\mathbb{Z}_\infty B$. This can be checked on stalks and is hence reduced to the affine case. From now, the proof works precisely as in the affine case, see [Kr] Cor. 3.2. □

$K_0(s\mathbf{T})$ is a λ–ring, i.e., the axioms in [Kr] Def. 4.1 are satisfied. If R is a $K_0(s\mathbf{T})$–algebra, then it is called a $K_0(s\mathbf{T})$–λ–algebra if it is equipped with operators λ^i for $i \geq 1$ such that $K_0(s\mathbf{T}) \oplus R$ is a λ–ring (cf. [Kr] 5.). Note that λ^0 has to have the constant value 1. If R itself does not have a unity, then it cannot be a λ–ring.

Theorem B.2.10 (Gillet, Soulé). *Let Y be a K–coherent space. For $k \geq 1$ and $m \geq 0$ there are maps*

$$\lambda^k : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \longrightarrow H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}) \ .$$

They turn $H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$ into a $K_0(s\mathbf{T})$–λ–algebra.

Proof. This is essentially [GSo1] Prop. 8. Put $G = Gl_n$ in the previous lemma. Let $\tilde{\mathbb{Z}}^n = [\mathbb{Z}^n_{id}] - [n \cdot 1] \in R_{\mathbb{Z}}(Gl_n)$ where $\mathbb{Z}^n_{id}$ is the canonical representation of Gl_n on $\mathbb{Z}^n$ and 1 is the trivial representation. We define $\lambda^k_n = r(\lambda^k(\tilde{\mathbb{Z}}^n))$. By composition it induces a map $\lambda^k_n : H^{-m}_{s\mathbf{T}}(Y, \mathbf{K}^n) \to H^{-m}_{s\mathbf{T}}(Y, \mathbf{K})$. These form a projective system and hence define an operation on K–cohomology of a K–coherent space. Well-definedness and all properties of a λ–ring are checked on the universal level (i.e., on $\mathbf{K}^n$ for varying n) and hence as in the affine case [Kr] Thm 5.1. For example, we want to show

$$\lambda^k(x + y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^j(y) \ .$$

Assume that x, y are represented by elements in $[Y, \mathbf{K}^n]$. On $R_{\mathbb{Z}}(Gl_n \times Gl_n)$ we have the λ-ring identity

$$\lambda^k \circ \bigoplus = \sum_{i=0}^{k} \lambda^i \otimes \lambda^j \ .$$

We evaluate this identity in $\tilde{\mathbb{Z}}^n$ and get an equality of elements in $R_{\mathbb{Z}}(Gl_n \times Gl_n)$. By the previous lemma it induces the same equality of elements in $[\mathbf{K}^n \times \mathbf{K}^n, \mathbf{K}]$. Composed with (x, y) this is the required equality. $\square$

Remark: A more conceptual proof was suggested to us by Soulé and the referee. One should use the integral completion functor constructed by Goerss and Jardine [GoeJr]. It has a universal property similar to the one of the +-construction and hence allows to copy directly Kratzer's arguments.
Technical Remark: When we try to define λ^0 in the same way, then we still get a map

$$\lambda^0 : \mathbb{Z}_\infty BGl_N \longrightarrow \mathbb{Z} \times \mathbb{Z}_\infty BGl \ .$$

It does not extend to the factor $\mathbb{Z}$ because $\lambda^0 : \mathbb{Z} \to \mathbb{Z}$ does not respect the base point - in fact it maps 0 to 1. This reflects the fact that the ring $K_0(Y)$ does not have a unity for a general space Y. The most striking example is $Y = C(i)$ where $i : Z \to X$ is a morphism between regular schemes (cf. [Sou4] 4.3). Then $K_0(Y) = \operatorname{Ker}(K_0(X) \to K_0(Z))$ does not contain 1.
Gillet and Soulé ([GSo1] Prop. 8) consider the structure as a $H^0_{s\mathbf{T}}(Y, \mathbf{K})$–$\lambda$–algebra. This only makes sense if $H^0_{s\mathbf{T}}(Y, \mathbf{K})$ happens to have a unity. However, we can check

in general that the operation of $H^0_{s\mathbf{T}}(Y,\mathbf{K})$ on $H^{-m}_{s\mathbf{T}}(Y,\mathbf{K})$ is compatible with the $K_0(s\mathbf{T})$–λ–algebra structure of both groups.

Note that the λ-structure is compatible with the contravariant functoriality of K-cohomology. This means that the long exact sequences for relative K-theory are compatible with the λ-operation where it is defined.

Once we have λ-operations we get as usual a γ-filtration and Adams-operators on the λ-module $H^n_{s\mathbf{T}}(Y,\mathbf{K})$ for $n \leq 0$. If the γ-filtration is locally finite, then we have in particular the Chern character

$$ch : H^n_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}} \longrightarrow \bigoplus_{j\in\mathbb{N}_0} \mathrm{Gr}^j_\gamma\, H^n_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}} \qquad \text{for } n \leq 0,$$

which is an isomorphism. For a quick survey cf. [T] pp. 117–123.

Definition B.2.11. *Let Y be a K-coherent space. Suppose that the γ-filtration is locally finite and hence that rationally K-cohomology splits into Adams-eigenspaces. Then we put for $j \geq n/2$*

$$H^n_{\mathcal{M}}(Y,j) = \mathrm{Gr}^j_\gamma\, H^{n-2j}_{s\mathbf{T}}(Y,\mathbf{K})_{\mathbb{Q}}\ ,$$

the motivic cohomology of the space Y. If $i : X \to Y$ is a morphism of spaces then we define relative motivic cohomology by

$$H^n_{\mathcal{M}}(Y \ \mathit{rel}\ X, j) = H^n_{\mathcal{M}}(\mathrm{Cone}(i), j)\ .$$

Remark: We restrict to this range of indices because we did not define Adams-eigenspaces for K-cohomology with positive indices (=K-theory with negative indices). However, if these K-groups vanish we can simply define the corresponding motivic cohomology groups to be zero. This is the case if X is a regular scheme.

The long exact sequence for relative cohomology (B.1.6) together with the above remarks on the λ-operation give a long exact sequence for relative motivic cohomology

$$\to H^{-m}_{\mathcal{M}}(Y,A) \to H^{-m}_{\mathcal{M}}(X,A) \to H^{-m+1}_{\mathcal{M}}(Y \text{ rel } X, A) \to H^{-m+1}_{\mathcal{M}}(Y,A)\ .$$

Lemma B.2.12. *Let X be a space degenerate above some simplicial degree. We assume the conditions of the previous definition. Fix an integer j. There is a cohomological spectral sequence with starting terms*

$$E_1^{s,t} = \begin{cases} H^t_{\mathcal{M}}(X_s, j) & \text{for } s \geq 0,\ 2j \geq t, \\ 0 & \text{else.} \end{cases}$$

It converges to $H^{s+t}_{\mathcal{M}}(X,j)$ for $2j \geq s+t$.

Proof. Consider the skeletal spectral sequence B.1.7.a) with coefficients in the space $\mathbf{K}$. It reads

$$E_1^{p,q} = H^{-q}_{s\mathbf{T}}(X_p,\mathbf{K}) \Rightarrow H^{-(q-p)}_{s\mathbf{T}}(X,\mathbf{K})$$

for $p \geq 0$. By carefully checking the construction of the spectral sequence, we see that all differentials $d_r^{p,q}$ are induced by functoriality in the first argument. Hence they

are morphisms of λ-modules. For $q - p \geq 0$ the limit terms are also λ-modules and by construction the morphisms $e_\infty^{p,q} \to E_\infty^{p,q}$ are compatible with this structure. They are isomorphisms for $q > p$. Note, however, that we do not get enough information on the limit terms on the $p = q$-line. Convergence only implies that $e_\infty^{p,p}$ injects into $E_\infty^{p,p}$. We want to show that it is even a bijection. In order to see this we consider the skeletal spectral sequence with coefficients in the spectrum $\mathbf{K}$. The spectral sequences agree where the first is defined, in particular convergence of the second spectral sequence implies our isomorphism. (There is an issue here with the H-group structure. A priori the two spectral sequences use different group laws. But on all initial terms they give the same addition and hence also on all higher terms.)
Now we take Adams-eigenspaces. By re-indexing $s = p, t = -q + 2j$ we get a cohomological spectral sequence as stated. Note that we use the terms below the $p = q$-diagonal to compute the terms on it but we do not consider their limit terms. □

The same spectral sequence also shows that the conditions in the definition of motivic cohomology hold if X is a space constructed from schemes and degenerate above some degree.

The next thing we need is pushout at least for certain closed immersions and a Riemann-Roch theorem. Over a field push-forward was defined by de Jeu in [Jeu] 2.2. We adapt his method to more general bases and formalize the geometric situation.

Definition B.2.13. *Let S be a regular irreducible Noetherian affine scheme. Let X be smooth and quasi-projective over S. A finite diagram $\mathcal{D}_X$ over X is a category of finitely many smooth quasi-projective S-schemes with final object X such that all $\mathrm{Mor}_{\mathcal{D}_X}(Y, Y')$ are finite sets and such that all morphisms in $\mathcal{D}_X$ are of finite Tor-dimension.*
By the small Zariski site $\mathrm{Zar}_{\mathcal{D}_X}$ we mean the category of all finite disjoint unions of open subschemes of objects in $\mathcal{D}_X$ with the induced morphisms between them. It is equipped with the Zariski-topology. The corresponding topos will be denoted $\mathbf{T}_X$.

An easy case of such a diagram is a single morphism $Y \to X$ that meets the conditions.

We consider the following situation: Let $i : Z \to X$ be a closed immersion of smooth quasi-projective S-schemes and $\mathcal{D}_X$ a finite diagram over X. We assume the following conditions, corresponding to the ones formulated by de Jeu in [Jeu] 2.2:

(TC) For all X' in $\mathcal{D}_X$, the pullback $X' \times_X Z$ is S–smooth. If $f : X_1 \to X_2$ is a morphism in $\mathcal{D}_X$, then in the cartesian diagram

$$\begin{array}{ccc} Z_1 = X_1 \times_X Z & \longrightarrow & X_1 \\ {\scriptstyle f\times_X Z}\downarrow & & \downarrow{\scriptstyle f} \\ Z_2 = X_2 \times_X Z & \xrightarrow{i} & X_2 \end{array}$$

the maps f and i are tor-independent, i.e.,

$$\underline{\mathrm{Tor}}^k_{\mathcal{O}_{X_2}}(\mathcal{O}_{Z_2}, \mathcal{O}_{X_1}) = 0$$

for $k > 0$. ($\underline{\mathrm{Tor}}^k$ denotes the sheaf of tor-groups.)

Lemma B.2.14. *The pullback $\mathcal{D}_Z$ of $\mathcal{D}_X$ by Z satisfies the conditions for a finite diagram over Z.*

Proof. Finite Tor-dimension in $\mathcal{D}_Z$ follows from Tor-independence and the same property in $\mathcal{D}_X$. □

Let $Y_.$ be a space in $s\mathbf{T}_X$. Let $j : U \to X$ be the open complement of Z in X. Let $Y_. \times_X U$ be the pointed version of $j_! j^* Y_.$, i.e., the sheaf associated to the presheaf

$$V \mapsto \begin{cases} Y_.(V) & \text{if } V \to U \subset X, \\ 0 & \text{else.} \end{cases}$$

It is a space in $s\mathbf{T}_X$. Let $Y_. \times_X Z = i^{-1} Y_.$, a space in $s\mathbf{T}_Z$. If $Y_.$ is constructed from schemes, then so are $Y_. \times_X U$ and $Y_. \times_X Z$. The scheme components are given by the base change with U or Z respectively. Note that $i^{-1}(Y_. \times_X U)$ is empty, i.e., only consists of the base point.

Proposition B.2.15 (de Jeu). *Let $i : Z \to X$ be a closed immersion with open complement U. Let $\mathcal{D}_X$ be a finite diagram over X such that (TC) holds with respect to i. Then for $Y_. \in s\mathbf{T}$:*

a) *There is a natural pushout map*

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \longrightarrow H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}).$$

b) *Let $Y_.$ be a space in $s\mathbf{T}_X$ which is constructed from schemes. We assume that it is degenerate above some simplicial degree. Then*

$$Y_. \times_X Z = C(Y_. \times_X U \subset Y_.) \times_X Z$$

and the pushout

$$H^k_{s\mathbf{T}_Z}(Y_. \times_X Z, \mathbf{K}) \longrightarrow H^k_{s\mathbf{T}_X}(Y_. \ rel\, Y_. \times_X U, \mathbf{K})$$

is an isomorphism.

Proof. For an object V of the site $\mathrm{Zar}_{\mathcal{D}_X}$ let $M(V)$ be the category of all coherent sheaves on V. In it let $P(V, \mathcal{D}_X)$ be the subcategory of those sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_V}(\mathcal{O}_{V'}, \mathcal{F}) = 0$$

for all $j > 0$ and all $V' \to V$ in $\mathcal{D}_X$. Note that there are only finitely many conditions as our diagram is finite. The nice thing about $P(V, \mathcal{D}_X)$ is that it is contravariantly functorial. Hence Quillen's $\Omega BQP(\,\cdot\,, \mathcal{D}_X)$ (loop space of the classifying space of the Q-construction) defines a presheaf of simplicial sets on the site by [Q2] §7 2.5. It is here where we use the fact that all schemes are quasi-projective. Let $\Omega BQP'_X$ be the space in $s\mathbf{T}_X$ defined by its sheafification. By Quillen's Resolution Theorem ([Q2] Thm 3, Cor 3, p. 27) there is a weak equivalence of spaces $\Omega BQP'_X \to \mathbf{K}_X$. (Basically this is the fact that K'-theory and K-theory agree for regular schemes.)

We also have the space $\Omega BQP'_Z$ in $s\mathbf{T}_Z$. For the closed immersion $i : V \times_X Z \to V$ the pushout i_* is exact on the category of coherent sheaves. Because of (TC), it maps the the subcategory $P(V \times Z, \mathcal{D}_Z)$ to $P(V, \mathcal{D}_X)$. In fact we get a morphism of spaces in $s\mathbf{T}_X$

$$i_*(\Omega BQP'_Z) \xrightarrow{i_*} \Omega BQP'_X \,.$$

Using the weak equivalences to $\mathbf{K}_?$ this defines a map in the homotopy category

$$i_*(\mathbf{K}_Z) \xrightarrow{i_*} \mathbf{K}_X \,.$$

If $Y_.$ is a space in $s\mathbf{T}_X$, then we get the map in a) as

$$H^k_{s\mathbf{T}_Z}(i^{-1}Y_., \mathbf{K}_Z) \longrightarrow H^k_{s\mathbf{T}_X}(i_*i^{-1}Y_., i_*\mathbf{K}_Z) \longrightarrow H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}_X) \,.$$

In the special case of a scheme Y part b) is nothing but Quillen's pushout isomorphism

$$K_n(i^{-1}Y) \longrightarrow K_n(Y \text{ rel } Y \times_X U)$$

for regular schemes [Q2] §7 Prop. 3.2 (recall that all schemes in the site are regular). This generalizes to the case of spaces constructed from schemes by the skeletal spectral sequence. □

Lemma B.2.16. *Consider a cartesian diagram of smooth quasi-projective S-schemes*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle f_Z}\downarrow & & {\scriptstyle f_X}\downarrow \\ Z & \xrightarrow{i} & X \end{array}$$

where i is a closed immersion. Let $\mathcal{D}_X$ be a finite diagram on X. Assume that the pullback $\mathcal{D}_{X'}$ defines a finite diagram over X' and that both i and i' satisfy (TC). We also assume that for all V in $\mathcal{D}_X$ the maps

$$V \times_X X' \longrightarrow V$$

and

$$V \times_X Z \longrightarrow V$$

are tor-independent.

Then for all spaces $Y_.$ in $s\mathbf{T}_X$ there is a commutative diagram

$$\begin{array}{ccc} H^k_{s\mathbf{T}_{Z'}}(f_Z^* i^* Y_., \mathbf{K}) & \xrightarrow{i'_*} & H^k_{s\mathbf{T}_{X'}}(f_X^* Y_., \mathbf{K}) \\ {\scriptstyle f_Z^*}\uparrow & & \uparrow{\scriptstyle f_X^*} \\ H^k_{s\mathbf{T}_Z}(i^* Y_., \mathbf{K}) & \xrightarrow{i_*} & H^k_{s\mathbf{T}_X}(Y_., \mathbf{K}) \end{array} \,.$$

Proof. We have to refine the categories $P(V, \mathcal{D}_Z)$ used in the proof of B.2.15 further. Let $P''(V, \mathcal{D}_Z)$ be the subcategory of $P'(V, \mathcal{D}_Z)$ of those coherent sheaves $\mathcal{F}$ satisfying

$$\underline{\mathrm{Tor}}^j_{\mathcal{O}_Z}(\mathcal{O}_{Z'}, \mathcal{F}) = 0 \,.$$

The induced space $\Omega BQP''_Z$ is again weakly equivalent to $\mathbf{K}_Z$. By [Q2] §7 2.11 there is a commutative diagram of spaces in $s\mathbf{T}_X$

$$\begin{array}{ccc} i'_* f_{X,*}\Omega BQP'_{Z'} & \longrightarrow & f_{X,*}\Omega P'_{X'} \\ \uparrow & & \uparrow \\ i_*\Omega BQP''_Z & \longrightarrow & \Omega BQP'_X \end{array} .$$

This proves the lemma. ∎

We also need the following lemma from algebraic geometry.

Lemma B.2.17. *Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

of smooth S-schemes where i is a closed embedding, then the blow-up of X' in Z' is the base change by f of the blow-up of X in Z provided i and f are tor-independent.

Proof. In order to see this, note that by [EGAII] 3.5.3 we have to check that $f^*(\mathcal{I}^n) = \mathcal{I}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is isomorphic to $\mathcal{J}^n$ where $\mathcal{I}$ is the sheaf of ideals of Z in X and $\mathcal{J}$ the one of Z' in X'. This follows from tor-independence in the case $n = 1$. Note that in general we have a surjection $f^*\mathcal{I}^n \to \mathcal{J}^n$. Let K_n be the kernel. Pull-back by f^* is right exact, i.e., we have an exact sequence

$$f^*\mathcal{I}^2 \to \mathcal{J} \to f^*(\mathcal{I}/\mathcal{I}^2) \to 0 .$$

Together with the above surjectivity this implies $f^*(\mathcal{I}/\mathcal{I}^2) \cong \mathcal{J}/\mathcal{J}^2$. As X respectively X' are regular and Z respectively Z' are locally given by regular sequences, the structural theorem [Ha] II Theorem 8.21A e) implies

$$f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \cong \mathcal{J}^n/\mathcal{J}^{n+1} .$$

By the snake lemma $K_{n+1} \to K_n$ is surjective and hence $f^*(\mathcal{I}^n/\mathcal{I}^{n+k}) \cong \mathcal{J}^n/\mathcal{J}^{n+k}$ for all k. But then

$$\mathcal{J}^n \cong \varprojlim \mathcal{J}^n/\mathcal{J}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\operatorname{im} f^*\mathcal{I}^{n+k} \cong \varprojlim f^*\mathcal{I}^n/\mathcal{J}^k f^*\mathcal{I}^n \cong f^*\mathcal{I}^n .$$

∎

Push-forward is not a λ-ring morphism but it does respect the γ-filtration up to a shift, at least under good conditions. This is made precise in the following Riemann-Roch Theorem, which is a slight generalization of de Jeu's in [Jeu] 2.3. He considers a special type of diagram and restricts to a base field. De Jeu imitates the proof in [T] Theorem 1.1, which is over a field. However, his arguments work for our base as well. Indeed, the original article [Sou4] Thm 3 treated the more general case.

Theorem B.2.18 (Grothendieck-Riemann-Roch). *Let S be a regular irreducible Noetherian affine scheme S. Let $i : Z \to X$ be a closed immersion of constant codimension d of quasi-projective smooth S-schemes. For $? = X, Z$ let $td(?) \in \mathrm{Gr}^*_\gamma K_0(?)_{\mathbb{Q}}$ be the usual Todd classes (e.g. [T] p. 135). Let a finite diagram $\mathcal{D}_X$ be given that satisfies the conditions (TC) with respect to i. Finally let $Y_.$ be a space constructed from schemes in $s\mathbf{T}_X$.*

a) *The homomorphism $i_* : K_n(i^{-1}Y_.)_{\mathbb{Q}} \to K_n(Y_.)_{\mathbb{Q}}$ has degree $-d$ with respect to the γ-filtration, i.e.,*

$$F^j K_n(i^{-1}Y_.)_{\mathbb{Q}} \xrightarrow{i_*} F^{j-d} K_n(Y_.)_{\mathbb{Q}} \quad .$$

b) *The following diagram commutes:*

$$\begin{tikzcd}
K_n(i^{-1}Y_.)_{\mathbb{Q}} \arrow[r, "td(Z)ch"] \arrow[d, "i_*"'] & \mathrm{Gr}^*_\gamma K_n(i^{-1}Y_.)_{\mathbb{Q}} \arrow[d, "i_*"] \\
K_n(Y_.)_{\mathbb{Q}} \arrow[r, "td(X)ch"] & \mathrm{Gr}^*_\gamma K_n(Y_.)_{\mathbb{Q}}
\end{tikzcd}$$

Remark: $td(?)$ is a unit with augmentation 1. Hence the horizontal maps in b) are isomorphisms.

Proof. We essentially have to prove classical Riemann-Roch for the inclusion $Z \to X$. The conditions on our situation are chosen in a way that the diagrams we drag along do not make any difficulties. Note also that we can replace $Y_.$ by the cone of $Y_. \times U \to Y_.$, i.e., we can assume that all pushout maps are isomorphisms. Having observed this we can follow de Jeu's arguments in [Jeu] 2.3.

The first step is to prove the analogue of [T] Theorem 1.2 or [Jeu] Proposition 2.5 ("Riemann-Roch without denominators"). We only sketch the idea: Because of functoriality B.2.16 and the homotopy property of K'-theory we can make the transformation to the normal cone. Hence we can assume without loss of generality that i is a section of a projective bundle over Z. The existence of the projection p which is a left-inverse of i allows to make explicit calculations. All details of the argument can be found in [Jeu] 2.5 when replacing $K_0(Y_0)$ ($= K_0(X_0)$ there) by $K_0(X) = K_0(s\mathbf{T}_X)$. The necessary compatibility of blow-up and base change is guaranteed by the previous lemma.

We then show that up to multiplication with the appropriate Todd class i_* has the required behaviour with respect to Adams eigenspaces. The argument is the same as in [Jeu] Proposition 2.3 or [T] Lemma 2.2. Now the theorem follows by the same formal manipulations as in the proof of [T] Lemma 2.3. □

Corollary B.2.19. *Let $i : Z \to X$ (closed immersion of constant codimension d) and $Y_.$ be as in the theorem. Let $U = X \smallsetminus Z$. Then there is a natural localization sequence*

$$\begin{aligned}\dots \longrightarrow K_m(Z \times_X Y_.)_{\mathbb{Q}} \longrightarrow K_m(Y_.)_{\mathbb{Q}} \longrightarrow K_m(U \times_X Y_.)_{\mathbb{Q}} \\ \longrightarrow K_{m-1}(Z \times_X Y_.)_{\mathbb{Q}} \longrightarrow \dots\end{aligned}$$

or in terms of motivic cohomology

$$\begin{aligned}\dots \longrightarrow H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d) \longrightarrow H^i_{\mathcal{M}}(Y_., j) \longrightarrow H^i_{\mathcal{M}}(U \times Y_., j)\\ \longrightarrow H^{i-2d+1}_{\mathcal{M}}(Z \times_X Y_., j-d) \longrightarrow \dots\end{aligned}$$

Proof. Part b) of Theorem B.2.18 implies that

$$i_* : \bigoplus_{j\in\mathbb{N}_0} \mathrm{Gr}^j_\gamma K_m(Y_. \text{ rel } Y_. \times U) \longrightarrow \bigoplus_{j\in\mathbb{N}_0} \mathrm{Gr}^{j-d}_\gamma K_m(Y_. \times Z)$$

is an isomorphism, i.e., $H^i_{\mathcal{M}}(Y_. \text{ rel } Y_. \times U, j) \cong H^{i-2d}_{\mathcal{M}}(Z \times_X Y_., j-d)$.
We consider the long exact sequence of relative K-cohomology or relative motivic cohomology for the open embedding $U \times Y_. \subset Y_.$. We can use i_* to identify the relative cohomology with cohomology of the closed complement. □

Only a few K-groups are known. However, the ranks of the K-groups of number fields are understood.

Theorem B.2.20 (Borel). *Let K be a number field with ring of S-integers $\mathfrak{o}_S$ where S is a finite set of primes of K. Let $B = \mathrm{Spec}\,\mathfrak{o}_S$. As usual r_1 is the number of real places of K and r_2 the number of complex places. Then the motivic cohomology has the following ranks:*

$H^0_{\mathcal{M}}(B,0)$	1	
$H^1_{\mathcal{M}}(B,1)$	$\#S + r_1 + r_2 - 1$	
$H^1_{\mathcal{M}}(B,n)$	r_2	$n > 1$, *even* ;
$H^1_{\mathcal{M}}(B,n)$	$r_1 + r_2$	$n > 1$, *odd* ;
$H^i_{\mathcal{M}}(B,j)$	0	*else* .

Proof. The computation of $K_0(B)$ and $K_1(B)$ is classical ([Ba] Ch. IX, Prop. 3.2 and Ch. X, Cor. 3.6). The higher K-groups for the ring of integers $\mathfrak{o}_K$ were calculated by Borel ([Bo1], Prop 12.2). It follows from Quillen's computation of the K-groups of finite fields that the ranks are not changed by localizing at finite primes. □

B.3 Cohomology of Abelian Sheaves

We now show how the usual cohomology theories fit in the set-up of generalized cohomology. This is well documented in the literature [BrG], [G], [Jeu]. In the case of a cohomology theory defined by a pseudo-flasque complex of presheaves $\mathcal{F}$, we compare the different possible points of view. These are Zariski-cohomology of the associated complex of sheaves, generalized cohomology of the associated space or simply cohomology of the sections. We always get the same cohomology groups (B.3.2 and B.3.4). If the complex of presheaves $\mathcal{F}$ is part of a twisted duality theory (B.3.7), we define Chern classes from K–cohomology of spaces to cohomology with coefficients in $\mathcal{F}$. Finally we check compatibility of the localization sequence in K–cohomology with the one for cohomology of spaces with coefficients in $\mathcal{F}$ (B.3.8).

By a complex we always mean a cohomological complex. Of course it can also be considered as a homological complex by inverting the signs of the indices.

The Dold-Puppe functor [M] Thm 22.4 attaches to a complex of abelian groups G which is concentrated in non-positive degrees a simplicial abelian group $K(G)$ pointed by 0 whose homotopy groups $\pi_i(K(G),0)$ agree with the cohomology groups $h^{-i}(G)$. It induces an equivalence between the homotopy category of simplicial abelian groups and the homotopy category of complexes of abelian groups concentrated in non-positive degrees. By construction of the functor K there is a natural weak equivalence of spaces

$$\mathrm{Cone}(K(G)\to *)\longrightarrow K(\mathrm{Cone}(G\to 0))=K(G[1])$$

and hence a natural map $\Omega K(G[1])\to K(G)$ in the homotopy category of pointed simplicial sets, which is a homotopy equivalence. If G is an arbitrary complex of abelian groups, let $\tau_{\leq N}G$ be the canonical sub-complex in degrees less or equal to N. We put

$$K(G)_N=K(\tau_{\leq N}G[N])\ .$$

The natural map $\tau_{\leq N-1}G[N]\to\tau_{\leq N}G[N]$ induces

$$K(G)_{N-1}\cong\Omega K(\tau_{\leq N-1}G[N])\longrightarrow\Omega K(G)_N\ ,$$

which is a weak equivalence. This means the $K(G)_N$ form an infinite loop spectrum whose homotopy groups reflect all cohomology groups of the complex.

Definition B.3.1. *Let $\mathcal{G}$ be a cohomological complex of sheaves of abelian groups on the big Zariski site. The sheafified version of the above construction yields an infinite loop spectrum of spaces $K(\mathcal{G})$ with*

$$h^{-i}(\mathcal{G})\cong\underline{\pi}_i(K(\mathcal{G}),0)$$

where the right hand side is the sheafification of the presheaf

$$U\mapsto\pi_i(K(\mathcal{G})(U),0)\ .$$

As a spectrum $K(\mathcal{G})$ defines generalized cohomology groups with indices in $\mathbb{Z}$ for any space X.

Proposition B.3.2. *Let $\mathcal{G}$ be a bounded below complex of sheaves on the big Zariski site. Let X be a scheme. Then*

$$H^i_{s\mathbf{T}}(X,K(\mathcal{G}))\cong H^i_{\mathrm{ZAR}}(X,\mathcal{G})\ .$$

Proof. As $\mathcal{G}$ is bounded below it has a bounded below resolution by flasque sheaves. Now the proof proceeds as in [BrG] Prop. 2. The main ingredient is that $K(\mathcal{I})$ is a fibrant space if $\mathcal{I}$ is a flasque sheaf. □

Definition B.3.3. a) *Following [BrG], Sect. 2 a complex $\mathcal{F}$ of abelian presheaves on the big Zariski site is called pseudo-flasque if it has the Mayer-Vietoris property, i.e., for open subschemes U and V of some scheme X, we have a long exact sequence of abelian groups*

$$\begin{aligned}\ldots\longrightarrow h^i\left(\mathcal{F}(U\cup V)\right)\longrightarrow h^i\left(\mathcal{F}(U)\oplus\mathcal{F}(V)\right)\longrightarrow h^i\left(\mathcal{F}(U\cap V)\right)\\ \longrightarrow h^{i+1}\left(\mathcal{F}(U\cup V)\right)\longrightarrow\ldots\end{aligned}$$

More precisely, the square

$$\begin{array}{ccc} \mathcal{F}(U \cap V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cup V) \end{array}$$

is homotopically cartesian.

b) *Let $\mathcal{F}$ be a complex of abelian presheaves. For the object $\star \amalg U$ in $\mathbf{T}$ where U is a scheme, we put*

$$\mathcal{F}(\star \amalg U) = \mathcal{F}(U) .$$

Let X be a space constructed from schemes. Then we put

$$\mathcal{F}(X) = \mathrm{Tot}_i \, \mathcal{F}(X_i) .$$

the total complex of the cosimplicial complex $\mathcal{F}(X_i)_{i \in \mathbb{N}_0}$.

Taking the total complex of a bicomplex as in b) of course involves a choice of signs which we fix once and for all. Different choices of signs differ by a canonical isomorphism of the total complex.

Lemma B.3.4. *Let $\mathcal{F}$ be a bounded below pseudo-flasque complex of abelian presheaves. Let $\tilde{\mathcal{F}}$ be its sheafification. Then*

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) = h^i(\mathcal{F}(X))$$

for all spaces X constructed from schemes.

Proof. Let $\mathcal{I}$ be a (bounded below) flasque resolution of $\tilde{\mathcal{F}}$. This is in particular a pseudo-flasque complex of presheaves that is quasi-isomorphic to $\mathcal{F}$ as a complex of presheaves because both compute Zariski-cohomology of $\tilde{\mathcal{F}}$. As in the proof of [BrG] Theorem 4, the simplicial sheaf $K(\mathcal{I})$ is a fibrant resolution of $K(\tilde{\mathcal{F}})$. Hence we can assume without loss of generality that $\mathcal{F}$ itself is a complex of flasque sheaves.
For the case of a scheme X the lemma is the reformulation of [BrG] Theorem 4 in the easier case of simplicial presheaves that come from a complex of abelian presheaves. In the general case

$$\begin{aligned} H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}})) &= \pi_{-i} \operatorname{Hom}_{.}(X, K(\tilde{\mathcal{F}})) \\ &= \pi_{-i} \operatorname{Hom}_{.}(\operatorname{hocolim} X_j, K(\tilde{\mathcal{F}})) \\ &= \pi_{-i} \operatorname{holim} \operatorname{Hom}_{.}(X_j, K(\tilde{\mathcal{F}})) && \text{[BouK] XII Prop. 4.1} \\ &= h^i(\operatorname{Tot} \mathcal{F}(X_i)) = h^i(\mathcal{F}(X)) \ . \end{aligned}$$

□

This means if we define a cohomology theory by a pseudo-flasque complex of presheaves on the big Zariski site we can freely change from the point of view of generalized cohomology to ordinary Zariski-cohomology or cohomology of the sections of the presheaf.

If $X \to Y$ is a morphism of schemes, we consider as usual its Čech-nerve $\operatorname{cosk}_0(X/Y)$, i.e., the simplicial Y-scheme given by

$$\operatorname{cosk}_0(X/Y)_n = (X \times_Y \cdots \times_Y X) \qquad n+1\text{-fold product}$$

with the natural boundary and degeneracy morphisms.

Definition B.3.5. *We say that a morphism $X \to Y$ of schemes has cohomological descent for the cohomology theory given by the complex of abelian Zariski-sheaves $\mathcal{G}$ if the natural morphisms*

$$H^i_{s\mathbf{T}}(Y, K(\mathcal{G})) \longrightarrow H^i_{s\mathbf{T}}(\operatorname{cosk}_0(X/Y), K(\mathcal{G}))$$

are isomorphisms for all $i \in \mathbb{Z}$.

This is of course a very special case of the general notion of cohomological descent.

Lemma B.3.6. *Let $j : U \to X$ be an open immersion with closed complement Y. Let $\mathcal{F}$ be a pseudo-flasque complex of presheaves on ZAR_X with sheafification $\tilde{\mathcal{F}}$.*

a) *There are natural isomorphisms*

$$H^i_{s\mathbf{T}}(X \ rel\, Y, K(\tilde{\mathcal{F}})) \longrightarrow H^i_{\mathrm{ZAR}}(X, j_!j^*\tilde{\mathcal{F}}))\ .$$

b) *If $\tilde{Y} \to Y$ is a morphism with cohomological descent for $\tilde{\mathcal{F}}$, then we get a natural isomorphism*

$$H^i_{s\mathbf{T}}(X \ rel\ \operatorname{cosk}_0(\tilde{Y}/Y), K(\tilde{\mathcal{F}})) \xrightarrow{\cong} H^i_{\mathrm{ZAR}}(X, j_!j^*\tilde{\mathcal{F}})\ .$$

Proof. By B.3.4 the left-hand side of a) is canonically isomorphic to the cohomology of

$$\mathcal{F}(C(Y \xrightarrow{i} X)) \cong \operatorname{Cone}\left(\mathcal{F}(X) \xrightarrow{\mathcal{F}(i)} \mathcal{F}(Y)\right)[-1]$$

where the right hand side is the cone in the category of cohomological complexes. We assume without loss of generality that $\tilde{\mathcal{F}}$ is a flasque complex. The key point is the short exact sequence of complexes of sheaves on X

$$0 \longrightarrow j_!j^*\tilde{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}} \longrightarrow i_*i^*\tilde{\mathcal{F}} \longrightarrow 0\ \ .$$

It induces a canonical quasi-isomorphism of complexes

$$j_!j^*\tilde{\mathcal{F}} \longrightarrow \operatorname{Cone}\left(\tilde{\mathcal{F}} \to i_*i^*\tilde{\mathcal{F}}\right)[-1]\ .$$

We now take $R\Gamma_{\mathrm{Zar}}(X,\ \cdot\ )$ of the right-hand side. Because $\mathcal{F}$ was assumed to be pseudo-flasque the morphism

$$\operatorname{Cone}(\mathcal{F}(X) \to \mathcal{F}(Y)) \longrightarrow \operatorname{Cone}\left(\tilde{\mathcal{F}}(X) \to \tilde{\mathcal{F}}(Y)\right)\ .$$

is a quasi-isomorphism. This last fact follows from B.3.4 and B.3.2. (Of course it can also be proved, even more easily, in terms of complexes of abelian groups rather than simplicial abelian groups.) In the case of a morphism $\tilde{Y} \to Y$ with cohomological descent the left hand side of the statement is by B.3.4 given by the cohomology of

$$\mathrm{Cone}\left(\mathcal{F}(X) \to \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y)\right)[-1] \ .$$

The natural morphism $\mathcal{F}(Y) \longrightarrow \mathcal{F}(\mathrm{cosk}_0(\tilde{Y}/Y))$ is a quasi-isomorphism by definition and Lemma B.3.4. $\square$

Theorem B.3.7 (Gillet, de Jeu). *Let $\mathcal{F} = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}(i)$ be a pseudo-flasque complex of abelian presheaves on the big Zariski site. Assume that $\mathcal{F}$ defines a twisted duality theory, i.e., the extra data of [G] Def. 1.1 exist and all conditions of loc. cit. Def. 1.2 are fulfilled. Then:*

- *There are Chern class maps of spaces*

$$c_j : \mathbf{K} \longrightarrow K(\tilde{\mathcal{F}}(j)[2j]) \ .$$

They induce morphisms

$$c_j : H^i_{s\mathbf{T}}(Y, \mathbf{K}) \longrightarrow H^{i+2j}_{s\mathbf{T}}(Y, K(\tilde{\mathcal{F}}(j)))$$

for all spaces Y in $s\mathbf{T}$.

- *If Y is a K-coherent space, then the total Chern class c_Γ is a morphism of λ-algebras on K–cohomology of Y.*

- *Let $i : Z \to X$ a closed immersion of smooth S–schemes with open complement U. The map $i_! : i_*\mathcal{F}(r)\mid_Z \to \mathcal{F}(r+d)\mid_X [2d]$ required in [G] Def. 1.2. induces push-forward on generalized cohomology. If $Y_.$ is a space over X as in B.2.18, then the diagram*

$$\begin{array}{ccc} \mathrm{Gr}^j_\gamma K_n(Y_. \times_X Z)_\mathbb{Q} & \xrightarrow{i_*} & \mathrm{Gr}^{j+d}_\gamma K_n(Y_.)_\mathbb{Q} \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_{j+d}} \\ H^{2j-n}_{s\mathbf{T}}(Y_. \times_X Z, K(\tilde{\mathcal{F}}(j)))_\mathbb{Q} & \xrightarrow{i_!} & H^{2j+2d-n}_{s\mathbf{T}}(Y_., K(\tilde{\mathcal{F}}(j+d)))_\mathbb{Q} \end{array}$$

is commutative.

Proof. The construction of the Chern classes is [G] Thm 2.2. Gillet's formulation is for schemes but he constructs in fact a morphism of spaces (loc. cit. p. 225) so the results hold for more general spaces (see also [GSo1] 4.1). The assertion on the λ-ring structure is [GSo1] Thm. 7. We sketch the idea: Everything is defined on the level of coefficients, so it does not depend on Y. Compatibility with multiplication is [G] 2.3.2. Compatibility with γ-operators can be checked on the level of universal Chern classes, i.e., for elements $C_{i,N} \in H^{2i}_{s\mathbf{T}}(BGl_n, \tilde{\mathcal{F}}(i))$. Now use the splitting principle ([G] 2.4).
The last part of the proposition is a generalization of Gillet's Riemann-Roch Theorem [G] 4.1 to spaces of our special type. The proof carries over by the same method as in the proof of Riemann-Roch for K-cohomology B.2.18. Mutis mutanda the statement can be found in [Jeu] Lemma 2.13. $\square$

Remark: This will allow to define regulator maps from K-cohomology to the cohomology theories we are interested in.

Corollary B.3.8. *Let X, Z, d, $Y_.$ and $\mathcal{F}$ be as in the theorem. In addition assume that $\mathcal{F}$ is pseudo-flasque. Let U be the complement of Y in X. We abbreviate $Y_U = Y_. \times_X U$, $Y_Z = Y_. \times_X Z$ and $F_j = K(\tilde{\mathcal{F}}(j))$. Then there is a natural morphism of long exact sequences*

$$\begin{array}{ccccccc}
H^{i-1}_{\mathcal{M}}(Y_U, j) & \longrightarrow & H^{i-2d}_{\mathcal{M}}(Y_Z, j-d) & \longrightarrow & H^i_{\mathcal{M}}(Y_., j) & \longrightarrow & H^i_{\mathcal{M}}(Y_U, j) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{i-1}_{s\mathbf{T}}(Y_U, F_j) & \longrightarrow & H^{i-2d}_{s\mathbf{T}}(Y_Z, F_{j-d}) & \longrightarrow & H^i_{s\mathbf{T}}(Y_., F_j) & \longrightarrow & H^i_{s\mathbf{T}}(Y_U, F_j) \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
h^{i-1}\mathcal{F}(j)(Y_U) & \longrightarrow & h^{i-2d}\mathcal{F}(j-d)(Y_Z) & \longrightarrow & h^i\mathcal{F}(j)(Y_.) & \longrightarrow & h^i\mathcal{F}(j)(Y_U)
\end{array}$$

Proof. We start with the long exact sequences for relative cohomology (B.1.7) with coefficients in the spectrum $\mathbf{K}$ and in the spectrum $K(\tilde{\mathcal{F}})$. Their compatibility is nothing but functoriality. Relative cohomology is replaced by cohomology of $Y_. \times_X Z$ using B.3.7. Finally we pass to graded pieces of the γ-filtration. Note that the indices in the definition of motivic cohomology are chosen in a way that they agree with the indices of other cohomology theories under Chern class maps. Equality of the last two lines is B.3.4 □

Note that the last line has nothing to do with generalized cohomology or spaces.

B.4 Continuous Etale Cohomology

There are different ways of defining continuous étale cohomology. We will see that they all give the same thing.

Fix a number field K and a prime l. Let B be an open subscheme of $\operatorname{Spec} \mathfrak{o}_K[1/l]$ where $\mathfrak{o}_K$ is the ring of integers of K.

Proposition B.4.1 (Deligne, Ekedahl). *Let $f : Y \to X$ be a morphism of B-schemes of finite type. Then there are triangulated categories $D^b_c(X - \mathbb{Z}_l)$ and $D_c(Y - \mathbb{Z}_l)$ admitting the following: there is a t-structure whose heart are the constructible l-adic systems. There are functors*

$$f_!, f_* : D^b_c(Y - \mathbb{Z}_l) \longrightarrow D^b_c(X - \mathbb{Z}_l)$$

and

$$f^*, f^! : D^b_c(X - \mathbb{Z}_l) \longrightarrow D^b_c(Y - \mathbb{Z}_l)$$

having all the usual properties of Grothendieck functors.

Proof. This is [Ek] Thm 6.3. In the case $B = \operatorname{Spec} \mathfrak{o}_K[1/l]$ the category was already constructed in [D4], 1.1.2. □

Remark: $D^b_c(X-\mathbb{Z}_l)$ should be thought of as the bounded derived categories of constructible l-adic sheaves on X_{et}. By Ekedahl's construction $D^b_c(X-\mathbb{Z}_l)$ is a subcategory of a localization of a subcategory of the derived category of the abelian category $(X_{et})^{\mathbb{N}}-\mathbb{Z}_l$. By this notation Ekedahl means the category of projective systems of étale sheaves on X ringed by the projective system $\mathbb{Z}/l^n$. The four functors are defined on the level of this last derived category. Ekedahl then shows that they induce well-defined functors on $D^b_c(X-\mathbb{Z}_l)$. In the case B open in $\operatorname{Spec}\mathfrak{o}_K[1/l]$, we get away with Deligne's more straightforward construction.

Definition B.4.2 (1. Version). a) *For $k\in\mathbb{Z}$ let $\mathbb{Z}_l(k)$ be the constructible l-adic sheaf on B given by the projective system $\mu_{l^n}^{\otimes k}$.*

b) *We define continuous étale cohomology of $s: X\to B$ by*

$$H^i_{cont}(X,k)=\operatorname{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0),s^*\mathbb{Z}_l(k)[i]) \quad .$$

c) *If $j:U\to X$ is an open immersion with complement Y we define relative continuous étale cohomology by*

$$H^i_{cont}(X \text{ rel } Y,k)=\operatorname{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0),j_!(s\circ j)^*\mathbb{Z}_l(k)[i]) \quad .$$

d) *More generally, let $\mathcal{M}$ be an object of $D^b_c(X-\mathbb{Z}_l)$. We define continuous étale cohomology of X with coefficients in $\mathcal{M}$ as*

$$H^i_{cont}(X,\mathcal{M})=\operatorname{Hom}_{D^b_c(X-\mathbb{Z}_l)}(s^*\mathbb{Z}_l(0),\mathcal{M}[i]) \quad .$$

This definition allows to derive all the usual spectral sequences from the calculus of the Grothendieck functors.

Remark: As checked in [H2] §4 this definition coincides with Jannsen's original one in [Jn1] sect. 3. In our case continuous étale cohomology with coefficients in a constructible l-adic sheaf $(\tilde{\mathcal{F}}_n)_n$ is nothing but the naive $\varprojlim H^n_{et}(X,\tilde{\mathcal{F}}_n)$ because all $H^n_{et}(X,\tilde{\mathcal{F}}_n)$ are finite.

Let us now define continuous étale cohomology in a way that fits in with the setting of the previous section.

Definition B.4.3 (2. Version). *Consider the projective system of sheaves $\left(\mu_{l^n}^{\otimes k}\right)_{n\in\mathbb{N}}$ on the big étale site over B. Let $\mathcal{I}$ be an injective resolution in the category of projective systems. It is given by a projective system $\mathcal{I}_n$ of injective resolutions of $\mu_{l^n}^{\otimes k}$ on the big étale site with split surjective transition morphisms ([Jn1] 1.1). By taking sections we get a projective system of complexes of Zariski-presheaves $R\Gamma\left(\mu_{l^n}^{\otimes k}\right)_{n\in\mathbb{N}}$. The functor $R\varprojlim$ turns it into a complex $\mathcal{F}_l(k)$ of Zariski-presheaves. For any space X put*

$$H^i_{cont}(X,k)=H^i_{s\mathbf{T}}(X,K(\tilde{\mathcal{F}}_l(k))) \quad .$$

In particular if $\iota: Y\to X$ is a morphism of spaces, then we put

$$H^i_{cont}(X \text{ rel } Y,k)=H^i_{s\mathbf{T}}(C(\iota),K(\tilde{\mathcal{F}}_l(k))) \quad .$$

Lemma B.4.4. *If X is a B-scheme, then both versions of the definition of continuous étale cohomology agree canonically. If $Z \to X$ is a closed immersion, then the same is true for both definitions of relative continuous étale cohomology.*

Proof. $\mathcal{F}_l(X)$ is nothing but an explicit version of the derived functor $R\varprojlim R\Gamma(X, \cdot)$ from the derived category of projective systems of étale sheaves to the derived category of abelian groups. Hence the complex $\mathcal{F}_l(X)$ computes the first version of continuous étale cohomology. In particular it has the Mayer-Vietoris property. Hence we can apply the lemmas of the previous section (B.3.4) and get

$$H^i_{s\mathbf{T}}(X, K(\tilde{\mathcal{F}}_l)) = h^i(\mathcal{F}_l(X)) \quad .$$

To extend the result to relative étale cohomology we use essentially the same argument as in B.3.6.b). $\square$

Remark: When we say that the isomorphism is canonical, we think in particular of the following situation: The cartesian diagram of schemes

$$\begin{tikzcd}
U' \arrow[r, "j'"] \arrow[d] & X' \arrow[d, "f"] & Y' \arrow[l, "i'"'] \arrow[d] \\
U \arrow[r, "j"] & X & Y \arrow[l, "i"'] \\
U'' \arrow[r, "j''"] \arrow[u] & X'' \arrow[u, "g"] & Y'' \arrow[l, "i''"'] \arrow[u]
\end{tikzcd}$$

(f, j open, g, i closed complements) induces a map

$$H^i_{cont}(X \text{ rel } Y, n) \xrightarrow{f^*} H^i_{cont}(X' \text{ rel } Y', n) \quad ,$$

which is compatible with the identification. If all schemes are smooth and X'' intersects Y transversally, then we also get the same long exact sequence

$$\begin{aligned}
\cdots \to H^{i-2d}_{cont}(X'' \text{ rel } Y'', n-d) \to H^i_{cont}(X \text{ rel } Y, n) \to H^i_{cont}(X' \text{ rel } Y', n) \\
\to H^{i+1-2d}_{cont}(X'' \text{ rel } Y'', n-d) \to \cdots
\end{aligned}$$

using either definition of relative cohomology.

Lemma B.4.5. *If $\tilde{Y} \to Y$ is a proper covering (i.e., a proper and surjective map), then it has cohomological descent for continuous étale cohomology. In particular if $Y \to X$ is a closed embedding and $\tilde{Y}$ a proper covering of Y, then there is a natural isomorphism*

$$H^i_{cont}(X \text{ rel } Y, j) \longrightarrow H^i_{cont}(X \text{ rel } \operatorname{cosk}_0(\tilde{Y}/Y), j)$$

where the right hand side is taken in the sense of spaces.

Proof. Cohomological descent is a consequence of the same descent for étale cohomology with torsion coefficients prime to the characteristic of the schemes ([SGA4,II], Exp. Vbis, 4.1.6). By B.3.6.b) the second part follows. $\square$

Proposition B.4.6. *On the Zariski site of smooth schemes over B, the presheaf $\tilde{\mathcal{F}}_l$ has the properties of a twisted duality theory. There are regulator maps from K-cohomology to continuous étale cohomology*

$$H^i_{\mathcal{M}}(Y,j) \longrightarrow H^i_{cont}(Y,j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f : Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y',j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y,j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{cont}(Y',j) & \xrightarrow{f^*} & H^i_{cont}(Y,j) \end{array} .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_.$ a space constructed form schemes over X as in B.2.18, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc} H^{n-2d}_{\mathcal{M}}(Y_. \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_.,j) \\ {\scriptstyle c_{j-d}}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^{n-2d}_{cont}(Y_. \times_X Z, j-d) & \xrightarrow{i_!} & H^n_{cont}(Y_.,j) \end{array}$$

is commutative.

Proof. We restrict to smooth schemes for simplicity. We have to define the extra-structure from [G] 1.1 and 1.2. We put

$$H_i(X,j) = H^{2d-i}_{cont}(X, d-j)$$

for a d-dimensional smooth connected scheme. Pull-back on cohomology and pushout on homology are induced from the functors on sheaves on the étale site. We do not work out the details. For a single étale sheaf μ_{l^n} this is actually one of Gillet's examples 1.4 (iii). □

There is really only one case when this regulator is understood.

Lemma B.4.7. *Let K be a number field, $\mathfrak{o}_K$ be its ring of integers and l a prime. Assume $2i-k \geq 2$, then Soulé's l-adic regulator*

$$K_{2i-k}(\mathfrak{o}_K[1/l]) \otimes \mathbb{Z}_l \longrightarrow H^k_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/l], i)$$

agrees with the one obtained from Prop. B.4.6.

Proof. Put $A = \mathfrak{o}_K[1/l]$. Soulé's definition in [Sou2] is the composition

$$K_{2i-k}(A) \longrightarrow \varprojlim K_{2i-k}(A, \mathbb{Z}/l^\nu) \xrightarrow{\varprojlim \overline{c}_{i,k}} \varprojlim H^k_{et}(A, \mathbb{Z}/l^\nu(i))$$

where $\overline{c}_{i,k}$ is as in [Sou1] II.2.3. There is a natural map of presheaves $\mathcal{F}_l(i) \to R\Gamma(\,\cdot\,, \mathbb{Z}/l^\nu(i))$. Hence in Gillet's definition of Chern classes, we get a commutative diagram

$$\begin{array}{ccc} K_{2i-k}(A) & \xrightarrow{c_i} & H^k_{cont}(\operatorname{Spec} A, i) \\ & \searrow & \downarrow \\ & & H^k_{et}(\operatorname{Spec} A, \mathbb{Z}/l^\nu(i)) \end{array} .$$

Hence we only have to consider finite coefficients. Furthermore in this simple case of a regular commutative ring, we do not really need to consider the sheafified versions and generalized cohomology. Gillet's construction boils down to a composition of the Hurewicz-map with universal Chern classes.
For $2i - k \geq 2$, the map $\overline{c}_{i,k}$ is defined by the same type of composition ([Sou2] II 2.3.) with the same universal Chern classes.
By the definition of K-theory with coefficients, we have a commutative diagram (loc. cit. II.2.2) with $X = \mathbb{Z}_\infty BGl(A)$:

$$\begin{array}{ccccccc} \longrightarrow & \pi_n(X) & \xrightarrow{\times l^\nu} & \pi_n(X) & \longrightarrow & \pi_n(X, \mathbb{Z}/l^\nu) & \longrightarrow \\ & h\downarrow & & h\downarrow & & \downarrow h_q & \\ \longrightarrow & H_n(X, \mathbb{Z}) & \xrightarrow{\times l^\nu} & H_n(X, \mathbb{Z}) & \longrightarrow & H_n(X, \mathbb{Z}/q) & \longrightarrow \end{array} .$$

For the prime 2 compare also [We]. $\square$

Theorem B.4.8 (Soulé). *Let K be a number field, $\mathfrak{o}_K$ be its ring of integers and l any prime. Let S' be a finite set of prime ideals of $\mathfrak{o}_K$ and $S = S' \cup \{l\}$. Let $\mathfrak{o}_S$ be the localization of $\mathfrak{o}_K$ at S. The regulator map*

$$c_j : H^i_{\mathcal{M}}(\operatorname{Spec} \mathfrak{o}_{S'}, j) \otimes_{\mathbb{Q}} \mathbb{Q}_l \longrightarrow H^i_{cont}(\operatorname{Spec} \mathfrak{o}_S, j)_{\mathbb{Q}}$$

is always injective and an isomorphism for $i = 1$ and $j > 1$. We have the following behaviour for pairs of indices (i, j):

$(0,j)$	$j \in \mathbb{Z}$	*isomorphism*
$(1,j)$	$j < 1$	*mot. coh. vanishes, l-adic does not in general*
$(1,1)$		*injective of finite codimension*
$(1,j)$	$j > 1$	*isomorphism*
$(2,j)$	$j < 1$	*conjectured to be isom., i.e., etale coh. to vanish*
$(2,1)$		*injective of finite codimension*
$(2,j)$	$j > 1$	*isomorphism, i.e., both vanish*
(i,j)	*else*	*both vanish*

Proof. We have

$$H^i_{cont}(\operatorname{Spec} \mathfrak{o}_S, j)_{\mathbb{Q}} = H^i(G_S, \mathbb{Q}_l(j))$$

where G_S is the Galois group of the maximal extension of K that is unramified outside of S. We first check that these groups vanish for $i > 2$: By [Mi] I Cor. 4.15 all $H^i(G_S, \mu_{l^n}^{\otimes j})$ are finite. This means that the projective systems for varying n are

Artin-Rees. We do not get a $\varprojlim^1$-contribution to continuous cohomology. Moreover, by loc. cit. I. 4.10.c) the $H^i(G_S, \mu_{l^n}^{\otimes j})$ for $i \geq 3$ are 2-torsion. This implies that their projective limit is 2-torsion. In total we have vanishing cohomology $H^i(G_S, \mathbb{Q}_l(j))$ for $i \geq 3$.
The case $i = 0$ is trivial. $H^1(G_S, \mathbb{Q}_l(1)) = E_S \otimes \mathbb{Q}_l$ where E_S are the S-units, while $H^1_{\mathcal{M}}(\operatorname{Spec} \mathfrak{o}_{S'}, 1) = \mathfrak{o}^*_{S'} \otimes \mathbb{Q}_l$. For $H^2(G_S, \mathbb{Q}_l(1))$ (the S-Brauer-group) the codimension is the same as in the $(1,1)$–case by Euler-Poincaré duality (cf. the discussion in [Jn2] Lemma 2 and Cor. 1.). In the remaining cases, neither motivic (B.2.20) nor continuous étale cohomology ([Jn3] Lemma 4) is changed by the inversion of S', at least up to torsion. We assume $S' = \emptyset$. For odd l, the cases $(1, j)$ and $(2, j)$ for $j > 1$ are Soulé's result in [Sou2] Theorem 1. Note that we are in the range where the previous lemma applies.
For $l = 2$, we have to refine the argument. On the level of $\mathbb{Q}_2$-coefficients we may, by Galois descent, assume that K contains $\sqrt{-1}$ – note that the only prime which could possibly ramify in this quadratic extension has been inverted, and hence we get an étale extension of rings. By [DwF], Theorem 8.7 and the succeeding remark, we have surjectivity even for $l = 2$.
To conclude, we need to show that the $\mathbb{Q}_2$-vector spaces have the right dimension. Let $j > 1$. By [Jn2], proof of Lemma 1, the dimension of

$$H^i_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], j)_{\mathbb{Q}}$$

equals the corank of

$$H^i_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j))\,.$$

By [Sou3], 1.2 and Proposition 2, this corank, for $i = 1$, equals the rank of the K–group if and only if

$$H^2_{cont}(\operatorname{Spec} \mathfrak{o}_K[1/2], \mathbb{Q}_2/\mathbb{Z}_2(j))$$

is torsion. This in turn follows from [We], Theorem 7.3. □

Finally we want to discuss Soulé's elements in K-theory with coefficients. Everything is in the setting of simplicial sets and spectra in the usual sense. Generalized cohomology does not enter. Let Σ be the sphere spectrum and l^r a prime power. By definition of the Moore spectrum there is a cofibration sequence

$$\Sigma \xrightarrow{l^r} \Sigma \xrightarrow{i_{l^r}} M_{l^r} \xrightarrow{j_{l^r}} S\Sigma\ .$$

Recall that for the ring of integers in a number field A

$$K_n(A, \mathbb{Z}_l) = \varprojlim K_n(A, \mathbb{Z}/l^\nu) = \varprojlim \pi_n(\mathbf{K} \wedge M_{l^\nu})\ .$$

The Moore spectrum has a unique product for $l > 2$. For $l = 2, r \geq 2$ there are two projective systems of regular product structures on M_{l^r} ([O], Theorem 2 (a), (b) and Lemma 5). Together with the product structure on $\mathbf{K}$ this defines a product on $K_*(A, \mathbb{Z}_l)$ for $l \geq 2$.

For $d \geq 2$, we define $R = \mathbb{Z}(\mu_d, 1/dl)$. Recall ([Sou2], Lemma 1, [Sou5], 4.1–4.3) Soulé's construction of maps

$$\varphi_l : \text{primitive elements of } \mu_d \longrightarrow K_{2n+1}(R, \mathbb{Z}_l) = K_{2n+1}(R) \otimes_{\mathbb{Z}} \mathbb{Z}_l \ .$$

The original statement is for odd primes l, but using the above 2-adic product the construction works without any changes for $l = 2$. For a primitive d-th root of unity ω, choose some $(\alpha_r)_{r\geq 1} \in \varprojlim \mu_{dl^r}$ satisfying $\alpha_1^l = \omega$. Let $(\beta_r)_{r\geq 1} \in \varprojlim K_2(R, \mathbb{Z}/l^r)$ be the projective system of Bott elements with $j_{2^r}(\beta_r) = \alpha_r \in K_1(R)$. Using the formalism of norm compatible units developed in [Sou2], one lets $\varphi_l(\omega)$ denote the projective system

$$\left(N_r((1-\alpha_r) \cup (\beta_r^d)^{\cup n})\right)_r \in \varprojlim K_{2n+1}(R, \mathbb{Z}/l^r\mathbb{Z}) \ .$$

Remark: It is not clear to the authors whether the 2-adic Soulé elements depend on the choice of product on the Moore spectrum. By [O] pp. 263–264, the difference between the two regular products μ and μ' on M_{2^r} is given by

$$M_{2^r} \wedge M_{2^r} \xrightarrow{j_{2^r} \wedge j_{2^r}} S\Sigma \wedge S\Sigma \xrightarrow{\eta^2} \Sigma \xrightarrow{i_{2^r}} M_{2^r} \ .$$

Lemma B.4.9. *Let ζ be a root of unity and $n \geq 0$. The restriction map from $H^1_{cont}(\mathbb{Q}(\zeta), \mathbb{Q}_l(n+1))$ into*

$$\begin{aligned} & H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta), \mathbb{Q}_l(n+1))^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \\ = & \left(\varprojlim_{r\geq 1} \left(H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta), \mu_{l^r}) \otimes \mu_{l^r}^{\otimes n}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \\ = & \left(\varprojlim_{r\geq 1} \left(\mathbb{Q}(\mu_{l^\infty}, \zeta)^* / (\mathbb{Q}(\mu_{l^\infty}, \zeta)^*)^{l^r} \otimes \mu_{l^r}^{\otimes n}\right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \right)^{\mathrm{Gal}(\mathbb{Q}(\mu_{l^\infty},\zeta)/\mathbb{Q}(\zeta))} \end{aligned}$$

is injective.

Proof. Note that the argument given in the discussion preceding [WiIV], Theorem 4.5 is incorrect since the transition maps

$$H^1_{cont}\left(\mathbb{Q}(\mu_{l^m}, \zeta), \mu_{l^r}^{\otimes(n+1)}\right) \longrightarrow H^1_{cont}\left(\mathbb{Q}(\mu_{l^{m+1}}, \zeta), \mu_{l^r}^{\otimes(n+1)}\right)$$

are in general not injective. The kernel of the restriction map is given by

$$H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\zeta), \mathbb{Q}_l(n+1)) \ .$$

Since $[\mathbb{Q}(\mu_l, \zeta) : \mathbb{Q}(\zeta)]$ is prime to l, we have to show that

$$H^1_{cont}(\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\mu_l, \zeta), \mathbb{Z}_l(n+1))$$

is torsion. But the Galois group G of $\mathbb{Q}(\mu_{l^\infty}, \zeta)/\mathbb{Q}(\mu_l, \zeta)$ is isomorphic to $\mathbb{Z}_l$, and hence its first cohomology equals the functor of coinvariants. Our claim follows since $n \geq 0$. $\square$

Proposition B.4.10. *Let ζ be a fixed d-th root of unity. The l-adic regulator*

$$r_l : K_{2n+1}(R) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^1_{cont}(\mathbb{Q}(\mu_d), \mathbb{Q}_l(n+1))$$

takes $\varphi_l(\zeta^b)$ to the cyclotomic element in continuous Galois cohomology

$$\left(\sum_{\alpha^{l^r}=\zeta^b} [1-\alpha] \otimes (\alpha^d)^{\otimes n} \right)_r$$

(in the description of the last lemma) defined by Soulé and Deligne (cf. [Sou2], page 384, [D5], 3.1, 3.3).

Proof. If l is odd, then this is [Sou1], Théorèmes 1 and 2. For $l = 2$ the same is true using the properties of the 2-adic regulator (see [We]). □

B.5 Absolute Hodge Cohomology

Let $B = \operatorname{Spec} \mathbb{C}$ or $B = \operatorname{Spec} \mathbb{R}$ in this section.

In A.1.9 a definition of absolute Hodge cohomology and relative cohomology for general varieties over $\mathbb{C}$ was given. The variant over $\mathbb{R}$ was A.2.6.

By A.1.10 resp. A.2.7 absolute Hodge cohomology of smooth varieties is given functorially by Beilinson's complexes $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B, n)$.

Lemma B.5.1. *These form a pseudo-flasque complex of presheaves on the Zariski site of smooth B-schemes.*

Proof. By construction [B1] they form a presheaf on pairs $(U, \overline{U})$ where $\overline{U}$ is a compactification with complement an NC-divisor. (For more details cf. [H1] Prop. 8.3.3.) Taking the limit over all choices of $\overline{U}$ we get the desired presheaf. To say it is pseudo-flasque means that absolute Hodge cohomology has the Mayer-Vietoris property. In the context of A.1.9 and A.2.6 it is a formal consequence of the existence of triangles $(i_*i^!, id, j_*j^*)$ for open immersions j with closed complement i. In the context of [B1] it follows from the Mayer-Vietoris property of De Rham-cohomology and singular cohomology. □

We now consider the corresponding generalized cohomology.

Definition B.5.2 (2. Version). *If X is a space over B, then we define absolute Hodge cohomology by*

$$H^i_{\mathfrak{H}^p}(X/B, n) = H^i_{s\mathbf{T}}(X, K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B, n)) \,.$$

If $f : Z \to X$ is a morphism of spaces, then we define relative cohomology

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/B, n) = H^i_{s\mathbf{T}}(\operatorname{Cone}(f), K(\widetilde{R\Gamma}_{\mathfrak{H}^p}(\,\cdot\,/B, n)) \,.$$

Lemma B.5.3. *There is a functorial isomorphism between both definitions of absolute Hodge cohomology for a smooth variety X. If $Y \to X$ is a closed immersion of smooth schemes, then the same is true for relative cohomology.*

Proof. Lemma B.3.4 and Lemma B.3.6.a). □

In order to get the same equalities at least for some singular varieties we have to check a descent property for Hodge modules. For this we need functoriality of i_*i^* with values in complexes of Hodge modules rather than objects in the derived category.

Lemma B.5.4. *Let $X/\mathbb{C}$ be smooth and $i : Y \to X$ a closed reduced subscheme of pure codimension 1. Let $Y = \bigcup_{i=0}^n Y_i$. For $I \subset \{0, \dots, n\}$ and $M \in \mathrm{MHM}_F(X)$ let*

$$i_I : Y_I = \bigcap_{i \in I} Y_i \longrightarrow X$$

$$j_I : U_I = X \smallsetminus \bigcup_{i \in I} Y_i \longrightarrow X$$

$$M_I = j_{I!} j_I^* M \in \mathrm{MHM}_F(X) \ .$$

All Y_I are equipped with the reduced structure. Then $i_{I*}i_I^*M$ defines a functor

$$\{\textit{subsets of}\ \{0, \dots, n\}\} \longrightarrow C^b(\mathrm{MHM}_F(X)) \ .$$

Proof. As j_I is affine both j_I^* and $j_{I!}$ map Hodge modules to such. Note that locally each Y_i is given by a function f_i on X. The functor $i_{I*}i_I^*$ has an explicit description for closed subschemes of the type Y_I given in the proof of [S2] Prop. 2.19. In fact

$$i_{I*}i_I^*M = \ldots \longrightarrow \bigoplus_{I' \subset I; |I'|=2} M_{I'} \longrightarrow \bigoplus_{I' \subset I; |I'|=1} M_{I'} \longrightarrow M$$

where the complex sits in degrees less or equal to zero. □

Proposition B.5.5. *Let $X/\mathbb{C}$ be smooth and $i : Y \to X$ a closed subscheme as in the lemma. Let $\tilde{Y} = Y_0 \amalg \cdots \amalg Y_n$ and*

$$\tilde{Y}_\cdot = \mathrm{cosk}_0(\tilde{Y}/Y) \xrightarrow{s} Y \ ,$$

i.e.,

$$\tilde{Y}_k = \tilde{Y} \times_Y \cdots \times_Y \tilde{Y} \qquad (k+1 \textit{ factors}) \ .$$

Then the functor s_*s^* defined by the total complex of the cosimplicial complex $(s_{n*}s_n^*)_{n \in \mathbb{N}_0}$ is isomorphic to i_*i^*.

Proof. Note that

$$\tilde{Y}_k = \coprod_{I \in \{0, \dots, n\}^{k+1}} Y_I$$

where $Y_I = Y_{\{i_0, \dots, i_k\}}$ in the notation of the previous lemma. Let M be in $\mathrm{MHM}_F(X)$. By the previous lemma we get indeed a cosimplicial complex hence s_*s^*M is a well-defined complex of Hodge modules. Let $\tilde{Y}_\cdot^{\leq}$ be the simplicial subscheme given by

$$\tilde{Y}_k^{\leq} = \coprod_{I = (i_0 \leq i_1 \leq \cdots \leq i_k)} Y_I \xrightarrow{s_k^{\leq}} Y \ .$$

By the Hodge module version of the combinatorial Lemma B.6.2, the morphism $s_*s^*M \to s_*^{\leq} s^{\leq *}M$ is a quasi-isomorphism. By definition ([S2] 2.19)

$$i_*i^*M = M_{\{0,\ldots,n\}} \to M \ ,$$

and this complex is canonically quasi-isomorphic to the total complex of the constant cosimplicial complex i_*i^*M. It is easy to see that the natural morphism

$$\operatorname{Tot} i_*i^*M \longrightarrow s_*^{\leq} s^{\leq *}M$$

is a quasi-isomorphism. $\square$

Corollary B.5.6. *Let X/B be smooth. Suppose $Y \to X$ is an NC-divisor over B all of whose irreducible components are smooth over B. Then the group $H^i_{\mathfrak{H}^p}(Y/B,j)$ as defined in A.1.9 resp. A.2.6 is isomorphic to the generalized cohomology group $H^i_{\mathfrak{H}^p}(\tilde{Y}_\cdot/B,j)$ and to the same noted group in [B1].*

Proof. The condition on Y ensures that $\tilde{Y}_\cdot$ is indeed a smooth simplicial scheme. It gives rise to a space over B. Cohomological descent for the coefficients as in B.5.5 implies cohomological descent for their global sections in the sense of B.3.5. We can use $\tilde{Y}_\cdot$ as the smooth proper hyper-covering needed in Beilinson's definition. Equality to the generalized cohomology version is again B.3.4. $\square$

This is of course cohomological descent for a closed Čech-covering. We have restricted to this case which is built into the very definition of Hodge modules for simplicity. There is no reason why there should not be cohomological descent in the same generality as for constructible sheaves.

Lemma B.5.7. *Let X/B be smooth, and $Z \subset X$ a closed immersion of an NC-divisor all of whose irreducible components are smooth over B. Let $\tilde{Z}_\cdot$ be the smooth simplicial scheme of B.5.5, then there is a canonical isomorphism*

$$H^i_{\mathfrak{H}^p}(X \text{ rel } Z/B,n) = H^i_{\mathfrak{H}^p}(X \text{ rel } \tilde{Z}_\cdot/B,n)$$

where we use the original definition on the left and the second on the right.

Proof. This follows by the general method of B.3.6.b) from the descent property that we have just established. $\square$

Remark: If we had checked cohomological descent in general, then we would get B.5.6 for arbitrary varieties and B.5.7 for arbitrary closed immersions.

Theorem B.5.8. *On the site of smooth schemes over B, the presheaves $R\Gamma_{\mathfrak{H}^p}(\,\cdot\,/B,n)$ have the properties of a twisted duality theory. There are regulator maps from K-cohomology to absolute Hodge cohomology*

$$H^i_{\mathcal{M}}(Y,j) \longrightarrow H^i_{\mathfrak{H}^p}(Y/B,j)$$

for all K–coherent spaces Y. They are compatible with pullback, i.e., if $f: Y \to Y'$ is a map of K–coherent spaces, we get commutative diagrams

$$\begin{array}{ccc} H^i_{\mathcal{M}}(Y',j) & \xrightarrow{f^*} & H^i_{\mathcal{M}}(Y,j) \\ {\scriptstyle c_j}\downarrow & & \downarrow{\scriptstyle c_j} \\ H^i_{\mathfrak{H}^p}(Y'/B,j) & \xrightarrow{f^*} & H^i_{\mathfrak{H}^p}(Y/B,j) \end{array} \quad .$$

If $i : Z \to X$ is a closed immersion of smooth schemes (constant codimension d) with open complement U and $Y_{\cdot}$ a space constructed form schemes over X as in B.2.18, then the regulator is compatible with pushout, i.e., the diagram

$$\begin{array}{ccc} H^{n-2d}_{\mathcal{M}}(Y_{\cdot} \times_X Z, j-d) & \xrightarrow{i_*} & H^n_{\mathcal{M}}(Y_{\cdot}, j) \\ \downarrow{\scriptstyle c_{j-d}} & & \downarrow{\scriptstyle c_j} \\ H^{n-2d}_{\mathfrak{H}^p}(Y_{\cdot} \times_X Z/B, j-d) & \xrightarrow{i_!} & H^n_{\mathfrak{H}^p}(Y_{\cdot}/B, j) \end{array}$$

is commutative.

Proof. We use Gillet's method B.3.7. All axioms of a twisted duality theory hold e.g. [H1] Ch. 15. Granted this the proof proceeds as in the l-adic case (B.4.6). □

Remark: Recall ([N], (7.1)) that there is a natural transformation from absolute Hodge to Deligne cohomology. The composition of the above regulator with this transformation was already constructed in [Jeu], 2.5.

Theorem B.5.9 (Borel). *Let K be a number field with r_1 real and r_2 pairs of complex embeddings into $\mathbb{C}$. We consider the ring of integers $\mathfrak{o}_K$ as a scheme over $\mathbb{Z}$. Then the Beilinson regulator*

$$H^i_{\mathcal{M}}(\operatorname{Spec} \mathfrak{o}_K, j) \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^i_{\mathfrak{H}^p}((\operatorname{Spec} \mathfrak{o}_K)_{\mathbb{R}}/\mathbb{R}, j)$$

is an isomorphism for all pairs $(i,j) \neq (0,0),(1,1)$. It is injective of codimension r_1+r_2-1 for $(i,j)=(0,0)$, and injective of codimension one in the case $(i,j)=(1,1)$.

Proof. Note that the cohomological dimension of the category of Hodge structures is 1. The case $i=0$ is trivial, and the case $(1,1)$ is Dirichlet's classical result. In [Bo2], the claim (and much more) is proved for the Borel regulator instead of the Beilinson regulator. By [Rp], Corollary 4.2, the two regulators coincide up to a non–vanishing rational factor. □

B.6 A Combinatorial Lemma

This section gives a purely combinatorial proof why two conceivable definitions of the Čech-nerve of a covering are homotopically equivalent. This is well-known at least for open coverings and Čech-cohomology (and probably in general). But for lack of finding an appropriate reference we work out the combinatorics here.

Let $C(n)$ be the following simplicial set:

$$C(n)_k = \{1, \dots, n\}^{k+1}$$

with the obvious face and degeneracy maps. Let $C(n)^{\leq}$ be the simplicial subset of simplices whose entries are ordered by $\leq$. In fact this is the simplicial version of the n-simplex.

Suppose we are given a covariant functor from the category of subsets of $\{1, \dots, n\}$ to the category of sets. We get simplicial sets by setting

$$A(n)_k = \dot{\bigcup}_{I \in C(n)_k} A_I$$

$$A(n)^{\leq}_k = \dot{\bigcup}_{I \in C(n)^{\leq}_k} A_I$$

where A_I is the value of our functor on the set $I = \{i_0, \ldots, i_k\}$. Note that the elements of $C(n)_k$ are ordered tuples but the value of A_I does not depend on the ordering.

Lemma B.6.1. *If the functor has constant value A, then both simplicial sets have the homotopy*

$$\pi_i \left(A(n)^?_\cdot, \star\right) = \begin{cases} A & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$$

Proof. Obviously it is enough to consider the case $A = \star$, i.e., of the simplicial sets $C(n)^{\leq} \to C(n)$ themselves. Both simplicial sets satisfy the extension condition [M] 1.3 rather trivially. Hence we can use the combinatorial computation of the homotopy groups given in [M] Def. 3.6. We immediately get the result. □

Proposition B.6.2. *For a general functor A the injection $A(n)^{\leq} \to A(n)$ of simplicial sets is a weak homotopy equivalence.*

Proof. We filter the simplicial sets $C(n)^?$ by the simplicial subsets $F^iC(n)^?$ of simplices in which at most i different integers occur. This induces a filtration of the simplicial sets $A(n)^?$. Let $G^iA(n)^?$ be the cofibre of the cofibration $F^{i-1}A(n)^? \subset F^iA(n)^?$. It consists of simplices in which precisely i different integers occur. We argue by induction on i for all functors A at the same time. There is a long exact homotopy sequence attached to the cofibration sequence

$$F^{i-1}A(n)^? \longrightarrow F^iA(n)^? \longrightarrow G^iA(n)^? \, .$$

By induction it suffices to show that all cofibres $G^iA(n)/(G^iA(n)^{\leq})$ are weakly equivalent to the final object $\star$. The cofibre decomposes into a union of simplicial sets corresponding to a different choice of i elements in $\{1, \ldots, n\}$ each. If suffices to prove acyclicity for one choice e.g for the subset $\{1, \ldots, i\}$. Hence we only have to consider $G^iA(i)/G^iA(i)^{\leq}$. But this last cofibre is isomorphic to $G^iB(i)/G^iB(i)^{\leq}$ where B is the functor with constant value $A_{\{1,\ldots,i\}}$. For $i > 1$ it is easy to see that $\pi_0 G^iB(i)/G^iB(i)^{\leq} = \star$. By B.6.1 the quotients $B(n)/B(n)^{\leq}$ are acyclic for all n. Using the same cofibration sequence as for A and the inductive hypothesis this implies that all $G^iB(i)/G^iB(i)^{\leq}$ are acyclic. □

Note that A could also be a functor to the category of abelian groups or to the dual of the category of abelian groups.

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An Alternative Proof of Scheiderer's Theorem on the Hasse Principle for Principal Homogeneous Spaces

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Abstract. We give an alternative proof of the Hasse principle for principal homogeneous spaces defined over fields of virtual cohomological dimension at most one which is based on a special decomposition of elements in Chevalley groups.

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1. Introduction

Let Y be a smooth irreducible projective curve defined over the real number field $\mathbb{R}$ and $k = \mathbb{R}\,(Y)$ be the field of $\mathbb{R}$-rational functions on Y. For a point $P \in Y(\mathbb{R})$ we denote the completion of k at the point P by k_P. The present paper is devoted to the Hasse principle for the existence of a rational point on principal homogeneous spaces of a connected linear algebraic group G defined over k. It was Colliot-Thélène who conjectured ([CT], Conjecture 2.9) that for any such space X the Hasse principle holds relative to all local fields k_P, $P \in Y(\mathbb{R})$, i.e. $X(k) \neq \emptyset$ iff $X(k_P) \neq \emptyset$ for each $P \in Y(\mathbb{R})$. Since principal homogeneous spaces of G are in natural one-to-one correspondence with elements of the set $H^1(k, G)$ the latter statement is equivalent to the following: the natural map of pointed sets

$$H^1(k, G) \longrightarrow \prod_{P \in Y(\mathbb{R})} H^1(k_P, G) \tag{1}$$

has trivial kernel ([S]).

In [CT] Colliot-Thélène proved the Hasse principle for algebraic k-tori and reduced the general case to that of a simple simply connected algebraic group G. The case of an arbitrary connected k-group G has been studied by Scheiderer ([Sch1]).

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To prove the Hasse principle he first made an important observation (which eventually turned out to be crucial) that local objects k_P can be replaced by real closures k_ξ of k, $\xi \in \Omega_k$, where Ω_k denotes the set of all orderings of k. Indeed, using the description of orderings of k and the so-called Artin-Lang homomorphism theorem ([Srl], Theorem 3.1) it is easy to show that the condition $X(k_P) \neq \emptyset$ for each real point P on Y implies $X(k_\xi) \neq \emptyset$ for each ordering ξ of k and hence the triviality of the kernel of (1) follows immediately from the triviality of the kernel of

$$\theta: \ H^1(k, G) \longrightarrow \prod_{\xi \in \Omega_k} H^1(k_\xi, G) \tag{2}$$

The question whether θ is injective makes sense not only for the function fields of curves but also for an arbitrary field k and it turned out that θ is indeed injective if k has virtual cohomological dimension (vcd) at most 1 (recall that function fields in one variable over $\mathbb{R}$ are such). We have even more.

Theorem 1. (Scheiderer, [Sch1]) *Let K be any field of virtual cohomological dimension ≤ 1. Then the Hasse principle holds for any homogeneous K-space X of a connected linear algebraic K-group G.*

Scheiderer's proof can be divided into two parts. In the first one it is proved that for X as in the theorem (here G may even be not connected) there exists a principal homogeneous space Z which is everywhere locally trivial and dominates X. The strategy of the proof in this part going back to Springer ([S], [Sp]) consists of replacing X by a homogeneous space which dominates X and has a smaller stabilizer. It is worth mentioning that in this part most arguments do not use specific properties of K and so most of them are valid over an arbitrary perfect field.

The second part of Scheiderer's proof is devoted to the case of a principal homogeneous space. To treat such a space Scheiderer first constructs a locally constant sheaf of sets $\mathcal{H}^1(G)$ on Ω_K whose stalks are just the sets $H^1(K_\xi, G)$. Then he shows that there exists a natural bijection between the set of global sections of $\mathcal{H}^1(G)$ and $H^1(K, G)$. As a whole the proof in this part is quite complicated. It is based on using étale machinery and, in particular, strongly relies on results of the book [Sch2].

The aim of this paper is to provide a simpler and shorter self-contained proof which is based only on the Bruhat decomposition in semisimple algebraic groups and the so-called strong approximation property (SAP) of fields (see § 3). We show that in fact the Hasse principle follows immediately modulo two facts. Informally speaking one of them says that the kernel of the natural map $H^1(K, T) \to H^1(K, G)$, where G is an (absolutely) simple simply connected linear K-group and T is a K-torus splitting over $K(\sqrt{-1})$, can be parametrized by "good" rational functions (see § 2) and the other says that any field of virtual cohomological dimension ≤ 1 is an SAP field.

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2. Algebraic groups splitting over quadratic extensions

Throughout the section K denotes an arbitrary field of characteristic 0. Let G be an (absolutely) simple simply connected algebraic group of rank n defined over K

and splitting over quadratic extension $L = K(\sqrt{d}\,)$. Let

$$\Theta = \mathrm{Gal}\,(L/K) = \langle \tau \mid \tau^2 = 1 \rangle.$$

Consider a Borel L-subgroup B such that $T = B \cap \tau(B)$ is a maximal torus which will be assumed for simplicity to be K-anisotropic. Since T is splitting over L, one has

$$T \simeq \mathrm{R}^{(1)}_{L/K}(\mathrm{G}_m) \times \ldots \times \mathrm{R}^{(1)}_{L/K}(\mathrm{G}_m).$$

To prove the Hasse principle we need to describe $\mathrm{Ker}\,[H^1(\Theta, T(L)) \to H^1(\Theta, G(L))]$. This description can be easily extracted from [Ch]. However this paper is written in Russian and the translation made by the AMS is unreadable and contains a lot of misprints. So for the sake of expository completeness and the reader's convenience we include here details.

First recall some basic facts about the structure of the group $G(L)$ (for details see [St1]). Let $\Sigma = R(T, G)$ be the root system of G relative to T. The Borel subgroup B determines an ordering on the set Σ and hence a system of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. If $\alpha = \sum n_i\alpha_i \in \Sigma^+$, then the number $\mathrm{ht}(\alpha) = \sum n_i$ is called the height of α. If $\{X_\alpha, \alpha \in \Sigma; H_{\alpha_1}, \ldots, H_{\alpha_n}\}$ is a Chevalley basis of the Lie algebra of G, then $G(L)$ is generated by the corresponding root subgroups $G_\alpha = \langle x_{\pm\alpha}(t) \mid t \in L \rangle$, where

$$x_\alpha(t) = \sum_{n=0}^{\infty} t^n X_\alpha^n\,/n!\ ,$$

and the torus T is generated by $T_\alpha = T \cap G_\alpha = \langle h_\alpha(t) \rangle$, where $h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}$ and $w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$.

Furthermore, since G is simply connected the following relations hold in G (cf. [St1], Lemma 28 b), Lemma 20 c), Lemma 15):

A) $T = \langle h_{\alpha_1}(t_1) \rangle \times \cdots \times \langle h_{\alpha_n}(t_n) \rangle$ and for $\alpha \in \Sigma$ we have

$$h_\alpha(t) = \prod_{i=1}^{n} h_{\alpha_i}(t)^{n_i}, \qquad \text{where } H_\alpha = \sum_{i=1}^{n} n_i H_{\alpha_i}; \tag{3}$$

B) For $\alpha, \beta \in \Sigma$ let $\langle \beta, \alpha \rangle = 2\,(\beta, \alpha)/(\alpha, \alpha)$. Then we have

$$h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta, \alpha \rangle} u) \tag{4}$$

C) For all $u, v \in L$ such that $1 + uv \neq 0$ we have

$$x_{-\alpha}(u) x_\alpha(v) = x_\alpha(v(1+uv)^{-1}) h_\alpha(1+uv)^{-1} x_{-\alpha}(u(1+uv)^{-1}) \tag{5}$$

D) For all $\alpha, \beta \in \Sigma$, $\beta \neq -\alpha$, we have

$$x_\alpha(v) x_\beta(u) x_\alpha(v)^{-1} x_\beta(u)^{-1} = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{i,j} v^i u^j) \tag{6}$$

where the product on the right hand side is taken over all roots of the form $i\alpha + j\beta$ and the $c_{i,j}$ are integers which depend on α, β and on the chosen ordering of the roots but do not depend on v and u.

Since T is K-defined, τ acts on the root system Σ. More exactly, for any $\alpha \in \Sigma$ the character $\alpha + \tau(\alpha)$ is K-defined and hence is zero, i.e. $\tau(\alpha) = -\alpha$, since, by assumption, T is K-anisotropic. It follows that there exists $c_\alpha \in L^*$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$; in particular, the subgroup G_α is K-defined.

The constants c_α actually lie in K and $c_{-\alpha} = c_\alpha^{-1}$. Indeed, for rank one groups, i.e. of the form $\mathrm{SL}\,(1, D)$, where D is a quaternion K-algebra, this fact can be verified directly. The general case easily reduces to the rank one case since G_α is a simple simply connected K-group of rank 1. Thus, we have

Lemma 1. *There exists constant $c_\alpha \in K^*$ such that for any $u \in L$ one has $\tau(x_\alpha(u)) = x_{-\alpha}(c_\alpha \tau(u))$. Moreover, $G_\alpha \simeq SL(1, D)$, where D is a quaternion algebra over K of the form $D = (d, c_\alpha)$.*

Proof: Straightforward computations. □

Lemma 2. *The positive roots $\Sigma^+ = \{\beta_1, \dots, \beta_m\}$ can be ordered in such a way that the following two properties hold:*
1) for any pair of roots β_i, β_j, for which $i < j$ and $\beta_i + \beta_j = \beta_k \in \Sigma^+$, the root β_k is between β_i and β_j, i.e. $i < k < j$;
2) if Σ is a root system of type either A_{2n-1} or D_n or E_6 and σ is the outer automorphism of Σ induced by the non-trivial automorphism of order 2 (resp. 3) of the corresponding Dynkin diagram, then for any root $\beta_i \in \Sigma^+$ the roots β_i and $\sigma(\beta_i)$ (resp. β_i, $\sigma(\beta_i)$, $\sigma^2(\beta_i)$) are neighbours.

Proof. a) Let $\Sigma = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le 2n\}$ be a root system of type A_{2n-1}. Let $\alpha_1 = \varepsilon_1 – \varepsilon_2, \dots, \alpha_{2n-1} = \varepsilon_{2n-1} – \varepsilon_{2n}$ be a basis of Σ and Σ_1 be the subsystem generated by the roots $\alpha_2, \dots, \alpha_{2n-2}$. By induction, we can pick an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the required properties. Let $\gamma = \alpha_1 + \cdots + \alpha_{2n-1}$. We number the remaining roots $\Sigma^+ \setminus \{\Sigma_1^+ \cup \gamma\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$ in the order of decreasing height. If β_i denotes the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$ such that $\mathrm{ht}\,(\beta_i) \ge n$, then the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k,\, \beta_{k+1}, \dots, \beta_i,\, \gamma,\, \beta_{i+1}, \dots, \beta_{m-1}\}$$

is as required.

b) Σ is a root system of type $A_{2n}, B_n, C_n, D_n, E_7$. It follows from the description of root systems of these types that there exists a subsystem Σ_1 generated by $n-1$ simple roots, say $\alpha_1, \dots, \alpha_{n-1}$, such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ can be written as a sum $\beta = m_1\alpha_1 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n$. If Σ is of type D_n and $|\sigma| = 2$, we may assume in addition that the set $\{\alpha_1, \dots, \alpha_{n-1}\}$ is stable under σ. The root system Σ_1 has rank $n-1$ and so by induction, there exists an ordering of the required type on the set $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$. We number the remaining roots $\Sigma^+ \setminus \Sigma_1^+ = \{\beta_{k+1}, \dots, \beta_m\}$ in the order of decreasing height. Then the ordering $\{\beta_1, \dots, \beta_m\}$ is as required.

c) Σ is a root system of type E_6, E_8, F_4, G_2. Here one can argue as in case a). Namely, there exists a subsystem Σ_1 generated by simple roots $\alpha_1, \dots, \alpha_{n-1}$ such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ is of the form $\beta = m_1\alpha_1 + \cdots + m_{n-1}\alpha_{n-1} + \alpha_n$ except for the maximal root $\tilde{\alpha}$ and $\tilde{\alpha}$ is of the form $\tilde{\alpha} = m_1\alpha_1 + \cdots + m_{n-1}\alpha_{n-1} + 2\alpha_n$. Let $b = \mathrm{ht}\,(\tilde{\alpha})$. Again, applying induction we can find an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the desired properties and then we number the roots $\Sigma^+ \setminus \{\Sigma_1^+ \cup \tilde{\alpha}\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$ in the order of decreasing height. If Σ has type E_6, we may assume in addition that β and $\sigma(\beta)$ are neighbours for all $\beta \in \Sigma^+$. Let β_i be the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$

such that $\mathrm{ht}\,(\beta_i) \geq b/2$. We claim that the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k, \, \beta_{k+1}, \dots, \beta_i, \, \tilde{\alpha}, \, \beta_{i+1}, \dots, \beta_{m-1}\}$$

has the desired properties. Indeed, if $\beta_j = \beta_s + \beta_t$, where $s < t$ and $j \in \{k+1, \dots, m-1\}$, then clearly β_s belongs to Σ_1^+. It follows that β_j lies between β_s and β_t, since $\mathrm{ht}\,(\beta_j) \geq \mathrm{ht}\,(\beta_s), \mathrm{ht}\,(\beta_t)$. Now let $\tilde{\alpha} = \beta_s + \beta_t$, $s < t$,. Then $s, t \in \{k+1, \dots, m-1\}$ and $\mathrm{ht}\,(\beta_s) \geq b/2$, $\mathrm{ht}\,(\beta_t) < b/2$ (we use the fact that $\mathrm{ht}(\tilde{\alpha})$ is odd), implying $\tilde{\alpha}$ is also between β_s and β_t.

d) Σ has type D_4 and $|\sigma| = 3$. Let $\alpha_1, \dots, \alpha_4$ be simple roots such that σ permutes $\alpha_1, \alpha_3, \alpha_4$. Then the required ordering is as follows: first we place α_2, then all roots of the height 2, then the maximal root and then the roots of heights 3, 4, 1 respectively. □

Corollary 1. *Let β_i, β_j, $j < i$, be any two positive roots. Then for any positive root β_k of the form $\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k < j$. Analogously, for any negative root of the form $-\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k > i$.*

Proof. We distinguish three cases.

a) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type A_2. Then $r = l = 1$ and hence if $\beta_k = \beta_j - \beta_i$ is a positive root then $\beta_k + \beta_i = \beta_j$, implying $k < j < i$. Analogously, if $\beta_j - \beta_i = -\beta_k$ then we have $j < i < k$.

b) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type B_2. Then either $r = l = 1$ or $r = 1$ and $l = 2$ or $r = 2$ and $l = 1$. The case $r = l = 1$ was already handled in part a). Now let $\beta_k = \beta_j - 2\beta_i$. Then $\beta_j - \beta_i = \beta_s$ is also a positive root implying $s < j$. Futhermore, $\beta_k = \beta_s - \beta_i$ and $s < j < i$. So again we have $k < s < j$. The remaining cases can be handled in a similar way.

c) $\langle \beta_i, \beta_j \rangle_{\mathbb{Q}} \cap \Sigma$ has type G_2. Here the proof is similar to that of case b) and we omit it. □

Proposition 1. *Fix an order in Σ^+ as in Lemma 2. Then the regular map*

$$\omega : \mathbb{G}_m^n \times \mathbb{A}^{2m} \longrightarrow G \,, \qquad (t_1, \dots, t_n, u_1, \dots, u_m, v_1, \dots, v_m) \longrightarrow$$
$$\prod_{i=1}^n h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

is birational over L.

Remark 1. This statement is also true in positive characteristic. There is the only place which require additional work: one need additionally to check that ω is a separable map.

Proof. Both sides have the same dimension and hence it suffices to prove the injectivity of ω on some Zariski open subset, since $\mathrm{char}\, K = 0$.

First we show that for any integer i and any parameters $u_1, \dots, u_i$ and $v_1, \dots, v_i$ from some Zariski open subset the element

$$A_i = x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_i}(u_i) x_{\beta_i}(v_i)$$

of the group G can be written in the form

$$A_i = \prod_{k=1}^n h_{\alpha_k}(f_k) \prod_{j=1}^m x_{-\beta_j}(r_j) \prod_{j=1}^{i-1} x_{\beta_j}(s_j)\, x_{\beta_i}(v_i),$$

where f_k, r_j, s_j are rational functions depending on $u_1, \dots, u_i$, $v_1, \dots, v_{i-1}$.

If $i = 1$ there is nothing to prove. By induction, we may write A_{i-1} in the form

$$\prod_{k=1}^{n} h_{\alpha_k}(f_k) \prod_{j=1}^{m} x_{-\beta_j}(r_j) \prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1}).$$

To write $A_i = A_{i-1}\, x_{-\beta_i}(u_i)\, x_{\beta_i}(v_i)$ in the same form we have to transpose $x_{-\beta_i}(u_i)$ with each factor in the product $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})$. By (6) and by Corollary 1, every time doing so we obtain additional factors $x_{\beta_s}(\,)$ or $x_{-\beta_s}(\,)$, where $s < i-1$ in the first case and $s > i$ in the second case. Collecting together all these factors corresponding to negative roots we can write the element $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})\, x_{-\beta_i}(u_i)$ in the form

$$\prod_{k=1}^{n} h_{\alpha_i}(\tilde{f}_k) \prod_{j=1}^{m} x_{-\beta_j}(\tilde{r}_j) \prod_{j=1}^{i-1} x_{\beta_j}(\tilde{s}_j)$$

and so our claim follows.

Now we are ready to prove the injectivity of ω. Suppose that

$$\omega(t_1, \dots, t_n,\, u_1, \dots, u_m,\, v_1, \dots, v_m) = \omega(\tilde{t}_1, \dots, \tilde{t}_n,\, \tilde{u}_1, \dots, \tilde{v}_m) \tag{7}$$

From the above argument and the Bruhat decomposition we get immediately $v_m = \tilde{v}_m$. To show that $u_m = \tilde{u}_m$, we use (4), (5). Namely, it follows from (4), (5) that the left hand side of (7) may be written in the form

$$\prod_{i=1}^{n} h_{\alpha_i}(f_i)\, [x_{\beta_1}(s_1)\, x_{-\beta_1}(r_1)] \cdots [x_{\beta_{m-1}}(s_{m-1})\, x_{-\beta_{m-1}}(r_{m-1})]$$

$$x_{\beta_m}[v_m(1 + u_m v_m)]\, x_{-\beta_m}[u_m(1 + u_m v_m)^{-1}]\,,$$

where $f_1, \dots, f_n$, $s_1, \dots, s_{m-1}$, $r_1, \dots, r_{m-1}$ are rational functions. Rewriting the right hand side of (7) in the same form we conclude that

$$u_m(1 + u_m v_m)^{-1} = \tilde{u}_m(1 + \tilde{u}_m \tilde{v}_m)^{-1},$$

hence $u_m = \tilde{u}_m$. After cancelling the factor $x_{-\beta_m}(u_m)\, x_{\beta_m}(v_m)$ in (7) the same argument shows that $v_{m-1} = \tilde{v}_{m-1}$, $u_{m-1} = \tilde{u}_{m-1}$ and so on. $\square$

Now we are in position to formulate the main result of the section.

THEOREM 2. *Let $g \in G(L)$ be such that $g^{1-\tau} \in T(L)$. Then there exist quaternion algebras $D_1, \dots, D_m$ over K and elements $w_1, \dots, w_m \in K$ which are reduced norm of $D_1, \dots, D_m$ respectively and elements $t_1, \dots, t_n \in L$ such that*

$$g^{1-\tau} = \prod_{i=1}^{n} h_{\alpha_i}(t_i \tau(t_i)) \prod_{i=1}^{m} h_{\beta_i}(w_i)$$

PROOF. If $g^{1-\tau} \in T(L)$, then for any $x \in G(K)$ one has $g^{1-\tau} = (gx)^{1-\tau}$. Since $G(K)$ is Zariski dense in G, we may always assume that our element g is in "generic" position by which we mean point in some Zariski open subset $U \subset G$ which can be easily specified from the argument. So let

$$g = \prod_{i=1}^{n} h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

where t_i, u_i, $v_i \in L$. Denote $t = \prod_{i=1}^n h_{\alpha_i}(t_i)$ and $g_i = x_{-\beta_i}(u_i)x_{\beta_i}(v_i)$, $i = 1, \ldots, m$. Let also $t' = g^{1-\tau}$, so that

$$t \cdot g_1 \cdots g_m = t' \cdot \tau(t) \cdot \tau(g_1) \cdots \tau(g_m) \tag{8}$$

By Lemma 1, we have $\tau(g_i) \in G_{\beta_i}$. Then applying Proposition 1 we conclude that g_m and $\tau(g_m)$ coincide modulo $T_{\beta_m}(L) = T(L) \cap G_{\beta_m}$ and so the element $g_m^{\tau-1}$ is of the form $h_{\beta_m}(w_m)$ for some parameter w_m. We claim that $w_m \in K$ and it is a reduced norm of the quaternion K-algebra $D_m = (d, d_{\beta_m})$, where $d_{\beta_m} = c_{\beta_m}$. Indeed, by construction the cocycle $(g_m^{\tau-1}) \in Z^1(\Theta, T_{\beta_m}(L))$ is trivial in $Z^1(\Theta, G_{\beta_m}(L))$ and by Lemma 1, $G_{\beta_m} \simeq SL(1, D_m)$, hence our claim follows.

Substituting $\tau(g_m) = h_{\beta_m}(w_m) \cdot g$ in (8) and cancelling g, we have then

$$t \cdot g_1 \cdots g_{m-1} = t' \cdot \tau(t) \cdot h_{\beta_m}(w_m) \cdot [h_{\beta_m}(w_m)^{-1}\tau(g_1)h_{\beta_m}(w_m)] \cdots$$
$$\cdots [h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)]$$

Applying again Proposition 1 and arguing analogously we have

$$[h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)] = h_{\beta_{m-1}}(w_{m-1}) \cdot g_{m-1}$$

for some parameter w_{m-1}, which is again a reduced norm of the quaternion K-algebra $D_{m-1} = (d, d_{\beta_{m-1}})$, where

$$d_{\beta_{m-1}} = c_{\beta_{m-1}} w_m^{\langle \beta_{m-1}, \beta_m \rangle}.$$

To see it, let $\tilde{g}_{m-1} = h_{\beta_m}(w_m)^{-1}\tau(g_{m-1})h_{\beta_m}(w_m)$. Using (4) we have

$$\tilde{g}_{m-1} = x_{\beta_{m-1}}(c_{\beta_{m-1}}^{-1} w_m^{-\langle \beta_{m-1}, \beta_m \rangle}\tau(u_m)) \cdot x_{\beta_{m-1}}(c_{\beta_{m-1}} w_m^{\langle \beta_{m-1}, \beta_m \rangle}\tau(v_m)).$$

It follows that $(h_{\beta_{m-1}}(w_{m-1})) = (\tilde{g}_{m-1} \cdot g_{m-1}^{-1})$ can be viewed as a trivial cocycle in an K-group of rank 1 whose K-structure, i.e. action of τ, is given by the constant $d_{\beta_{m-1}}$. This fact combined with Lemma 1 implies w_{m-1} is a reduced norm of D_{m-1}, as claimed, and so on. Theorem 2 is proved. □

In § 4 we will also deal with a simple simply connected algebraic K-group G which is quasi-split over a quadratic extension L/K and for such a group we also need to describe elements of the form $g^{1-\tau} \in T(L)$, where $g \in G(L)$.

Clearly, K-groups of type ${}^2A_{2n}$ split over a quadratic extension of K. Since this case has been already handled, we may assume that G is an outer form of type not A_{2n}. As above, let B be an L-Borel subgroup B of G such that $T = B \cap \tau(B)$ is a maximal K-anisotropic torus.

Let F/K be the minimal extension over which G is an inner form and let $E = F \cdot L$. Let τ and σ be non-trivial automorphisms of E/K such that $\tau|_F = 1$ and $\sigma|_L = 1$ respectively. In the case ${}^{3,6}D_4$ by σ we denote any automorphism of order 3.

Clearly, σ induces an outer automorphism of the root system $\Sigma = \mathrm{R}\,(T, G)$ which will be denoted by the same letter. Let $\Lambda = \{\gamma_1, \ldots, \gamma_s\} \subset \Sigma^+$ (resp. Λ') be a set of representatives of all orbits of σ in Σ^+ (resp. in Π). We divide Λ into two parts: $\Lambda_1 = \{\gamma_i \in \Lambda \mid \sigma(\gamma_i) = \gamma_i\}$ and $\Lambda_2 = \Lambda \setminus \Lambda_1$. Let also $\Lambda_i' = \Lambda' \cap \Lambda_i$, $i = 1, 2$. For $\gamma_i \in \Lambda_1$ (resp. Λ_2) we denote by H_i the subgroup in G generated by G_{γ_i} (resp. G_{γ_i}, $G_{\sigma(\gamma_i)}$ and $G_{\sigma^2(\gamma_i)}$, if $|\sigma| = 3$).

Lemma 3. *H_i is a simple simply connected K-group of type A_1 (resp. $A_1 \times A_1$ or $A_1 \times A_1 \times A_1$) if $\gamma_i \in \Lambda_1$ (resp. $\gamma_i \in \Lambda_2$ and $|\sigma| = 2$ or $|\sigma| = 3$).*

Proof. It suffices to note that τ acts on Σ as either -1, if Σ has type D_{2n}, or $-\sigma$ otherwise, since it permutes positive and negative roots. Moreover, the combination $\beta_i \pm \sigma(\beta_i)$ is not a root, hence G_{γ_i} and $G_{\sigma(\gamma_i)}$ commute. $\square$

Theorem 3. *Let $g \in G(L)$ be such that $g^{1-\tau} \in T(L)$. Then there exist quaternion algebras $D_1, \dots, D_s$ and elements $w_1, \dots, w_s$ which are reduced norm of $D_1, \dots, D_s$ respectively and elements $t_1, \dots, t_p$ such that:*

1) If Σ is not of type ${}^{3,6}D_4$, then

$$g^{1-\tau} = \prod_{\alpha_i \in \Lambda_1'} h_{\alpha_i}(t_i\tau(t_i)) \prod_{\alpha_i \in \Lambda_2'} h_{\alpha_i}(t_i\tau(t_i)) h_{\sigma(\alpha_i)}[\sigma(t_i)(\tau\circ\sigma)(t_i)] \cdot$$
$$\prod_{\gamma_i \in \Lambda_1} h_{\gamma_i}(w_i) \prod_{\gamma_i \in \Lambda_2} h_{\gamma_i}(w_i) h_{\sigma(\gamma_i)}(\sigma(w_i))$$

2) If Σ is of type ${}^{3,6}D_4$, then

$$g^{1-\tau} = \prod_{\alpha_i \in \Lambda_2'} h_{\alpha_i}(t_i\tau(t_i)) h_{\sigma(\alpha_i)}[\sigma(t_i)(\tau\circ\sigma)(t_i)] h_{\sigma^2(\alpha_i)}[\sigma^2(t_i)(\tau\circ\sigma^2)(t_i)] \cdot$$
$$\prod_{\alpha_i \in \Lambda_1'} h_{\alpha_i}(t_i\tau(t_i)) \prod_{\gamma_i \in \Lambda_1} h_{\gamma_i}(w_i) \prod_{\gamma_i \in \Lambda_2} h_{\gamma_i}(w_i) h_{\sigma(\gamma_i)}(\sigma(w_i)) h_{\sigma^2(\gamma_i)}(\sigma^2(w_i))$$

Here D_i is over K (resp. over F) and $w_i \in K$ (resp. F), if $\gamma_i \in \Lambda_1$ (resp. $\gamma_i \in \Lambda_2$), and $t_i \in L$ (resp. E), if $\alpha_i \in \Lambda_1'$ (resp. $\alpha_i \in \Lambda_2'$).

Proof. As in the L-split case first we may assume that g is in "generic" position and so by property 2 in Lemma 2 and by Proposition 1, it can be written in the form $g = t\, g_1 \cdots g_s$, where $t \in T$, $g_i \in H_i$, $i = 1, \dots, s$. Then the rest of the proof works exactly as in the L-split case, since by Lemma 3 all subgroups H_i are of the form $\mathrm{R}_{K'/K}(\mathrm{SL}(1, D))$, where D is a quaternion algebra over K' and K' is either F or K. $\square$

3. Some cohomological computations

From now on we assume that $\mathrm{vcd}\,(K) \le 1$ and we let $L = K(\sqrt{-1}\,)$. We also assume that the set Ω_K of all orderings on K is non-empty; this means, in particular, that $\mathrm{char}\, K = 0$. Recall ([Srl]) that there is a canonical topology on Ω_K under which Ω_K is compact and totally disconnected.

Remark 2. If $\Omega_K = \emptyset$, then -1 is a sum of squares in K and so $\mathrm{cd}\,(K) = \mathrm{cd}\,(K(\sqrt{-1})) \le 1$ ([S], Ch. 2, Prop. 10$'$). Therefore, if $\Omega_K = \emptyset$, then by Steinberg's theorem ([St2]) one has $H^1(K, G) = 1$ for any connected linear algebraic K-group G.

To reduce the proof of the Hasse principle to the case of simply connected semisimple groups we need two auxiliary cohomological statements (Propositions 2 and 4 below) which are very particular cases of the general Theorem 12.13 in [Sch2]. Since we do not need to consider such a generality as in [Sch2] we include here the straightforward proofs of these statements.

Let A be a discrete Γ-module, where $\Gamma = \mathrm{Gal}\,(\overline{K}/K)$, and let

$$\varphi_i : H^i(K, A) \to \prod_{\xi \in \Omega_K} H^i(K_\xi, A)$$

be the canonical map induced by res_{K_ξ}. We want to describe $\operatorname{Ker}\varphi_i$, $i \geq 2$, and $\operatorname{Im}\varphi_1$. To do so first remind that there is not a canonical way of choosing a real closure of K at $\xi \in \Omega_K$. If K_ξ and K'_ξ are two real closures of K at ξ, then by the theorem of Artin-Schreier ([Srl] Ch. 3, Theorem 2.1) there is a unique K-isomorphism $K_\xi \simeq K'_\xi$, hence there is an element $g \in \Gamma$ such that $g\tau_\xi g^{-1} = \tau'_\xi$, where τ_ξ (resp. τ'_ξ) is the involution (= element of order 2) in Γ corresponding to K_ξ (resp. K'_ξ) (in other words, there is a natural one-to-one correspondence between points of the set Ω_K and conjugacy classes of involutions in Γ).

The element g induces a natural map $\lambda_{i,g} : H^i(K_\xi, A) \to H^i(K'_\xi, A)$ and obviously we have $res_{K'_\xi} = \lambda_{i,g} \circ res_{K_\xi}$. It follows that the question on whether φ_i is injective does not depend on a choice of real closures K_ξ, $\xi \in \Omega_K$.

Clearly, any cocycle from $Z^1(K_\xi, A)$ is determined by the single element $a \in A$ such that $a\tau_\xi(a) = 1$. We will say that an element $\{a_\xi\}_{\xi\in\Omega_K} \in \prod_{\xi\in\Omega_K} H^1(K_\xi, A)$ is locally constant if there are a decomposition $\Omega_K = U_1 \cup \ldots \cup U_l$ into disjoint clopen (= open and closed) sets and elements $\{a_1, \ldots , a_l\}$ of A for which the following condition holds: for any $\xi \in U_i$ there are a cocycle c_ξ representing a_ξ and $g_\xi \in \Gamma$ such that the cocycle $\lambda_{1,g_\xi}(c_\xi)$ is determined by a_i. Analogously, for any $i \geq 1$ one defines the subset of elements in $\prod_{\xi\in\Omega_K} H^i(K_\xi, A)$ which are locally constant. We denote this subset by $\left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lc}$. Since for any $\zeta \in H^i(K, A)$ the element $\varphi_i(\zeta)$ is locally constant we denote by the same letter the canonical map

$$\varphi_i : H^i(K, A) \longrightarrow \left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lc} \subset \prod_{\xi\in\Omega_K} H^i(K_\xi, A)$$

Proposition 2. *If A is a finite discrete Γ-module, then the maps φ_i are injective for all integers $i \geq 2$.*

Proof. Since $H^i(L, A) = 1$, $i \geq 2$, the *"res-cores"* argument shows that $H^i(K, A)$ has exponent 2. So replacing A, if necessary, by its 2-Sylow subgroup we may assume that A is a 2-group. First examine the case $A = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4. *Let $A = \mathbb{Z}/2\mathbb{Z}$. Then φ_i is surjective if $i \geq 1$ and injective if $i \geq 2$.*

Proof. Recall ([L], §17) that a field F is said to be an SAP field (strong approximation property) if for any two disjoint closed subsets $A, B \subset \Omega_F$ there exists an element $f \in F$ such that f is positive at all orderings in A, but negative at all orderings in B. We need

Proposition 3. ([L], Theorem 17.9) *If* $\operatorname{vcd}(K) \leq 1$, *then K is a SAP field.*

Surjectivity of φ_i, $i \geq 1$. In view of the periodicity of $H^i(K_\xi, \mathbb{Z}/2\mathbb{Z})$ it suffices to consider the cases $i = 1, 2$. If $i = 1$ then $H^1(K, \mathbb{Z}/2\mathbb{Z}) = K^*/K^{*2}$, hence the surjectivity of φ_1 follows immediately from Proposition 3. Furthermore, any element from $H^2(K, \mathbb{Z}/2\mathbb{Z})$ splits over L and so can be represented by a quaternion algebra having L as a maximal subfield. Then clearly, the surjectivity of φ_2 again follows from Proposition 3.

Injectivity of φ_i, $i \geq 2$. The proof is similar to that of [B-P], Lemma 2.3. Namely, by Arason's theorem ([A1], Satz 3), local triviality of $\zeta \in H^i(K, \mathbb{Z}/2\mathbb{Z})$ implies that $\zeta \cup (-1)^r = 0$ for some integer r, where $\cup$ denotes the cup product. On the other hand from the exact sequence

$$H^i(L,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{cor} H^i(K,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup(-1)} H^{i+1}(K,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{res} H^{i+1}(L,\mathbb{Z}/2\mathbb{Z})$$

([A2], Corollary 4.6) and from the equalities

$$H^i(L,\mathbb{Z}/2\mathbb{Z}) = H^{i+1}(L,\mathbb{Z}/2\mathbb{Z}) = 1, \quad i \geq 2$$

we conclude that the product $\cup(-1)$ is an isomorphism. Therefore, $\zeta = 1$, as required. Lemma 4 is proved. □

We come back to an arbitrary finite 2-primary module A. Let Γ_2 be a Sylow 2-subgroup of Γ. Since the restriction map $H^i(K,A) \to H^i(\Gamma_2,A)$ is injective, after replacing Γ by Γ_2 we may assume that Γ is a pro-2-group. But for such a group any irreducible module is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ ([S], §4, Proposition 20). Therefore there exists a submodule $A' \subset A$ such that $A/A' = \mathbb{Z}/2\mathbb{Z}$. It induces the commutative diagram

$$\begin{array}{ccccc}
H^i(K,\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^{i+1}(K,A') & \longrightarrow \\
\Big\downarrow \theta_1 & & \Big\downarrow \theta_2 & \\
\left(\prod\limits_{\xi\in\Omega_K} H^i(K_\xi,\mathbb{Z}/2\mathbb{Z})\right)^{lc} & \longrightarrow & \left(\prod\limits_{\xi\in\Omega_K} H^{i+1}(K_\xi,A')\right)^{lc} & \longrightarrow \\
H^{i+1}(K,A) & \longrightarrow & H^{i+1}(K,\mathbb{Z}/2\mathbb{Z}) & \\
\Big\downarrow \theta_3 & & \Big\downarrow \theta_4 & \\
\left(\prod\limits_{\xi\in\Omega_K} H^{i+1}(K_\xi,A)\right)^{lc} & \longrightarrow & \left(\prod\limits_{\xi\in\Omega_K} H^{i+1}(K_\xi,\mathbb{Z}/2\mathbb{Z})\right)^{lc} &
\end{array}$$

By what has been proved above, θ_1 (resp. θ_4) is surjective (resp. injective) and by induction, θ_2 is injective. It follows that θ_3 is injective as well. Proposition 2 is proved. □

PROPOSITION 4. *If A is a finite discrete Γ-module, then φ_1 is surjective.*

PROOF. Since φ_i, $i \geq 2$, are injective, one can easily verify that if the statement holds both for a submodule $A' \subset A$ and the quotient A/A', then it also holds for A. So we may assume, if necessary, that A is irreducible. It suffices to prove that for a given $\xi \in \Omega_K$ and an element $a \in A$ for which $a\tau_\xi(a) = 1$ there exist a small clopen neighbourhood $U \subset \Omega_K$ of ξ and a cocycle $\zeta \in Z^1(K,A)$ such that for a proper real closure $K_{\xi'}$ of K at ξ' the cocycle $\mathrm{res}_{K_{\xi'}}(\zeta)$ is determined by the element a if $\xi' \in U$, and is trivial otherwise.

We need the following simple property of orderings of K (see [Srl]):

if F/K is an extension of odd degree then for any ordering $\xi \in \Omega_K$ there is an extension of ξ to F; moreover, the restriction map $\phi : \Omega_F \to \Omega_K$ is a local homeomorphism.

Let E be a finite Galois extension of K over which A is a trivial module and let $F \subset E$ be the subfield corresponding to a Sylow 2-subgroup of $\mathrm{Gal}\,(E/K)$. Denote $\Delta = \mathrm{Gal}\,(\overline{K}/F)$. Let $\phi^{-1}(\xi) = \{\xi_1, \ldots, \xi_t\} \subset \Omega_F$, where, as above, $\phi : \Omega_F \to \Omega_K$ is the restriction map.

By construction, $\phi(\xi_i) = \xi$. So we can pick a small clopen neighbourhood $U \subset \Omega_K$ of ξ and disjoint small clopen neighbourhoods $U_i \subset \Omega_F$ of ξ_i, $i = 1, \dots, t$, such that the restriction map $\phi|_{U_i} : U_i \to U$ is a homeomorphism and $\phi^{-1}(U) = U_1 \cup \dots \cup U_t$. Taking smaller neighbourhoods, if necessary, one can additionally assume that for any $\xi' \in U_1$ there is an involution $\tau_{\xi'} \in \Delta$ corresponding to ξ' for which the following property holds:

$$\begin{aligned}&\text{if } g \in \Gamma \backslash \Delta \text{ be such that } \tilde{\tau}_{\xi'} = g\,\tau_{\xi'}\,g^{-1} \in \Delta \text{ then the point of } \Omega_F \\ &\text{corresponding to the involution } \tilde{\tau}_{\xi'} \text{ does not lie in } U_1.\end{aligned} \tag{9}$$

Indeed, let $I_\Delta \subset \Delta$ be a subset of involutions and $\tau \in I_\Delta$ be an involution which corresponds to ξ_1. Assume the contrary. Since I_Δ, Γ are compact and totally disconnected there exist then in Δ a sequence of involutions $(\tau_1, \tau_2, \dots)$ converging to τ and a converging sequence of elements $(g_1, g_2, \dots)$ in $\Gamma \backslash \Delta$ such that $g_i\,\tau_i\,g_i^{-1} \in \Delta$. Letting $g = \lim g_i$, one has $g \in \Gamma \backslash \Delta$ and $\tau' = g\,\tau\,g^{-1} \in \Delta$. But by assumption, the point ξ' of Ω_F corresponding to τ' lies in U_1 and $\phi(\xi') = \xi$. This means that $\xi' = \xi_1$, hence there is $\delta \in \Delta$ such that $\tau' = \delta\,\tau\,\delta^{-1}$, implying $g^{-1}\delta$ lies in the centralizer $C_\Gamma(\tau)$. But every involution in Γ is self-centralizing, i.e. $C_\Gamma(\tau) = \langle \tau \rangle$,– a contradiction.

The map φ_1 is clearly surjective for the field F, since A can be viewed as $\mathrm{Gal}\,(E/F)$-module and $\mathrm{Gal}\,(E/F)$ is a 2-group, implying that any irreducible $\mathrm{Gal}\,(E/F)$-module is of the form $\mathbb{Z}/2\mathbb{Z}$. Therefore, we can pick $\zeta' \in Z^1(F, A)$ such that for proper real closures the cocycle $res_{F_{\xi'}}(\zeta')$ is determined by the element a if $\xi' \in U_1$ and is trivial otherwise. We claim that the cocycle $\zeta = cor\,_K^F\,(\zeta')$ has the same property. To verify it we need

PROPOSITION 5. ([Br], Ch. III, Proposition 9.5) *Let A be a Γ-module and $\Theta \subset \Delta \subset \Gamma$ be subgroups. If $[\Gamma : \Delta] < \infty$ and $z \in H^*(\Delta, A)$ then we have*

$$\operatorname{res}_\Theta^\Gamma \circ \operatorname{cor}_\Delta^\Gamma(z) = \sum_{g \in \Lambda} \operatorname{cor}_{\Theta \cap g\Delta g^{-1}}^\Theta \circ \operatorname{res}_{\Theta \cap g\Delta g^{-1}}^{g\Delta g^{-1}}(\hat{g}(z)),$$

where Λ is a set of representatives of double cosets $\Theta\,g\,\Delta$ and

$$\hat{g} : H^*(\Delta, A) \to H^*(g\,\Delta\,g^{-1}, A)$$

is the natural map induced by pair $(\operatorname{int}(g^{-1}),\ g)$.

To prove our claim first take $\eta \in U$. Let $\xi' = \phi^{-1}(\eta) \cap U_1$ and let $\tau_{\xi'} \in \Delta$ be an involution corresponding to ξ' and satisfying (9). Then applying Proposition 5 we have

$$\operatorname{res}_{K_{\xi'}}(\zeta) = \sum \operatorname{res}_{\Theta_{\xi'} \cap g\Delta g^{-1}}^{g\Delta g^{-1}}(\hat{g}(\zeta')) = \sum \operatorname{res}_{g^{-1}\Theta_{\xi'}\,g \cap \Delta}^{\Delta}(\zeta') = \operatorname{res}_{\Theta_{\xi'}}^{\Delta}(\zeta')$$

where $\Theta_{\xi'} = \langle \tau_{\xi'} \rangle$, hence $\operatorname{res}_{K_{\xi'}}(\zeta)$ is defined by a. Analogously, one shows that $\operatorname{res}_{K_\eta}(\zeta)$ is trivial if $\eta \notin U$. Proposition 4 is proved. □

COROLLARY 2. *Let A be a commutative connected linear algebraic K-group. Then φ_2 is injective.*

PROOF. One has $H^i(L, A) = 1$, $i \geq 1$. So $H^i(K, A)$ has exponent 2 and hence the map $H^i(K, {}_2A) \to H^i(K, A)$ is surjective, where ${}_2A$ consists of all elements of A killed by 2. By Proposition 4, it gives the surjectivity of φ_1 for A. Then the result follows from the injectivity of φ_2 for ${}_2A$. □

Corollary 3. *The Hasse principle holds for algebraic K-tori.*

Proof. Let T be a K-torus. There exists K-quasi-split torus S and its connected K-subtorus H such that $T = S/H$. Then the commutative diagram

$$\begin{array}{ccccc} H^1(K,S)=1 & \longrightarrow & H^1(K,T) & \longrightarrow & H^2(K,H) \\ \Big\downarrow{\scriptstyle\theta_1} & & \Big\downarrow{\scriptstyle\theta_2} & & \Big\downarrow{\scriptstyle\theta_3} \\ \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,S)=1 & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,T) & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^2(K_\xi,H) \end{array}$$

shows that the injectivity of θ_2 follows from that of θ_3. □

4. The Hasse principle for principal homogeneous spaces

Let us keep the notations of § 3. In particular, we assume that K is a field with $\operatorname{vcd}(K) \le 1$, $L = K(\sqrt{-1}\,)$ and $\Omega_K \neq \emptyset$. Let also τ be the non-trivial element of $\operatorname{Gal}(L/K)$. Using the results of the previous sections we may produce a simple proof of the triviality of the kernel of (2).

a) Let G' be a connected linear algebraic K-group, $Z \le G'$ be a finite central K-subgroup and let $G = G'/Z$.

Lemma 5. *If the Hasse principle holds for G' then it also holds for G.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} H^1(K,Z) & \longrightarrow & H^1(K,G') & \longrightarrow & H^1(K,G) & \xrightarrow{\lambda_1} \\ \Big\downarrow{\scriptstyle\theta_1} & & \Big\downarrow{\scriptstyle\theta_2} & & \Big\downarrow{\scriptstyle\theta_3} & \\ \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,Z) & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,G') & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,G) & \xrightarrow{\lambda_2} \\ & & & \xrightarrow{\lambda_1} & H^2(K,Z) & \\ & & & & \Big\downarrow{\scriptstyle\theta_4} & \\ & & & \xrightarrow{\lambda_2} & \prod\limits_{\xi\in\Omega_K} H^2(K_\xi,Z) & \end{array}$$

By assumption and by Proposition 2, the maps θ_2, θ_4 are injective. Then from the above diagram and from Proposition 4 we have $\operatorname{Ker}\theta_3 = 1$. □

b) *Reduction to semisimple groups.* Since unipotent K-groups have trivial cohomology we may assume without loss of generality that G is reductive. Then $G = T \cdot H$ is an almost direct product of the central torus T and the semisimple group $H = [G,G]$. Let $G' = T \times H$. Clearly, the kernel of the natural morphism $G' \to G$ is finite and by induction and by Corollary 3, the Hasse principle holds for H and T. So by Lemma 5, it holds for G as well.

c) *Reduction to simple simply connected groups.* One can again apply Lemma 5 to a simply connected covering G' of G.

d) Let G be an (absolutely) simple simply connected K-group. By Steinberg's theorem ([St2]), G has a Borel subgroup B over L. We may assume that $T = B \cap \tau(B)$ is a maximal K-torus of G. Since $H^1(L,G) = 1$, the map $H^1(L/K,G(L)) \to H^1(K,G)$ is surjective. By Lemma 6.28 [Pl-R], the map $H^1(L/K,T(L)) \to H^1(L/K,G(L))$ is surjective as well, hence any class $[\zeta] \in$

$H^1(K,G)$ can be represented by a cocycle $\zeta' \in Z^1(L/K,T(L))$. Let S be a maximal K-split subtorus of T.

First let $S \neq 1$. Then $C_G(S)$ is a proper connected subgroup of G. Since $C_G(S)$ is a reductive part of some parabolic K-subgroup, one has $\operatorname{Ker}(H^1(E,C_G(S)) \to H^1(E,G)) = 1$ for any extension E/K ([Pr-R], Lemma 5.1). So if in addition $\zeta \in \operatorname{Ker}\theta$, then for each $\xi \in \Omega_K$ the element $\operatorname{res}_{K_\xi}(\zeta')$ is trivial as an element of $H^1(K_\xi, C_G(S))$, hence the claim follows by induction.

e) $S = 1$*, i.e.* T *is a* K*-anisotropic torus.* By Steinberg's theorem, G is either split or quasi-split over L. We examine the L-splitting case only, since the L-quasi-splitting case can be handled analogously. Identify $Z^1(\Theta,T(L))$ with $(K^*)^n$. Arguing as in d) we get that any element from $\operatorname{Ker}\theta$ can be represented by a cocycle $\zeta \in Z^1(\Theta,T(L))$. We claim that there exist a maximal K-torus $T' \subset G$ isomorphic to T over K and a cocycle $\zeta' \in Z^1(\Theta,T'(L))$ equivalent to ζ in $Z^1(\Theta,G(L))$ such that ζ' is everywhere locally positive. By Corollary 3, the last would mean that ζ' is trivial as an element of $H^1(\Theta,T'(L))$, hence ζ is trivial in $H^1(\Theta,G(L))$ as well.

To show it, we proceed as in Theorem 2. Namely, we construct inductively quaternion algebras $D_1,\dots,D_m$ over K and elements $g_i \in G_{\beta_i}(L)$ such that for $g = g_1\cdots g_m$ the element $g^{1-\tau} \in T(L)$ and the components of the cocycles $(g^{1-\tau})$ and ζ everywhere locally have the same signs.

As in Theorem 2, we begin with $D_m = (-1, d_{\beta_m})$, where $d_{\beta_m} = c_{\beta_m}$. For $\xi \in \Omega_K$ let $g_\xi \in G(\overline{K}_\xi)$ be such that $\zeta = (g_\xi^{1-\tau})$ (note that T is still anisotropic over K_ξ). We may assume that g_ξ is in "generic" position and so we may write g_ξ as a product $g_\xi = t_\xi\, g_{\xi,1}\cdots g_{\xi,m}$, where $t_\xi \in T$, $g_{\xi,i} \in G_{\beta_i}$, $i = 1,\dots,m$.

We have already known that $\tau(g_{\xi,m}) = h_{\beta_m}(w_{\xi,m})\, g_{\xi,m}$ for some parameter $w_{\xi,m} \in K_\xi$. By virtue of the facts that our field K has the property SAP and the Hasse principle holds for groups of type A_1 ([B-P], [Sch1]) we can pick $w_m \in K$, which has everywhere locally the same sign as $w_{\xi,m}$, and $g_m \in G_{\beta_m}(L)$ such that $h_{\beta_m}(w_m) = g_m^{1-\tau}$.

Next consider the quaternion K-algebra $D_{m-1} = (-1, d_{\beta_{m-1}})$, where

$$d_{\beta_{m-1}} = c_{\beta_{m-1}} w_m^{\langle \beta_{m-1},\beta_m\rangle}.$$

Let $w_{\xi,m-1} \in K_\xi$ be such that $h_{\beta_{m-1}}(w_{\xi,m-1})\, h_{\beta_m}(w_{\xi,m}) = (g_{\xi,m-1}\, g_{\xi,m})^{1-\tau}$ Again we can pick $w_{m-1} \in K$ such that for all $\xi \in \Omega_K$ the elements w_{m-1} and $w_{\xi,m-1}$ have the same sign. By construction, the equation $h_{\beta_{m-1}}(w_{m-1})\, h_{\beta_m}(w_m) = (x\, g_m)^{1-\tau}$, where $x \in G_{\beta_{m-1}}(L)$, has solution everywhere locally, so it has solution g_{m-1} globally, and so on.

Thus, there exists $g \in G(L)$ such that the components of both cocycles $(g\tau(g^{-1}))$ and ζ have the same signs in K_ξ for each $\xi \in \Omega_K$. To complete the proof of the theorem it remains to notice that the cocycle $\zeta' = \tau(g)^{-1}\zeta\, g$ is equivalent to ζ in $Z^1(\Theta,G(L))$, takes values in the K-defined and L-splitting torus $T' = \tau(g)^{-1}T\tau(g)$ and ζ' is everywhere locally positive. $\square$

Remark 3. The same argument shows that θ is still injective if we replace Ω_K by a dense set of orderings.

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Partition Regular Systems of Linear Inequalities

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Introduction

In 1930 Ramsey published his paper *On a problem in formal logic* [12]. He established a result, nowadays known as Ramsey's Theorem:

> Let k and r be positive integers. Then for every r-coloring of the k-element subsets of ω there exists an infinite subset $S \subseteq \omega$ such that all k-element subsets of S are colored the same.

Already in 1927 van der Waerden published his theorem on arithmetic progressions [15]. He proved that for every coloring of the natural numbers with finitely many colors there exists a monochromatic arithmetic progression of given length. Van der Waerden's result can be seen in the context of Schur's investigations [14] on the distribution of quadratic residues and nonresidues. Schur knew about the existence of monochromatic solutions of $x + y = z$. He worked on such problems in order to resolve Fermat's conjecture, which was proved by Wiles in 1994.

The above mentioned work of Ramsey [12] and van der Waerden [15] gave rise to the part of discrete mathematics, known as Ramsey Theory or Partition Theory. An important contribution was made by Rado [10] in 1933. Working on his dissertation, supervised by Schur, he was able to prove a common generalization of Schur's and van der Waerden's results by introducing the concept of *regularity*: A system of linear equations $A\vec{x} = \vec{0}$ is called regular over a ring R if it has monochromatic solutions for every coloring of R with finitely many colors. In his *Studien zur Kombinatorik* (1933) [10] Rado gave a complete characterization of all regular systems of linear equations over the rational numbers. The property Rado used in order to describe regular systems of linear equations is an syntactical property of the matrix. It is

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characterized by certain linear dependences of the columns of the matrix A and is called *column property.*

It is possible to generalize the concept of regularity to systems of linear inequalities. We call a system of linear inequalities $A\vec{x} \leq \vec{0}$ partition regular if for every coloring of the natural numbers with finitely many colors there exists a monochromatic solution of $A\vec{x} \leq \vec{0}$. Rado considered systems of linear inequalities only incidentally. He stated the following proposition which is easy to prove:

> Let the system $\sum_{j=1}^{n} a_{ij}x_j = 0, \quad 1 \leq i \leq m$ be partition regular and assume that the following system of inequalities has a solution in the natural numbers:
>
> $$(*) \qquad \sum_{j=1}^{n} a_{ij}x_j \begin{cases} = 0 & \text{for} \quad 1 \leq i \leq m_1, \\ > 0 & \text{for} \quad m_1 < i \leq m. \end{cases}$$
>
> Then also $(*)$ is partition regular.

Of course this observation is far away from being a characterization of partition regular systems of inequalities but it can be taken as a starting point for our investigations.

The characterization of all partition regular systems of linear inequalities is a central goal of this paper. In the first chapter we define a generalized column property called *cpi*, which can be used to characterize partition regular systems of linear inequalities. It is an interesting feature of Rado's proof that the linear system $A\vec{x} = \vec{0}$ is already regular if there exists a monochromatic solution with respect to one (number theoretic) type of coloring. Systems of inequalities let things tend to be more difficult.

Several years after finishing his *Studien zur Kombinatorik*, Rado [11] considered systems of linear equations with coefficients in $\mathbb{R}$ and he also extended the set of partitioned numbers to the field of real numbers. It turned out that it is possible to carry over the previous results from the natural numbers to the reals. We will show in chapter 1 that our arguments can also be used if we consider real systems of inequalities partitioning the set of reals.

As well as for homogeneous systems the column property can be used to describe partition regularity of inhomogeneous systems of inequalities. We will give a complete characterization of those systems which are partition regular, over the natural numbers, over the set of integers and over the rationals.

The column property for systems of inequalities as well as the column property in the sense of Rado is a syntactical property of the matrix and does not explicitly refer to the set of solutions of the system. In 1973 Deuber [1] gave a semantical characterization of partition regular systems of equations. The approach is by a description of the arithmetic structure of the sets of solutions of regular linear systems $A\vec{x} = \vec{0}$. The central notion is the one of $(m, p, c)-$sets. He proved the following theorem:

> A system $A\vec{x} = \vec{0}$ is partition regular if and only if there exist positive integers m, p, c such that every $(m, p, c)-$set contains a solution of $A\vec{x} = \vec{0}$.

In chapter two we will show that $(m, p, c)-$sets can also be used to characterize solution spaces of partition regular systems of linear inequalities.

Starting with results of Erdös and Rado [4] another part of partition theory was developed, which is nowadays known as *Canonical Ramsey Theory.* In Canonical Ramsey Theory one considers colorings with no restriction on the number of colors. The first result is a canonical version of Ramsey's theorem. Later Erdös and Graham [3] proved a generalization of van der Waerden's theorem:

> For every coloring Δ of the natural numbers with arbitrary many colors there exists an arithmetic progression, which is colored monochromatic or injective with respect to Δ.

A canonical analogue of the Rado-Deuber-Theorem on regular systems of equations and (m, p, c)-sets was proved by Lefman [7]. His result states:

> Let $A\vec{x} = \vec{0}$ be a partition regular system of linear equations. For every coloring Δ of the natural numbers with arbitrary many colors there exists a solution of the system $A\vec{x} = \vec{0}$ such that Δ restricted to this solution is either monochromatic, injective or a *block-coloring.*

The third case is related to the partitioning of the columns of A into blocks, corresponding to the column property and to the rows of the (m, p, c)-sets. In chapter 3. we prove a canonical partition theorem for systems of inequalities.

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1. Systems of Homogeneous, Linear Inequalities

Notations By $\mathbb{N} = \{1, 2, 3, \ldots\}$ we denote the set of positive integers; $[n] = \{1, 2, \ldots, n\}$ is the set of the natural numbers less or equal than n. A matrix A with m rows and n columns is denoted by $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$, where a_{ij} is the entry of A which belongs to the ith row and jth column. For $i, j \le n$ the jth column of a matrix A is denoted by $a^{(j)}$ the ith row by $a_{(i)}$. For a matrix $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ the system

$$\sum_{j=1}^{n} a_{ij}x_j \le 0, \quad 1 \le i \le m$$

is abbreviated as $A\vec{x} \le \vec{0}$. For a given matrix $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$, $k \le n$ and $\epsilon > 0$ by $A^k(\epsilon) = (a^k_{ij}(\epsilon))_{1\le i\le m, 1\le j\le n}$ we denote the following matrix:

$$\begin{pmatrix} a_{11} & \ldots & a_{1k-1} & a_{1k} - \epsilon & a_{1k+1} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mk-1} & a_{mk} - \epsilon & a_{mk+1} & \ldots & a_{mn} \end{pmatrix},$$

obtained from A by subtracting ϵ in column k.

For $k, l \in [n]$, $k < l$ and $\epsilon > 0$ the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k-1} & a_{1k}-\epsilon & a_{1k+1} & \dots & a_{1l-1} & a_{1l+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk-1} & a_{mk}-\epsilon & a_{mk+1} & \dots & a_{ml-1} & a_{ml+1} & \dots & a_{mn} \end{pmatrix}$$

obtained by deleting column l in $A^k(\epsilon)$, is denoted by $A^k_l(\epsilon)$ and the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1k-1} & a_{1k}+a_{1l}-\epsilon & a_{1k+1} & \dots & a_{1l-1} & a_{1l+1} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mk-1} & a_{mk}+a_{ml}-\epsilon & a_{mk+1} & \dots & a_{ml-1} & a_{ml+1} & \dots & a_{mn} \end{pmatrix},$$

obtained from $A^k(\epsilon)$ by adding the kth and the lth column, is denoted by $A^{(k)+(l)}(\epsilon)$.

Rado considered systems of linear equations over $\mathbb{Q}$. In his paper, published in 1933 [10], Rado gives a characterization of all systems of linear homogeneous equations which have for every coloring of the natural numbers with finitely many colors a solution in one color class. Rado called those systems regular. The central definition in this context is the following:

Definition 1.1. *Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be a matrix with m rows an n columns and with entries $a_{ij} \in \mathbb{Z}$. A has the column property if there exists $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \dots I_l$ of the column indices such that*

1. *for all $1 \le i \le m$ we have $\sum_{j\in I_0} a_{ij} = 0$ and*
2. *for all $k < l, j \in \cup_{s\le k} I_s$ there exist $c_k, c_{kj} \in \mathbb{N}$ such that for all $1 \le i \le m$ we have*
$$\sum_{j\in\cup_{s\le k} I_s} c_{jk}a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} = 0.$$

Rado proved the following theorem:

Theorem 1.1. (Rado 1933) *A system of homogeneous linear equations $A\vec{x} = \vec{0}$ is regular if and only if A has the column property.*

In the following we will consider systems of linear inequalities rather than systems of linear equations. First we define partition regularity for systems of inequalities.

Definition 1.2. *Let $A = (a_{ij})_{1\le i\le m 1\le j\le n}$ be a rational matrix and let $\vec{b} = (b_1, \dots, b_m) \in \mathbb{Q}^m$. The system*

$$(*) \qquad \sum_{j=1}^{n} a_{ij}x_j \le b_i, \quad 1 \le i \le m$$

is called partition regular over $\mathbb{N}$ *if for every* $c \in \mathbb{N}$ *and every c-coloring of the natural numbers* $\Delta : \mathbb{N} \to [c]$ *there exists a solution* $\vec{x} = (x_1, \dots x_n) \in \mathbb{N}^n$ *of* $(*)$ *such that* $\Delta\big|_{\{x_1,\dots,x_n\}} = const.$

In the following section we will give a characterization of all systems of homogeneous linear inequalities which are partition regular over $\mathbb{N}$. It turns out that a natural generalization of Rado's column property can be used to describe these systems.

DEFINITION 1.3. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix. A has the column property for systems of inequalities (abbreviated as cpi) over* $\mathbb{N}$ *if there exists* $l \in \mathbb{N}$ *and a partition* $[n] = I_0 \cup I_1 \dots \cup I_l$ *such that*

1. *for all* $1 \le i \le m$ *we have* $\sum_{j\in I_0} a_{ij} \le 0$ *and*
2. *for all* $k < l, j \in \cup_{s\le k} I_s$ *there exist* $c_k, c_{jk} \in \mathbb{N}$ *such that for all* $1 \le i \le m$ *we have*
$$\sum_{j\in\cup_{s\le k} I_s} c_{kj} a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} \le 0.$$

If a matrix A has the column property (in the sense of Rado) [10] the system $A\vec{x} \le \vec{0}$ obviously is partition regular. But there are many other systems of inequalities which are partition regular without A having Rado's column property. For example the matrix
$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$
has *cpi* but not the column property.

THEOREM 1.2. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix. The system of inequalities* $(*)$ $A\vec{x} \le \vec{0}$ *is partition regular over* $\mathbb{N}$ *if and only if* A *has cpi over* $\mathbb{N}$.

Both implications stated in theorem 1.5. are not completely trivial to prove. We start by showing that *cpi* implies partition regularity. This part of the proof proceeds along the general lines of the corresponding proof for systems of equations [10]. The following lemma combines arithmetic progressions and partition regular systems of linear inequalities:

LEMMA 1.1. *Let* $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ *be a rational matrix,* $A\vec{x} \le \vec{0}$ *a partition regular system of inequalities and let* $p \in \mathbb{N}$. *Then for every* $c \in \mathbb{N}$ *and every c-coloring* $\Delta : \mathbb{N} \to [c]$ *there exists* $\vec{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ *and* $d \in \mathbb{N}$ *such that*

1. $A\vec{x} \le \vec{0}$ *and*
2. *for all* $i, j \le n$, *for all* $k, l \le p$ *we have* $\Delta(x_i + ld) = \Delta(x_j + kd)$.

PROOF OF LEMMA 1.1.: $A\vec{x} \le \vec{0}$ is partition regular. Thus by compactness [6] for every $c \in \mathbb{N}$ there exists $N^* = N^*(c) \in \mathbb{N}$ such that for every c-coloring $\Delta : [N^*] \to [c]$ there exists a monochromatic solution $\vec{x} = (x_1 \dots x_n)$ of $A\vec{x} \le \vec{0}$ such that for all $1 \le i \le n$ we have $x_i \le N^*$.

Let $\Delta : \mathbb{N} \to [c]$ be an arbitrary c-coloring. Define the following coloring $\Delta^* : \mathbb{N} \to [r^{N^*}]$ by

$$\Delta^*(x) = (\Delta(ix))_{1 \le i \le N^*}.$$

By van der Waerden's theorem [15] there exists a "long" arithmetic progression which is monochromatic with respect to Δ^*, i. e. there exist $a', d' \in \mathbb{N}$ such that for all $l \le pN^{*^{n-1}}$ we have $\Delta^*(a' + ld')$ =const.
Define $\Delta^{**} : \mathbb{N} \to [c]$ by

$$\Delta^{**}(x) = \Delta(a'x).$$

By the choice of N^* there exists a solution $\vec{x'} = (x'_1, \ldots, x'_n) \in [N^*]^n$ of $A\vec{x} \le \vec{0}$ which is monochromatic for Δ^*. For all $i \le n$ let $x_i = x'_i a'$. By homogeneity $\vec{x} = (x_1, \ldots x_n)$ is a solution of $A\vec{x} \le \vec{0}$ and because of the definition of Δ^{**} for all $i, j \le n$ we have $\Delta(x_i a') = \Delta(x_j a')$.
Let $d = d'x'_1 \ldots x'_n$. Then for $i \le n$ and $l \le p$ we have:

$$x'_i a' + ld = x'_i(a' + ld'x'_1 \ldots x'_{i-1}x'_{i+1} \ldots x'_n).$$

Hence by the definition of a', d' and Δ^* for all $l \le p$ we have $\Delta(x'_i a' + ld)$ =const.
$\Box_{lemma}$ 1.6.

PROOF OF THEOREM 1.2. (FIRST PART): First we show that if A has *cpi* over $\mathbb{N}$ then $(*)$ is partition regular. We know by assumption that there is some $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \ldots \cup I_l$ such that

1. for all $1 \le i \le m$ we have $\sum_{j \in I_0} a_{ij} \le 0$ and
2. for all $k < l$, for all $j \in \cup_{s \le k} I_s$ there exist $c_{kj}, c_k \in \mathbb{N}$, such that for all $1 \le i \le m$ we have
$$\sum_{j \in \cup_{s \le k} I_s} c_{kj} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \le 0.$$

To prove that $(*)$ is partition regular we will use a double induction. We proceed by main induction on the number of colors c and by subsidiary induction on l, the number of column classes.

Let $A_k = (a_{ij})_{1 \le i \le m, j \in \cup_{s \le k} I_s}$ be the submatrix of A which only consists of the columns belonging to block 1 up to k. We will show by induction that for all $k \le l$ A_k is partition regular.

For $k = 0$ there is nothing to show because every singleton forms a solution of the system $A_0\vec{x} \le \vec{0}$. Assume that $A_k\vec{x} \le \vec{0}$ is partition regular for some $k \ge 0$ (which will be kept fix by now), i. e. (by compactness) for every $c \in \mathbb{N}$ there exists $R(c, A_k) \in \mathbb{N}$ such that for every c-coloring $\Delta : [R(c, A_k)] \to [c]$ there exists a monochromatic solution $(x_j)_{j \in \cup_{s \le k} I_s}$, such that $A_k\vec{x} \le \vec{0}$ and for all $j \in \cup_{s \le k} I_s$ we have $x_j \le R(c, A_k)$. We will show that A_{k+1} is partition regular, i. e. for all $c \in \mathbb{N}$ there exists $R(c, A_{k+1}) \in \mathbb{N}$

First we observe that $x_j = c_{jk}$ for $j \in \cup_{s\leq k} I_s$ and $x_j = c_k$ for $j \in I_{k+1}$ form a solution of the system $A_{k+1}\vec{x} \leq \vec{0}$. So we are done if only one color is used for the coloring, i. e. there exists $R(1, A_{k+1})$. Now assume that $R(c, A_{k+1})$ exists for some (fixed) $c \geq 1$. We will show that $R(c+1, A_{k+1})$ exists.

Let $\Delta : \mathbb{N} \to [c+1]$ be an arbitrary (c+1)-coloring. Use lemma 1.6. for the (by assumption) partition regular system $A_k\vec{x} \leq \vec{0}$ with
$p = R(c, A_{k+1}) \cdot (max_{j\in\cup_{s\leq k} I_s}\{c_{kj}\})$. Hence there exists $(y_j)_{j\in\cup_{s\leq k} I_s}$, such that for all $1 \leq i \leq m$ we have

$$\sum_{j\in\cup_{s\leq k} I_s} a_{ij} y_j \leq 0$$

and there exists $d \in \mathbb{N}$ such that for all $j \in \cup_{s\leq k} I_s$ and $t \leq p$ we have

$$\Delta(y_j + td) = const.$$

for all $1 \leq i \leq m$ and $t \in [R(c, A_{k+1})]$ it follows

$$\begin{aligned} & \sum_{j\in\cup_{s\leq k} I_s} (y_j + c_{kj}td)a_{ij} + \sum_{j\in I_{k+1}} c_k t d a_{ij} \\ = & \sum_{j\in\cup_{s\leq k} I_s} y_j a_{ij} + td\Big(\sum_{j\in\cup_{s\leq k} I_s} c_{kj} a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij}\Big) \quad \leq 0. \end{aligned}$$

Further for all $j \in \cup_{s\leq k} I_s$ and $t \leq p$ we have

$$\Delta(y_j + c_{kj}td) = const.$$

Say $\Delta(y_j + c_{kj}td) = c+1$.

We distinguish the following cases:

1. There exist $t \in [R(c, A_{k+1})]$ such that $\Delta(c_k td) = c+1$. Then we are done.

2. For all $t \in [R(c, A_{k+1})]$ the relation $\Delta(c_k td) \in [c]$ holds. Then consider the c-coloring: $\Delta' : [R(c, A_{k+1})] \to [c]$ which is defined by

$$\Delta'(x) = \Delta(c_k x d).$$

   By definition of $R(c, A_{k+1})$ there exists a solution $(t_j)_{j\in\cup_{s\leq k} I_s}$ of the system $A_{k+1}\vec{x} \leq \vec{0}$ which is monochromatic for Δ'. Hence $(c_k d t_j)_{j\in\cup_{s\leq k+1} I_s}$ forms a solution of $A_{k+1}\vec{x} \leq \vec{0}$ which is monochromatic with respect to Δ.

$\Box_{theorem \quad 1.2.(first \quad part)}$

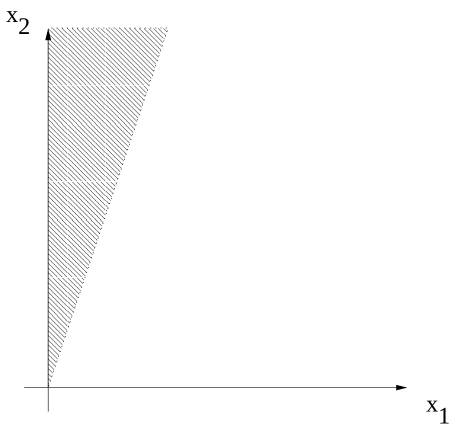
In order to demonstrate the structure of the proof of the second part of theorem 1.5. we will give a short overview. For his characterization of regular systems of linear equations Rado [10] had to prove that for each systems $A\vec{x} = \vec{0}$, which is regular, A has the column property. It is an interesting feature of Rado's proof that a system $A\vec{x} = \vec{0}$ is regular if there exists a monochromatic solution with respect to one type of coloring. For systems of linear inequalities $A\vec{x} \leq \vec{0}$ with A having only two columns there also exists a certain type of coloring such that $A\vec{x} \leq \vec{0}$ is partition regular if it has a monochromatic solution with respect to this type of coloring. In lemma 1.12. we will show, that a system $(*)\quad a \leq \frac{x_1}{x_2} \leq b$, where $a, b \in \mathbb{Q}$ and $1 < a \leq b$, is not partition regular. It is easy to see that essentially each system $A\vec{x} \leq \vec{0}$ with A having only two columns can be transformed into a system $(*)$ for suitable a and b. If such a system is partition regular this means that one of the following cases holds:

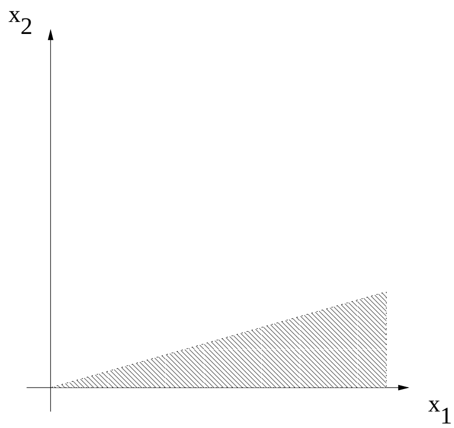
1. $a \leq 0$ and $b > 0$ or
2. $a \leq 1$ and $b \geq 1$.

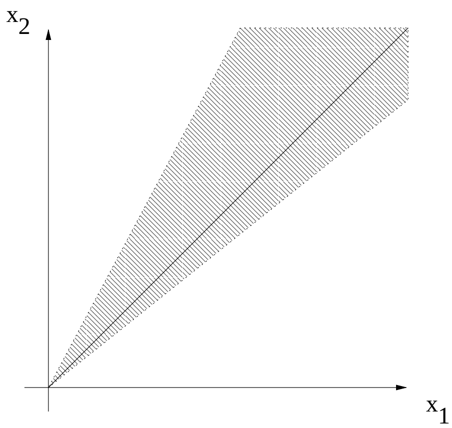
It is not difficult to see that these conditions exactly lead to *cpi*. If we visualize a partition regular system

$$(**)\begin{cases} a_{11}x_1 + a_{12}x_2 \leq 0 \\ a_{21}x_1 + a_{22}x_2 \leq 0 \end{cases}$$

geometrically then obviously the solutions are bounded by two straight lines. Three typical cases occur, i.e. one of the axes is a limiting line or the diagonal is contained in the solution space:







We will prove theorem 1.5. by induction on the number of columns of A. In order to start the induction we described the situation for $n = 2$. Let us consider a rational matrix A with n columns. Assume that the system

$$A\vec{x} \leq \vec{0} \quad (***)$$

is partition regular. Under certain assumptions we can transform the system $A\vec{x} \leq \vec{0}$ for each choice of k, l with $1 \leq k < l \leq n$ into the following system:

$$-\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\neq l,k}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l} \leq \frac{x_k}{x_l} \leq -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\neq l,k}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l}$$

for all s with $a_{sk} < 0$ and for all t with $a_{tk} > 0$. Thus we have a similar situation as in $(*)$ except that the fraction $\frac{x_k}{x_l}$ is not bounded by constant terms a and b but by terms which depend on $x_1 \ldots x_{k-1}, x_{k-2}, \ldots x_n$. Thus we cannot directly apply lemma 1.12. Consider this situation for fixed k and l. Assume that there are colorings of the natural numbers with finitely many colors such that for each monochromatic solution $x_1, \ldots x_n$ of the system $(***)$ either

1. there exists $\epsilon_1 > 0$ and $r \in \mathbb{N}$ such that $1 + \epsilon_1 \leq \frac{x_k}{x_l} \leq r$ or
2. there exists $\epsilon_2 > 0$ and $\epsilon_3 > 0$ such that $\epsilon_2 \leq \frac{x_k}{x_l} \leq 1 - \epsilon_3$.

Then again by lemma 1.12. $(***)$ cannot be partition regular. To avoid such situations the terms $-\frac{a_{sl}}{a_{sk}} - \sum_{j=1,j\neq l,k}^{n} \frac{a_{sj}}{a_{sk}} \frac{x_j}{x_l}$ and $-\frac{a_{tl}}{a_{tk}} - \sum_{j=1,j\neq l,k}^{n} \frac{a_{tj}}{a_{tk}} \frac{x_j}{x_l}$ have to fulfill certain conditions for every coloring. This is what is shown in lemma 1.13. With this kind of arguments it is possible to show that for every choice of k and l with $1 \leq k < l \leq n$ either for all $\epsilon > 0$ the system $A_l^k(\epsilon)$ is partition regular or for all $\epsilon > 0$ the system $A_k^l(\epsilon)$ is partition regular, if the system $A\vec{x} \leq \vec{0}$ is partition regular. By induction we can conclude that either for all $\epsilon > 0$ the matrix $A_l^k(\epsilon)$ has *cpi* or for all $\epsilon > 0$ the matrix $A_k^l(\epsilon)$ has *cpi*. Therefore we define:

Definition 1.4. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix. A has the ϵ-property if the following conditions are satisfied:*

1. *The system $A\vec{x} \leq \vec{0}$ has a solution in the natural numbers and*
2. *For all $1 \leq k < l \leq n$ one of the following conditions is satisfied:*
   (a) *For all $\epsilon > 0$ the matrix $A^k(\epsilon)$ has cpi over $\mathbb{N}$,*
   (b) *for all $\epsilon > 0$ the matrix $A^l(\epsilon)$ has cpi over $\mathbb{N}$,*

   i.e. for at most one r with $1 \leq r \leq n$ there is an $\epsilon_0 > 0$ such that $A^r(\epsilon_0)$ has not cpi.

Note that if the matrix $A^k(\epsilon_0)$ has *cpi* for some $\epsilon_0 > 0$ then for all $\epsilon \geq \epsilon_0$ $A^k(\epsilon)$ has *cpi*.

Remark 1.1. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix, such that $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Let $1 \leq k < l \leq n$.*

1. *If the matrix $A^k_l(\epsilon)$ has cpi then $A^k(\epsilon)$ has cpi.*

   A^k_l has cpi. Let $I_0, \dots, I_r$ be the corresponding partition of the column indices. Define $I_{r+1} = \{l\}$. Then $I_0, \dots, I_{r+1}$ is a partition of $[n]$ which proves cpi for $A^k(\epsilon)$.

2. *If the matrix $A^{(k)+(l)}(\epsilon)$ has cpi then the matrices $A^k(\epsilon)$ and $A^l(\epsilon)$ have cpi.*

   Let the blocks for $A^{(k)+(l)}(\epsilon)$ be $I'_0, \dots, I'_q$ and assume that the column

$$a^{(k')}(\epsilon) = \begin{pmatrix} a_{1k} + a_{1l} - \epsilon \\ a_{2k} + a_{2l} - \epsilon \\ \vdots \\ a_{mk} + a_{ml} - \epsilon \end{pmatrix}$$

   belongs to the block I'_p. Then $A^k(\epsilon)$ and $A^l(\epsilon)$ have cpi with the corresponding blocks being $I_r = I'_r$ for $r \neq p$ and $I_p = I'_p - \{k'\} \cup \{k, l\}$.

Up to now we did not succeed in proving that A has *cpi*, but we know that if we transform A only a little then the transformed matrix has *cpi* and it is possible to do this transformations in nearly each column. What we will show in lemma 1.9. is that the property *cpi* is continuous in a certain manner.

Lemma 1.2. *If $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is a rational matrix, which satisfies the ϵ-property, then A has cpi.*

In order to prove lemma 1.9. we need the following lemma:
Lemma 1.3. *Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix such that for all $1 \leq i \leq m$ the entries of row i sum up to zero, i.e. $\sum_{j=1}^n a_{ij} = 0$. Let $s_1, \dots, s_m \in \mathbb{Q}$. For all $\epsilon > 0$ let $A'(\epsilon) = (a'_{ij}(\epsilon))_{1 \leq i \leq m, 1 \leq j \leq n+1}$, be the matrix with entries $a'_{ij}(\epsilon) = a_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$ and $a_{in+1} = s_i - \epsilon$ for $1 \leq i \leq m$. Further let $A' = A'(0)$.*
If for all $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$, then the system $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$.

Proof of lemma 1.3.: Let A, $A'(\epsilon)$ and A' be as in the assumptions of lemma 1.10. Assume that for all $1 \leq i \leq m$ we have $\sum_{j=1}^n a_{ij} = 0$. Thus the system $A\vec{x} \leq \vec{0}$ can be transformed into the following system

$$(*) \qquad \sum_{j=1}^{n-1} a_{ij}(x_j - x_n) \leq 0, \quad 1 \leq i \leq m,$$

which will be abbreviated in the following as $A^*\vec{y} \leq \vec{0}$, where $A^* = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n-1}$, and $y_j = x_j - x_n$ for $1 \leq j \leq n-1$.
The system $A\vec{x} \leq \vec{0}$ (resp. $A\vec{x} < \vec{0}$) has a solution in $\mathbb{N}$ if and only if $(*)$ (resp. $A^*\vec{y} < 0$) has a solution in $\mathbb{Z}$.

In the following we will consider A^* instead of A. (The entries of A^* will be denoted without $*$.) Assume that the set of rows of A^* is linear independent over $\mathbb{Q}$. Then there exists $\vec{y} = (y_1, \dots y_{n-1}) \in \mathbb{Q}^{n-1}$ such that $A^*\vec{y} < \vec{0}$. Multiplication with the least common multiple of the denominators of y_j yields a solution $\vec{y}' = (y'_1 \dots y'_{n-1}) \in \mathbb{Z}^{n-1}$ of the system $A^*\vec{y} < 0$. Thus the system $A\vec{x} < \vec{0}$ has a solution in $\mathbb{N}$ and therefore $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Hence we are done in this case.

Next we consider the case where the set of rows of A^* is not linear independent. Assume that A^* consists of the rows $a_{(1)}, \dots a_{(k)}, b_{(k+1)}, \dots, b_{(m)}$ for some $k \geq 0$, where $a_{(1)}, \dots, a_{(k)}$ are linear independent and for all $k+1 \leq i \leq m$ we have $b_{(i)} = \sum_{s=1}^{k} c_s^i a_{(s)}$ for suitable $c_s^i \in \mathbb{Q}$.

We will prove the lemma by induction on k. If $k = 0$ then A^* is the zero-matrix. Hence A is the zero-matrix and therefore the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ if and only if for all $1 \leq i \leq m$ we have $s_i - \epsilon \leq 0$. This is true for all $\epsilon > 0$ by assumption and therefore for all $1 \leq i \leq m$ we have $s_i \leq 0$.
If $k = 1$ for all $2 \leq i \leq m$ we have $b_{(i)} = c_1^i a_{(i)}$ for suitable $c_1^i \in \mathbb{Q}$. We distinguish the following cases:

1. for all $2 \leq i \leq m$ we have $c_1^i > 0$.

   If $a_{(1)}\vec{y} < 0$ holds then for all $2 \leq i \leq m$ we have $b_{(i)}\vec{y} < 0$. Because $a_{(1)}$ is not the zero-vector there exists a solution $\vec{y} \in \mathbb{Z}^n$ such that $A^*\vec{y} < \vec{0}$ and hence we are done in this case.

2. There exists i such that $c_1^i = 0$.

   In this case we have $b_{(i)} = \vec{0}$ and the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution only if $s_i - \epsilon \leq 0$. Because this is true for every $\epsilon > 0$, we have $s_i \leq 0$. Hence $(b_{(i)} s_i)\vec{x} \leq \vec{0}$ is true for every choice of $\vec{x}$ where $x_{n+1} \geq 0$. Therefore the matrix keeps its properties if we omit the row $b_{(i)}$.

3. There exists i such that $c_1^i < 0$.

   Let i be arbitrary with $c_1^i < 0$. By assumption we know that for every $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution. Let $\vec{x}(\epsilon) = (x_1(\epsilon), \dots, x_n(\epsilon)), x(\epsilon)$ be one specific solution of the system $A'(\epsilon)\vec{x} \leq \vec{0}$, i. e.

   $$a_{(1)}\vec{x}(\epsilon) + (s_1 - \epsilon)x(\epsilon) \leq 0,$$

   which is equivalent to

   $$\sum_{j=1}^{n} a_{1j}x_j(\epsilon) \leq -(s_1 - \epsilon)x(\epsilon)$$

   and correspondingly we have

   $$b_{(i)}\vec{x}(\epsilon) + (s_i - \epsilon)x(\epsilon) \leq 0,$$

   which is equivalent to

   $$c_1^i(\sum_{j=1}^{n} a_{1j}x_j(\epsilon)) \leq -(s_i - \epsilon)x(\epsilon)$$

Dividing by $c_1^i > 0$ we obtain

$$\sum_{j=1}^{n} a_{1j}x_j(\epsilon) \geq -\frac{s_i - \epsilon}{c_1^i}x(\epsilon).$$

Hence a solution $x_1(\epsilon), \ldots, x_n(\epsilon)$ exists if and only if

$$-\frac{s_i - \epsilon}{c_1^i} \leq s_1 - \epsilon,$$

which means

$$s_i \leq -c_1^i s_i + (c_1^i - 1)\epsilon.$$

This is true for all $\epsilon > 0$ and hence

$$s_i \leq -c_1^i s_1$$

holds.

Thus the statement is true for $k = 1$.

Assume that our statement is true for some (fixed) $k \geq 1$. Let A^* consist of the rows $a_{(1)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(m)}$, where $a_{(i)}$ are linear independent and for $k+1 \leq i \leq m$ let

$$b_{(i)} = \sum_{s=1}^{k+1} c_s^i a_{(s)}$$

for suitable $c_s^i \in \mathbb{Q}$. Further assume that for every $\epsilon > 0$ the system $A'(\epsilon)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. We distinguish the following cases:

1. There exists $1 \leq s \leq k+1$ such that for all $k+2 \leq i \leq m$ we have $c_s^i > 0$.

   Let $c = max_{k+1<i<m, 1\leq l\leq k, l\neq s}|c_l^i|$. $a_{(1)}, \ldots, a_{(k+1)}$ are linearly independent by assumption. Hence there exists $\vec{y} = (y_1, \ldots, y_n)$ such that for all $1 \leq i \leq k$ we have $a_{(i)}\vec{y} < 0$ and

   $$min_{k+1\leq i\leq m}|c_s^i(a_{(s)}\vec{y})| > c \cdot (max_{1\leq l\leq k, l\neq s}|a_{(l)}\vec{y}|)(k-1).$$

   Then $y_1, \ldots y_{n-1}$ form a solution for the whole system $A^*\vec{y} < \vec{0}$ and hence $A'\vec{x} \leq \vec{0}$ has a solution.

2. There exists s such that for all $k+1 \leq i \leq m$ we have $c_s^i \geq 0$ and $c_s^i = 0$ for at least one i.

   Without loss of generality let $s = 1$ and $c_1^i > 0$ for $k+1 \leq i \leq l$ and $c_1^i = 0$ for $l < i \leq m$. Then the matrix which consists of the rows $a_{(1)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(l)}$ is dealt within case 1. But the rows $b_{(l+1)}$ up to $b_{(m)}$ only depend on the $k-1$ generators $a_{(2)}$ up to $a_{(k+1)}$. Hence by induction we obtain a solution $y_1, \ldots, y_n$ for the rows $a_{(2)}, \ldots, a_{(k+1)}, b_{(k+2)}, \ldots, b_{(m)}$ which are independent of $a_{(1)}$. Thus we also obtain a solution for the whole system.

3. For $a \leq i \leq k+1$ we define

$$c_j^i = \begin{cases} 1 & \text{for} \quad j = i \\ 0 & \text{for} \quad j \neq i. \end{cases}$$

Then it remains to consider the case where there exist $1 \leq i_1, i_2 \leq m$ and there exists $1 \leq s \leq k+1$ such that $c_s^{i_1} > 0$ and $c_s^{i_2} < 0$.

Without loss of generality let $s = 1$. Further we can divide the entries of each row i by $|c_1^i|$, if $|c_1^i| \neq 0$, such that we may assume that $|c_1^i| = 1$ for each i, where $|c_1^i| \neq 0$.

For every $\epsilon > 0$ the system $A^*(\epsilon)'\vec{x} \leq \vec{0}$ has a solution. Let $\vec{y}^\epsilon = (y_1^\epsilon, \ldots, y_{n-1}^\epsilon), x^\epsilon$ be such a solution, i. e.

$$\sum_{s=1}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon) + (s_i - \epsilon)x^\epsilon \leq 0 \quad \text{for } k+2 \leq i \leq m$$

and

$$a_{(i)}\vec{y}^\epsilon \leq -(s_i - \epsilon)x^\epsilon \quad \text{for } 1 \leq i \leq k+1.$$

Thus we have

$$\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y})^\epsilon + (s_i - \epsilon)x^\epsilon \leq -c_1^i a_{(1)}\vec{y}^\epsilon.$$

Dividing by $-c_1^i$ leads to

$$\sum_{s=2}^{k+1} c_s^r(a_{(s)}\vec{y}^\epsilon) + (s_r - \epsilon)x^\epsilon \leq a_{(1)}\vec{y}^\epsilon \leq -\sum_{s=2}^{k+1} c_s^j(a_{(s)}\vec{y}^\epsilon) - (s_j - \epsilon)x^\epsilon$$

for all r with $c_1^r = -1$ and j with $c_1^j = 1$. Further we know that $a_{(1)}\vec{y} \leq -(s_1 - \epsilon)x^\epsilon$. Hence we additionally obtain:

$$\sum_{s=2}^{k+1} c_s^r(a_{(s)}\vec{y}^\epsilon) \leq -(s_1 - \epsilon)x^\epsilon$$

for all i satisfying $c_1^i = -1$ and

$$\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon) \leq -(s_i - \epsilon)x^\epsilon$$

for all i satisfying $c_1^i = 0$. Transforming these inequalities we get the following system of inequalities:

$$(***) \begin{cases} a_{(i)}\vec{y}^\epsilon + (s_i - \epsilon)x^\epsilon \leq 0 & 2 \leq i \leq k+1 \\ (\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon)) + (s_i - \epsilon)x^\epsilon \leq 0 & \text{for all } i \text{ with} \\ & c_1^i = 0 \\ (\sum_{s=2}^{k+1} c_s^i(a_{(s)}\vec{y}^\epsilon)) + (s_i + s_1 - 2\epsilon)x^\epsilon \leq 0 & \text{for all } i \text{ with} \\ & c_1^i = -1 \\ (\sum_{s=2}^{k+1}(c_s^i + c_s^j)(a_{(s)}\vec{y}^\epsilon)) + (s_i + s_j - 2\epsilon)x^\epsilon \leq 0 & \text{for all } i, j \text{ with} \\ & c_1^i = -1, c_1^j = 1 \end{cases}$$

By assumption we know that for all $\epsilon > 0$ the system $A^*(\epsilon)'\vec{x} \le \vec{0}$ has a solution. Hence the system $(***)$ has a solution for every $\epsilon > 0$. In system $(***)$ only k row vectors are linear independent, namely $a_{(2)}, \ldots, a_{(k+1)}$. Thus we can use induction to show that the system $(***)$ has a solution for $\epsilon = 0$. Thus the system $A'\vec{x} \le \vec{0}$ has a solution in $\mathbb{N}$.

$\square_{lemma}$ 1.2.

Claim 1.1. *Let $A = (a_{ij})_{1\le i\le m, 1\le j\le n}$ be a rational matrix which has cpi with the first block being $I_0^A = \{1, \ldots k\}$ and $\sum_{j=1}^k a_{ij} = 0$. Let*

$$B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{l1} & b_{l2} & \ldots & b_{ln} \\ & & A & \end{pmatrix}$$

such that for all $1 \le i \le l$ the relation $\sum_{j=1}^k b_{ij} < 0$ holds. Then B has cpi.

Proof of claim 1.1.: Obviously $I_0^B = \{1, \ldots, k\}$ satisfies the first condition of *cpi*. Let $I_0^A, \ldots, I_v^A$ be the partition of columns of A and for $1 \le r < v, j \in \cup_{s\le r} I_s$ let $c_{rj}^A, c_r^A \in \mathbb{N}$ be the corresponding coefficients . Let the parameters $b(r), \delta, B(r), c(r) \quad 1 \le r \le v$ be "big enough", in particular we define:

$$\begin{array}{lcl} b(r) & = & max_{1\le i\le l}\{\sum_{j\in I_{r+1}^A} b_{ij},\} \\ \delta & = & max_{1\le i\le l}\{\sum_{j=1}^k b_{ij}\} \quad (<0), \\ B(r) & = & max_{1\le i\le l}\{\sum_{j\in\cup_{w\le r} I_w^A} |b_{ij}|\}, \\ c(r) & = & max_{j\in\cup_{w\le r} I_w^A}\{c_{rj}^A, c_r^A\}. \end{array}$$

and let $a(r) \in \mathbb{N}$ be minimal such that

$$a(r)\delta \le -(c(r)B(r) + c_r b(r)).$$

Such an a=a(r) exists because δ is negative. Let $c_{rj}^B = c_{rj}^A + a$ if $j \le k$ and $c_{rj}^B = c_{rj}$ otherwise. For $1 \le r \le v$ let $c_r^B = c_r^A$ and $I_r^B = I_r^A$. Then for all $1 \le i \le l$ we have:

$$\sum_{j=1}^k (a + c_{rj})b_{ij} + \sum_{j\in\cup_{w\le r} I_w^A, j>k} c_{rj}b_{ij} + c_r \sum_{j\in I_{r+1}^A} b_{ij}$$

$$= a\sum_{j=1}^k b_{ij} + \sum_{j=1}^k c_{rj}b_{ij} + \sum_{j\in\cup_{w\le r} I_w^A, j>k} c_{rj}b_{ij} + c_r \sum_{j\in I_{r+1}^A} b_{ij}$$

$$\leq a\delta + c(r)B(r) + c_r b(r) \leq 0.$$

Further for all $1 \leq i \leq l$ we have:

$$\begin{aligned}\sum_{j=1}^{k}(a+c_{ij})a_{ij} &= (\sum_{j=1}^{k} c_{rj}a_{ij}) + a(\sum_{j=1}^{k} a_{ij}) \\ &= \sum_{j=1}^{k} c_{rj}a_{ij}.\end{aligned}$$

Hence B has *cpi*. $\Box_{claim}$ 1.1.

Proof of lemma 1.2.: Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix which has the ϵ-property, i. e. for all $1 \leq k < l \leq n$ either $A^k(\epsilon)$ or $A^l(\epsilon)$ has *cpi* for every $\epsilon > 0$. We will prove that A has *cpi*. If the matrix $A^k(\epsilon)$ has *cpi* for some $k \leq n$, let $I_0^k, \ldots, I_{l_k}^k$ be a partition of columns of $A^k(\epsilon)$, which certifies *cpi*. We can assume that the partition of $[n]$ into blocks does not depend on ϵ because there are only finitely many possibilities of partitioning $[n]$ into blocks. By the pigeonhole principle at least one partition has to occur for arbitrary small $\epsilon > 0$. But if a matrix $A^k(\epsilon_0)$ has *cpi* with blocks $I_0^k(\epsilon_0), \ldots I_{l_k}^k(\epsilon_0)$ then for all $\epsilon > \epsilon_0$ the matrix $A^k(\epsilon)$ has *cpi* with the same blocks.

We will prove lemma 1.3. by a downward induction on the size of the block I_0^k which is maximal for $k \leq n$, for which the matrix $A^k(\epsilon)$ has *cpi* for all $\epsilon > 0$. To illustrate the main idea of the proof we first show the theorem for matrices with one and two columns.

$n = 1:$

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}.$$

The system $A\vec{x} \leq \vec{0}$ has a solution $x \in \mathbb{N}$. Therefore we have $a_{i1} \leq 0$ and thus A has *cpi* with $I_0 = \{1\}$.

$n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}$$

There are only three (finitely many) possibilities to arrange the columns of A into blocks. Hence we can assume that there is an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ the partition of the columns of $A^i(\epsilon)$ into blocks is the same.

1. $I_0^1 = \{1\}$ or $I_0^2 = \{2\}$ resp.

   For all $1 \leq i \leq m$ and all $\epsilon > 0$ we have $a_{i1} - \epsilon \leq 0$. Hence for all $1 \leq i \leq m$ the relation $a_{i1} \leq 0$ holds. So the first condition of *cpi* is satisfied with $I_0 = \{1\}$.

   Further by the definition of the ϵ-property the system $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$, Let x_1^*, x_2^* be such a solution. Then the second condition is fulfilled with $c_{11} = x_1^*$ and $c_1 = x_2^*$, i. e. for all $1 \leq i \leq m$ we have $c_{11}a_{i1} + c_1 a_{i2} \leq 0$. Hence A has *cpi*.

2. $I_0^1 = \{1, 2\}$

   In this case for all $1 \leq i \leq m$ and for all $\epsilon > 0$ we have $a_{i1} + a_{i2} - \epsilon \leq 0$. Hence for all $1 \leq i \leq m$ we have $a_{i1} + a_{i2} \leq 0$. Therefore A has *cpi* with $I_0 = \{1, 2\}$.

Now we will prove the lemma for matrices of arbitrary size.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Let $1 \leq k < l \leq n$. We know by assumption that for all $\epsilon > 0$ either $A^k(\epsilon)$ or $A^l(\epsilon)$ has *cpi*. As mentioned above we can assume that the partition of $[n]$ into blocks does not depend on ϵ. In order to start the induction we consider the case where we can find some $1 \leq k \leq n$ such that $A^k(\epsilon)$ has *cpi* for every $\epsilon > 0$ and $|I_0^k|$=n, i. e. the sum over all columns of $A^k(\epsilon)$ is less of equal to zero. In this case for all $1 \leq i \leq m$ and every $\epsilon > 0$ we have

$$a_{i1} + a_{i2} + \ldots + a_{in} - \epsilon \leq 0.$$

Hence for all $1 \leq i \leq m$ we have

$$a_{i1} + a_{i2} + \ldots + a_{in} \leq 0$$

and therefore A has *cpi* with $I_0 = [n]$.
Next we consider the case where we can find some k, $1 \leq k \leq n$ such that $A^k(\epsilon)$ has *cpi* for every $\epsilon > 0$ and $|I_0^k| = n - 1$. First assume that $k \in I_0^k$. Then for all $1 \leq i \leq m$ and all $\epsilon > 0$ we have:

$$(\sum_{j \in I_0^k} a_{ij}) - \epsilon \leq 0.$$

In this case for all $1 \leq i \leq m$ we obtain

$$\sum_{j \in I_0^k} a_{ij} \leq 0.$$

If $k \notin I_0^k$ for all $1 \leq i \leq m$ we also have

$$\sum_{j \in I_0^k} a_{ij} \leq 0.$$

Thus in both cases the first condition of *cpi* is satisfied choosing $I_0 = I_0^k$.
Let $I_1 = [n] - I_0$. Note that $|I_1| = 1$ and assume $p \in I_1$. We know that the system $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Let $x_1^*, \ldots x_n^*$ be such a solution. Then for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0} c_{1j} a_{ij} + c_1 a_{ip} \leq 0,$$

if we choose $c_{1j} = x_j^*$ for $j \in I_0$ and $c_1 = x_p^*$.

Assume inductively that the following is true for some (fixed) $k \leq n-1$: Let A be a rational matrix with m rows and n columns which has the ϵ- property. If there exists a column s, such that for all $\epsilon > 0$ $A^s(\epsilon)$ has *cpi* and $|I_0^s| \geq k$, then A has *cpi*.

In the following we will show that if A is a rational matrix which has the ϵ-property and there exists a column s, such that for all $\epsilon > 0$ $A^s(\epsilon)$ has *cpi* and $|I_0^s| = k-1$, then A has *cpi*. Without loss of generality we can assume that $I_0^s = \{1, \ldots k-1\}$ for some (fixed) s. For $k-1 \leq n-2$, we have $|[n] - I_0^s| \geq 2$. A has the ϵ- property, therefore either $A^k(\epsilon)$ or $A^{k+1}(\epsilon)$ has *cpi* for all $\epsilon > 0$. Without loss of generality we can assume that $A^k(\epsilon)$ has *cpi*. We will consider several cases:

1. $I_0^k \not\subseteq I_0^s$.

   In this case for all $\epsilon > 0$ and all $1 \leq i \leq m$ we have

   $$(\sum_{j=1}^{k-1} a_{ij}) - \epsilon \leq 0$$

   and therefore

   $$\sum_{j=1}^{k-1} a_{ij} \leq 0.$$

   Further for all $1 \leq i \leq m$ we have

   $$\sum_{j \in I_0^k} a_{ij} \leq 0.$$

   We distinguish the following cases:

   (a) $I_0^k \cap I_0^s = \emptyset$

   Then we have

   $$\sum_{j \in I_0^k \cup I_0^s} a_{ij} \leq 0.$$

   Let $I_0 = I_0^k \cup I_0^s$ and $I_l = I_l^k - I_0^s$. Because of the definition of I_l^k for all $\epsilon > 0$ and for all $j \in \cup_{s \leq l} I_s^k$ there exists $c_{lj}^k(\epsilon)$ and $c_l^k(\epsilon)$ such that for all $1 \leq i \leq m$ we have

   $$\sum_{j \in \cup_{s \leq l} I_s^k} c_{lj}^k(\epsilon) a_{ij}^k(\epsilon) + c_l^k(\epsilon) \sum_{j \in I_{l+1}^k} a_{ij}^k(\epsilon) \leq 0$$

and therefore

$$\sum_{j\in\cup_{s\leq l}I_s^k} c_{lj}^k(\epsilon)a_{ij}^k(\epsilon) + \sum_{j\in(I_0^s - I_{l+1}^k)} a_{ij}^k(\epsilon) + \sum_{j\in I_0^s\cap I_{l+1}^k} (1+c_l^k(\epsilon))a_{ij}^k(\epsilon) +$$

$$\sum_{j\in(I_{l+1}^k - I_0^s)} c_l(\epsilon)a_{ij}^k(\epsilon) \leq 0.$$

Hence we conclude that we can choose $I_0 = I_0^k \cup I_0^s$ to prove *cpi* and $|I_0| > |I_0^s| = k-1$. So we are done by induction.

(b) $I_0^k \cap I_0^s \neq \emptyset$

Without loss of generality we can assume that $I_0^k \cap I_0^s = \{1,\ldots,l\}$. Consider the matrix

$$B = \begin{pmatrix} 2a_{11} & 2a_{12} & \ldots & 2a_{1l} & a_{1l+1} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2a_{m1} & 2a_{m2} & \ldots & 2a_{ml} & a_{ml+1} & \ldots & a_{mn} \end{pmatrix} = (b_{ij})_{1\leq i\leq m, 1\leq j\leq n}.$$

We claim that B has the ϵ-property. This is true because

(i) the system $A'\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ for if $x_1,\ldots,x_n$ is a solution of $A\vec{x} \leq \vec{0}$, then $x_1,\ldots,x_l, 2x_{l+1},\ldots,2x_n$ forms a solution of $B\vec{x} \leq \vec{0}$.

(ii) Let $1 \leq p \leq n$ such that $A^p(\epsilon)$ has *cpi* for every $\epsilon > 0$ with blocks $I_0^p, I_1^p, \ldots$. Let $I_0'^p = I_0^k \cup I_0^s$. Then for all $1 \leq i \leq m$ the following is true:

$$0 \geq \sum_{j\in I_0^k} a_{ij}^k(\epsilon) + \sum_{j\in I_0^s} a_{ij}^s(\epsilon) = \sum_{j\in I_0'^p} b_{ij}^p(\epsilon).$$

Let $I_r'^p = I_{r-1}^p - (I_0^k \cup I_0^s)$. $A^p(\epsilon)$ has *cpi* for every $\epsilon > 0$. Hence there exist $c_{r-1j}^p = c_{r-1j}^p(\epsilon), c_{r-1}^p = c_{r-1}^p(\epsilon)$ such that for all $1 \leq i \leq m$ we have

$$\sum_{j\in\cup_{q\leq r-1}I_q^p} c_{r-1j}a_{ij}^p(\epsilon) + c_{r-1}\sum_{j\in I_r^p} a_{ij}^p(\epsilon) \leq 0.$$

Hence we have

$$\begin{aligned}&\sum_{j=1}^{l} b_{ij}^p(\epsilon) + \sum_{j\in I_0'^p - \{1,\ldots,l\}} 2b_{ij}^p(\epsilon) + \sum_{j\in\cup_{q\leq r-1}I_q^p\cap(I_0^k\cup I_0^s)} c_{r-1j}2b_{ij}^p(\epsilon) + \\ &\sum_{j\in(\cup_{q\leq r-1}I_q^p)-(I_0^k\cup I_0^s)} c_{r-1j}2b_{ij}^p(\epsilon) + \sum_{j\in I_r^p\cap(I_0^k\cup I_0^s)} c_{r-1}2b_{ij}^p(\epsilon) + \sum_{j\in I_r'^p} c_{r-1}2b_{ij}^p(\epsilon) \\ &\leq 0.\end{aligned}$$

Hence $B^p(\epsilon)$ has *cpi* if $A^p(\epsilon)$ has *cpi*. Therefore B has the ϵ-property and $|I_0'^p| \geq k$. Hence B has *cpi* by induction.

We claim that if B has *cpi* then A has *cpi*.

Let the partition into blocks for B be $I_0^B, I_1^B, \ldots, I_v^B$. Let $I_0 = \{1, \ldots, k-1\}$. We know that

$$\sum_{j=1}^{k-1} a_{ij} \leq 0.$$

Let $I_1 = (I_0^k \cup I_0^s) - \{1, \ldots, k-1\}$, let $c_{01} = \ldots = c_{0l} = 2, c_{0l+1} = \ldots = c_{0k-1} = 1$ and $c_0 = 1$. Then for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0} c_{j0} a_{ij} + c_0 \sum_{j \in I_1} a_{ij} \leq 0.$$

Let $I_r = I_{r-2}^B - (I_0^k \cup I_0^s)$. We know that there exist c_{r-2j}^B, c_{r-2}^B such that we have

$$\sum_{j \in \cup_{w \leq r-3} I_w^B} c_{r-2j}^B b_{ij} + c_{r-2}^B \sum_{j \in I_{r-2}^B} b_{ij} \leq 0$$

and thus

$$\begin{aligned} &\sum_{j=1}^{l} 2a_{ij} + \sum_{j \in (I_0^k \cup I_0^s) - \{1,\ldots,l\}} a_{ij} + \sum_{j \in (\cup_{w \leq r-3} I_w^B) - (I_0^k \cup I_0^s)} c_{r-2j}^B a_{ij} + \\ &\sum_{j \in (\cup_{w \leq r-3} I_w^B) \cap (I_0^k \cup I_0^s)} c_{r-2j}^B a_{ij} + c_{r-2}^B \sum_{j \in I_{r-2}^B \cap (I_0^k \cup I_0^s)} a_{ij} + c_{r-2}^B \sum_{j \in I_r} a_{ij} \\ &\leq 0. \end{aligned}$$

Hence A has *cpi*.

2. $I_0^k \subseteq I_0^s = \{a^{(1)}, \ldots a^{(k-1)}\}$.

   (If $I_0^s \subset I_0^k$, we would have $|I_0^k| \geq k$ and we were done by induction.)

   Without loss of generality we can assume that $I_0^k = I_0^s$, because otherwise it is possible to choose I_0^k as the first block for the matrix $A^s(\epsilon)$. We distinguish the following cases:

   (a) $k \notin I_1^k$

   In this case there exist $c_{1j} \in \mathbb{N}, c_1 \in \mathbb{N}$ such that for all $1 \leq i \leq m$ we have

$$\sum_{j \in I_0^k} c_{1j} a_{ij} + c_1 \sum_{j \in I_1^k} a_{ij} \leq 0.$$

   Consider the following matrix $B = (b_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where for all $1 \leq i \leq m$ b_{ij} is defined by

$$b_{ij} = \begin{cases} c_{1j} a_{ij} & \text{for} \quad 1 \leq j \leq k-1 \\ c_1 a_{ij} & \text{for} \quad j \in I_1^k \\ a_{ij} & \qquad \text{otherwise.} \end{cases}$$

   We claim that B has the ϵ-property.

(i) The system $B\vec{x} \le \vec{0}$ has a solution, for if $x_1, \dots x_n$ is a solution of the system $A\vec{x} \le \vec{0}$, then define a solution of the system $B\vec{y} \le \vec{0}$, $\vec{y} = (y_1, \dots, y_n)$ by

$$y_j = \begin{cases} \frac{1}{c_{1j}} x_j & \text{if } 1 \le j \le k-1 \\ \frac{1}{c_1} x_j & \text{if } j \in I_1^k \\ x_j & \text{otherwise.} \end{cases}$$

If we multiply $\vec{y}$ by the least common multiple of c_{1j}, c_1 we obtain a solution of the system $B\vec{x} \le \vec{0}$ in $\mathbb{N}$.

(ii) Let $1 \le p \le n$ be given such that for every $\epsilon > 0$ $A^p(\epsilon)$ has *cpi* and let $I_0^p, \dots, I_l^p$ be the blocks and $c_{rj}^p(\epsilon), c_r^p(\epsilon)$ the corresponding coefficients, such that for all $1 \le i \le m$ we have

$$\sum_{j \in I_0^p} a_{ij}^p(\epsilon) \le 0$$

and

$$\sum_{j \in \cup_{w \le r} I_w^p} c_{rj}^p(\epsilon) a_{ij}^p(\epsilon) + c_r^p(\epsilon) \sum_{j \in I_{r+1}^p} a_{ij}^p(\epsilon) \le 0.$$

Now we will show that $B^p(\epsilon)$ has *cpi* for all $\epsilon > 0$. Let $I_0'^p = I_0^k \cup I_1^k$ and $I_r'^p = I_{r-1}^p - I_0'^p$. Then for all $1 \le i \le m$ we have

$$\sum_{j \in I_0'^p} b_{ij} \le 0$$

and

$$\sum_{j \in \cup_{w \le r-1} I_w^p} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + c_{r-1}^p(\epsilon) \sum_{j \in I_r^p} a_{ij}^p(\epsilon) \le 0.$$

It follows that

$$\sum_{j \in I_0'^p} b_{ij}^p(\epsilon) + \sum_{j \in ((\cup_{w \le r-1} I_w^p) \cap I_0'^p)} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + \sum_{j \in I_{r-1}^p \cap I_0^p} c_r^p(\epsilon) a_{ij}^p(\epsilon) +$$
$$\sum_{j \in (\cup_{w \le r-1} I_w^p) - I_0'^p} c_{r-1j}^p(\epsilon) a_{ij}^p(\epsilon) + \sum_{j \in I_r'^p} c_r^p(\epsilon) a_{ij}^p(\epsilon) \le 0.$$

Hence B has the ϵ-property. Thus B has *cpi* by induction. Let the corresponding partition of blocks be $I_0^B, \dots, I_l^B$ and let c_{rj}^B, c_r^B be the corresponding coefficients. We claim that A has *cpi*.

Let $I_0 = I_0^k, I_1 = I_1^k, I_r = I_{r-2}^B - (I_0^k \cup I_1^k)$. Obviously for all $1 \le i \le m$ we have

$$\sum_{j \in I_0} a_{ij} \le 0.$$

For $2 \le r \le l-1$, for all $1 \le i \le m$ we have

$$\sum_{j \in \cup_{w \le r-2} I_w^B} c_{r-2j}^B b_{ij} + c_{r-1}^B \Big(\sum_{j \in I_{r-1}^B} b_{ij} \Big) \le 0.$$

Thus for all $1 \leq i \leq m$ the following is true

$$\sum_{j\in I_0^k} a_{ij} + \sum_{j\in I_1^k} a_{ij} + \sum_{j\in(\cup_{w\leq r-2} I_w^B \cap (I_0^k \cup I_1^k))} c_{r-2j}^B b_{ij}$$
$$+ \sum_{j\in I_{r-1}^B \cap (I_0^k\cup I_1^k)} c_{r-2}^B b_{ij} + \sum_{j\in \cup_{w\leq r-2} I_w^B - (I_0^k \cup I_1^k)} c_{r-2j}^B b_{ij} + c_{r-2}^B (\sum_{j\in I_{r+1}} b_{ij})$$
$$\leq 0.$$

Hence A has *cpi*.

(b) $k \in I_1^k$.

Without loss of generality we can assume that $I_1^k = \{k,\dots,r\}$. For all $1 \leq i \leq m$ we know that $\sum_{j=1}^{k-1} a_{ij} \leq 0$. It is no restriction to assume that

$$\sum_{j=1}^{k} a_{ij} \begin{cases} = 0 & \text{for} \quad 1 \leq i \leq m_1 \\ < 0 & \text{for} \quad m_1 < i \leq m \end{cases}$$

for some $m_1 \leq m$. In claim 1.11. we have shown that it is enough to consider the first m_1 rows of A. Let

$$B = (a^{(1)},\dots,a^{(k-1)})$$

be the matrix which consists of the first $k-1$ columns of A. Let

$$B'(\epsilon) = \begin{pmatrix} a_{11} & \dots & a_{1k-1} & (\sum_{j=k}^r a_{1j} - \epsilon) \\ \vdots & \vdots & \vdots & \vdots \\ a_{m_11} & \dots & a_{m_1k-1} & (\sum_{j=k}^r a_{m_1j} - \epsilon) \end{pmatrix}.$$

Obviously adding up the columns of B we get the zero vector. Further for all $\epsilon > 0$ the system $B'(\epsilon)\vec{x} \leq \vec{0}$ has a solution. Hence we can apply lemma 1.10. to show that the system $B'(0)\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$. Assume that $c_{11},\dots,c_{1k-1},c_1$ is such a solution, hence for all $1 \leq i \leq m$ we have

$$\sum_{j=1}^{k-1} a_{ij}c_{1j} + c_1 \sum_{k}^{r} a_{ij} \leq 0.$$

Then we consider the matrix $B = (b_{ij})_{1\leq i\leq m, 1\leq j\leq n}$

$$b_{ij} = \begin{cases} c_{1j}a_{ij} & \text{for} \quad 1 \leq j \leq k-1 \\ c_1 a_{ij} & \text{for} \quad k \leq j \leq r \\ a_{ij} & \text{otherwise.} \end{cases}$$

As in case a) it is now possible to show that B has the ϵ-property. Then by induction B has *cpi* which again implies as in case a) that A has *cpi*.

$\Box_{lemma}$ 1.3.

LEMMA 1.4. *Let $a, b \in \mathbb{Q}$ and let the following system of inequalities be given:*

$$(*) \qquad a \leq \frac{x_1}{x_2} \leq b.$$

Let

1. $1 < a \le b$ *or*

2. $0 < a \le b < 1$.

Then $(*)$ *is not partition regular over* $\mathbb{N}$.

Proof of lemma 1.4.:

1. Assume that $1 < a \le b$.

   Let $n \in \mathbb{N}$ be minimal such that $a^n > b$. Consider the following coloring: $\Delta^{a,b} : \mathbb{N} \to [n+1]$ which is defined by

$$(**) \quad \Delta^{a,b}(x) = (\lfloor log_a(x) \rfloor mod(n+1)) + 1.$$

   In the following we will show that $(*)$ has no monochromatic solution for $\Delta^{a,b}$.

   Assume on the contrary that x_1, x_2 form a solution of $(*)$ which is monochromatic with respect to $\Delta^{a,b}$. Let $log_a(x_1) = \mu_{x_1}$ and $log_a(x_2) = \mu_{x_2}$. Then we have

$$\mu_{x_1} \equiv \mu_{x_2} mod(n+1).$$

   Say $\mu_{x_1} = k_{x_1}(n+1) + r$ and $\mu_{x_2} = k_{x_2}(n+1) + r$ for some $0 \le r \le n$. Because x_1, x_2 forms a solution of $(*)$ we have

$$a \le \frac{x_1}{x_2} \le b$$

   and thus

$$a \le \frac{x_1}{x_2} < \frac{a^{\mu_{x_1}+1}}{a^{\mu_{x_2}}} = a^{(k_{x_1}-k_{x_2})(n+1)+1}.$$

   Therefore we have

$$(k_{x_1} - k_{x_2})(n+1) + 1 > 1$$

   and hence

$$k_{x_1} - k_{x_2} > 0.$$

   On the other hand we have:

$$a^n > b \ge \frac{x_1}{x_2} \ge \frac{a^{\mu_{x_1}}}{a^{\mu_{x_2}+1}} = a^{(k_{x_1}-k_{x_2})(n+1)-1},$$

   which implies

$$(k_{x_1} - k_{x_2})(n+1) - 1 < n$$

   and hence

$$k_{x_1} - k_{x_2} < 1.$$

   which is in contradiction to $(**)$.

2. Assume that $0 < a \le b < 1$. Consider the following system of inequalities which is equivalent to $(*)$:

$$\frac{1}{a} \ge \frac{x_2}{x_1} \ge \frac{1}{b}.$$

   Then we have $1 < \frac{1}{b} \le \frac{1}{a}$ and we can follow the arguments of case 1.

$\square_{lemma}$ 1.4.

Lemma 1.5. *Let $z \in \mathbb{N}$ be given. Let $n \geq 2$ and let $f_i(x_2, \ldots, x_n) : \mathbb{R}^{n-1} \to \mathbb{R}$, $g_i(x_2, \ldots, x_n) : \mathbb{R}^{n-1} \to \mathbb{R}$ for $1 \leq i \leq z$ be given. Consider the following system of inequalities:*

$$f_i(x_2, \ldots, x_n) \leq \frac{x_1}{x_2} \leq g_i(x_2, \ldots, x_n). \qquad (*)$$

Let $(*)$ satisfy the following conditions:

1. *$\exists i_1, 1 \leq i_1 \leq z, \exists \epsilon_1, 0 < \epsilon_1 < 1, \exists c_1 \in \mathbb{N}$ and $\exists \Delta^1 : \mathbb{N} \to [c_1]$ such that $(*)$ has no solution $x_1, \ldots, x_n$ which is monochromatic with respect to Δ^1 and*
$$f_{i_1}(x_2, \ldots, x_n) \leq \epsilon_1.$$
2. *$\exists i_2, 1 \leq i_2 \leq z, \exists \epsilon_2, \epsilon_3, 0 < \epsilon_2, \epsilon_3 < 1, \exists c_2 \in \mathbb{N}$ and $\exists \Delta^2 : \mathbb{N} \to [c_2]$ such that $(*)$ has no solution $x_1, \ldots, x_n$ which is monochromatic with respect to Δ^2 and*
$$f_{i_2}(x_2, \ldots, x_n) \leq 1 + \epsilon_2$$
or there is no solution $x_1, \ldots, x_n$ which is monochromatic with respect to Δ^2 and
$$g_{i_2}(x_1, \ldots, x_n) \geq \epsilon_3.$$
3. *$\exists k \in \mathbb{N}, \exists c_3 \in \mathbb{N}$ and $\exists \Delta^3 : \mathbb{N} \to [c_3]$ such that $(*)$ has no solution $x_1, \ldots, x_n$ which is monochromatic with respect to Δ^3 and*
$$\frac{x_1}{x_2} \geq k.$$

Then there exists $c^* \in \mathbb{N}$ and a coloring $\Delta^* : \mathbb{N} \to [c^*]$, such that $(*)$ has no solution which is monochromatic for Δ^*.

Proof of lemma 1.5.: Let $\epsilon_1, \epsilon_2, \epsilon_3, k, c_1, c_2, c_3$ and $\Delta^1, \Delta^2, \Delta^3$ be defined as in the assumptions of lemma 1.13. Consider colorings of the form $\Delta^{a,b}$ which are defined as in the proof of lemma 1.11. $(**)$ with appropriate a and b, namely:

$$\Delta^4 = \Delta^{\frac{1}{1-\epsilon_3}, \frac{1}{\epsilon_1}} : \mathbb{N} \to [c_4],$$

where $c_4 \in \mathbb{N}$ is minimal such that $\frac{1}{1-\epsilon_3}^{(c_4-1)} > \frac{1}{\epsilon_1}$ and

$$\Delta^5 = \Delta^{1+\epsilon_2, k} : \mathbb{N} \to [c_5],$$

where $c_5 \in \mathbb{N}$ is minimal such that $(1+\epsilon_2)^{(c_5-1)} > k$.
Then define Δ^* as follows:

$$\Delta^* : \mathbb{N} \to \prod_{j=1}^{5} [c_j],$$

$$\Delta^*(x) = (\Delta^1(x), \Delta^2(x), \Delta^3(x), \Delta^4(x), \Delta^5(x)).$$

We claim that $(*)$ has no solution which is monochromatic for Δ^*.
Assume on the contrary that $x_1, \ldots, x_n$ is a solution of $(*)$ which is monochromatic with respect to Δ^*. Because $x_1, \ldots, x_n$ is monochromatic for Δ^* it is monochromatic for Δ^1. Hence we have

$$f_{i_1}(x_2, \ldots, x_n) \geq \epsilon_1,$$

which implies

$$\frac{x_1}{x_2} \geq \epsilon_1. \qquad (1)$$

Besides $x_1, \ldots, x_n$ is monochromatic for Δ^2. Hence we have

$$f_{i_2}(x_2, \ldots, x_n) \geq 1 + \epsilon_2$$

or

$$g_{i_2}(x_2, \ldots, x_n) \leq 1 - \epsilon_3,$$

which implies

$$\frac{x_1}{x_2} \geq 1 + \epsilon_2 \qquad (2)$$

or

$$\frac{x_1}{x_2} \leq 1 - \epsilon_3. \qquad (3)$$

Finally $x_1, \ldots, x_n$ is monochromatic for Δ_3 and therefore we have:

$$\frac{x_1}{x_2} \leq k. \qquad (4)$$

If we put together (1) and (3) and (2) and (4) respectively, we obtain:

$$\epsilon_1 \leq \frac{x_1}{x_2} \leq 1 - \epsilon_3 \qquad (5)$$

or

$$1 + \epsilon_2 \leq \frac{x_1}{x_2} \leq k. \qquad (6)$$

By lemma 1.12. (5) has no monochromatic solution for Δ^4 and (6) has no monochromatic solution for Δ^5. Hence $x_1, \ldots, x_n$ is not monochromatic for Δ^*. That is in contradiction to our assumption. $\square_{lemma}$ 1.13.

Now we are able to prove the second part of theorem 1.5., i.e. A has *cpi* if the system $A\vec{x} \leq \vec{0}$ is partition regular.

Proof of theorem 1.3. (second part): We will prove the theorem by induction on the number of columns of A. Note that a system, which is partition regular, necessarily has a solution.
$n = 1:$

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

The system $(a_{i1}x_1 \le 0)_{1\le i\le m}$ is partition regular. Hence it has a solution in $\mathbb{N}$, therefore for all $1 \le i \le m$ we have $a_{i1} \le 0$ and thus A has *cpi* with $I_0 = \{1\}$.
In order to demonstrate the idea of the proof we additionally consider the case $n = 2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}$$

We distinguish the following cases:

1. For each row I $1 \le i \le m$ the first entry is less or equal zero, i.e. $a_{i1} \le 0$.

   Let $I_0 = \{1\}$ and $I_1 = \{2\}$. Assume that $y_1, y_2 \in \mathbb{N}$ form a solution of the system $A\vec{x} \le \vec{0}$. Then for all $1 \le i \le m$ we have

   $$\sum_{j\in I_0} c_{1j}a_{ij} + c_1 \sum_{j\in I_1} a_{ij} = c_{11}a_{i1} + c_1 a_{i2} \le 0$$

   if we choose $c_{11} = y_1$ and $c_1 = y_2$.

2. For each row i $1 \le i \le m$ the first entry is greater or equal zero, i.e. $a_{i1} \ge 0$.

   In this case for all $1 \le i \le 0$ we have $a_{i2} \le 0$

   Then A has *cpi* with blocks $I_0 = \{2\}$ and $I_1 = \{1\}$.

3. There exist $s, t \in [m]$ such that $a_{s1} < 0$ and $a_{t1} > 0$.

   Then the system $A\vec{x} \le \vec{0}$ can be transformed as follows:

   $$-\frac{a_{t2}}{a_{t1}} \le \frac{x_1}{x_2} \le -\frac{a_{s2}}{a_{s1}}$$

   for all t with $a_{t1} < 0$ and for all s with $a_{s1} > 0$ and

   $$a_{t2}x_2 \le 0 \quad \text{for all } t \text{ with } a_{t1} = 0.$$

   By lemma 1.12. we know that one of the following cases holds:

   (a) $-\frac{a_{t2}}{a_{t1}} \le 0$ for all t with $a_{t1} < 0$ and $-\frac{a_{s2}}{a_{s1}} \ge 0$ for all s with $a_{s1} > 0$ and (obviously) $a_{t2} \le 0$ for all t with $a_{t1} = 0$. In this case for all $1 \le i \le m$ we obtain

   $$a_{t2} \le 0.$$

   Thus A has *cpi* with blocks $I_0 = \{2\}$ and $I_1 = \{1\}$.

   (b) $-\frac{a_{t2}}{a_{t1}} \le 1$ for all t with $a_{t1} < 0$ and $-\frac{a_{s2}}{a_{s1}} \ge 1$ for all s with $a_{s1} > 0$ and hence for all $1 \le t \le m$ with $a_{t1} \ne 0$ we have

   $$a_{t1} + a_{t2} \le 0$$

   and obviously for all $1 \le t \le m$ with $a_{t1} = 0$ we have

   $$a_{t2} \le 0$$

and hence

$$a_{t1} + a_{t2} \leq 0.$$

Thus A has *cpi* with $I_0 = \{1, 2\}$ in this case.

Hence we are done in the case $n = 2$.
Let us assume that the theorem is true for all matrices A with less than n columns for some (fixed) $n \geq 2$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}.$$

To prove the theorem we distinguish the following cases:

1. There exists $1 \leq j^* \leq n$ such that for all $1 \leq i \leq m$ the j^*th entry satisfies $a_{ij^*} < 0$.

   In this case let $I_0 = \{j^*\}$ and $I_1 = [n] - \{j^*\}$ and choose $c_{1j^*} > max_{1 \leq i \leq m}\{\frac{\sum_{s=1, s \neq j^*}^{n} |a_{is}|}{|a_{ij^*}|}\}, c_1 = 1$.

2. There exists $1 \leq j^* \leq n$ such that for all $1 \leq i \leq m$ the j^*th entry satisfies $a_{ij^*} \leq 0$.

   Without loss of generality assume $j^* = 1$ and $a_{i1} < 0$ for $1 \leq i \leq m_1$ and $a_{i1} = 0$ for $m_1 < i \leq m$ for some $m_1 \leq m$. Then we have:

$$A = \begin{pmatrix} a_{11} < 0 & \\ \vdots & * \\ a_{1m_1} < 0 & \\ 0 & \\ \vdots & A' \\ 0 & \end{pmatrix}.$$

   Hence A is partition regular if and only if A' is partition regular. By induction A' has *cpi*. Let the corresponding blocks be $I'_0, \dots I'_r$ for a suitable $r \in \mathbb{N}$ and for $1 \leq k \leq r$ and for $j \in \cup_{s \leq k} I_s$ let the coefficients be c'_{kj}, c'_k. Then A has *cpi* with blocks $I_0 = \{1\}, I_s = I'_{s-1}$ for $1 \leq s \leq r$ and coefficients

$$c_{k1} = \frac{max_{1 \leq i \leq m_1}\{\sum_{j \in \cup_{s \leq k} I'_s} c'_{kj} a_{ij} + c'_k \sum_{j \in I'_{k+1}} a_{ij}\}}{min_{1 \leq i \leq m_1} |a_{1i}|}$$

   for $2 \leq k \leq r$ and

$$c_{11} = \frac{max_{1 \leq i \leq m_1} \sum_{j \in I'_1} a_{ij}}{min_{1 \leq i \leq m_1} |a_{1i}|},$$

   $c_1 = 1$ and $c_{kj} = c'_{k-1j}$ for all $j \neq 1$ and all $1 \leq k \leq r$.

3. There exists j^* such that for all $1 \leq i \leq m$ we have $a_{ij^*} \geq 0$.

   In this case obviously $A' = A - \{a^{(j^*)}\}$, the matrix which we obtain from A by omitting the column j^*, is partition regular and has *cpi* by induction. Let the blocks of A' be $I'_0, \ldots I'_r$ and define for all $1 \leq s \leq r$ $I_s = I'_s$ and $I_{r+1} = \{j^*\}$. Further let $y_1, \ldots, y_n \in \mathbb{N}$ be a solution of the system $A\vec{x} \leq \vec{0}$. Then A has *cpi* with coefficients $c_{rj} = y_j$ for $j \neq j^*$, and $c_r = y_{j^*}$.

4. Each column has both positive and negative entries.

   Let $1 \leq k < l \leq n$ be given. Then the system $A\vec{x} \leq \vec{0}$ can be transformed as follows:

$$(*) \begin{cases} -\dfrac{a_{sk}}{a_{sl}} - \displaystyle\sum_{j=1, j\notin\{k,l\}}^{n} \dfrac{a_{sj}}{a_{sl}} \dfrac{x_j}{x_k} \leq \dfrac{x_l}{x_k} \leq -\dfrac{a_{tk}}{a_{tl}} - \displaystyle\sum_{j=1, j\notin\{k,l\}}^{n} \dfrac{a_{sj}}{a_{sl}} \dfrac{x_j}{x_k} \\ \text{for all } s, t \text{ with } a_{sl} < 0 \text{ and } a_{tl} > 0, \\ \sum_{j=1, j\neq l}^{n} a_{ij} x_j \leq 0 \\ \text{for all } i \text{ with } a_{il} = 0. \end{cases}$$

   By lemma 1.13. we know that one of the following cases holds:

   (a) For all $\epsilon > 0$ the following system of inequalities is partition regular:

$$-\frac{a_{sk}}{a_{sl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sl}} \frac{x_j}{x_k} \leq \epsilon \qquad \text{for all } s \text{ with } a_{sl} < 0$$

$$-\frac{a_{tk}}{a_{tl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tl}} \frac{x_j}{x_k} \geq 0 \qquad \text{for all } t \text{ with } a_{tl} > 0$$

   and

$$\sum_{j=1, j\neq l}^{n} a_{ij} x_j \leq 0 \qquad \text{for all } i \text{ with } a_{il} = 0.$$

   That means that for every $\epsilon > 0$ the system

$$A_l^k(\epsilon)\vec{y} \leq \vec{0}$$

   is partition regular and has *cpi* by induction. Hence by remark 1.8. $A^k(\epsilon)$ has *cpi* for all $\epsilon > 0$.

   (b) For all $r > 0$ and each coloring of the natural numbers with finitely many colors the system $(*)$ has a monochromatic solution $x_1, \ldots, x_n$ such that

$$\frac{x_l}{x_k} > r,$$

   which is equivalent to

$$\frac{x_k}{x_l} < \frac{1}{r}.$$

We transform the system $A\vec{x} \leq \vec{0}$ as in $(*)$ exchanging k and l. Then we obtain:

$$-\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sk}}\frac{x_j}{x_l} \leq \frac{x_k}{x_l} \leq -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tk}}\frac{x_j}{x_l}$$

for all s, t with $a_{sk} < 0$ and $a_{tk} > 0$ and

$$\sum_{j=1, j\neq k}^{n} a_{ij}x_j \leq 0$$

for all i with $a_{ik} = 0$. Therefore the following system is partition regular for each $r > 0$:

$$\begin{cases} -\frac{a_{sl}}{a_{sk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sk}}\frac{x_j}{x_l} \leq \frac{1}{r} \\ \text{for all } 1 \leq s \leq m \text{ with } a_{sk} < 0 \\ -\frac{a_{tl}}{a_{tk}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tk}}\frac{x_j}{x_l} \geq 0 \\ \text{for all } 1 \leq t \leq m \text{ with } a_{tk} > 0 \text{ and} \\ \sum_{j=1, j\neq k}^{n} a_{ij}x_j \leq 0 \\ \text{for all } 1 \leq i \leq m \text{ with } a_{ik} = 0. \end{cases}$$

Hence the system $A_k^l(\frac{1}{r})$ is partition regular for every $r > 0$ and has *cpi* by induction. Therefore by remark 1.8. the system $A^l(\frac{1}{r})$ has *cpi* for every $r > 0$.

(c) For all $\epsilon > 0$ the following system is partition regular:

$$(*)\begin{cases} -\frac{a_{sk}}{a_{sl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{sj}}{a_{sl}}\frac{x_j}{x_k} \leq 1+\epsilon \\ \text{for all } 1 \leq s \leq m \text{ with } a_{sl} < 0 \\ -\frac{a_{tk}}{a_{tl}} - \sum_{j=1, j\notin\{k,l\}}^{n} \frac{a_{tj}}{a_{tl}}\frac{x_j}{x_k} \geq 1-\epsilon \\ \text{for all } 1 \leq t \leq m \text{ with } a_{tl} > 0 \\ \sum_{j=1, j\neq l}^{n} a_{ij}x_j \leq 0 \\ \text{for all } 1 \leq i \leq m \text{ with } a_{il} = 0. \end{cases}$$

Then for every $\epsilon > 0$ the system $A^{(k)+(l)}\vec{y} \leq \vec{0}$ is partition regular and has *cpi* by induction, therefore by remark 1.8. $A^l(\epsilon)$ and $A^k(\epsilon)$ have *cpi*.

The system $A\vec{x} \leq \vec{0}$ has a solution in $\mathbb{N}$ because otherwise it could not be partition regular and hence A has the ϵ-property. Therefore by lemma 1.9. A has *cpi*.

$\square_{theorem}$ 1.5.

In the following we will generalize the set of partitioned numbers. We will first state results over $\mathbb{Z}$ and $\mathbb{Q}$ and finally we will consider real matrices and generalize the set of partitioned numbers to the reals.

DEFINITION 1.5. *Let $K \subset \mathbb{R} - \{0\}$ be a set. Let $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ be a matrix with entries in $\mathbb{R}$. A has the column property for systems of inequalities (cpi) over K if there exists $l \in \mathbb{N}$ and a partition $[n] = I_0 \cup I_1 \cup \ldots \cup I_l$ of the column indices such that*

1. *There exists $c \in K$ such that for all $1 \le i \le m$ we have $c \sum_{j \in I_0} a_{ij} \le 0$ and*
2. *for all $k < l, j \in \cup_{s \le k} I_s$ there exist $c_k, c_{kj} \in K$ such that for all $1 \le i \le m$ we have*
$$\sum_{j \in \cup_{s \le k} I_s} c_{jk} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \le 0.$$

And correspondingly we define:

DEFINITION 1.6. *Let $K \subset \mathbb{R} - \{0\}$ be a set. Let $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ be a real matrix. Let $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$. The system $A\vec{x} \le \vec{b}$ is called partition regular over K, if for every $c \in \mathbb{N}$ and every c-coloring of K $\Delta : K \to [c]$ there exists a solution $x_1, \ldots x_n \in K$ of $A\vec{x} \le \vec{b}$ such that $\Delta \big|_{\{x_1 \ldots x_n\}} = const.$*

LEMMA 1.6. *Let $K \subset \mathbb{R} - \{0\}$ and $K = K_1 \cup K_2$ such that $K_1 \cap K_2 = \emptyset$. Let $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ be a real matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \le \vec{0}$ is partition regular over K.*
2. *The system $A\vec{x} \le \vec{0}$ is partition regular over K_1 or the system is partition regular over K_2.*

PROOF OF LEMMA 1.6.: If the system $A\vec{x} \le \vec{0}$ is partition regular over K_1 or over K_2 then it is clearly partition regular over K. For the opposite direction assume that the system $A\vec{x} \le \vec{0}$ is neither partition regular over K_1 nor over K_2, i. e. there exists $c_1 \in \mathbb{N}$ and a coloring $\Delta_1 : K_1 \to [c_1]$ and there exists $c_2 \in \mathbb{N}$ and a coloring $\Delta_2 : K_2 \to [c_2]$, such that $A\vec{x} \le \vec{0}$ has no monochromatic solution in K_1 for Δ_1 and no monochromatic solution in K_2 with respect to Δ_2. Define the following coloring: $\Delta : K \to [max\{c_1, c_2\}] \times [2]$ by

$$\Delta(x) = \begin{cases} (\Delta_1(x), 1) & \text{if} \quad x \in K_1 \\ (\Delta_2(x), 2) & \text{if} \quad x \in K_2. \end{cases}$$

Obviously the system $A\vec{x} \le \vec{0}$ has no monochromatic solution with respect to the coloring Δ which is a contradiction to the partition regularity. $\Box_{lemma}$ 1.6.

If we use lemma 1.16. together with theorem 1.5. we obtain the following theorem:

THEOREM 1.4. *Let $A = (a_{ij})_{1 \le i \le m, 1 \le j \le n}$ be a rational matrix. The system $A\vec{x} \le \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$ if and only if A has cpi either over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$.*

Proof of theorem 1.4.: By lemma 1.16. we know that the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$ iff it is either partition regular over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$. The first case is equivalent to A having *cpi* over $\mathbb{N}$ by theorem 1.5. In the second case consider $(*)$ $-A\vec{x} \leq \vec{0}$ where $-A = (-a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$. $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z}^- - \{0\}$ iff $-A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{N}$. This is equivalent to $-A$ having *cpi* over $\mathbb{N}$, which is equivalent to A having *cpi* over $\mathbb{Z}^- - \{0\}$.

$\Box_{theorem\quad 1.4.}$

Theorem 1.5. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Q} - \{0\}$.*
2. *A has cpi over $\mathbb{Q}^+ - \{0\}$ or over $\mathbb{Q}^- - \{0\}$.*
3. *A has cpi over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$.*

Proof of theorem 1.5. :
1. implies 2.:
It is enough to show that if $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Q}^+ - \{0\}$ then it has *cpi* over $\mathbb{Q}^+ - \{0\}$. This can be shown following the arguments of the second part of the proof of theorem 1.5. using $Q^+ - \{0\}$ instead of $\mathbb{N}$.
2. implies 3.:
Assume that A has *cpi* over $\mathbb{Q}^+ - \{0\}$, i. e. there exists a partition of the columns of A into blocks $[n] = I_0 \cup \ldots \cup I_l$ such that

1. There exists $q \in \mathbb{Q}^+ - \{0\}$ such that for all $1 \leq i \leq m$ we have $q \sum_{j\in I_0} a_{ij} \leq 0$, i. e. $\sum_{j\in I_0} a_{ij} \leq 0$.
2. For $k < l, j \in \cup_{s\leq k} I_s$ there exist $c_{kj}, c_k \in \mathbb{Q}^+ - \{0\}$ such that for all $1 \leq i \leq m$ we have
$$\sum_{j\in \cup_{s\leq k} I_s} c_{kj} a_{ij} + c_k \sum_{j\in I_{k+1}} a_{ij} \leq 0.$$
By multiplying the above inequality with the common divisor of c_{kj}, c_k we obtain positive integer coefficients.

3. implies 1.:
If A has *cpi* over $\mathbb{Z}^+ - \{0\}$ or over $\mathbb{Z}^- - \{0\}$ then by theorem 1.17. the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{Z} - \{0\}$. Hence it is partition regular over $\mathbb{Q} - \{0\}$.

$\Box_{theorem\quad 1.5.}$

Theorem 1.6. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a real matrix. Then the following statements are equivalent:*

1. *The system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R} - \{0\}$.*
2. *A has cpi over $\mathbb{R}^+ - \{0\}$ or over $\mathbb{R}^- - \{0\}$.*

Proof of theorem 1.6. :
1. implies 2.:
It is enough to show that if the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R}^+ - \{0\}$ then A has *cpi* over $\mathbb{R}^+ - \{0\}$. This can be shown following the arguments of the second part of the proof of theorem 1.5. using $\mathbb{R}^+ - \{0\}$ instead of $\mathbb{N}$.
2. implies 1. :
Again it is enough to show that if A has *cpi* over $\mathbb{R}^+ - \{0\}$ then the system $A\vec{x} \leq \vec{0}$ is partition regular over $\mathbb{R}^+ - \{0\}$. To prove this we employ a generalized environment lemma using the multidimensional version of van der Waerden's Theorem which is due independently to Gallai (see [10]) and Witt [16] instead of van der Waerden's Theorem [15]:

Lemma 1.7. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a real matrix such that the system $A\vec{x} \leq \vec{0}$ is partition regular over $R^+ - \{0\}$. Let $t \in \mathbb{N}$ and $W \subset \mathbb{R}$, $W = \{w_1, \dots w_t\}$ be given. Let $c \in \mathbb{N}$. Then for every c-coloring $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ there exists $\vec{x} = (x_0, \dots, x_n) \in (\mathbb{R}^+ - \{0\})^n$ and there exists $r \in R^+ - \{0\}$ such that*

1. *$A\vec{x} \leq \vec{0}$ and*
2. *For all j, k with $1 \leq j \leq n$, $1 \leq k \leq t$ we have $\Delta(x_j + rw_k) = const.$*

Proof of lemma 1.7.: Assume that A is partition regular. Hence by compactness [6] there exists a finite set $V = V(A, c) \subset \mathbb{R}^+ - \{0\}$ such that for every c-coloring of V there exists a monochromatic solution of the system $A\vec{x} \leq \vec{0}$ in V. Let $V = \{v_1, \dots, v_t\}$.
Let $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ be an arbitrary coloring. Define a coloring $\Delta^* : \mathbb{R}^+ - \{0\} \to [c^t]$ by

$$\Delta^*(x) = (\Delta(xv_i))_{1\leq i\leq t}.$$

Define a finite set $W = \{w | w = \prod_{s=1}^n v_{j_s}, j_s \in [t]\}$. By Gallai-Witt's Theorem there exists a homothetic copy of the set W which is monochromatic with respect to Δ^*, say $W' = a' + r'W = \{a' + r'w | w \in W\}$. Consider another coloring Δ^{**} :V$\to [c]$ which is defined by $\Delta^{**}(x) = \Delta(a'x)$. By definition of V there exists a monochromatic solution of the system $A\vec{x} \leq \vec{0}$ in V with respect to Δ^{**}, say $x'_1, \dots, x'_n$. Then $(x'_1 a', \dots, x'_n a')$ is a solution and for all $1 \leq j \leq n$ we have $\Delta(x'_j a') = const.$
Let $r = r' x'_1 \dots x'_n$. Then we have:

$$x'_i a' + rv_j = x'_i(a' + r' v_j x'_1 \cdot \dots \cdot x'_{i-1} x'_{i+1} \cdot \dots \cdot x'_n)$$

and by the definition of W

$$(v_j x'_1 \cdot \dots \cdot x'_{i-1} x'_{i+1} \cdot \dots \cdot x'_n) \in W.$$

Hence for all $1 \leq i \leq n, 1 \leq j \leq t$ we finally have

$$\Delta(x'_i a') = \Delta(x'_i a' + rv_j).$$

$\square_{lemma}$ 1.7.

Now we are able to prove the second part of theorem 1.19.:

Let A be a real matrix which has *cpi* over $\mathbb{R}^+ - \{0\}$. Let $[n] = I_0 \cup \ldots \cup I_l$ be the corresponding partition. We prove theorem 1.19. by main induction over the number of colors and by subsidiary induction over the number of blocks. In both cases the start of the induction is easy to obtain: The system $A\vec{x} \leq \vec{0}$ has a solution (just take the coefficients c_{jl-1}, c_l). If only one color is used every solution is monochromatic. If $l = 0$ every singleton provides a solution.

Let $A_k = (a^{(j)} | j \in \cup_{j \leq k} I_k)$ be the submatrix of A which only consists of the columns belonging to the first k blocks. Assume that A_k is partition regular over $\mathbb{R}^+ - \{0\}$ for some $k \geq 0$ and assume that for every coloring with $c-1$ colors the system $(*)$ $A_{k+1}\vec{x} \leq \vec{0}$ has a monochromatic solution, i. e. by compactness there exists a finite set $V_{c-1} \subset \mathbb{R}^+ - \{0\}$ such that for every $(c-1)$-coloring $(*)$ has a monochromatic solution in V_{c-1}.

Let $\Delta : \mathbb{R}^+ - \{0\} \to [c]$ be an arbitrary coloring. We define W, a finite subset of $\mathbb{R}$, by $W = \{w = vu | v \in V, u \in \{c_{jk}, c_k | 1 \leq j \leq n, 1 \leq k < l\}\}$. We apply lemma 1.20. to A_k and W. Thus there exists a solution $(y_i)_{i \in \cup_{s \leq k} I_s}$ of the system $A_k \vec{y} \leq \vec{0}$ and $r \in \mathbb{R}^+ - \{0\}$ such that for all $i \in \cup_{s \leq k} I_s$ and all $w \in W$ we have $\Delta(y_i + rw) = const.$ Combining *cpi* and the fact that the y_i form a solution for every $v \in V$ we obtain:

$$\sum_{j \in \cup_{s \leq k} I_s} a_{ij}(y_j + c_{kj} r v) + \sum_{j \in I_{k+1}} a_{ij} c_k r v \leq 0.$$

Without loss of generality we may assume that $\Delta(y_i + r c_{kj} v) = c$ for all $i \in \cup_{s \leq k} I_s$ and $v \in V$.
If now one of the numbers $c_k r v$ is also colored in c we have found a monochromatic solution of the system $A_{k+1}\vec{x} \leq \vec{0}$. Otherwise the coloring

$$\Delta^* : V \to [c-1]$$

defined by

$$\Delta^*(x) = \Delta(x r c_k)$$

is well defined. Therefore by induction on the number of colors and the definition of V there exists a monochromatic solution of $A_{k+1}\vec{x} \leq \vec{0}$ with respect to Δ^*, say $(x_i^*)_{i \in \cup_{s \leq k+1} I_s}$. Then $(x_i^* r c_k)_{i \in \cup_{s \leq k+1} I_s}$ forms a solution which is monochromatic with respect to Δ. $\square_{theorem\ 1.6.}$

In his dissertation [10] Rado also considered systems of inhomogeneous equations. As well as for homogeneous systems the columns property plays an important role for the characterization of partition regular systems of inhomogeneous inequalities. We are able to give a complete characterization of those systems which are partition regular over the natural numbers, over the set of integers and over the rationals.

Theorem 1.7. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix, let $\vec{b} = (b_1, \ldots, b_m) \in \mathbb{Q}^m$. The system of inequalities $A\vec{x} \subseteq 0$ is partition regular over $\mathbb{N}$ if and only if one of the following conditions is satisfied:

1. There exists $a \in \mathbb{N}$ such that $A \begin{pmatrix} a \\ \vdots \\ a \end{pmatrix} \leq \overrightarrow{b}$

2. A has cpi and there exists $\overrightarrow{x} = (x_1, \dots, x_n) \in \mathbb{N}^n$ and there exists $I \subseteq [m]$, such that $\sum_{j=1}^{n} a_{ij} x_j \begin{cases} < 0 & \text{for} \quad i \in I \\ \leq 0 & \text{for} \quad i \in [m] - I. \end{cases}$

   and there exists $a \in \mathbb{Z}$ such that for all $i \in [m] - I$ we have $\sum_{j=1}^{n} a_{ij} a \leq b_i$

The proof of theorem 1.21 is a little bit tricky and in its main parts very technical. The interested reader can find the complete proof in [17].

THEOREM 1.8. Set $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix and $\vec{b} = (b_1, \dots, b_m) \in \mathbb{Q}^m$. The system $A\vec{x} \leq \vec{b}$ is partition regular over $\mathbb{Q} - \{0\}$ if and only if $A\vec{x} \in \vec{b}$ is partition regular over $\mathbb{N}$ or the system $-\vec{A}\vec{x} \leq \vec{b}$ with $-A = (-a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is partition regular over $\mathbb{N}$.

If we partition the set $\mathbb{Q} - \{0\}$ the situation is different:

THEOREM 1.9. Let $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ be a rational matrix, let $\vec{b} = (b_1, \dots, b_n) \in \mathbb{Q}^n$. The system $(x)A\vec{x} \leq \vec{b}$ is partition regular over $\mathbb{Q}$ if and only if one of the following cases is valid:

1. There exists $a^* \in \mathbb{Q}$ such that for all $1 \leq i \leq m$ we have $\sum_{j=1}^{n} a_{ij} a^* \leq b_i$

2. There exists $I \subseteq [m]$ such that $b_i \geq 0$ for $i \in I, b_i > 0$ for $i \in [m]J - I$ and the matrix $A_I = (a_{ij})_{i \in I, 1 \leq j \leq n}$ has cpi over $\mathbb{Q}^+ - \{0\}$.

3. A has cpi over $\mathbb{Q}^+ - \{0\}$ and there exists $I \subseteq [m]$ and there exists $\vec{x}(x_1 - x_n) \in (\mathbb{Q}^+ - \{0\}^n$ such that
   $\sum_{j=1}^{n} a_{ij} x_j \begin{cases} < 0 & \text{for} \quad i \in I \\ \leq 0 & \text{for} \quad i \in [m] - I. \end{cases}$

   and there exists $a^x \in \mathbb{Q}^+ - \{0\}$ such that for all $i \in [m] - I$ we have $\sum_{j=1}^{n} a_{ij} a^* \leq b_i$.

4. $-A = (-a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ fulfills condition 1, 2, or 3.

2. (m, p, c)-SETS

In 1973 Deuber [1] gave a semantical characterization of partition regular system of linear equations. The nature of this characterization is somewhat different form Rado's approach. Deuber described the arithmetic structure of the sets of solutions of partition regular linear systems $A\vec{x} = \vec{0}$. The central definition is that of $(m, p, c,)$-sets, which are m-fold arithmetic progressions together with c-fold differences:

DEFINITION 2.1. *Let $m, p, c \in \mathbb{N}$. A set $D \subseteq \mathbb{N}$ is an (m, p, c)-set if there exist $d_0, \ldots, d_m \in \mathbb{N}$ such that $D = D_{p,c}(d_0 \ldots d_m)$ consists of all numbers of the following list:*

$$\begin{array}{ccccccccc} cd_0 & + & l_1 d_1 & + & l_2 d_2 & + & \ldots & + & l_m d_m, \\ & & cd_1 & + & l_2 d_2 & + & \ldots & + & l_m d_m, \\ & & & & cd_2 & + & \ldots & + & l_m d_m, \\ & & & & & & & & \vdots \\ & & & & & & & & cd_m, \end{array}$$

where $l_i \in [-p, p]$, i. e.

$$D_{p,c}(d_0, \ldots, d_m) = \{cd_i + \sum_{j=i+1}^{m} l_j d_j | i \leq m, l_j \in [-p, p]\}.$$

In particular a $(1, k, c)$-set is a $(2k+1)$-term arithmetic progressions together with its differences. Deuber proved the following theorem [1]:

THEOREM 2.1. (DEUBER 1973) *A linear system $A\vec{x} = \vec{0}$ is partition regular if and only if there exist positive integers m, p, c such that every $(m, p, c)-$set D contains a solution of $A\vec{x} = \vec{0}$.*

(m, p, c)-sets not only describe the arithmetic structure of sets of solutions of partition regular systems of linear equations but they can also be used to characterize sets of solutions of systems of linear inequalities.

THEOREM 2.2. *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be a rational matrix. Let $A\vec{x} \leq \vec{0}$ be a partition regular system of linear inequalities. Then there exist $m, p, c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution of the system $A\vec{x} \leq \vec{0}$.*

PROOF OF THEOREM 2.2.: By theorem 1.5. we know that A has *cpi*, i. e. there exists $m \in \mathbb{N}$ and a partition $I_0 \cup \ldots \cup I_m = [n]$ such that

1. for all $1 \leq i \leq l$ we have $\sum_{j \in I_0} a_{ij} \leq 0$ and
2. for $k \leq m$ and $j \in \cup_{s \leq k} I_s$ there exist $c_{kj}, c_k \in \mathbb{N}$ such that for every $k < m$ and for all $1 \leq i \leq l$ we have

$$\sum_{j \in \cup_{s \leq k} I_s} c_{kj} a_{ij} + c_k \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

Let c be the least common multiple of $\{c_k | 1 \leq k < m\}$. Multiply each inequality by $\frac{c}{c_k}$ such that for all $1 \leq i \leq l$ we have

$$\sum_{j \in \cup_{s \leq k} I_s} c'_{kj} a_{ij} + c \sum_{j \in I_{k+1}} a_{ij} \leq 0.$$

Further let $p = max_{1\leq i\leq l, 1\leq k<m}|c'_{kj}|$. We claim that these m, p, c have the desired properties. Let $A_k = (a_{ij})_{1\leq i\leq m, j\in\cup_{s\leq k} I_s}$ be the submatrix of A which only consists of the columns of A belonging to the blocks one up to k. We will prove the claim by induction on m.
Let $m = 0$. Hence $A = A_0$, i. e. for all $1 \leq i \leq l$ we have $\sum_{j=1}^{n} a_{ij} \leq 0$. Thus every singleton forms a solution of the system $A\vec{x} \leq \vec{0}$ and $D_{pc}(d_0) = \{cd_0\} \neq \emptyset$. Assume that the statement is true for some $k \geq 0$. Consider a $(k+1, p, c)$-set $D = D_{p,c}(d_0, \ldots, d_{k+1})$. By induction we know that the (k, p, c)-set $D_{p,c}(d_0, \ldots, d_k)$ contains a solution of the system $A_k\vec{x} \leq \vec{0}$. Let $(y_i)_{i\in\cup_{s\leq k} I_s}$ be such a solution, i. e. $y_i \in D_{p,c}(d_0, \ldots, d_k)$ and for all $1 \leq i \leq l$ we have

$$\sum_{j\in\cup_{s\leq k} I_s} a_{ij} y_j \leq 0,$$

which implies

$$\underbrace{\sum_{j\in\cup_{s\leq k} I_s} a_{ij} y_j}_{\leq 0} + \underbrace{d_{k+1}(\sum_{j\in\cup_{s\leq k} I_s} c_{kj} a_{ij} + c\sum_{j\in I_{k+1}} a_{ij})}_{\leq 0} \leq 0.$$

Hence for all $1 \leq i \leq l$ we have

$$\sum_{j\in\cup_{s\leq k} I_s} a_{ij}(y_j + d_{k+1}c_{kj}) + \sum_{j\in I_{k+1}} c d_{k+1} a_{ij} \leq 0.$$

For $y_j \in D_{p,c}(d_0, \ldots, d_k)$ and $|c_{kj}| \leq p$ we have

$$y_i + c_{kj} d_{k+1} \in D_{p,c}(d_0, \ldots, d_{k+1}) \text{ and}$$

$$c d_{k+1} \in D_{p,c}(d_0, \ldots, d_{k+1}).$$

Hence we found a solution of the system $A_{k+1}\vec{x} \leq \vec{0}$ in the arbitrary chosen $(k+1, p, c)$-set $D_{p,c}(d_0, \ldots, d_{k+1})$. $\square_{Theorem}$ 2.2.

Theorem 2.3. *Let $A = (a_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ be a rational matrix. If there exist $m, p, c \in \mathbb{N}$ such that every $(m, p, c)-$set contains a solution of the system $A\vec{x} \leq \vec{0}$ then the system $A\vec{x} \leq \vec{0}$ is partition regular.*

Proof of Theorem 2.3.: Let $m, p, c \in \mathbb{N}$ be given such that every $(m, p, c)-$set contains a solution of the system $A\vec{x} \leq \vec{0}$. By Deuber's theorem [1] we know that for every coloring Δ of the natural numbers with finitely many colors there exist $d_0 \ldots d_m$ such that the (m, p, c)-set $D = D_{p,c}(d_0, \ldots, d_m)$ is monochromatic with respect to Δ. For every (m, p, c)-set contains a solution of the system $A\vec{x} \leq \vec{0}$, so does D and hence $A\vec{x} \leq \vec{0}$ is partition regular. $\square_{theorem}$ 3.4.

Deuber [1] also proved a partition theorem for $(m, p, c)-$sets in order to resolve the following conjecture Rado stated 1933 [10].

Call a subset $S \subseteq \mathbb{N}$ *partition regular* if every partition regular system of linear equations can be solved in S. Rado conjectured that coloring a partition regular set S there is one color class which is again partition regular.

Theorem 2.4. (Deuber 1973) *Let m, p, c and r be positive integers. Then there exist positive integers n, q, d such that for every (n, q, d)-set $D \subseteq \mathbb{N}$ and every r-coloring $\Delta \to [r]$ there exists a monochromatic $(m, p, c)-$set $D' \subseteq D$.*

We can enlarge the definition of a partition regular set [1] to systems of linear inequalities:

Definition 2.2. *Call a subset $S \subseteq \mathbb{N}$ partition regular for systems of inequalities (pri) if every partition regular system of inequalities $A\vec{x} \leq \vec{0}$ can be solved in S.*

Note that for matrices A and B having *cpi* over $\mathbb{N}$ also the direct sum

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

has *cpi* over $\mathbb{N}$.

Theorem 2.5. *For every coloring of a pri set with finitely many colors at least one of the color classes again is partition regular for inequalities.*

Proof of Theorem 2.5.: Assume that the statement is false, i. e. there exists a set $S \subseteq \mathbb{N}$ which is *pri* and there exists $r \in \mathbb{N}$ and a coloring $\Delta : S \to [r]$ such that no color class of Δ is *pri*. Thus for each color class i there exists a matrix A_i such that the system $A_i\vec{x} \leq \vec{0}$ is partition regular but has no solution in $\Delta^{-1}(i)$. Consider the system

$$(*) \qquad \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & \ddots & 0 \\ 0 & \dots & & 0 & A_r \end{pmatrix} \vec{x} \leq \vec{0}.$$

$(*)$ is partition regular therefore there exist $m, p, c \in \mathbb{N}$ such that every $(m, p, c)-$set contains a solution of $(*)$. By Deuber's theorem [1] there exist $n, q, d \in \mathbb{N}$ such that each coloring of an arbitrary $(n, q, d)-$set with finitely many colors contains a monochromatic $(m, p, c)-$set. For S is *pri*, it contains a $(n, q, d)-$set. Hence there is some $(m, p, c)-$set in S which is monochromatic with respect to Δ and thus there exists a monochromatic solution of $(*)$ in S which contradicts the definition of $(*)$.

$\Box_{theorem}$ 2.5.

3. Canonical Results

In this chapter we want to extend our considerations to colorings with an unlimited number of colors. Call a coloring Δ of a set S *canonical* if Δ is either

1. monochromatic, i. e. for all $s, t \in S$ it holds $\Delta(s) = \Delta(t)$ or
2. distinct, i.e. for all $s, t \in S$ with $s \neq t$ it holds $\Delta(s) \neq \Delta(t)$.

In 1950 Erdös and Rado [4] proved a canonical version of Ramsey's theorem:

Theorem 3.1. (Erdös, Rado 1950) *If an infinite set S is colored then some infinite subset T is canonically colored. For all $k \in \mathbb{N}$ if $|S| > (k-1)^2 + 1$ and S colored there exists a subset $T \subseteq S$, $|T| = k$ which is canonically colored.*

Later Erdös and Graham [3] proved a canonical version of van der Waerden's theorem, i. e. for every $k \in \mathbb{N}$ and every coloring of the positive integers there exists a canonically colored k-term arithmetic progression. In 1986 Lefmann [7] extended the Erdös-Graham canonical theorem for arithmetic progressions to a canonical partition theorem for $(m, p, c)-$sets and partition regular systems of linear equations.

Let $D = D_{p,c}(d_0, \ldots, d_m) = \{cd_i + \sum_{j=i+1}^{m} l_j d_j | i \leq m, l_j \in [-p, p]\}$. Say that the elements of the form $cd_i + l_{i+1}d_{i+1} + \ldots + l_m x_m$ *belong to the ith row of the $(m, p, c)-$set* $D_{p,c}(d_0, \ldots, d_m)$. Let us further say that $\Delta : D_{p,c}(d_0, \ldots, d_m) \to \omega$ is a *row-coloring* provided that any two numbers $a, b \in D_{p,c}(d_0, \ldots, d_m)$ are colored the same if and only if they belong to the same row of $D_{p,c}(d_0, \ldots, d_m)$.
Lefmann proved the following theorem [7]:

Theorem 3.2. (Lefmann 1986) *Let $m, p, c \in \mathbb{N}$. Then there exists a least positive integer $L(m, p, c)$ with the following property: For every coloring $\Delta : [L(m, p, c)] \to \omega$ there exists a (m, p, c)-set $D_{p,c}(d_0, \ldots, d_m) \subseteq [L(m, p, c)]$ such that $\Delta\big|_{D_{p,c}(d_0,\ldots,d_m)}$ either is a canonical coloring or a row-coloring.*

As a corollary Lefmann [7] proved a canonical version of Rado's theorem:

Corollary 3.1. (Lefmann) *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be an integer valued matrix having the column property, i. e. the system of linear equations $A\vec{x} = \vec{0}$ is partition regular. Let $I_0 \cup \ldots \cup I_m = [n]$ be the corresponding partition of the columns of A into blocks. Then there exists a positive integer $N \in \mathbb{N}$ such that for every coloring $\Delta : [N] \to \omega$ there exists a solution $\vec{x} = (x_1 \ldots x_n)$ such that one of the following cases holds:*

1. *$\Delta\big|_{\{x_1,\ldots,x_n\}}$ is a canonical coloring.*
2. *Each two elements x_i, x_j of $\{x_1, \ldots, x_n\}$ are colored the same if and only if $\{i, j\} \subseteq I_k$ for some $k \leq m$.*

In the following we will prove a canonical theorem for systems of linear inequalities, which is similar to the above canonical version of Rado's theorem.

Theorem 3.3. *Let $A = (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$ be a rational matrix and let the system $A\vec{x} \leq \vec{0}$ be partition regular, i. e. A has cpi. Let $I_0 \cup \ldots \cup I_m = [n]$ be the corresponding* *partition of the columns of A into blocks. Then for every coloring $\Delta : \mathbb{N} \to \omega$ of the natural numbers there exists a solution $\vec{x} = (x_1 \ldots, x_n) \in \mathbb{N}^n$ such that one of the following cases is valid:*

1. $\Delta \big|_{\{x_1,\ldots,x_n\}}$ *is a canonical coloring*
2. $\Delta(x_i) = \Delta(x_j)$ *for some $i, j \in [n]$ if and only if there exists some $k \leq m$ such that $i, j \in I_k$.*

Proof of theorem 3.3.: The system $A\vec{x} \leq \vec{0}$ is partition regular. Thus by theorem 3.3. there exist positive integers m, p, c such that every $(m,p,c)-$set contains a solution of the system $A\vec{x} \leq \vec{0}$. In the proof of lemma 3.3. in chapter 3 we saw that a solution of $A\vec{x} \leq \vec{0}$ in an arbitrary $(m,p,c)-$set D can be constructed in such a way that for $i \in I_l$ x_i comes from the lth row of D. Let $\Delta : \mathbb{N} \to \omega$ be given. Theorem 4.2. gives us a $(m,p,c)-$set $D_{p.c}(d_0,\ldots,d_m)$ such that $\Delta \big|_{D_{p,c}(d_0,\ldots,d_m)}$ either is a canonical or a row-coloring. Let $\vec{y} = (y_1 \ldots y_n)$ be a solution of the system $A\vec{x} \leq \vec{0}$ such that for all $1 \leq i \leq n$ we have $y_i \in D_{p,c}(d_0,\ldots,d_m)$ and for $i \in I_k$ y_i belongs to the kth row of $D_{p,c}(d_0,\ldots,d_m)$. If $D_{p,c}(d_0,\ldots,d_m)$ is canonically colored then $\Delta \big|_{\{y_1,\ldots,y_n\}}$ is a canonical coloring and if $\Delta \big|_{D_{p,c}(d_0,\ldots,d_m)}$ is a row coloring then $\Delta(y_i) = \Delta(y_j)$ if and only if y_i and y_j belong to the same row of $D_{p,c}(d_0,\ldots,d_m)$, i. e. if and only if i and j belong to the same block I_k for some $k \leq m$. $\Box_{theorem}$ 3.3.

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On 14-dimensional quadratic forms in I^3, 8-dimensional forms in I^2, and the common value property

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Abstract. Let F be a field of characteristic $\neq 2$. We define certain properties $D(n)$, $n \in \{2, 4, 8, 14\}$, of F as follows: F has property $D(14)$ if each quadratic form $\varphi \in I^3F$ of dimension 14 is similar to the difference of the pure parts of two 3-fold Pfister forms; F has property $D(8)$ if each form $\varphi \in I^2F$ of dimension 8 whose Clifford invariant can be represented by a biquaternion algebra is isometric to the orthogonal sum of two forms similar to 2-fold Pfister forms; F has property $D(4)$ if any two 4-dimensional forms over F of the same determinant which become isometric over some quadratic extension always have (up to similarity) a common binary subform; F has property $D(2)$ if for any two binary forms over F and for any quadratic extension E/F we have that if the two binary forms represent over E a common nonzero element, then they represent over E a common nonzero element in F. Property $D(2)$ has been studied earlier by Leep, Shapiro, Wadsworth and the second author. In particular, fields where $D(2)$ does not hold have been known to exist.

In this article, we investigate how these properties $D(n)$ relate to each other and we show how one can construct fields which fail to have property $D(n)$, $n > 2$, by starting with a field which fails to have property $D(2)$ and then passing to transcendental field extensions. Particular emphasis is devoted to the situation where K is a field with a discrete valuation with residue field k of characteristic $\neq 2$. Here, we study how the properties $D(n)$ behave when one passes from K to k or vice versa. We conclude with some applications and an explicit and detailed example involving rational function fields of transcendence degree at most four over the rationals.

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1 Introduction

After Pfister [P] proved his structure results on quadratic forms of even dimension ≤ 12 and of trivial signed discriminant and Clifford invariant (cf. Theorem 2.1(i)–(iv) in this paper) over a field F of characteristic $\neq 2$, there have been various attempts to extend and generalize his results. Merkurjev's theorem [Me 1] implies that even-dimensional forms of trivial signed discriminant and Clifford invariant are exactly the forms whose Witt classes lie in I^3F, the third power of the fundamental ideal IF of even-dimensional forms in the Witt ring WF of F. But there have been no further results concerning the explicit characterization of such forms of a given dimension ≥ 14 until Rost [R] gave a description of 14-dimensional forms with trivial invariants as being transfers of scalar multiples of pure parts of 3-fold Pfister forms defined over a quadratic extension of the base field (cf. Theorem 2.1(v) in this paper). It remained open whether such 14-dimensional forms can always be written up to similarity as the difference of the pure parts of two 3-fold Pfister forms over F. It turns out that this question is related to the question whether 8-dimensional forms in I^2F whose Clifford invariant is given by the class of a biquaternion algebra are always isometric to a sum of scalar multiples of two 2-fold Pfister forms.

Izhboldin suggested a method to construct counterexamples to the second question which then leads to counterexamples to the first one (after a ground field extension). One crucial step to make his approach work depended on the construction of examples of two quaternion algebras over a suitable field F such that there exists a quadratic extension E/F over which these two quaternion algebras have a common slot, but no such common slot over E can be chosen to be an element in F. In this paper, we reduce this existence problem to the existence of quadratic field extensions which do not have a certain property $CV(2,2)$ defined by Leep [Le] (see also [SL]). This property has been studied in [STW], where it is shown that generally quadratic extensions do not have this property $CV(2,2)$. As a consequence, both questions above concerning 14-dimensional forms in I^3F and 8-dimensional forms in I^2F have negative answers in general.

It should be noted that the examples in [STW] of quadratic extensions not having $CV(2,2)$ are all in characteristic 0. Independently, Izhboldin and Karpenko [IK 2] found a method to construct counterexamples to the common slot problem above which is of a very general nature and works in all characteristics, thus also leading to counterexamples to the above questions on quadratic forms and incidentally also providing counterexamples to $CV(2,2)$ for quadratic extensions. Needless to say that they employ machinery quite different from what is used in [STW].

In the next section, we will recall the known results on forms in I^3F and prove certain others which are crucial in the understanding of 14-dimensional forms in I^3F. In section 3 we will then investigate the relations between the questions raised above. We will state these results in terms of certain properties $D(n)$ of the ground field F which describe the behaviour of certain forms of dimension $n \in \{2,4,8,14\}$ over F. In section 4, we consider the situation of a discrete valuation ring R with residue field k of characteristic not 2 and quotient field K. The purpose is to determine how the properties $D(n)$ for k and K relate to each other. These results can then be used to show that starting with a field F which does not have property $D(2)$, one obtains fields which do not have property $D(n)$, $n \in \{4,8,14\}$, by passing to rational field extensions. In section 5, we exhibit the properties $D(n)$ for fields with finite Hasse number and for their power series extensions. Finally, in section 6, we derive some further consequences and exhibit in all detail an example, starting over $\mathbf{Q}(x)$, which will then lead (after going up to rational field extensions over $\mathbf{Q}(x)$) to the explicit construction of counterexamples to all the problems touched upon in this article.

The standard references for those results in the theory of quadratic forms and division algebras which we will need in this paper are Lam's book [L 1] and Scharlau's book [S]. Most of the notations we will use are also borrowed from these two sources.

Fields are always assumed to be of characteristic $\neq 2$, and we only consider nondegenerate finite dimensional quadratic forms. Let φ and ψ be two quadratic forms over a field F. We write $\varphi \simeq \psi$ (resp. $\varphi \sim \psi$) to denote that the two forms are isometric (resp. equivalent in the Witt ring WF). The forms φ and ψ are said to be similar if there exists some $a \in F^\times$ such that $\varphi \simeq a\psi$. We call ψ a subform of φ, and write $\psi \subset \varphi$, if ψ is isometric to an orthogonal summand of φ. The hyperbolic plane $\langle 1, -1\rangle$ is denoted by $\mathbf{H}$. We write $d_\pm(\varphi)$ for the signed discriminant of a form φ, and $c(\varphi)$ for its Clifford invariant. For a field extension E/F, we write $D_E(\varphi)$ to denote the set of elements in $E^\times$ represented by φ_E, the form obtained from φ by scalar extension to E.

We use the convention $\langle\langle a_1, \cdots, a_n\rangle\rangle$ to denote the n-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ over F. By P_nF (resp. GP_nF) we denote the set of all forms over F which are isometric (resp. similar) to n-fold Pfister forms.

Forms of dimension 6 with trivial signed discriminant are called *Albert forms*, in reference to the following theorem of Albert:

> The biquaternion algebra $(a_1, a_2)_F \otimes (a_3, a_4)_F$ is a division algebra if and only if the quadratic form $\langle -a_1, -a_2, a_1a_2, a_3, a_4, -a_3a_4\rangle$ is anisotropic.

For a proof, see [A, Th. 3] or [P, p. 123].

2 Pfister's and Rost's results and some consequences

We begin by stating the results of Pfister and Rost on even-dimensional forms with trivial signed discriminant and Clifford invariant. Pfister proved the results on forms of dimension ≤ 12 in [P, Satz 14, Zusatz] (our statement of the 12-dimensional case is a little different but can easily be deduced from Pfister's original proof). The 14-dimensional case is due to Rost [R].

THEOREM 2.1 *Let φ be an even-dimensional form over F with $d_\pm\varphi = 1$ and $c(\varphi) = 1$.*

(i) *If $\dim\varphi < 8$ then φ is hyperbolic.*

(ii) *If $\dim\varphi = 8$ then $\varphi \in GP_3F$.*

(iii) *If $\dim\varphi = 10$ then $\varphi \simeq \pi \perp \mathbf{H}$ with $\pi \in GP_3F$.*

(iv) *If $\dim\varphi = 12$ then $\varphi \simeq \alpha \otimes \beta$ for some Albert form α and some binary form β or, equivalently, there exist $r, s, t, u, v, w \in F^\times$ such that $\varphi \sim r(\langle\langle s, t, u\rangle\rangle - \langle\langle s, v, w\rangle\rangle)$ in WF.*

(v) *If $\dim\varphi = 14$ and φ is anisotropic, then there exists a quadratic extension $L = F(\sqrt{d})$ and some $\pi \in P_3L$ such that φ is the trace of $\sqrt{d}\pi'$, where π'*

denotes the pure part of π. (Here, "trace" means the transfer defined via the trace map.)

Part (i) of the following corollary can also easily be deduced from the classifications given in [H 2, Th. 4.1, Th. 5.1]. We will give a self-contained proof. Part (ii) is an observation due to Karpenko [K, Cor. 1.3].

Corollary 2.2 *Let φ be a form over F.*

(i) *If $\dim\varphi = 10$ and there exists $\sigma \in P_2F$ such that $\varphi \equiv \sigma \pmod{I^3F}$, then there exist $r \in F^\times$ and $\pi \in GP_3F$ such that $\varphi \sim \pi + r\sigma$.*

(ii) *If $\dim\varphi = 14$ and $\varphi \in I^3F$ then there exists an Albert form α such that $\alpha \subset \varphi$.*

Proof. (i) Let $s \in F^\times$ such that $\varphi \simeq \langle s\rangle \perp \varphi'$, and let σ' be the pure part of σ. Let $\psi := (\varphi' \perp -s\sigma')_{\mathrm{an}}$. Note that $\dim\psi \le 12$. We have

$$\psi \equiv \varphi \perp -s\sigma \equiv \sigma \perp -s\sigma \equiv 0 \pmod{I^3F}.$$

If $\dim\psi \le 10$ then by Th. 2.1 there exists $\pi \in GP_3F$ (possibly hyperbolic) such that $\psi \sim \pi$ in WF. Thus, $\varphi \sim \psi + s\sigma \sim \pi + s\sigma$ in WF and we put $r = s$.

So suppose that $\dim\psi = 12$. Then, by Th. 2.1(iv), there exists a quadratic extension $E = F(\sqrt{d})$ such that ψ_E is hyperbolic, i.e. $\varphi'_E \sim s\sigma'_E$, and comparing dimensions yields that $i_W(\varphi'_E) \ge 3$. In particular, there exist $x, y, z \in F^\times$ such that $\varphi' \simeq \langle 1, -d\rangle \otimes \langle x, y, z\rangle \perp \varphi''$ with $\dim\varphi'' = 3$ (cf. [S, Ch. 2, Lemma 5.1]). Consider $\pi := \langle 1, -d\rangle \otimes \langle x, y, z, xyz\rangle \in GP_3F$ and $\alpha := -xyz\langle 1, -d\rangle \perp \varphi'' \perp \langle s\rangle$. Then $\varphi - \pi \sim \alpha$ in WF and thus $\alpha \equiv \sigma \pmod{I^3F}$. Note that α is an Albert form with $c(\alpha) = c(\sigma)$. It follows from Jacobson's theorem (see, e.g., [MaS]) that there exists $r \in F^\times$ such that $\alpha \sim r\sigma$ and therefore $\varphi \sim \pi + r\sigma$ in WF.

(ii) Any isotropic form of dimension ≥ 7 contains some Albert form as a subform as can readily be verified. Thus, if φ is isotropic, it contains some Albert form (which also follows from Th. 2.1(iv)). So assume that φ is anisotropic. By Th. 2.1(v), there exists a quadratic extension $E = F(\sqrt{d})$ and some form $\langle\langle u, v, w\rangle\rangle \in P_3E$ such that $\varphi \simeq \mathrm{tr}(\sqrt{d}\langle\langle u, v, w\rangle\rangle')$. Let $\alpha := \mathrm{tr}(\sqrt{d}\langle -u, -v, uv\rangle)$. Clearly, $\langle -u, -v, uv\rangle \subset \langle\langle u, v, w\rangle\rangle'$ and thus $\alpha \subset \varphi$. Furthermore, $\dim\alpha = 6$, and we have by [S, Ch. 2, Th. 5.12] that, in $F^\times/F^{\times 2}$, $\det\alpha = d^3 N_{E/F}(\det(\sqrt{d}\langle -u, -v, uv\rangle)) = d^3 N_{E/F}(\sqrt{d}) = -d^4 = -1$. Therefore $\alpha \in I^2F$. Hence, α is an Albert subform of φ. □

Proposition 2.3 *Let φ be a form over F with $\dim\varphi = 14$ and $\varphi \in I^3F$. Then there exist forms $\pi_i \in GP_3F$, $i = 1, 2, 3$, such that $\varphi \sim \pi_1 + \pi_2 + \pi_3$ in WF. Furthermore, the following statements are equivalent:*

(i) *There exist $\tau_1, \tau_2 \in P_3F$ and $s_1, s_2 \in F^\times$ such that $\varphi \sim s_1\tau_1 + s_2\tau_2$ in WF.*

(ii) *There exist $\tau_1, \tau_2 \in P_3F$ and $s \in F^\times$ such that $\varphi \simeq s(\tau'_1 \perp -\tau'_2)$, where τ'_1 and τ'_2 are the pure parts of τ_1 resp. τ_2.*

(iii) *There exists $\sigma \in GP_2F$ such that $\sigma \subset \varphi$.*

Proof. Let φ be a 14-dimensional form if I^3F. By Cor. 2.2(ii), we can write $\varphi \simeq \alpha \perp \psi$ with an Albert form α and some $\psi \in I^2F$, $\dim\psi = 8$. After scaling, we may assume that $\alpha \sim \sigma_1 - \sigma_2$ in WF with $\sigma_1, \sigma_2 \in P_2F$. Let $x \in F^\times$ such that $\psi \simeq \langle -x \rangle \perp \psi'$ and consider the 10-dimensional form $\psi' \perp x\sigma_1'$. We then have

$$\psi' \perp x\sigma_1' \equiv \psi + x\sigma_1 \equiv \varphi - \alpha + x\sigma_1 \equiv \sigma_2 - \sigma_1 + x\sigma_1 \equiv \sigma_2 \pmod{I^3F}.$$

By Cor. 2.2(i), there exists $y \in F^\times$ and $\pi_3 \in GP_3F$ such that $\psi' \perp x\sigma_1' \sim \psi + x\sigma_1 \sim \pi_3 + y\sigma_2$ in WF. Let now $\pi_1 := \langle\langle x \rangle\rangle \otimes \sigma_1 \in P_3F$ and $\pi_2 := \langle\langle y \rangle\rangle \otimes \sigma_2 \in P_3F$. One checks readily that we have $\varphi \sim \pi_1 - \pi_2 + \pi_3$ in WF.

As for the equivalences, (ii) trivially implies (i), and the converse follows readily after comparing dimensions of φ and $s_1\tau_1 \perp s_2\tau_2$, implying that the latter form is isotropic, and then using the multiplicativity of the Pfister forms τ_1, τ_2.

(ii) implies (iii) since τ_1' as well as τ_2' clearly contain subforms in GP_2F.

Finally, let $\varphi \in I^3F$ with $\dim \varphi = 14$ and suppose there exists $\sigma \in GP_2F$ with $\varphi \simeq \sigma \perp \psi$. Then $\dim \psi = 10$ and $\psi \equiv -\sigma \pmod{I^3F}$. By Cor. 2.2, there exist $\pi_1 \in GP_3F$ and $x \in F^\times$ such that $\psi \sim \pi_1 - x\sigma$ in WF. Let $\pi_2 := \langle\langle x \rangle\rangle \otimes \sigma \in GP_3F$. We then have $\varphi \sim \psi + \sigma = \pi_1 + \pi_2$ in WF, which implies (i). □

The fact that each 14-dimensional form in I^3F is Witt equivalent to the sum of three forms in GP_3F has been noticed independently by Izhboldin. A somewhat different proof of the equivalence of the three statements above is given in [IK 2, Prop. 17.2].

Let us now turn our attention to 8-dimensional I^2-forms over a field F. It is well-known that if φ is such a form, then the Clifford invariant $c(\varphi)$ can be represented as the class of $Q_1 \otimes Q_2 \otimes Q_3$ for suitable quaternion algebras Q_i. In particular, its index is 1, 2, 4, or 8. Which of these cases occurs can be determined in terms of the splitting behaviour of φ over (multi)quadratic extensions of F. To this end, we will need results on the Scharlau transfer of certain quadratic forms.

LEMMA 2.4 (i) (See also [S, Ch. 2, Lemma 14.8].) *Let $E = F(\sqrt{d})$ and $\tau \in GP_2E$. Then there exist $a_1, a_2 \in F^\times$, $b_1, b_2, c \in E^\times$, such that in WE, one has $c\tau \sim \langle\langle a_1, b_1 \rangle\rangle - \langle\langle a_2, b_2 \rangle\rangle$.*

(ii) *Let $\varphi \in I^2F$ be anisotropic,* $\dim \varphi = 8$, *and suppose that* $\operatorname{ind} c(\varphi) = 4$. *Then there exists a quadratic extension $E = F(\sqrt{d})$ and some $\tau \in GP_2E$ such that $\varphi \simeq \operatorname{tr}(\tau)$, where "tr" denotes the transfer defined via the trace map (cf. also Theorem 2.1(iv)).*

Proof. (i) After scaling, we may assume that $\tau \simeq \langle\langle x_1, x_2 \rangle\rangle$ with x_1, $x_2 \in E^\times$. If x_1 or x_2 lies in F, then obviously we are done. So let us assume that $x_1, x_2 \notin F$. Since E is 2-dimensional over F, the elements 1, x_1, x_2 are not linearly independent over F, hence we may find a_1, $a_2 \in F^\times$ such that $a_1x_1 + a_2x_2 = 0$ or 1. The form $\langle\langle a_1x_1, a_2x_2 \rangle\rangle$ is then hyperbolic. Multiplying by $\langle a_1, -a_1a_2x_2 \rangle$ both sides of

$$\langle 1, -a_1x_1 \rangle \sim \langle a_1, -a_1x_1 \rangle + \langle 1, -a_1 \rangle$$

we get

$$\langle\langle x_1, a_2x_2 \rangle\rangle \simeq \langle\langle a_1, a_2x_2 \rangle\rangle.$$

Substituting $\langle 1, -a_2x_2 \rangle \sim \langle a_2, -a_2x_2 \rangle + \langle 1, -a_2 \rangle$ in the left side, we obtain

$$a_2\langle\langle x_1, x_2 \rangle\rangle \sim \langle\langle a_1, a_2x_2 \rangle\rangle - \langle\langle a_2, x_1 \rangle\rangle.$$

We may thus choose $b_1 = a_2x_2$ and $b_2 = x_1$.

Part (ii) is due to Izhboldin and Karpenko [IK 2, Th. 16.10], and its proof (which we will omit) is based on Rost's result on 14-dimensional I^3-forms. $\square$

Proposition 2.5 *Let φ be an 8-dimensional form in I^2F. Then* $\operatorname{ind} c(\varphi) \in \{1,2,4,8\}$ *and there exists a multiquadratic extension L/F of degree 1, 2, 4 or 8 such that $\varphi_L \sim 0$. Moreover, for $i = 0$, 1, 2, 3, we have* $\operatorname{ind} c(\varphi) \leq 2^i$ *if and only if there exists a multiquadratic extension L/F of degree $\leq 2^i$ such that $\varphi_L \in GP_3L$. For $i = 1$, 2, 3, this condition is also equivalent to the existence of a multiquadratic extension L'/F of degree $\leq 2^i$ such that $\varphi_{L'} \sim 0$.*

Proof. Write $\varphi \simeq \beta_1 \perp \beta_2 \perp \beta_3 \perp \beta_4$, where the β_i are binary forms with $d_\pm\beta_i = d_i \in F^\times/F^{\times 2}$. Then $d_4 = d_1d_2d_3$ as $\varphi \in I^2F$, and for $L = F(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3})$, we obviously have $(\beta_i)_L \sim 0$ and thus $\varphi_L \sim 0$. Hence, we also have that $c(\varphi_L) = 0$ in $\mathrm{Br}L$. Thus, $c(\varphi)_L$ is split and it follows readily that $\operatorname{ind} c(\varphi) \in \{1,2,4,8\}$. (Of course, this also follows from the fact mentioned above that $c(\varphi)$ can be represented as the class of some triquaternion algebra.)

As for the remaining statements, the case $i = 0$ follows from Theorem 2.1(ii).

If $\varphi_L \in GP_3L$ for some quadratic extension L/F, then $c(\varphi_L) = 0$ in $\mathrm{Br}L$. We then have $\operatorname{ind} c(\varphi) \leq 2$, hence $c(\varphi) = [Q]$ for some quaternion algebra Q over F. It is well-known that in this case φ is divisible by some binary form β (see for example [H 2, Th. 4.1]). With $d = d_\pm\beta$ and $L' = F(\sqrt{d})$, we get $\varphi_{L'} \sim 0$. Finally, if $\varphi_{L'} \sim 0$ for some quadratic extension L'/F, then $\varphi_{L'} \in GP_3L'$, as it is isometric to the hyperbolic 3-fold Pfister form over L'.

Similarly as above, the existence of a biquadratic extension L'/F such that $\varphi_{L'} \sim 0$ trivially implies the existence of a biquadratic extension L/F with $\varphi_L \in GP_3L$, which in turn implies that $\operatorname{ind} c(\varphi) \leq 4$. It remains to show that $\operatorname{ind} c(\varphi) \leq 4$ implies the existence of L' as above. We may assume by (ii) that $\operatorname{ind} c(\varphi) = 4$. By Lemma 2.4(ii), there exists a quadratic extension $E = F(\sqrt{d})$ and a form $\tau \in GP_2E$ such that $\varphi \simeq \mathrm{tr}(\tau)$. By Lemma 2.4(i), there exist $a_1, a_2 \in F^\times$ and binary forms β_1, β_2 over E such that $\tau \sim \langle\langle a_1\rangle\rangle \otimes \beta_1 + \langle\langle a_2\rangle\rangle \otimes \beta_2$ in WE. By [S, Ch. 2, Th. 5.6], we get

$$\varphi \sim \mathrm{tr}(\tau) \sim \langle\langle a_1\rangle\rangle \otimes \mathrm{tr}(\beta_1) + \langle\langle a_2\rangle\rangle \otimes \mathrm{tr}(\beta_2) \ .$$

Let $L' = F(\sqrt{a_1}, \sqrt{a_2})$. Then $\langle\langle a_i\rangle\rangle_{L'} \sim 0$ and hence $\varphi_{L'} \sim 0$. $\square$

Remark 2.6 Using Rost's description of 14-dimensional I^3-forms as certain transfers, one can prove, similarly as in part (iii) of the previous proposition, that every 14-dimensional I^3-form becomes hyperbolic over some multiquadratic extension of degree ≤ 4. Another way of proving this is as follows. Let $\varphi \in I^3F$, $\dim\varphi = 14$. By Cor. 2.2, we can write $\varphi \simeq \psi \perp \alpha$ for some Albert form α. Let $a \in F^\times$ such that $\psi \perp a\alpha$ is isotropic. Note that the anisotropic part of $\psi \perp a\alpha$ has dimension ≤ 12, and it is again in I^3F. By Theorem 2.1, there exists $b \in F^\times$ such that this anisotropic part is divisible by $\langle\langle b\rangle\rangle$. Thus, for $E = F(\sqrt{a}, \sqrt{b})$ we get

$$\varphi_E \sim (\psi \perp \alpha)_E \sim (\psi \perp a\alpha)_E \sim 0 \ .$$

3 Forms of dimension 14 in I^3, of dimension 8 in I^2, and the property $CV(2,2)$

Let E/F be a field extension. Then E/F is said to have the common value property for pairs of forms of dimension n and m, property $CV(n,m)$ for short, if for any pair of forms φ and ψ over F with $\dim\varphi = n$ and $\dim\psi = m$ we have that if φ_E and ψ_E represent a common element over E, then they already represent a common element of $F^\times$ over E, i.e., if $D_E(\varphi)\cap D_E(\psi)\neq\emptyset$, then $D_E(\varphi)\cap D_E(\psi)\cap F^\times\neq\emptyset$. This definition is originally due to Leep [Le]. Trivially, the property $CV(1,n)$ holds for all n and all extensions E/F. We are interested in the case where E/F is a quadratic extension. The following was shown in [STW, Lemma 2.7].

Lemma 3.1 *Let E/F be a quadratic extension. Then E/F has property $CV(2,2)$ iff E/F has property $CV(n,m)$ for all pairs of positive integers n,m.*

We now define certain properties of a field F pertaining to quadratic forms and quaternion algebras and we will investigate the relationships among them.

Property $D(14)$: Every 14-dimensional form in I^3F is similar to the difference of two forms in P_3F or, equivalently by Prop. 2.3, contains a subform in GP_2F.

Property $D(8)$: Every 8-dimensional form $\varphi\in I^2F$ whose Clifford invariant $c(\varphi)$ can be represented by a biquaternion algebra contains a subform in GP_2F.

Property $D(4)$: Suppose φ_1 and φ_2 are 4-dimensional forms over F with $d_\pm\varphi_1 = d_\pm\varphi_2$. If there is a quadratic extension E/F such that $(\varphi_1)_E\simeq(\varphi_2)_E$, then there is a binary form β over F which is similar to a subform of both φ_1 and φ_2.

Property CS: Suppose Q_1 and Q_2 are quaternion algebras over F and E/F is a quadratic extension. If $(Q_1)_E$ and $(Q_2)_E$ have a common slot over E, then such a slot can be chosen in F, i.e., if there exist $u,v,w\in E^\times$ such that $(Q_1)_E\simeq(u,v)_E$ and $(Q_2)_E\simeq(u,w)_E$, then there exists $u'\in F^\times$, $v',w'\in E^\times$ such that $(Q_1)_E\simeq(u',v')_E$ and $(Q_2)_E\simeq(u',w')_E$.

Property $D(2)$: Every quadratic extension E/F has property $CV(2,2)$.

(The notation $D(n)$ alludes to the fact that the thus-labelled property describes a certain behaviour of certain forms of dimension n over the field in question.)

Remark 3.2 (i) As for property $D(8)$, if there exist a biquaternion algebra B over F and an 8-dimensional form $\varphi\in I^2F$ such that $c(\varphi)=[B]$ in $\mathrm{Br}F$ and such that φ does not contain a subform in GP_2, then B is necessarily a division algebra and φ is anisotropic.

For if φ were isotropic, one could readily find 4-dimensional subforms of determinant 1 as φ would contain the universal form $\mathbf{H}$ as a subform. Furthermore, if B were not a division algebra, then there would exist a quaternion algebra Q such that $c(\varphi)=[B]=[Q]$. By Prop. 2.5, φ would become hyperbolic over some quadratic extension $F(\sqrt{d})$ and would therefore be divisible by $\langle\langle d\rangle\rangle$. The existence of a subform in GP_2F would follow immediately.

(ii) As for property $D(4)$, if there exist forms φ_1 and φ_2 over F with $\dim\varphi_1 = \dim\varphi_2 = 4$ and $d_\pm\varphi_1 = d_\pm\varphi_2 = d$ and a quadratic extension E/F such that $(\varphi_1)_E\simeq$

$(\varphi_2)_E$, but there does not exist a binary form β over F such that β is similar to a subform of both φ_1 and φ_2, then the quadratic extension cannot be given by $F(\sqrt{d})$.

In fact, Wadsworth [W] showed that if two 4-dimensional forms over F of the same determinant d become similar over the extension $F(\sqrt{d})$, then they are already similar over F. In view of this result, it is even more remarkable that there are fields where property $D(4)$ fails.

Furthermore, if the two forms φ_1 and φ_2 are as above, then necessarily $d \notin F^{\times 2}$, i.e. $\varphi_1, \varphi_2 \notin GP_2F$. In fact, suppose that $\varphi_1 \simeq r\langle\langle a,b\rangle\rangle$ and $\varphi_2 \simeq s\langle\langle u,v\rangle\rangle$, and let $\alpha \simeq \langle -a,-b,ab,u,v,-uv\rangle$. If there exists a quadratic extension $E = F(\sqrt{e})/F$, $e \in F^\times \backslash F^{\times 2}$, such that $(\varphi_1)_E \simeq (\varphi_2)_E$, then it follows readily that $\langle\langle a,b\rangle\rangle_E \simeq \langle\langle u,v\rangle\rangle_E$ and hence that α_E is hyperbolic. Suppose that α is anisotropic over F. Then there exists a 3-dimensional form γ over F such that $\alpha \simeq \langle\langle e\rangle\rangle \otimes \gamma$ and therefore $d_\pm \alpha = e$, a contradiction. Hence, α is isotropic and there exists $x \in F^\times$ such that $-x$ is represented by $\langle -a,-b,ab\rangle$ and $\langle -u,-v,uv\rangle$. In particular, there exist $y,z \in F^\times$ such that $\langle\langle a,b\rangle\rangle \simeq \langle\langle x,y\rangle\rangle$ and $\langle\langle u,v\rangle\rangle \simeq \langle\langle x,z\rangle\rangle$. It follows that $\beta := \langle\langle x\rangle\rangle$ is similar to a subform of both φ_1 and φ_2.

The following observation provides a useful criterion as for when an 8-dimensional I^2-form whose Clifford invariant can be represented by a biquaternion algebra contains a subform in GP_2F. We will use it in various proofs involving property $D(8)$ (see also [IK 2, Prop. 16.4]).

Lemma 3.3 *Let φ be an 8-dimensional form in I^2F such that $c(\varphi) = [A]$ for some biquaternion algebra A over F with associated Albert form α. The following are equivalent:*

(i) *φ contains a subform in GP_2F.*

(ii) *There exists a quadratic extension $L = F(\sqrt{d})$ such that φ_L is isotropic and A_L is not a division algebra.*

(iii) *There exists a quadratic extension $L = F(\sqrt{d})$ such that φ_L and α_L are both isotropic.*

(iv) *There exists a binary form over F which is similar to a subform of both φ and α.*

Proof. The equivalence of (ii) and (iii) is clear by Albert's theorem, and the equivalence of (iii) and (iv) is also rather obvious. In view of Remark 3.2(i), we may assume that φ is anisotropic and that A is a division algebra, i.e. α is anisotropic. It remains to show (i) $\Longleftrightarrow$ (ii).

Suppose that (i) holds. Then $\varphi \simeq \psi_1 \perp \psi_2$ with $\psi_i \in GP_2F$. Let $L = F(\sqrt{d})$ be any quadratic extension such that ψ_2 becomes isotropic and hence hyperbolic over L. Then we have $c(\varphi_L) = c((\psi_1)_L) = [A_L]$. Since $\psi_1 \in GP_2F$, there exists a quaternion algebra Q over F such that $c(\psi_1) = [Q]$. Hence, $[Q_L] = [A_L]$, which implies that A_L cannot be a division algebra.

Conversely, suppose that there exists a quadratic extension $L = F(\sqrt{d})$ with φ_L isotropic and A_L not division. Since φ_L is isotropic and in I^2L, there exists a 6-dimensional form $\psi \in I^2L$ with $\varphi_L \sim \psi$, in particular, $c(\psi) = c(\varphi_L) = [A_L]$. By Albert's theorem, ψ must be isotropic, hence the Witt index of φ over L is ≥ 2. Thus, there exists a binary form β over F such that $\langle\langle d\rangle\rangle \otimes \beta \subset \varphi$ (cf. [S, Ch. 2, Lemma 5.1]). (i) now follows as $\langle\langle d\rangle\rangle \otimes \beta \in GP_2F$. ◻

Theorem 3.4

$$D(2) \Rightarrow CS \iff D(4) \qquad \textit{and} \qquad D(8) \Rightarrow D(14).$$

Proof. $D(2) \Rightarrow CS$: It is well-known that $(a,b)_F \simeq (a',b')_F$ iff $\langle -a,-b,ab\rangle \simeq \langle -a',-b',a'b'\rangle$. Suppose that F does not have property CS, and let $(a,b)_F$ and $(u,v)_F$ be quaternion algebras over F and let E/F be a quadratic extension such that the quaternion algebras have a common slot over E but such that no common slot over E can be given by an element in F. By the remark above, the fact that they have a common slot over E translates into $D_E(\langle -a,-b,ab\rangle)\cap D_E(\langle -u,-v,uv\rangle)\neq\emptyset$, and the fact that such a common slot cannot be chosen in F translates into $D_E(\langle -a,-b,ab\rangle)\cap D_E(\langle -u,-v,uv\rangle)\cap F^\times=\emptyset$. We conclude that E/F does not have property $CV(3,3)$, which, by Lemma 3.1, yields that F does not have property $D(2)$.

$CS \iff D(4)$: Suppose F does not have property CS and let $(a,b)_F$ and $(u,v)_F$ be quaternion algebras over F such that they have a common slot over $L=F(\sqrt{d})$, but no such common slot can be chosen in F. Let

$$\psi_1 := \langle d,-a,-b,ab\rangle \quad\text{and}\quad \psi_2 := \langle d,-u,-v,uv\rangle\ .$$

We first show that there does not exist a binary form β such that β is similar to a subform of ψ_1 and ψ_2. Then we show that there exists a quadratic extension $E=F(\sqrt{e})$ and some $x\in F^\times$ such that $(\psi_1)_E\simeq(x\psi_2)_E$. This then implies that property $D(4)$ fails.

Suppose there exists a binary form β with, say, $d_\pm\beta=s$ such that β is similar to a subform of ψ_1 and ψ_2. Then the forms $(\psi_1)_L\simeq\langle\langle a,b\rangle\rangle_L$ and $(\psi_2)_L\simeq\langle\langle u,v\rangle\rangle_L$ are, over $L(\sqrt{s})$, isotropic and hence hyperbolic, or, equivalently, the quaternion algebras $(a,b)_L$ and $(u,v)_L$ are split over $L(\sqrt{s})$. Hence, there exist $t,w\in L^\times$ such that $(a,b)_L\simeq(s,t)_L$ and $(u,v)_L\simeq(s,w)_L$, which yields the common slot $s\in F^\times$, a contradiction.

Let now $r\in F^\times$ and consider $\psi_1\perp -r\psi_2\in I^2F$. We then have in WF

$$\begin{array}{rcl}\psi_1\perp -r\psi_2 & \sim & \langle d,-rd\rangle+\langle -a,-b,ab\rangle - r\langle -u,-v,uv\rangle\\ & \sim & \langle -1,r,d,-rd\rangle+\langle 1,-a,-b,ab\rangle - r\langle 1-u,-v,uv\rangle\\ & \sim & \langle\langle a,b\rangle\rangle - r\langle\langle u,v\rangle\rangle - \langle\langle d,r\rangle\rangle\ ,\end{array}$$

which yields $c(\psi_1\perp -r\psi_2)=[(a,b)_F(u,v)_F(d,r)_F]$. Now $(a,b)_F$ and $(u,v)_F$ have a common slot over $L=F(\sqrt{d})$, i.e. $(a,b)_F(u,v)_F$ is not a division algebra over L and thus there exist $x,y,z\in F^\times$ such that $(a,b)_F(u,v)_F\simeq(d,x)_F(y,z)_F$, by [LLT, Prop. 5.2]. The above computation then shows that $c(\psi_1\perp -x\psi_2)=[(y,z)_F]$. Hence, $\psi_1\perp -x\psi_2$ is an 8-dimensional form in I^2F whose Clifford invariant is given by the class of a quaternion algebra, thus there exists a quadratic extension $E=F(\sqrt{e})/F$ such that $(\psi_1\perp -x\psi_2)_E$ is hyperbolic (cf. also Rem. 3.2(i)), i.e. $(\psi_1)_E\simeq(x\psi_2)_E$.

As for the converse, suppose that F does not have property $D(4)$ and let φ_1 and φ_2 be two 4-dimensional forms such that $d_\pm\varphi_1=d_\pm\varphi_2=d$ and that there exists a quadratic extension E/F such that $(\varphi_1)_E\simeq(\varphi_2)_E$, but there does not exist $\beta\in P_1F$ similar to a subform of both φ_1 and φ_2. After scaling, we may assume that there exist $a,b,u,v,x\in F^\times$ such that

$$\varphi_1\simeq\langle d,-a,-b,ab\rangle \quad\text{and}\quad \varphi_2\simeq x\langle d,-u,-v,uv\rangle\ .$$

Similar to above, we have that $\varphi_1 \perp -\varphi_2 \in I^2F$ and that $c(\varphi_1 \perp -\varphi_2) = [(a,b)_F(u,v)_F(d,x)_F]$. On the other hand, $\varphi_1 \perp -\varphi_2$ is hyperbolic over the quadratic extension E of F. Hence, the index of the Clifford algebra of $\varphi_1 \perp -\varphi_2$ can be at most 2, which implies that the Clifford invariant can be represented by a quaternion algebra, say, $c(\varphi_1 \perp -\varphi_2) = [(y,z)_F]$, $y,z \in F^\times$. In particular, $(a,b)_F(u,v)_F \simeq (d,x)_F(y,z)_F$, and it follows that $(a,b)_F(u,v)_F$ is not a division algebra over $L = F(\sqrt{d})$, i.e. $(a,b)_L$ and $(u,v)_L$ have a common slot. To show that property CS fails, it suffices to show that this common slot cannot be in F.

Suppose there exist $r \in F^\times$ and $s,t \in L^\times$ such that $(a,b)_L \simeq (r,s)_L$ and $(u,v)_L \simeq (r,t)_L$. Let $K = F(\sqrt{r})$. Since $(r,s)_L$ and $(r,t)_L$ split over $L(\sqrt{r}) = K(\sqrt{d})$, one sees easily that $(\varphi_1)_{K(\sqrt{d})}$ and $(\varphi_2)_{K(\sqrt{d})}$ are hyperbolic. On the other hand, $d_\pm\varphi_1 = d_\pm\varphi_2 = d$, and it is well-known and easy to show that an anisotropic 4-dimensional form stays anisotropic over the field obtained by adjoining the square root of the determinant of the form. Hence, $(\varphi_1)_K$ and $(\varphi_2)_K$ are both isotropic, which yields that both φ_1 and φ_2 contain subforms similar to $\langle 1, -r\rangle$, a contradiction.

$D(8) \Rightarrow D(14)$: If F does not have property $D(14)$, there exists a form $\varphi \in I^3F$ with $\dim\varphi = 14$ such that φ does not contain a subform in GP_2F. By Cor. 2.2, we can write $\varphi \simeq \alpha \perp \psi$ with an Albert form α and some 8-dimensional form $\psi \in I^2F$. Clearly $\psi \equiv \alpha \pmod{I^3F}$ and therefore $c(\psi) = c(\alpha)$. Since α is an Albert form, there exists a biquaternion algebra B over F such that $c(\alpha) = c(\psi) = [B]$ in $\mathrm{Br}F$. Furthermore, ψ does not contain a subform in GP_2F as φ does not contain such a subform, hence F does not have property $D(8)$. $\square$

We do not know whether $D(4)$ implies $D(8)$ or not.

4 The properties $D(n)$ over fields with a discrete valuation

Let R be a discrete valuation ring with residue class field k and quotient field K. Suppose that $\mathrm{char}\, k \neq 2$, and let π be a uniformizing element of R. For each form φ over K, there exist forms φ_1 and φ_2 which have diagonalizations containing only units in $R^\times$ such that $\varphi \simeq \varphi_1 \perp \pi\varphi_2$. The residue forms $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are called the *first* and *second* residue forms respectively; they are uniquely determined by φ (see [S, Ch. 6, Def. 2.5]). If $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are both anisotropic, then φ is anisotropic. The converse holds if R is 2-henselian, by Springer's theorem [S, Ch. 6, Cor. 2.6]. A typical example of such a discrete valuation ring in the equal characteristic case is $R = k[[t]]$, the power series ring in one variable t.

Our aim is to investigate how the properties $D(n)$, $n \in \{2,4,8,14\}$, behave after going down from K to k or going up from k to K (under the extra hypothesis that R is 2-henselian).

We first go down from K to k, assuming that the residue map $R \to k$ has a section, hence that k can be viewed as a subfield of K. (For instance, K may be an intermediate field between the field of rational fractions $k(t)$ and the power series field $k((t))$, and R the t-adic valuation ring.)

Theorem 4.1 *Suppose the residue map $R \to k$ has a section, and view k as a subfield of R.*

(i) *If K has property $D(4)$, then k has property $D(2)$ (hence also $D(4)$).*

(ii) *If K has property $D(8)$, then k has properties $D(4)$ and $D(8)$.*

(iii) *If K has property $D(14)$, then k has property $D(8)$ (hence also $D(14)$).*

Proof. (i) Suppose that k does not have property $D(2)$. It will suffice to show that K does not have property CS, since Theorem 3.4 shows that CS and $D(4)$ are equivalent. Let $a, b, c \in k^\times$ and let $E = k(\sqrt{e})/k$ be a quadratic extension such that $D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \neq \emptyset$ but $D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \cap k^\times = \emptyset$. Let $L = K(\sqrt{e})$. Then $D_L(\langle -a, -\pi, a\pi\rangle) \cap D_L(\langle -c, -b\pi, bc\pi\rangle) \neq \emptyset$ as these 3-dimensional subforms contain $-\pi\langle 1, -a\rangle_L$ and $-\pi\langle b, -bc\rangle_L$, respectively. We will show that $D_L(\langle -a, -\pi, a\pi\rangle) \cap D_L(\langle -c, -b\pi, bc\pi\rangle) \cap K^\times = \emptyset$, which, by the remark at the beginning of the proof of $D(2) \Rightarrow CS$ in Theorem 3.4, implies that $(a, \pi)_K$ and $(c, b\pi)_K$ have a common slot over L, but no such common slot can be chosen in K, which then shows that property CS fails for K.

In order to do this, we may replace K by its 2-henselization (or by its completion) for the discrete valuation. Then L is 2-henselian with residue field E, and it follows from Springer's theorem (cf. [S, Ch. 6, Cor. 2.6]) that if $D_L(\langle -a, -\pi, a\pi\rangle) \cap D_L(\langle -c, -b\pi, bc\pi\rangle) \cap K^\times \neq \emptyset$, then $D_E(\langle -a\rangle) \cap D_E(\langle -c\rangle) \cap k^\times \neq \emptyset$, which actually implies that $ac \in E^{\times 2}$, or $D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \cap k^\times \neq \emptyset$. The latter can be ruled out by our choice of $a, b, c \in k^\times$. Suppose that $ac \in E^{\times 2}$. Then $\langle 1, -a\rangle_E \simeq \langle 1, -c\rangle_E$. Since $D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \neq \emptyset$, there exists $r \in E^\times$ such that $\langle 1, -a\rangle_E \simeq r\langle 1, -a\rangle_E$ and $\langle b, -bc\rangle_E \simeq r\langle 1, -c\rangle_E$. These facts together yield

$$\langle b, -bc\rangle_E \simeq r\langle 1, -c\rangle_E \simeq r\langle 1, -a\rangle_E \simeq \langle 1, -a\rangle_E \ .$$

In particular, $1 \in D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \cap k^\times$, a contradiction.

(ii) Suppose k does not have property $D(4)$. Let φ_1 and φ_2 be 4-dimensional forms over k such that there exists a quadratic extension $E = k(\sqrt{e})/k$ with $(\varphi_1)_E \simeq (\varphi_2)_E$ but such that there does not exist a binary form β over k which is similar to a subform of both φ_1 and φ_2. Let $\varphi := \varphi_1 \perp -\pi\varphi_2 \in I^2K$. Then φ becomes hyperbolic over the biquadratic extension $K(\sqrt{e}, \sqrt{\pi})$. This shows that the index of the Clifford algebra of φ can be at most 4 and hence there exists a biquaternion algebra B such that $c(\varphi) = [B]$.

In order to prove that K does not have property $D(8)$, it remains to show that φ does not contain a subform in GP_2K. For this, we may replace K by its 2-henselization for the discrete valuation. Suppose $\sigma \in GP_2K$ is such that $\sigma \subset \varphi$. We may decompose $\sigma \simeq \sigma_1 \perp -\pi\sigma_2$, where σ_1 and σ_2 are even-dimensional forms which have a diagonalization containing only units in $R^\times$. By Springer's theorem, the residue forms $\overline{\sigma_1}$ and $\overline{\sigma_2}$ satisfy $\overline{\sigma_1} \subset \varphi_1$ and $\overline{\sigma_2} \subset \varphi_2$. If $\dim\sigma_1 = 0$ or $\dim\sigma_2 = 0$, then φ_2 or φ_1 lies in GP_2F, which is not possible (cf. Rem. 3.2). Therefore, $\dim\sigma_1 = \dim\sigma_2 = 2$. Since $d_\pm\sigma = 1$, there exists $s \in k^\times$ such that $\overline{\sigma_2} \simeq s\overline{\sigma_1}$, in which case $\overline{\sigma_1} \subset \varphi_1$ and $s\overline{\sigma_1} \subset \varphi_2$, a contradiction to the choice of φ_1 and φ_2. We conclude that φ does not contain a subform in GP_2K.

If k does not have property $D(8)$, there exists an 8-dimensional form $\psi \in I^2k$ such that $\operatorname{ind} c(\psi) \leq 4$ which does not contain any subform in GP_2k. As in the preceding argument, we may use residues and Springer's theorem to show that, viewed over K, the form ψ does not contain any subform in GP_2K. Therefore, K does not have property $D(8)$.

(iii) Suppose k does not have property $D(8)$, i.e. there exist an 8-dimensional form $\psi \in I^2k$ and a biquaternion algebra B over k such that $c(\psi) = [B]$, and such that

ψ does not contain a subform in GP_2k. Let α be an Albert form with $c(\alpha) = [B]$. By Remark 3.2, ψ and α are both anisotropic (in the case of α this follows after invoking Albert's theorem because B is a division algebra). In particular, α also does not contain a subform in GP_2k. Consider the form $\varphi := \alpha \perp \pi\psi$ over K. Obviously, $c(\varphi) = c(\alpha)c(\psi) = 1$ in $\mathrm{Br}K$ and thus $\varphi \in I^3K$ and $\dim\varphi = 14$. We will show that φ does not contain a subform in GP_2K which then implies that property $D(14)$ fails for K. For this, we may replace K by its 2-henselization for the discrete valuation.

Suppose there exists $\sigma \in GP_2K$ such that $\sigma \subset \varphi$. As in the proof of (ii) above, we decompose $\sigma \simeq \sigma_1 \perp \pi\sigma_2$ and obtain by Springer's theorem $\overline{\sigma_1} \subset \alpha$ and $\overline{\sigma_2} \subset \psi$. If $\dim\sigma_1 = 0$ or $\dim\sigma_2 = 0$, it follows that ψ or α contains a subform in GP_2k, a contradiction. Therefore, $\dim\sigma_1 = \dim\sigma_2 = 2$ and, since $d_\pm\sigma = 1$, we have $d_\pm\overline{\sigma_1} = d_\pm\overline{\sigma_2}$. Let $d \in k^\times$ be a representative of $d_\pm\overline{\sigma_1}$ and $E = k(\sqrt{d})$. Then α_E and ψ_E are isotropic and it follows from Lemma 3.3 that ψ contains a subform in GP_2k, a contradiction. □

Corollary 4.2 *Let k be a field and let K_i, $1 \le i \le 3$, be any field with $k(t_1, \cdots, t_i) \subset K_i \subset k((t_1)) \cdots ((t_i))$, where t_1, t_2, t_3 are independent variables over k. If k does not have property $D(2)$, then K_1 does not have property $D(4)$, K_2 does not have property $D(8)$, and K_3 does not have property $D(14)$.*

A more precise statement is in Corollary 6.2 below.

Remark 4.3 The hypothesis that the residue map has a section is used in the proof of Theorem 4.1 to find suitable lifts for quadratic forms over k. If the valuation is 2-henselian, this hypothesis is not needed. Indeed, in the proof of part (i) we may choose any lifts a', b', c', $e' \in R$ of a, b, c, e, and set $L = K(\sqrt{e'})$. Since $D_E(\langle 1, -a\rangle) \cap D_E(\langle b, -bc\rangle) \neq \emptyset$, the 2-henselian hypothesis ensures that $D_L(\langle 1, -a'\rangle) \cap D_L(\langle b', -b'c'\rangle) \neq \emptyset$, hence $D_L(\langle -a', -\pi, a'\pi\rangle) \cap D_L(\langle -c', -b'\pi, b'c'\pi\rangle) \neq \emptyset$. The rest of the proof holds without change.

Similarly, in the proof of part (ii), we may choose for φ the quadratic form over K whose first and second residues are φ_1 and φ_2 respectively, and use the henselian hypothesis to see that φ becomes hyperbolic over the biquadratic extension $L(\sqrt{\pi})$, where L is the quadratic extension of K with residue field E.

For the proof of (iii), choose for φ the quadratic form over K whose first and second residues are α and ψ respectively, and use Witt's theorem on the structure of $\mathrm{Br}K$ (which is a Brauer-group analogue of Springer's theorem) (see [Se, Ch. XII, §3]) to see that $c(\varphi) = 1$.

Our next goal is to lift properties $D(n)$ from k to K, assuming that the valuation is 2-henselian.

Theorem 4.4 *Suppose the valuation ring R is 2-henselian.*

(i) *If k has property $D(2)$, then K has property $D(2)$ (hence also $D(4)$).*

(ii) *If k has properties $D(4)$ and $D(8)$, then K has property $D(8)$.*

(iii) *If k has property $D(8)$, then K has property $D(14)$.*

Proof. (i) If k has property $D(2)$, then property $D(2)$ for K follows from [STW, Th. 3.10].

(ii) Assume that k has properties $D(4)$ and $D(8)$. Let $\varphi \in I^2K$, $\dim\varphi = 8$, such that $c(\varphi)$ can be represented by a biquaternion algebra. We want to show that φ contains a subform in GP_2K. By Remark 3.2(i), we may assume that φ is anisotropic. There exists an Albert form α over K such that $\varphi \equiv \alpha \pmod{I^3K}$. (Note that scaling φ resp. α does not affect this congruence.) With decompositions $\varphi \simeq \varphi_1 \perp \pi\varphi_2$ and $\alpha \simeq \alpha_1 \perp \pi\alpha_2$ as above, and using the fact that φ, $\alpha \in I^2K$, we obtain for the first and second residue forms, respectively, that $\overline{\varphi_i}, \overline{\alpha_i} \in Ik$, $i = 1, 2$, and that $d_\pm\overline{\varphi_1} = d_\pm\overline{\varphi_2}$ and $d_\pm\overline{\alpha_1} = d_\pm\overline{\alpha_2}$ in $k^\times/k^{\times 2}$. Furthermore, $(\varphi_1 \perp -\alpha_1) \perp \pi(\varphi_2 \perp -\alpha_2) \in I^3K$, hence $\overline{\varphi_i} \perp -\overline{\alpha_i} \in I^2k$, $i = 1, 2$, and thus in fact $d_\pm\overline{\varphi_1} = d_\pm\overline{\varphi_2} = d_\pm\overline{\alpha_1} = d_\pm\overline{\alpha_2}$.

If $\dim\varphi_1 = 0$ then $\overline{\varphi_2}$ is an 8-dimensional form in I^2k whose Clifford invariant can obviously be represented by some biquaternion algebra over k. Since k has property $D(8)$, $\overline{\varphi_2}$ contains some form in GP_2k as a subform. This subform can be lifted to a form in GP_2K which will be a subform of φ_2 and thus similar to a subform of φ. The case $\dim\varphi_2 = 0$ is treated in an analogous way. Thus, we may assume after scaling φ that $(\dim\varphi_1, \dim\varphi_2) \in \{(2, 6), (4, 4)\}$.

If $\dim\alpha_1 = 0$ or $\dim\alpha_2 = 0$, then $\overline{\alpha_i} \in I^2k$ which, by the above discriminant comparison, yields that $\overline{\varphi_1}$, $\overline{\varphi_2} \in I^2k$. In the case $\dim\varphi_1 = 2$, this forces $\overline{\varphi_1} \simeq \mathbf{H}$ which in turn implies that φ is isotropic, contrary to our assumption. If $\dim\varphi_1 = 4$, we have $\overline{\varphi_1} \in GP_2k$, and thus we even have $\varphi_1 \in GP_2K$. Hence, we may assume after scaling α that $\dim\alpha_1 = 2$, $\dim\alpha_2 = 4$, and that $\alpha_1 \perp -\varphi_1$ is isotropic.

If $\dim\varphi_1 = 2$, then the isotropy of $\alpha_1 \perp -\varphi_1$ together with $\overline{d_\pm\overline{\varphi_1}} = d_\pm\overline{\alpha_1} = \overline{d}$ for some $d \in R^\times$ implies that $\overline{\varphi_1} \simeq \overline{\alpha_1}$ which in turn is similar to $\overline{\langle 1, -d\rangle}$. Thus, over $\ell = k(\sqrt{\overline{d}})$, we get $(\overline{\alpha_2})_\ell \equiv (\overline{\varphi_2})_\ell \pmod{I^3\ell}$ and $(\overline{\alpha_2})_\ell$, $(\overline{\varphi_2})_\ell \in I^2\ell$. In particular, $(\overline{\varphi_2})_\ell$ is an Albert form, $(\overline{\alpha_2})_\ell \in GP_2\ell$, and $c((\overline{\varphi_2})_\ell) = c((\overline{\alpha_2})_\ell)$. Since $c((\overline{\alpha_2})_\ell)$ can be represented by a single quaternion algebra, this implies that the Albert form $(\overline{\varphi_2})_\ell$ is isotropic, and $\overline{\varphi_2}$ contains therefore a subform similar to $\overline{\langle 1, -d\rangle}$ over k. After lifting, we see that there exist $x, y \in R^\times$ such that $\varphi_1 \simeq x\langle 1, -d\rangle$ and $y\langle 1, -d\rangle \subset \varphi_2$. Hence, φ contains $\langle x, y\pi\rangle \otimes \langle 1, -d\rangle \in GP_2K$ as a subform.

Finally, suppose that $\dim\varphi_1 = 4$. The fact that φ_1 is anisotropic of dimension 4, $\dim\alpha_1 = 2$ and $\alpha_1 \perp -\varphi_1$ is isotropic imply that $\overline{\psi_1} = (\overline{\alpha_1} \perp -\overline{\varphi_1})_{\mathrm{an}}$ is not hyperbolic and of dimension ≤ 4. Since $d_\pm\overline{\varphi_1} = d_\pm\overline{\alpha_1}$, we also have $\overline{\psi_1} \in I^2k$. All this together yields $\overline{\psi_1} \in GP_2k$. Lifting $\overline{\psi_1}$ to a form $\psi_1 \in GP_2K$, we get by Springer's theorem

$$-\psi_1 + \pi(\varphi_2 \perp -\alpha_2) \sim (\varphi_1 \perp -\alpha_1) + \pi(\varphi_2 \perp -\alpha_2) \in I^3K\ ,$$

thus

$$\psi_1 \equiv \pi(\varphi_2 \perp -\alpha_2) \equiv \varphi_2 \perp -\alpha_2 \pmod{I^3K}\ ,$$

which obviously implies $\overline{\psi_1} \equiv \overline{\varphi_2} \perp -\overline{\alpha_2} \pmod{I^3k}$. Since $\overline{\varphi_2} \perp -\overline{\alpha_2}$ is an 8-dimensional I^2k-form whose Clifford invariant is the same as that of $\overline{\psi_1} \in GP_2k$, i.e., it can be represented by a single quaternion algebra, there exists $e \in R^\times$ such that $\overline{\varphi_2} \perp -\overline{\alpha_2}$ becomes hyperbolic over $k(\sqrt{\overline{e}})$ (see also Remark 3.2(i)), i.e., $\overline{\varphi_2}$ and $\overline{\alpha_2}$ are 4-dimensional forms which become isometric over the quadratic extension $k(\sqrt{\overline{e}})$. Since k has property $D(4)$, there exists $b \in R^\times$ such that $\overline{\langle 1, -b\rangle}$ is similar to a subform of both $\overline{\varphi_2}$ and $\overline{\alpha_2}$. After lifting, this shows that $\langle 1, -b\rangle$ is similar to a subform of both φ and α. It follows from Lemma 3.3 that φ contains a subform in GP_2K.

(iii) Suppose that k has property $D(8)$ and let φ be a 14-dimensional I^3-form over K, which we write as $\varphi \simeq \varphi_1 \perp \pi\varphi_2$ with first resp. second residue form $\overline{\varphi_1}$

resp. $\overline{\varphi_2}$ over k. To establish property $D(14)$, it suffices by Prop. 2.3 to show that φ contains a subform in GP_2K. This is obvious if φ is isotropic, so that we may assume that φ and hence $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are anisotropic. We have that $\overline{\varphi_1}, \overline{\varphi_2} \in I^2k$ as $\varphi \in I^3K$, and after scaling we may assume that $\dim \overline{\varphi_2} \in \{0, 2, 4, 6\}$.

If $\dim \overline{\varphi_2} = 0$, then $\varphi \simeq \varphi_1$ and we have in fact $\overline{\varphi_1} \in I^3k$. Since k has property $D(8)$, it has property $D(14)$ by Theorem 3.4, and by Prop. 2.3, $\overline{\varphi_1}$ contains a subform in GP_2k which can be lifted to a subform of φ in GP_2K.

If $\dim \overline{\varphi_2} = 2$, then $\overline{\varphi_2} \in I^2k$ implies that $\overline{\varphi_2}$ is isotropic, contrary to our assumption.

If $\dim \overline{\varphi_2} = 4$, then $\overline{\varphi_2} \in I^2k$ implies that $\overline{\varphi_2} \in GP_2k$, and after lifting we find again a subform of φ which is in GP_2K.

Finally, if $\dim \overline{\varphi_2} = 6$, then $\overline{\varphi_2}$ is an Albert form over k with associated biquaternion algebra A over k. Furthermore, $\overline{\varphi_1}$ is an 8-dimensional I^2-form over k and one has that $\overline{\varphi_1} \equiv \overline{\varphi_2} \pmod{I^3k}$, so that $c(\overline{\varphi_1}) = [A]$. Since k has property $D(8)$, it follows from Lemma 3.3 that there is a binary form $\overline{\beta}$ over k which is similar to both a subform of $\overline{\varphi_1}$ and of $\overline{\varphi_2}$. Lifting $\overline{\beta}$ to a binary form β over K, we see that φ_1 and φ_2 each contain a subform similar to β, say, $u\beta \subset \varphi_1$ and $v\beta \subset \pi\varphi_2$, $u, v \in K^\times$. Hence, φ contains $\langle u, v\rangle \otimes \beta \in GP_2K$ as a subform. □

Combining Remark 4.3 and Theorem 4.4, we obtain:

Corollary 4.5 (i) *k has property $D(2)$ iff K has property $D(2)$ iff K has property $D(4)$.*

(ii) *k has properties $D(4)$ and $D(8)$ iff K has property $D(8)$.*

(iii) *k has property $D(8)$ iff K has property $D(14)$.*

Note that for $n \in \{4, 8, 14\}$ it is generally *not* true that if $D(n)$ holds over k then $D(n)$ also holds over K, cf. Ex. 5.4 below.

Recall that a field F is called linked if the quaternion algebras over F form a subgroup in $\mathrm{Br}F$, in particular, any two quaternion algebras over F have a common slot and there are therefore no biquaternion division algebras. This readily implies that a linked field F always has properties $D(n)$, $n \in \{4, 8, 14\}$. We will encounter typical examples, like finite, local or global fields, etc., also in Cor. 5.1 below. But first, let us state the following immediate consequences of Theorem 4.4.

Corollary 4.6 *Let $K_0, K_1, K_2, \cdots$ be fields of characteristic $\neq 2$ such that K_{i+1} is the quotient field of a 2-henselian discrete valuation ring R_{i+1} with residue field K_i, $i \geq 0$. If K_0 has property $D(2)$, then K_i has property $D(2)$ for all $i \geq 0$.*

(i) *If K_0 has property $D(2)$ and $D(8)$, then K_i has property $D(n)$ for all $i \geq 0$ and all $n \in \{2, 4, 8, 14\}$.*

(ii) *If K_0 is linked, then K_0 has property $D(n)$ for $n \in \{4, 8, 14\}$, K_1 has properties $D(8)$ and $D(14)$, and K_2 has property $D(14)$.*

Proof. (i) follows by induction from Theorems 3.4 and 4.4, and (ii) is a consequence of the preceding remarks together with Theorem 4.4. □

5 Fields with finite Hasse number

For a field F, the Hasse number $\tilde{u}(F)$ is defined to be the supremum of the dimensions of anisotropic totally indefinite quadratic forms over F, where totally indefinite means indefinite with respect to each ordering on F. If F is not formally real, i.e., if F does not possess any orderings, then $\tilde{u}(F)$ is nothing but the supremum of the dimensions of anisotropic forms over F and coincides with the u-invariant $u(F)$, the supremum of the dimensions of anisotropic torsion forms. In the sequel, we investigate the properties $D(n)$, $n \in \{2,4,8,14\}$, over fields with finite Hasse number and of power series extensions of such fields.

For basic properties of fields with finite Hasse number, we refer the reader to [ELP]. Let us just mention that one always has $\tilde{u}(F) \neq 3,5,7$, and that F is a so-called SAP field if $\tilde{u}(F) < \infty$. Furthermore, using Merkurjev's index reduction formulas [Me 2], one can construct fields F with $\tilde{u}(F) = 2n$ for any integer $n \geq 0$, see for example [L 2], [Ho]. It is also well-known that fields of transcendence degree ≤ 1 over a real closed field have $\tilde{u} \leq 2$ (cf. [ELP, Th. I]), finite fields have $\tilde{u} = 2$, and local and global fields have $\tilde{u} = 4$ (for global fields, this is Meyer's theorem). Furthermore, if $\tilde{u}(F) \leq 4$, then F is linked. Conversely, if F is linked, then $\tilde{u}(F) \in \{0,1,2,4,8\}$ (cf. [EL], [E, Th. 4.7]).

Corollary 5.1 *Let F_0 be a field with $\tilde{u}(F_0) \leq 2$, or let F_0 be a local or global field. Let $F_i = F((t_1))\cdots((t_i))$ be the iterated power series field in i variables over F_0. Then F_i has property $D(n)$ for all $i \geq 0$ and all $n \in \{2,4,8,14\}$.*

Proof. By Cor. 4.6, it suffices to verify that F_0 has properties $D(2)$ and $D(8)$. For property $D(2)$, this follows from [STW, Ths. 3.6, 3.7]. Property $D(8)$ is a consequence of the fact that in each case, F_0 is a linked field (cf. [EL, § 1]). □

In the sequel, X_F denotes the space of orderings on F, and $\mathrm{sgn}_P(\varphi)$ denotes the signature of the form φ at the ordering $P \in X_F$.

Lemma 5.2 (i) *Let φ be an anisotropic form over F. Then*

$$\dim\varphi \leq \sup\{\tilde{u}(F), |\mathrm{sgn}_P(\varphi)|; P \in X_F\} .$$

(ii) *Let $\tilde{u}(F) \leq r$ and let φ_1, φ_2 be forms over F of dimension ≥ 3 such that $\dim\varphi_1 + \dim\varphi_2 \geq r+3$. Then there exists a binary form β which is similar to a subform of both φ_1 and φ_2.*

Proof. (i) If $\dim\varphi > \sup\{|\mathrm{sgn}_P(\varphi)|; P \in X_F\}$, then φ is totally indefinite, hence $\dim\varphi \leq \tilde{u}(F)$.

(ii) Since F is SAP, there exist $a_1, a_2 \in F^\times$ such that $\mathrm{sgn}_P(a_1\varphi_1), \mathrm{sgn}_P(a_2\varphi_2) \geq 0$ for all $P \in X_F$. Hence, $|\mathrm{sgn}_P(a_1\varphi_1 \perp -a_2\varphi_2)| \leq \dim\varphi_1 + \dim\varphi_2 - 3$ for all $P \in X_F$, and since $\dim\varphi_1 + \dim\varphi_2 - 3 \geq \tilde{u}(F)$, it follows from (i) that $\dim(a_1\varphi_1 \perp -a_2\varphi_2)_{\mathrm{an}} \leq \dim\varphi_1 + \dim\varphi_2 - 3$, which in turn yields for the Witt index that $i_W(a_1\varphi_1 \perp -a_2\varphi_2) \geq 2$. This shows that $a_1\varphi_1$ and $a_2\varphi_2$ have a common binary subform. □

We have seen above that iterated power series fields over fields with $\tilde{u} \leq 2$ always have the properties $D(n)$, $n \in \{2,4,8,14\}$. We now ask what happens if the base field has $\tilde{u} \geq 4$. Note that if $\tilde{u} \leq 4$, then F is linked as already mentioned above. (One can see this also by applying Lemma 5.2(ii), which shows that two 4-dimensional forms over F have always up to similarity a common binary subform, which, applied to 2-fold Pfister forms, implies linkage.) Of particular interest is the case $\tilde{u} = 4$ as will be illustrated by Ex. 5.4 below. For this reason, we state explicitly the following special case of Cor. 4.6(ii).

Corollary 5.3 *Let $F_i = F((t_1))\cdots((t_i))$ be the iterated power series field in i variables over a field F_0 with $\tilde{u}(F_0) = 4$.*

(i) *F_0 has property $D(n)$ for $n \in \{4, 8, 14\}$;*

(ii) *F_1 has property $D(n)$ for $n \in \{8, 14\}$;*

(iii) *F_2 has property $D(14)$.*

Example 5.4 Let $F = \mathbf{C}(x, y)$, the rational function field in two variables x, y over the complex numbers $\mathbf{C}$. It is well-known that $u(F) = \tilde{u}(F) = 4$. F does *not* have property $D(2)$ (cf. [STW, Remarks 4.18, 5.10]). But it has property $D(n)$, $n \in \{4, 8, 14\}$ by Cor. 5.3. It also shows that linked fields generally do not have property $D(2)$.

By Theorem 4.1, $F_1 = F((t_1))$ does not have property $D(4)$, but it has property $D(n)$ for $n \in \{8, 14\}$ by Cor. 5.3. Similarly, we see that $F_2 = F((t_1))((t_2))$ does not have property $D(8)$, but that it does have property $D(14)$.

All this shows that generally, the statements regarding the properties $D(n)$ in Cor. 5.3 cannot be strengthened. It shows furthermore for $n, m \in \{2, 4, 8, 14\}$, $m > n$, that generally $D(m) \not\Rightarrow D(n)$, so that the implications in Theorem 3.4 cannot be reversed without any further assumptions on the field in question.

For values of $\tilde{u}$ possibly bigger than 4, let us note the following.

Corollary 5.5 (i) *If $\tilde{u}(F) < 12$, then F has properties $D(8)$, $D(14)$, and $F((t))$ has property $D(14)$.*

(ii) *If $\tilde{u}(F) < 14$, then F has property $D(14)$.*

Proof. (i) Let φ be an 8-dimensional I^2-form over F such that $c(\varphi)$ can be represented by a biquaternion algebra A with associated Albert form α. To establish property $D(8)$, it suffices by Lemma 3.3 to show that φ and α have a common binary subform. Since $\tilde{u}(F) < 12$, this is an easy consequence of Lemma 5.2(ii). Property $D(14)$ for $F((t))$ follows from Theorem 4.4.

(ii) Let $\varphi \in I^3F$, $\dim\varphi = 14$. If F is not formally real, then $\tilde{u}(F) < 14$ implies that φ is isotropic and $D(14)$ follows easily. If F is formally real, then we first note that for each $P \in X_F$ we have $\mathrm{sgn}_P(\varphi) \equiv 0 \pmod 8$ because $\varphi \in I^3F$. Hence, $\mathrm{sgn}_P(\varphi) \in \{0, \pm 8\}$ as $\dim\varphi = 14$. By Lemma 5.2(i), $\dim\varphi_{\mathrm{an}} < 14$. Thus, again we have that φ is isotropic and we are done. □

Example 5.6 It is again interesting in this context to consider the example from above based on $\mathbf{C}(x, y)$. As was shown there, the field $F_1 = \mathbf{C}(x, y)((t_1))$ has property $D(8)$, but not $D(4)$, and $F_2 = F_1((t_2))$ has property $D(14)$, but not $D(8)$. $F_3 = F_2((t_3))$ does not even have property $D(14)$. One has $\tilde{u}(F_1) = u(F_1) = 8$, which shows that in part (i) of the above corollary, one cannot always expect that property $D(8)$ carries over to a power series extension. Also, F_2 is a field for which $D(8)$ fails, and we have $\tilde{u}(F_2) = u(F_2) = 16$, which is still a little higher than the bound given in part (i) above which assures that $D(8)$ holds. This naturally raises the question whether the bound given there is the best possible.

We note furthermore that $\tilde{u}(F_3) = u(F_3) = 32$. For F_3, we know that $D(14)$ fails, but its Hasse number is considerably higher than the bound in part (ii) of the above corollary, and therefore this example does not give an indication on how good this bound really is.

Knowing that $D(4)$ always holds if $\tilde{u}(F) \leq 4$ (see Corollaries 5.1 and 5.3) and that it can fail if $\tilde{u}(F) \geq 8$ (see Examples 5.4 and 5.6), it would be interesting to know if there exist fields F with $\tilde{u}(F) = 6$ for which $D(4)$ fails. We do know by Corollary 5.5 that $D(8)$ holds whenever $\tilde{u}(F) \leq 12$, so it holds in particular for all fields with $\tilde{u}(F) \leq 6$. In the following proposition, we establish property $D(8)$ for another class of fields which also contains all fields F with $\tilde{u}(F) \leq 6$.

In the sequel, $I_t^n F = I^n \cap W_t F$, where $W_t F$ denotes the torsion part of the Witt ring. If F is not formally real, then $WF = W_t F$, otherwise $W_t F$ consists of the classes of forms which have total signature zero (Pfister's local-global principle).

PROPOSITION 5.7 *Suppose that $I_t^3 F = 0$ and that F is SAP. Then F has property $D(8)$ (and hence also $D(14)$), and $F((t))$ has property $D(14)$.*

Proof. In view of Theorems 3.4 and 4.4, it suffices to establish property $D(8)$ for F.

Let $\varphi \in I^2 F$, $\dim \varphi = 8$ and $c(\varphi) = c(\alpha)$ with α an Albert form. We have to show that φ contains a subform in $GP_2 F$.

Suppose first that F is not formally real. By Merkurjev's theorem, we have $\varphi - \alpha \in I^3 F = I_t^3 F = 0$, hence $\varphi \sim \alpha$, and comparing dimensions yields that φ is isotropic and therefore contains a subform in $GP_2 F$ (see Remark 3.2(i)).

Hence, we may assume that F is formally real. Since $\varphi, \alpha \in I^2 F$, we have for all orderings $P \in X_F$ that $\mathrm{sgn}_P(\varphi), \mathrm{sgn}_P(\alpha) \equiv 0 \pmod 4$. Since $\dim \alpha = 6$ and $\dim \varphi = 8$, and since F is SAP, we may assume after scaling that $\mathrm{sgn}_P(\varphi) \in \{0, 4, 8\}$ and $\mathrm{sgn}_P(\alpha) \in \{0, 4\}$. On the other hand, we have $\varphi - \alpha \in I^3 F$ and thus $\mathrm{sgn}_P(\varphi \perp -\alpha) \equiv 0 \pmod 8$. Thus, we always have $\mathrm{sgn}_P(\varphi \perp -\alpha) \in \{0, 8\}$. Now if $\pi \in P_3 F$, then $\mathrm{sgn}_P(\pi) \in \{0, 8\}$, and since F is SAP, there exists $\pi \in P_3 F$ such that $\mathrm{sgn}_P(\pi) = \mathrm{sgn}_P(\varphi \perp -\alpha)$ for all $P \in X_F$. Hence, $\mathrm{sgn}_P(\varphi \perp -\alpha \perp -\pi) = 0$ for all $P \in X_F$, i.e. $\varphi \perp -\alpha \perp -\pi \in I^3 F \cap W_t F = I_t^3 F = 0$. Thus, $\varphi \perp -\pi \sim \alpha$, and comparing dimensions yields that the Witt index of $\varphi \perp -\pi$ is ≥ 5. In particular, φ contains a 5-dimensional Pfister neighbor of π as a subform. It is well-known that 5-dimensional Pfister neighbors always contain a subform in $GP_2 F$. Hence, φ contains a subform in $GP_2 F$. □

REMARK 5.8 (i) Note that the two classes of fields for which we established property $D(8)$, fields with $\tilde{u} < 12$ and SAP-fields with $I_t^3 F = 0$, respectively, are such that one class is not contained in the other. Indeed, using constructions similar to those in [L 2], [Ho], it is not difficult to construct fields F with $\tilde{u}(F) = 8$ or 10 and $I_t^3 F \neq 0$. On the other hand, to any positive integer n, there exist fields with $\tilde{u}(F) = 2n$ and $I_t^3 F = 0$ (cf. [Ho]). Since their Hasse number is finite, they are SAP-fields. Thus, there are SAP-fields with $I_t^3 F = 0$ for which $\tilde{u} \geq 12$.

(ii) We do not know whether $I_t^3F=0$ alone already suffices for property $D(8)$ (or maybe even $D(4)$) to hold, or whether we can replace SAP by some weaker property which together with $I_t^3F=0$ would imply property $D(8)$. Consider, for example, the field $F=\mathbf{R}((t_1))\cdots((t_i))$ with $i\geq 2$. We have $I_t^3F=0$ (in fact, we even have $W_tF=0$), and it is well-known that F is not SAP. However, F does have property $D(n)$, $n\in\{2,4,8,14\}$ by Corollary 5.1. Note also that $I_t^3F=0$ alone does not imply property $D(2)$ in general, as exemplified by the field $\mathbf{C}(x,y)$ (see Example 5.4).

(iii) It is well-known that a field F satisfies $I_t^2F=0$ and SAP if and only if $\tilde{u}(F)\leq 2$ (cf. [ELP, Theorems E, F]). In this case, F and its iterated power series extensions have property $D(n)$, $n\in\{2,4,8,14\}$ by Corollary 5.1.

6 Some further consequences and examples

A field extension K/F is said to be *excellent* if for every quadratic form φ over F there exists a form ψ over F such that $(\varphi_K)_{\mathrm{an}}\simeq\psi_K$, i.e. the anisotropic part of φ over K is defined over F. Izhboldin and Karpenko [IK 1, Part II] considered the question of excellence of extensions K/F where K is the function field of a Severi-Brauer variety $SB(A)$ of a central simple algebra A over F. One of the crucial cases in their investigations was the case where A was an algebra of exponent 2. In this situation, if the algebra is of index ≤ 2, then K/F is excellent as was shown by Arason in [ELW, App. II]. If the index is 8, then K/F is never excellent as was shown in [IK 1, Part II, Th. 3.10]. If the index is equal to 4, i.e. A is a biquaternion division algebra, examples are given in [IK 1] which show that both excellence and nonexcellence are possible for such an extension. Izhboldin himself noticed that if a field F does not have property $D(8)$, then one can readily find examples of biquaternion algebras A over F such that $F(SB(A))/F$ is nonexcellent.

In [Ma], Mammone gave counterexamples to a question raised by Knus concerning the product of a biquaternion algebra B and a quaternion algebra Q over F, both assumed to be division algebras: If $B\otimes_F Q$ is not a division algebra, does it follow that there exists a quadratic extension L/F over which both Q and B are not division (i.e. Q and B have a quadratic extension of F as a common subfield)? Again, if F does not have property $D(8)$ then a pair B, Q can be readily found which provides a counterexample.

The previous two implications for a field where property $D(8)$ fails are summarized in the following proposition.

Proposition 6.1 *Let F be a field where property $D(8)$ fails. Then the following holds:*

(i) (Izhboldin) *There exists a biquaternion division algebra A over F such that $F(SB(A))/F$ is nonexcellent.*

(ii) *There exist a biquaternion division algebra B over F and a quaternion division algebra Q over F which have the following properties:*

  (a) *$B\otimes_F Q$ is not a division algebra, and yet*

  (b) *there does not exist a quadratic extension L/F which is a common subfield of B and Q.*

Proof. Since F does not have property $D(8)$, there exist a biquaternion division algebra A over F and a form $\varphi \in I^2F$, $\dim\varphi = 8$ such that $c(\varphi) = [A]$ and such that φ does not contain a subform in GP_2F. After scaling, we may assume that $1 \in D(\varphi)$.

(i) Let $K = F(SB(A))$. By Rem. 3.2(i), φ is anisotropic and thus φ_K is also anisotropic (cf. [La, Th. 4]). In particular, φ_K is an anisotropic form in I^3K representing 1. Hence, $\varphi_K \in P_3K$. Let $\varphi \simeq \langle 1, -a, -b, \cdots\rangle$, $a, b \in F^\times$. It follows readily that there exists $c \in K^\times$ such that $\varphi_K \simeq \langle\langle a, b, c\rangle\rangle_K$. Suppose that K/F is excellent. Then, by [ELW, Prop. 2.11], we may assume that $c \in F^\times$ and we put $\pi := \langle\langle a, b, c\rangle\rangle \in P_3F$.

Let $\psi := (\varphi \perp -\pi)_{\mathrm{an}}$. We have $\psi \in I^2F$, $c(\psi) = c(\varphi) = [A]$, and $\dim\psi \le 10$. If $\dim\psi \le 6$ then φ and π have at least a 5-dimensional subform in common, i.e., φ contains a Pfister neighbor of π. Now each 5-dimensional Pfister neighbor contains a subform in GP_2F, thus φ contains a subform in GP_2F, a contradiction.

If $\dim\psi = 8$, then it follows again from [La, Th. 4] that ψ_K is anisotropic, a contradiction because we have by construction that ψ_K is hyperbolic.

Finally, suppose that $\dim\psi = 10$. Let $E = F(\psi)$. Then $\dim(\psi_E)_{\mathrm{an}} = 8$ or 6 (cf. [H 1, Cor. 1]). If $\dim(\psi_E)_{\mathrm{an}} = 8$, then, since $c(\psi_E) = [A_E]$ in $\mathrm{Br}E$, we have again that $(\psi_E)_{\mathrm{an}}$ stays anisotropic over $E(SB(A_E))$, obviously a contradiction to ψ becoming hyperbolic over $K = F(SB(A))$. Hence, $\dim(\psi_E)_{\mathrm{an}} = 6$, and by [H 2, Lemma 3.3] it follows that there exist a 6-dimensional form β and an anisotropic $\tau \in GP_4F$ such that $\psi \perp \beta \simeq \tau$. On the other hand, ψ and thus τ contain a 5-dimensional subform of $-\pi \in GP_3F$. Hence, τ becomes hyperbolic over $F(\pi)$. Using the multiplicativity of Pfister forms and the fact that $\tau \in W(F(\pi)/F)$ is anisotropic, we conclude readily that there exists $x \in F^\times$ such that $\tau \simeq -\pi \perp x\pi$. In the Witt ring, we thus get

$$\psi + \beta \sim \varphi - \pi + \beta \sim -\pi + x\pi$$

and hence $x\pi - \varphi \sim \beta$. Comparing dimensions yields that φ and $x\pi$ have a 5-dimensional subform in common, i.e., φ contains a Pfister neighbor of π and we get a contradiction as before.

(ii) After scaling, we may assume that $\varphi \simeq \langle -x, -y, xy\rangle \perp \varphi'$ for suitable $x, y \in F^\times$ and some form φ' over F with $\dim\varphi' = 5$ and $\det\varphi' = 1$. Now φ' does not represent $1 = \det\varphi'$ as φ' does not contain a subform in GP_2F. In particular, the Albert form $\beta := \varphi' \perp \langle -1\rangle$ is anisotropic, and therefore the biquaternion algebra B with $c(\beta) = [B]$ is a division algebra by Albert's theorem. Since $\langle -x, -y, xy\rangle$ is anisotropic, we also have that the quaternion algebra $Q = (x, y)_F$ is a division algebra. Furthermore, $\varphi \sim \langle\langle x, y\rangle\rangle + \beta$ in WF and therefore

$$[A] = c(\varphi) = c(\langle\langle x, y\rangle\rangle \perp \beta) = c(\langle\langle x, y\rangle\rangle)c(\beta) = [Q][B]$$

and it follows that $Q \otimes_F B$ is not a division algebra.

Suppose there exists a quadratic extension $L = F(\sqrt{d})/F$ such that Q_L and B_L are both not division. Then $\langle\langle x, y\rangle\rangle_L$ is hyperbolic and β_L is isotropic. It follows that φ_L is isotropic and A_L is not division. By Lemma 3.3, this implies that φ contains a subform in GP_2F, a contradiction. □

For an element $a \in F^\times$, let $N_F(a)$ denote the norm group $D_F(\langle 1, -a\rangle)$. Let now $a, b, c \in F^\times$ and let $E = F(\sqrt{c})$. Consider the following factor group:

$$N_1(a, b, c) = \frac{F^\times \cap N_E(a)N_E(b)}{(F^\times \cap N_E(a))(F^\times \cap N_E(b))}\ .$$

Corollary 6.2 *Let F be a field such that there exist $a,b,c \in F^\times$ with $N_1(a,b,c) \neq 1$. Let $E = F(\sqrt{c})$ and let $d \in F^\times \cap N_E(a)N_E(b) \setminus (F^\times \cap N_E(a))(F^\times \cap N_E(b))$. Let t_1, t_2, t_3 be independent variables over F and $F_i = F(t_1, \cdots, t_i)$ (or $F_i = F((t_1)) \cdots ((t_i))$), $i = 1, 2, 3$, and let $E_i = F_i(\sqrt{c})$.*

(i) *$\langle 1, -a\rangle$ and $d\langle 1, -b\rangle$ represent a common element over $E = F(\sqrt{c})$, but there does not exist an element in $F^\times$ which is represented by $\langle 1, -a\rangle$ and $d\langle 1, -b\rangle$ over $E = F(\sqrt{c})$.*

(ii) *The two quaternion algebras $(a, t_1)_{F_1}$ and $(b, t_1 d)_{F_1}$ have a common slot over E_1, but such a common slot cannot be chosen in F_1.*

(iii) *Let $\psi_1 := \langle c, -a, -t_1, t_1 a\rangle$ and $\psi_2 := \langle c, -b, -t_1 d, t_1 db\rangle$. Then there exist $u, v \in F_1^\times$ such that for $L = F_1(\sqrt{u})$ one has $(\psi_1)_L \simeq v(\psi_2)_L$, but there does not exist a binary form over F_1 which is similar to a subform of both ψ_1 and ψ_2.*

(iv) *The Clifford invariant of the form $\psi := \psi_1 \perp -t_2\psi_2 \in I^2 F_2$ can be represented by a biquaternion algebra A over F_2, but ψ does not contain any subform in $GP_2 F_2$.*

(v) *Let α be the Albert form over F_2 associated to A, and let $\varphi := \alpha \perp t_3\psi$. Then $\varphi \in I^3 F_3$, $\dim\varphi = 14$, but φ is not similar to the difference of the pure parts of two forms in $P_3 F_3$.*

Proof. Let $d = rs$, where $r \in N_E(a)$ and $s \in N_E(b)$. By multiplicativity of the norm form, we have $s^{-1} \in N_E(b)$, and the equality $r = ds^{-1}$ shows that $r \in D_E(\langle 1, -a\rangle)$ is represented by $d\langle 1, -b\rangle$. Suppose $D_E(\langle 1, -a\rangle) \cap D_E(d\langle 1, -b\rangle)$ contains an element $x \in F^\times$; then $x \in F^\times \cap N_E(a)$ and $x = dy$ for some $y \in N_E(b)$. Since $y = d^{-1}x \in F^\times$, we have $y \in F^\times \cap N_E(b)$. It follows that $d \in (F^\times \cap N_E(a))(F^\times \cap N_E(b))$ since $d = xy^{-1}$. This proves (i) (see also [STW, p. 69]). The remaining statements follow from Theorem 4.1 and its proof. □

Part (i) shows that property $D(2)$ fails for F if there exist $a, b, c \in F^\times$ with $N_1(a,b,c) \neq 1$. Actually, tracing back through the proof, it is easily seen that property $D(2)$ is equivalent to the vanishing of the group $N_1(a,b,c)$ for all $a,b,c \in F^\times$ (see [STW, Cor. 2.14]).

The group $N_1(a,b,c)$ occurs in [STW] as the homology group of a certain complex associated with the multiquadratic extension $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$. A more symmetric description of this group is given in [G, Prop. 3]:

$$N_1(a,b,c) \simeq \frac{N_F(a) \cap N_F(b) \cap N_F(c)}{F^{\times 2} N_{M/F}(M^\times)}.$$

As mentioned in the introduction, there exist fields F such that $D(2)$ fails, i.e., there exist $a,b,c \in F^\times$ with $N_1(a,b,c) \neq 1$. In [STW, Cor. 5.6 and 5.7], it is for example shown that $D(2)$ fails for finitely generated extensions of transcendence degree ≥ 2 (resp. ≥ 1) over any field of characteristic 0 (resp. over $\mathbf{Q}$).

Examples where $N_1(a,b,c) \neq 1$ arise in various contexts: in [LW], they are related to transfer ideals: for an arbitrary finite extension K/F, let $\mathcal{T}_{K/F}$ denote the image of the Witt ring WK in WF under the Scharlau transfer map associated with any nonzero linear form $s\colon K \to F$. Leep and Wadsworth show in [LW, Prop. 2.4] that if $N_1(a,b,c) \neq 1$, then for $M = F(\sqrt{a}, \sqrt{b}, \sqrt{c})$ we have

$$\mathcal{T}_{M/F} \neq \mathcal{T}_{F(\sqrt{a})/F} \cap \mathcal{T}_{F(\sqrt{b})/F} \cap \mathcal{T}_{F(\sqrt{c})/F}.$$

The group $N_1(a,b,c)$ is also related to problems in Galois cohomology and to the rationality problem for group varieties: over the field $L = F((t_1))((t_2))((t_3))$, consider the division algebra $D = (a,t_1)_L \otimes (b,t_2)_L \otimes (c,t_3)_L$ and the 8-dimensional quadratic form $q \in I^2L$ such that

$$q \sim \langle\langle a, t_1\rangle\rangle - \langle\langle b, t_2\rangle\rangle - a\langle\langle c, t_3\rangle\rangle.$$

Using the alternative description of $N_1(a,b,c)$ above, it is shown in [KLST, p. 283] and [Me 3, p. 329] that if $N_1(a,b,c) \neq 1$, then

$$L^{\times 2}\mathrm{Nrd}(D^\times) \neq \{x \in L^\times \mid (x) \cup (D) = 0 \text{ in } H^3(L, \mu_2)\},$$

where Nrd is the reduced norm, $(D) \in H^2(L,\mu_2)$ is the Galois cohomology class corresponding to D under the canonical isomorphism mapping $H^2(L,\mu_2)$ to the 2-torsion part of the Brauer group of L, and $(x) \in H^1(L,\mu_2)$ corresponds to $x \in L^\times$ under the canonical isomorphism $H^1(L,\mu_2) \simeq L^\times/L^{\times 2}$. On the other hand, under the same hypothesis, Gille shows in [G] that the adjoint group $\mathrm{PSO}(q)$ over L is not R-trivial, hence not stably L-rational.

To conclude, we illustrate Corollary 6.2 by an explicit example over $\mathbf{Q}(x)$ which is derived from the example given in [STW, Remark 5.4].

EXAMPLE 6.3 Let $F = \mathbf{Q}(x)$ be the rational function field in one variable over the rationals. Then it follows from [STW, Remark 5.4] that $N_1(x+4, x+1, x) \neq 1$ and that the two binary forms $\langle 1, -(x+4)\rangle$ and $2\langle 1, -(x+1)\rangle$ represent a common element over $E = F(\sqrt{x})$, but no element in $F^\times$ is represented by both these forms over E.

In fact, we have

$$\begin{aligned} \langle 1, -(x+4)\rangle \perp -2\langle 1, -(x+1)\rangle &\simeq \langle 2, -1, -(x+4), 2(x+1)\rangle \\ &\simeq \langle -1, x, 2(x+2)(x+4), -2x(x+1)(x+2)\rangle \ , \end{aligned}$$

which shows that the difference of these two binary forms becomes isotropic over $E = F(\sqrt{x})$, i.e., the two forms represent a common element over E. Indeed, we can compute such an element directly. We have that

$$\begin{aligned} (\sqrt{x}+2)^2 - (x+4) &= 4\sqrt{x} \in D_E(\langle 1, -(x+4)\rangle), \\ 2(\sqrt{x}+1)^2 - 2(x+1) &= 4\sqrt{x} \in D_E(2\langle 1, -(x+1)\rangle), \end{aligned}$$

and therefore $\sqrt{x} \in D_E(\langle 1, -(x+4)\rangle) \cap D_E(2\langle 1, -(x+1)\rangle)$.

Over $F_1 = F(t_1) = \mathbf{Q}(x, t_1)$, we now define the two 4-dimensional forms

$$\begin{aligned} \psi_1 &= \langle x, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle \\ \psi_2 &= \langle x, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle \end{aligned}$$

and the two quaternion algebras

$$\begin{array}{rcl} Q_1 & = & (x+4, t_1)_{F_1} \\ Q_2 & = & (x+1, 2t_1)_{F_1} \end{array}$$

over F_1. By our construction, we know that Q_1 and Q_2 have a common slot over $E_1 = F_1(\sqrt{x})$, but that no such common slot can be chosen in F_1. A common slot over E_1 is given by $\sqrt{x}t_1$.

Consider now the biquaternion algebra $B = Q_1 \otimes Q_2$ with associated Albert form

$$\beta \simeq \langle x+1, -(x+4)\rangle \perp t_1\langle 1, -x, -2(x+2)(x+4), 2x(x+1)(x+2)\rangle \sim \psi_1 \perp -\psi_2 \ .$$

We then get

$$\begin{array}{rcl} x(x+4)\beta & \simeq & \langle -x, -t_1(x+4), t_1x(x+4)\rangle \\ & & \perp \langle x(x+1)(x+4), -2t_1x(x+2), 2t_1(x+1)(x+2)(x+4)\rangle \end{array}$$

from which we conclude that

$$B = (x, t_1(x+4))_{F_1} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_1} \ .$$

As in the proof of $CS \iff D(4)$ in Theorem 3.4, we get for $u \in F_1^\times$ that $c(\psi_1 \perp -u\psi_2) = [B \otimes (u,x)_{F_1}]$, and by putting $u = t_1(x+4)$, we obtain

$$c(\psi_1 \perp -t_1(x+4)\psi_2) = [(x(x+1)(x+4), -2t_1x(x+2))_{F_1}] \ .$$

Now with $\langle x, -(x+1)\rangle \simeq \langle -1, x(x+1)\rangle$, we obtain

$$\begin{array}{rcl} \psi_1 \perp -t_1(x+4)\psi_2 & \simeq & \langle x, -(x+4), 2(x+4), -2(x+1)(x+4)\rangle \\ & & \perp t_1\langle -1, (x+4), (x+4), -x(x+1)(x+4)\rangle \ . \end{array}$$

Also, $\langle -1, x+4, x+4\rangle \simeq \langle x, -x(x+4), x+4\rangle$ represents $xx^2 + (x+4)x^2 = 2x^2(x+2)$. Hence,

$$\begin{array}{rcl} \langle x, 2t_1x^2(x+2)\rangle & \simeq & x\langle 1, 2t_1x(x+2)\rangle \\ & \subset & \psi_1 \perp -t_1(x+4)\psi_2 \ . \end{array}$$

Let $L = F_1(\sqrt{-2t_1x(x+2)})$. The above shows that $\psi_1 \perp -t_1(x+4)\psi_2$ becomes isotropic over L. On the other hand, $[(x(x+1)(x+4), -2t_1x(x+2))_L] = 0$, and it follows that $(\psi_1 \perp -t_1(x+4)\psi_2)_L$ is an isotropic 8-dimensional form in I^3L and hence hyperbolic. Thus, $(\psi_1)_L \simeq (t_1(x+4)\psi_2)_L$. However, by construction there does not exist a binary form over F_1 which is similar to a subform of both ψ_1 and ψ_2.

Let us now consider $\psi := \psi_1 \perp -t_2\psi_2$ over $F_2 = \mathbf{Q}(x, t_1, t_2)$. Then $\psi \in I^2F_2$ is of dimension 8, by construction it does not contain a subform in GP_2F_2, and for its Clifford invariant we get

$$c(\psi) = [B \otimes (t_2, x)_{F_2}] = [(x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2}] \ .$$

Consider the biquaternion algebra

$$A = (x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2} \ ,$$

which by our construction is necessarily a division algebra, and an associated Albert form

$$\begin{aligned}\alpha \quad \simeq \quad & \langle -x, -t_1t_2(x+4), t_1t_2x(x+4)\rangle \\ & \perp \langle x(x+1)(x+4), 2t_1(x+1)(x+2)(x+4), -2t_1x(x+2)\rangle \ .\end{aligned}$$

Then, over $F_3 = \mathbf{Q}(x, t_1, t_2, t_3)$, the form $\varphi := \alpha \perp t_3\psi$ is a 14-dimensional form in I^3F_3 which is not similar to the difference of the pure parts of two forms in P_3F_3.

We summarize the above results.

- The two forms $\langle 1, -(x+4)\rangle$ and $2\langle 1, -(x+1)\rangle$ over $\mathbf{Q}(x)$ both represent $\sqrt{x}$ over $\mathbf{Q}(x)(\sqrt{x})$, but there is no element in $\mathbf{Q}(x)^\times$ which is represented by both forms over $\mathbf{Q}(x)(\sqrt{x})$. In particular, $\mathbf{Q}(x)$ does not have property $D(2)$.
- The two quaternion algebras $(x+4, t_1)_{F_1}$ and $(x+1, 2t_1)_{F_1}$ over $F_1 = \mathbf{Q}(x, t_1)$ have a common slot over $\mathbf{Q}(x, t_1)(\sqrt{x})$, for example $t_1\sqrt{x}$, but no such common slot can be chosen in $\mathbf{Q}(x, t_1)$. In particular, $\mathbf{Q}(x, t_1)$ does not have property CS.
- The two forms $\psi_1 = \langle x, -(x+4)\rangle \perp -t_1\langle 1, -(x+4)\rangle$ and $\psi_2 = \langle x, -(x+1)\rangle \perp -2t_1\langle 1, -(x+1)\rangle$ over $\mathbf{Q}(x, t_1)$ do not simultaneously become isotropic over any quadratic extension of $\mathbf{Q}(x, t_1)$, i.e., there is no binary form over $\mathbf{Q}(x, t_1)$ which is similar to a subform of both ψ_1 and ψ_2. However, the forms ψ_1 and $t_1(x+4)\psi_2$ become isometric over $\mathbf{Q}(x, t_1)(\sqrt{-2t_1x(x+2)})$. In particular, $\mathbf{Q}(x, t_1)$ does not have property $D(4)$.
- The Clifford invariant of the 8-dimensional form $\psi = \psi_1 \perp -t_2\psi_2 \in I^2F_2$, where $F_2 = \mathbf{Q}(x, t_1, t_2)$, is represented by the biquaternion algebra

  $$A = (x, t_1t_2(x+4))_{F_2} \otimes (x(x+1)(x+4), -2t_1x(x+2))_{F_2} \ .$$

  However, ψ does not contain a subform in GP_2F_2. In particular, $\mathbf{Q}(x, t_1, t_2)$ does not have property $D(8)$.
- The extension $F_2(SB(A))/F_2$ is not excellent (cf. Prop. 6.1(i)).
- With

  $$\begin{aligned}\psi \quad \sim \quad & \langle -(x+4), -t_1, t_1(x+4), x, -t_2x, t_2\rangle \\ & - t_2\langle 1, -(x+1), -2t_1, 2t_1(x+1)\rangle\end{aligned}$$

  as above, and with

  $$\begin{aligned} c(\langle -(x+4), -t_1, t_1(x+4), x, -t_2x, t_2\rangle) &= [(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2}] \\ c(\langle 1, -(x+1), -2t_1, 2t_1(x+1)\rangle) &= [(x+1, 2t_1)_{F_2}] \ ,\end{aligned}$$

  we have that $(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2} \otimes (x+1, 2t_1)_{F_2}$ is not a division algebra, but $(x+4, t_1)_{F_2} \otimes (x, t_2)_{F_2}$ and $(x+1, 2t_1)_{F_2}$ have no proper common quadratic subextension of $F_2 = \mathbf{Q}(x, t_1, t_2)$ (cf. Prop. 6.1(ii)).
- With α an Albert form associated to A, the form $\alpha \perp t_3\psi$ of dimension 14 over $F_3 = \mathbf{Q}(x, t_1, t_2, t_3)$ is in I^3F_3, but it is not similar to the difference of the pure parts of two forms in P_3F_3. In particular, $\mathbf{Q}(x, t_1, t_2, t_3)$ does not have property $D(14)$.

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Metrics on States from Actions of Compact Groups

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Abstract. Let a compact Lie group act ergodically on a unital C^*-algebra A. We consider several ways of using this structure to define metrics on the state space of A. These ways involve length functions, norms on the Lie algebra, and Dirac operators. The main thrust is to verify that the corresponding metric topologies on the state space agree with the weak-$*$ topology.

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Connes [C1, C2, C3] has shown us that Riemannian metrics on non-commutative spaces (C^*-algebras) can be specified by generalized Dirac operators. Although in this setting there is no underlying manifold on which one then obtains an ordinary metric, Connes has shown that one does obtain in a simple way an ordinary metric on the state space of the C^*-algebra, generalizing the Monge-Kantorovich metric on probability measures [Ra] (called the "Hutchinson metric" in the theory of fractals [Ba]).

But an aspect of this matter which has not received much attention so far [P] is the question of when the metric topology (that is, the topology from the metric coming from a Dirac operator) agrees with the underlying weak-$*$ topology on the state space. Note that for locally compact spaces their topology agrees with the weak-$*$ topology coming from viewing points as linear functionals (by evaluation) on the algebra of continuous functions vanishing at infinity.

In this paper we will consider metrics arising from actions of compact groups on C^*-algebras. For simplicity of exposition we will only deal with "compact" non-commutative spaces, that is, we will always assume that our C^*-algebras have an identity element. We will explain later what we mean by Dirac operators in this setting (section 4). In terms of this, a brief version of our main theorem is:

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Theorem 4.2. *Let α be an ergodic action of a compact Lie group G on a unital C^*-algebra A, and let D be a corresponding Dirac operator. Then the metric topology on the state space of A defined by the metric from D agrees with the weak-$*$ topology.*

An important case to which this theorem applies consists of the non-commutative tori [Rf], since they carry ergodic actions of ordinary tori [OPT]. The metric geometry of non-commutative tori has recently become of interest in connection with string theory [CDS, RS, S].

We begin by showing in the first section of this paper that the mechanism for defining a metric on states can be formulated in a very rudimentary Banach space setting (with no algebras, groups, or Dirac operators). In this setting the discussion of agreement between the metric topology and the weak-$*$ topology takes a particularly simple form.

Then in the second section we will see how length functions on a compact group directly give (without Dirac operators) metrics on the state spaces of C^*-algebras on which the group acts ergodically. We then prove the analogue in this setting of the main theorem stated above.

In the third section we consider compact Lie groups, and show how norms on the Lie algebra directly give metrics on the state space. We again prove the corresponding analogue of our main theorem.

Finally, in section 4 we use the results of the previous sections to prove our main theorem, stated above, for the metrics which come from Dirac operators.

It is natural to ask about actions of non-compact groups. Examination of [Wv4] suggests that there may be very interesting phenomena there. The considerations of the present paper also make one wonder whether there is an appropriate analogue of length functions for compact quantum groups which might determine a metric on the state spaces of C^*- algebras on which a quantum group acts ergodically [Bo, Wn]. This would be especially interesting since for non-commutative compact groups there is only a sparse collection of known examples of ergodic actions [Ws], whereas in [Wn] a rich collection of ergodic actions of compact quantum groups is constructed. Closely related is the setting of ergodic coactions of discrete groups [N, Q]. But I have not explored any of these possibilities.

I developed a substantial part of the material discussed in the present paper during a visit of several weeks in the Spring of 1995 at the Fields Institute. I am appreciative of the hospitality of the Fields Institute, and of George Elliott's leadership there. But it took trying to present this material in a course which I was teaching this Spring, as well as benefit from [P, Wv1, Wv2, Wv3, Wv4], for me to find the simple development given here.

1. Metrics on states

Let A be a unital C^*-algebra. Connes has shown [C1, C2, C3] that an appropriate way to specify a Riemannian metric in this non-commutative situation is by means of a spectral triple. This consists of a representation of A on a Hilbert space $\mathcal{H}$, together with an unbounded self-adjoint operator D on $\mathcal{H}$ (the generalized Dirac operator), satisfying certain conditions. The set $\mathcal{L}(A)$ of Lipschitz elements of A consists of those $a \in A$ such that the commutator $[D, a]$ is a bounded operator. It is required that $\mathcal{L}(A)$ be dense in A. The Lipschitz semi-norm, L, is defined on $\mathcal{L}(A)$ just by the operator norm $L(a) = \|[D, a]\|$.

Given states μ and ν of A, Connes defines the distance between them, $\rho(\mu, \nu)$, by

$$\rho(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : a \in \mathcal{L}(A),\ L(a) \ \leq\ 1\} \ . \tag{1.1}$$

(In the absence of further hypotheses it can easily happen that $\rho(\mu, \nu) = +\infty$. For one interesting situation where this sometimes happens see the end of the discussion of the second example following axiom 4' of [C3].)

The semi-norm L is an example of a general Lipschitz semi-norm, that is [BC, Cu, P, Wv1, Wv2], a semi-norm L on a dense subalgebra $\mathcal{L}$ of A satisfying the Leibniz property:

$$L(ab) \leq L(a)\|b\| + \|a\|L(b) \ . \tag{1.2}$$

Lipschitz norms carry some information about differentiable structure [BC, Cu], but not nearly as much as do spectral triples. But it is clear that just in terms of a given Lipschitz norm one can still define a metric on states by formula (1.1).

However, for the purpose of understanding the relationship between the metric topology and the weak-$*$ topology, we do not need the Leibniz property (1.2), nor even that A be an algebra. The natural setting for these considerations seems to be the following very rudimentary one. The data is:

(1.3a) A normed space A, with norm $\| \ \|$, over either $\mathbb{C}$ or $\mathbb{R}$.

(1.3b) A subspace $\mathcal{L}$ of A, not necessarily closed.

(1.3c) A semi-norm L on $\mathcal{L}$.

(1.3d) A continuous (for $\| \ \|$) linear functional, η, on $\mathcal{K} = \{a \in \mathcal{L} : L(a) = 0\}$ with $\|\eta\| = 1$. (Thus, in particular, we require $\mathcal{K} \neq \{0\}$.)

Let A' denote the Banach-space dual of A, and set

$$S = \{\mu \in A' : \mu = \eta \text{ on } \mathcal{K},\ \text{and } \|\mu\| = 1\} \ .$$

Thus S is a norm-closed, bounded, convex subset of A', and so is weak-$*$ compact. In general S can be quite small; when A is a Hilbert space S will contain only one element. But in the applications we have in mind A will be a unital C^*-algebra, $\mathcal{K}$ will be the one-dimensional subspace spanned by the identity element, and η will be the functional on $\mathcal{K}$ taking value 1 on the identity element. Thus S will be the full state-space of A. (That $\mathcal{K}$ will consist only of the scalar multiples of the identity element in our examples will follow from our ergodicity hypothesis. We treat the case of general $\mathcal{K}$ here because this clarifies slightly some issues, and it might possibly be of eventual use, for example in non-ergodic situations.)

We do not assume that $\mathcal{L}$ is dense in A. But to avoid trivialities we do make one more assumption about our set-up, namely:

$$\text{(1.3e)} \qquad \mathcal{L} \quad \text{separates the points of} \quad S.$$

This means that given $\mu, \nu \in S$ there is an $a \in \mathcal{L}$ such that $\mu(a) \neq \nu(a)$. (Note that for $\mu \in S$ there exists $a \in \mathcal{L}$ with $\mu(a) \neq 0$, since we can just take an $a \in \mathcal{K}$ such that $\eta(a) \neq 0$.)

With notation as above, let $\tilde{\mathcal{L}} = \mathcal{L}/\mathcal{K}$. Then L drops to an actual norm on $\tilde{\mathcal{L}}$, which we denote by $\tilde{L}$. But on $\tilde{\mathcal{L}}$ we also have the quotient norm from $\| \ \|$ on $\mathcal{L}$, which we denote by $\| \ \|^{\sim}$. The image in $\tilde{\mathcal{L}}$ of $a \in \mathcal{L}$ will be denoted by $\tilde{a}$.

We remark that when $\mathcal{L}$ is a unital algebra (perhaps dense in a C^*-algebra), and when $\mathcal{K}$ is the span of the identity element, then the space of universal 1-forms Ω^1 over $\mathcal{L}$ is commonly identified [BC, Br, C2, Cu] with $\mathcal{L} \otimes \tilde{\mathcal{L}}$, and the differential $d : \mathcal{L} \to \Omega^1$ is given by $da = 1 \otimes \tilde{a}$. Thus in this setting our $\tilde{L}$ is a norm on the space of universal 1-coboundaries of $\mathcal{L}$. The definition of L which we will use in the examples of section 3 is also closely related to this view.

On S we can still define a metric, ρ, by formula (1.1), with $\mathcal{L}(A)$ replaced by $\mathcal{L}$. The symmetry of ρ is evident, and the triangle inequality is easily verified. Since we assume that $\mathcal{L}$ separates the points of S, so will ρ. But ρ can still take the value $+\infty$. We will refer to the topology on S defined by ρ as the "ρ-topology", or the "metric topology" when ρ is understood.

It will often be convenient to consider elements of A as (weak-$*$ continuous) functions on S. At times this will be done tacitly, but when it is useful to do this explicitly we will write $\hat{a}$ for the corresponding function, so that $\hat{a}(\mu) = \mu(a)$ for $\mu \in S$.

Without further hypotheses we have the following fact. It is closely related to proposition 3.1a of [P], where metrics are defined in terms of linear operators from an algebra into a Banach space.

1.4 Proposition. *The ρ-topology on S is finer than the weak-$*$ topology.*

Proof. Let $\{\mu_k\}$ be a sequence in S which converges to $\mu \in S$ for the metric ρ. Then it is clear from the definition of ρ that $\{\mu_k(a)\}$ converges to $\mu(a)$ for any $a \in \mathcal{L}$ with $L(a) \leq 1$, and hence for all $a \in \mathcal{L}$.

This says that $\hat{a}(\mu_k)$ converges to $\hat{a}(\mu)$ for all $a \in \mathcal{L}$. But $\hat{\mathcal{L}}$ is a linear space of weak-$*$ continuous functions on S which separates the points of S by assumption (and which contains the constant functions, since they come from any $a \in \mathcal{K}$ on which η is not 0). A simple compactness argument shows then that $\hat{\mathcal{L}}$ determines the weak-$*$ topology of S. Thus $\{\mu_k\}$ converges to μ in the weak-$*$ topology, as desired. □

There will be some situations in which we want to obtain information about $(\mathcal{L}, L)$ from information about S. It is clear that to do this S must "see" all of $\mathcal{L}$. The convenient formulation of this for our purposes is as follows. Let $\| \ \|_\infty$ denote the supremum norm on functions on S. Let it also denote the corresponding semi-norm on $\mathcal{L}$ defined by $\|a\|_\infty = \|\hat{a}\|_\infty$. Clearly $\|\hat{a}\|_\infty \leq \|a\|$ for $a \in \mathcal{L}$.

1.5 Condition. The semi-norm $\| \ \|_\infty$ on $\mathcal{L}$ is a norm, and it is equivalent to the norm $\| \ \|$, so that there is a constant k with

$$\|a\| \le k\|\hat{a}\|_\infty \qquad \text{for } \ a \in \mathcal{L}.$$

This condition clearly holds when A is a C^*-algebra, $\mathcal{L}$ is dense in A, and S is the state space of A, so that we are dealing with the usual Kadison functional representation [KR]. But we remark that even in this case the constant k above cannot always be taken to be 1 (bottom of page 263 of [KR]). This suggests that in using formula (1.1) one might want to restrict to using just the self-adjoint elements of $\mathcal{L}$, since there the function representation is isometric. But more experience with examples is needed.

We return to the general case. If we are to have the ρ-topology on S agree with the weak-$*$ topology, then S must at least have finite ρ-diameter, that is, ρ must be bounded. The following proposition is closely related to theorem 6.2 of [P].

1.6 Proposition. *Suppose there is a constant, r, such that*

$$\| \ \|^\sim \le r\tilde{L} \ . \tag{1.7}$$

Then ρ is bounded (by $2r$).

Conversely, suppose that Condition 1.5 holds. If ρ is bounded, (say by d), then there is a constant r such that (1.7) holds (namely $r = kd$ where k is as in 1.5).

Proof. Suppose that (1.7) holds. If $a \in \mathcal{L}$ and $L(a) \le 1$, then $\tilde{L}(\tilde{a}) \le 1$ and so $\|\tilde{a}\|^\sim \le r$. This means that, given $\varepsilon > 0$, there is a $b \in \mathcal{K}$ such that $\|a - b\| \le r + \varepsilon$. Then for any $\mu, \nu \in S$, we have, because μ and ν agree on $\mathcal{K}$,

$$|\mu(a) - \nu(a)| = |\mu(a - b) - \nu(a - b)| \le \|\mu - \nu\| \ \|a - b\| \le 2(r + \varepsilon) \ .$$

Since ε is arbitrarily small, it follows that $|\mu(a) - \nu(a)| \le 2r$. Consequently $\rho(\mu, \nu) \le 2r$.

Assume conversely that ρ is bounded by d. Fix $\nu \in S$, and choose $b \in \mathcal{K}$ such that $\eta(b) = 1$. Then for any $\mu \in S$ and any $a \in \mathcal{L}$ with $L(a) \le 1$ we have

$$d \ge \rho(\mu, \nu) \ge |\mu(a) - \nu(a)| = |\mu(a - \nu(a)b)| \ .$$

Suppose now that Condition 1.5 holds. We apply it to $a - \nu(a)b$. Thus, since S is compact, we can find μ such that

$$\|a - \nu(a)b\| \le k|\mu(a - \nu(a)b)| \ .$$

Consequently $\|a - \nu(a)b\| \le kd$, so that $\|\tilde{a}\|^\sim \le kd$. All this was under the assumption that $L(a) \le 1$. It follows that for general $a \in \mathcal{L}$ we have $\|\tilde{a}\|^\sim \le kd\tilde{L}(\tilde{a})$, as desired. □

We now turn to the question of when the ρ-topology and the weak-$*$ topology on S agree. The following theorem is closely related to theorem 6.3 of [P].

1.8 Theorem. *Let the data be as in* (1.3a–e)*, and let* $\mathcal{L}_1 = \{a \in \mathcal{L} : L(a) \leq 1\}$*. If the image of* $\mathcal{L}_1$ *in* $\mathcal{L}^{\sim}$ *is totally bounded for* $\| \ \|^{\sim}$*, then the* ρ*-topology on* S *agrees with the weak-*$*$ *topology.*

Conversely, if Condition 1.5 holds and if the ρ*-topology on* S *agrees with the weak-*$*$ *topology, then the image of* $\mathcal{L}_1$ *in* $\mathcal{L}^{\sim}$ *is totally bounded for* $\| \ \|^{\sim}$.

Proof. We begin with the converse, so that we see why the total-boundedness assumption is natural. If the ρ-topology gives the weak-$*$ t opology on S, then ρ must be bounded since S is compact. Thus by Proposition 1.6 there is a constant, r_o, such that $\| \ \|^{\sim} \leq r_o L^{\sim}$, since we assume here that Condition 1.5 holds. Choose $r > r_o$. Then $\|a\|^{\sim} < r$ if $a \in \mathcal{L}_1$. Consequently, if we let

$$\mathcal{B}_r = \{a \in \mathcal{L} : L(a) \leq 1 \text{ and } \|a\| \leq r\} ,$$

then the image of $\mathcal{B}_r$ in $\mathcal{L}^{\sim}$ is the same as the image of $\mathcal{L}_1$. Thus it suffices to show that $\mathcal{B}_r$ is totally bounded.

Let $a \in \mathcal{B}_r$ and let $\mu, \nu \in S$. Then

$$|\hat{a}(\mu) - \hat{a}(\nu)| = |\mu(a) - \nu(a)| \leq \rho(\mu, \nu) .$$

Thus $(\mathcal{B}_r)\hat{}$ can be viewed as a bounded family of functions on S which is equicontinuous for the weak-$*$ topology, since ρ gives the weak-$*$ topology of S. It follows from Ascoli's theorem [Ru] that $(\mathcal{B}_r)\hat{}$ is totally bounded for $\| \ \|_\infty$. By Condition 1.5 this means that $\mathcal{B}_r$ is totally bounded for $\| \ \|$ as a subset of A, as desired.

For the other direction we do not need Condition 1.5. We suppose now that the image of $\mathcal{L}_1$ in $\tilde{L}$ is totally bounded for $\| \ \|^{\sim}$. Let $\mu \in S$ and $\varepsilon > 0$ be given, and let $B(\mu, \varepsilon)$ be the ρ-ball of radius ε about μ in S. In view of Proposition 1.4 it suffices to show that $B(\mu, \varepsilon)$ contains a weak-$*$ neighborhood of μ. Now by the total boundedness of the image of $\mathcal{L}_1$ we can find $a_1, \ldots, a_n \in \mathcal{L}_1$ such that the $\| \ \|^{\sim}$-balls of radius $\varepsilon/3$ about the $\hat{a}_j$'s cover the image of $\mathcal{L}_1$. We now show that the weak-$*$ neighborhood

$$\mathcal{O} = \mathcal{O}(\mu, \{a_j\}, \varepsilon/3) = \{\nu \in S : |(\mu - \nu)(a_j)| < \varepsilon/3, \ 1 \leq j \leq n\}$$

is contained in $B(\mu, \varepsilon)$. Consider any $a \in \mathcal{L}_1$. There is a j and a $b \in \mathcal{K}$, depending on a, such that

$$\|a - a_j - b\| < \varepsilon/3 .$$

Hence for any $\nu \in \mathcal{O}$ we have

$$\begin{aligned} |\mu(a) - \nu(a)| &\leq |\mu(a) - \mu(a_j + b)| + |\mu(a_j + b) - \nu(a_j + b)| + |\nu(a_j + b) - \nu(a)| \\ &< \varepsilon/3 + |\mu(a_j) - \nu(a_j)| + \varepsilon/3 < \varepsilon . \end{aligned}$$

Thus $\rho(\mu, \nu) < \varepsilon$. Consequently $\mathcal{O} \subseteq B(\mu, \varepsilon)$ as desired. $\square$

Examination of the proof of the above theorem suggests a reformulation which provides a convenient subdivision of the problem of showing for specific examples that the ρ-topology agrees with the weak-$*$ topology. We will use this reformulation in the next sections.

1.9 Theorem. *Let the data be as in* (1.3a–e). *Then the ρ-topology on S will agree with the weak-$*$ topology if the following three hypotheses are satisfied:*

i) *Condition 1.5 holds.*
ii) *ρ is bounded.*
iii) *The set $\mathcal{B}_1 = \{a \in \mathcal{L} : L(a) \leq 1$ and $\|a\| \leq 1\}$ is totally bounded in A for $\| \ \|$.*

Conversely, if Condition 1.5 holds and if the ρ-topology agrees with the weak-$*$ topology, then the above three conditions are satisfied.

Proof. If conditions i) and ii) are satisfied, then, just as in the first part of the proof of Theorem 1.8, there is a constant r such that the image of $\mathcal{B}_r$ in $\tilde{\mathcal{L}}$ contains the image of $\mathcal{L}_1$. But $\mathcal{B}_r \subseteq r\mathcal{B}_1$. Thus if $\mathcal{B}_1$ is totally bounded then so is $\mathcal{B}_r$, as is then the image of $\mathcal{L}_1$. Then we can apply Theorem 1.8 to conclude that the ρ-topology agrees with the weak-$*$ topology.

Conversely, if the ρ-topology and the weak-$*$ topology agree, then condition ii) holds by Proposition 1.6. But by the first part of the proof of Theorem 1.8 there is then a constant r such that $\mathcal{B}_r$ is totally bounded. By scaling we see that $\mathcal{B}_1$ is also. $\square$

We remark that if we take any 1-dimensional subspace $\mathcal{K}$ of an infinite-dimensional normed space A, set $\mathcal{L} = A$, and let L be the pull-back to A of $\| \ \|^{\tilde{}}$ on $A/\mathcal{K}$, we obtain an example where ρ is bounded but the image of $\mathcal{L}_1$ in $\mathcal{L}^{\sim}$ is not totally bounded, nor is $\mathcal{B}_1$ totally bounded in A.

In the next sections we will find very useful the following:

1.10 Comparison Lemma. *Let the data be as in* (1.3a–e). *Suppose we have a subspace $\mathcal{M}$ of $\mathcal{L}$ which contains $\mathcal{K}$ and separates the points of S, and a semi-norm M on $\mathcal{M}$ which takes value* 0 *exactly on $\mathcal{K}$. Let ρ_L and ρ_M denote the corresponding metrics on S (possibly taking value $+\infty$). Assume that*

$$M \ \geq \ L \quad \text{on} \quad \mathcal{M},$$

in the sense that $M(a) \ \geq \ L(a)$ for all $a \in \mathcal{M}$. Then

$$\rho_M \ \leq \ \rho_L \ ,$$

in the sense that $\rho_M(\mu, \nu) \leq \rho_L(\mu, \nu)$ for all $\mu, \nu \in S$. Thus

i) *If ρ_L is finite then so is ρ_M.*
ii) *If ρ_L is bounded then so is ρ_M.*
iii) *If the ρ_L-topology on S agrees with the weak-$*$ topology then so does the ρ_M-topology.*

Proof. If $a \in \mathcal{M}$ and $M(a) \leq 1$ then $L(a) \leq 1$. Thus the supremum defining ρ_M is taken over a smaller set than that for ρ_L, and so $\rho_M \leq \rho_L$. Conclusions i) and ii) are then obvious. Conclusion iii) follows from the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. $\square$

For later use we record the following easily verified fact.

1.11 Proposition. *Let data be as above. Let t be a strictly positive real number. Set $M = tL$ on $\mathcal{L}$. Then $\rho_M = t^{-1}\rho_L$. Thus properties for ρ_L of finiteness, boundedness, and agreement of the ρ_L-topology with the weak-$*$ topology carry over to ρ_M.*

2. Metrics from actions and length functions

Let G be a compact group (with identity element denoted by e). We normalize Haar measure to give G mass 1. We recall that a length function on a group G is a continuous non-negative real-valued function, ℓ, on G such that

$$\ell(xy) \leq \ell(x) + \ell(y) \quad \text{for} \quad x, y \in G, \tag{2.1a}$$

$$\ell(x^{-1}) = \ell(x), \tag{2.1b}$$

$$\ell(x) = 0 \quad \text{exactly if} \quad x = e. \tag{2.1c}$$

Length functions arise in a number of ways. For example, if π is a faithful unitary representation of G on a finite-dimensional Hilbert space, then we can set $\ell(x) = \|\pi_x - \pi_e\|$. We will see another way in the next section. We will assume for the rest of this section that a length function has been chosen for G.

Let A be a unital C^*-algebra, and let α be an action (strongly continuous) of G by automorphisms of A. We let $\mathcal{L}$ denote the set of Lipschitz elements of A for α (and ℓ), with corresponding Lipschitz semi-norm L. That is [Ro1, Ro2], for $a \in A$ we set

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e\} ,$$

which may have value $+\infty$, and we set

$$\mathcal{L} = \{a \in A : L(a) < \infty\} .$$

It is easily verified that $\mathcal{L}$ is a $*$-subalgebra of A, and that L satisfies the Leibniz property 1.2. (More generally, for $0 < r < 1$ we could define L^r by

$$L^r(a) = \sup\{\|\alpha_x(a) - a\|/(\ell(x))^r : x \neq e\}$$

along the lines considered in [Ro1, Ro2]. For actions on the non-commutative torus this has been studied in [Wv2], but we will not pursue this here.)

It is not so clear whether $\mathcal{L}$ is carried into itself by α, but we do not need this fact here. (For Lie groups see theorem 4.1 of [Ro1] or the comments after theorem 6.1 of [Ro2].) Let us consider, however, the α-invariance of L. We find that

$$\begin{aligned} L(\alpha_z(a)) &= \sup\{\|\alpha_z(\alpha_{z^{-1}xz}(a) - a)\|/\ell(x) : x \neq e\} \\ &= \sup\{\|\alpha_x(a) - a\|/\ell(zxz^{-1}) : x \neq e\}. \end{aligned}$$

Thus if $\ell(zxz^{-1}) = \ell(x)$ for all $x, z \in G$, then L is α-invariant, and $\mathcal{L}$ is carried into itself by α. The metric ρ on S defined by L will then be α-invariant for the evident action on S. But we will not discuss this matter further here.

2.2 PROPOSITION. *The $*$-algebra $\mathcal{L}$ is dense in A.*

Proof. For $f \in L^1(G)$ we define α_f as usual by $\alpha_f(a) = \int f(x)\alpha_x(a)\,dx$. It is standard [BR] that as f runs through an "approximate delta-function", $\alpha_f(a)$ converges to a. Thus the set of elements of form $\alpha_f(a)$ is dense in A. Let λ denote the action of G by left translation of functions on G. A quick standard calculation shows that $\alpha_x(\alpha_f(a)) = \alpha_{\lambda_x(f)}(a)$. Thus

$$\|\alpha_x(\alpha_f(a)) - \alpha_f(a)\| = \|\alpha_{(\lambda_x f - f)}(a)\| \leq \|\lambda_x f - f\|_1 \|a\|,$$

where $\| \ \|_1$ denotes the usual L^1-norm. Thus we see that $\alpha_f(a) \in \mathcal{L}$ if $f \in Lip^1_\lambda$, the space of Lipschitz functions in $L^1(G)$ for λ (and ℓ).

Consequently it suffices to show that Lip^1_λ is dense in $L^1(G)$. We first note that it contains a non-trivial element, namely ℓ itself. For if $x, y \in G$, then

$$|(\lambda_x \ell)(y) - \ell(y)| = |\ell(x^{-1}y) - \ell(y)| \leq \ell(x),$$

where the inequality follows from 2.1a and 2.1b above. We momentarily switch attention to $C(G)$ with $\| \ \|_\infty$, and the action λ of G on it. Of course $\ell \in C(G)$. The above inequality then says that $\ell \in Lip^\infty_\lambda$, the space of Lipschitz functions in $C(G)$ for λ. But as mentioned earlier, Lip^∞_λ is easily seen to be a $*$-subalgebra of $C(G)$ for the pointwise product, and it contains the constant functions. Furthermore, a simple calculation shows that Lip^∞_λ is carried into itself by *right* translation. Since Lip^∞_λ contains ℓ, which separates e from any other point, it follows that Lip^∞_λ separates the points of G. Thus Lip^∞_λ is dense in $C(G)$ by the Stone-Weierstrass theorem. Since $\| \ \|_\infty$ dominates $\| \ \|_1$ for compact G, it follows that Lip^1_λ is dense in $L^1(G)$ as needed. $\square$

For simplicity of exposition we will deal only with the case in which we obtain metrics on the entire state space of the C^*-algebra A. For this purpose we want the subspace where L takes the value 0 to be one-dimensional. It is evident that L takes value 0 on exactly those elements of A which are α-invariant, and in particular on the scalar multiples of the identity element of A. Thus we need to assume that the action α is *ergodic*, in the sense that the only α-invariant elements are the scalar multiples of the identity.

The main theorem of this section is:

2.3 THEOREM. *Let α be an ergodic action of a compact group G on a unital C^*-algebra A. Let ℓ be a length function on G, and define $\mathcal{L}$ and L as above. Let ρ be the corresponding metric on the state space S of A. Then the ρ-topology on S agrees with the weak-$*$ topology.*

Proof. Because $\mathcal{L}$ is dense by Proposition 2.2, it separates the points of S. Consequently the conditions 1.3a–e are fulfilled (for the evident η). Thus L indeed defines a metric, ρ, on S (perhaps taking value $+\infty$).

Since G is compact, we can average α over G to obtain a conditional expectation from A onto its fixed-point subalgebra. Because we assume that α is ergodic, this conditional expectation can be viewed as a state on A. By abuse of notation we will denote it again by η, since it extends the evident state η on the fixed-point algebra. Thus

$$\eta(a) = \int_G \alpha_x(a)\,dx$$

for $a \in A$, interpreted as a complex number when convenient.

We will follow the approach suggested by Theorem 1.9. Now hypothesis (i) of that theorem is satisfied in the present setting, as discussed right after Condition 1.5 above. We now check hypothesis (ii), that is:

2.4 Lemma. *ρ is bounded.*

Proof. Let $\mu \in S$. Then for any $a \in \mathcal{L}$ we have

$$|\mu(a) - \eta(a)| = |\int \mu(a)dx - \mu(\int \alpha_x(a)dx)| = |\int \mu(a - \alpha_x(a))dx| \leq L(a) \int_G \ell(x)dx \ .$$

It follows that $\rho(\mu, \eta) \leq \int \ell(x)dx$. Thus for any $\mu, \nu \in S$ we have

$$\rho(\mu, \nu) \leq 2 \int_G \ell(x)dx \ ,$$

which is finite since ℓ is bounded. □

We now begin the verification of hypothesis (iii) of Theorem 1.9. For this we need the unobvious fact [HLS, Bo] that because G is compact and α is ergodic, each irreducible representation of G occurs with at most finite multiplicity in A. (In [HLS] it is also shown that η is a trace, but we do not need this fact here.) The following lemma is undoubtedly well-known, but I do not know a reference for it.

2.5 Lemma. *Let α be a (strongly continuous) action of a compact group G on a Banach space A. Suppose that each irreducible representation of G occurs in A with at most finite multiplicity. Then for any $f \in L^1(G)$ the operator α_f defined by*

$$\alpha_f(a) = \int_G f(x)\alpha_x(a)dx$$

is compact.

Proof. If f is a coordinate function for an irreducible representation π of G, then it is not hard to see (ch. IX of [FD]) that α_f will have range in the π-isotypic component of A, which we are assuming is finite-dimensional. Thus α_f is of finite rank in this case. But by the Peter-Weyl theorem [FD] the linear span of the coordinate functions for all irreducible representations is dense in $L^1(G)$. So any α_f can be approximated by finite rank operators. □

Proof of Theorem 2.3. We show now that $\mathcal{B}_1$, as in (iii) of Theorem 1.9, is totally bounded. Let $\varepsilon > 0$ be given. Since $\ell(e) = 0$ and ℓ is continuous at e, we can find $f \in L^1(G)$ such that $f \geq 0$, $\int_G f(x)dx = 1$, and $\int_G f(x)\ell(x)dx < \varepsilon/2$. By the previous lemma α_f is compact. Since $\mathcal{B}_1$ is bounded, it follows that $\alpha_f(\mathcal{B}_1)$ is totally bounded. Thus it can be covered by a finite number of balls of radius $\varepsilon/2$. But for any $a \in \mathcal{B}_1$ we have

$$\begin{aligned}\|a - \alpha_f(a)\| = \|a \int f(x)dx - \int f(x)\alpha_x(a)dx\| &\leq \int f(x)\|a - \alpha_x(a)\|dx \\ &\leq L(a) \int f(x)\ell(x)dx \leq \varepsilon/2 \ .\end{aligned}$$

Thus $\mathcal{B}_1$ itself can be covered by a finite number of balls of radius ε. □

3. Metrics from actions of Lie groups

We suppose now that G is a connected Lie group (compact). We let $\mathfrak{g}$ denote the Lie algebra of G. Fix a norm $\| \ \|$ on $\mathfrak{g}$. For any action α of G on a Banach space A we let A^1 denote the space of α-differentiable elements of A. Thus [BR] if $a \in A^1$ then for each $X \in \mathfrak{g}$ there is a $d_X a \in A$ such that

$$\lim_{t \to 0} (\alpha_{\exp(tX)}(a) - a)/t = d_X a \ ,$$

and $X \mapsto d_X a$ is a linear map from $\mathfrak{g}$ into A, which we denote by da. Since $\mathfrak{g}$ and A both have norms, the operator norm, $\|da\|$, of da is defined (and finite). A standard smoothing argument [BR] shows that A^1 is dense in A.

Suppose now that A is a C^*-algebra and that α is an action by automorphisms of A. We can set $\mathcal{L} = A^1$ and $L(a) = \|da\|$. It is easily verified that $\mathcal{L}$ is a $*$-subalgebra of A and that L satisfies the Leibniz property 1.2, though we do not need these facts here. Because G is connected, $L(a) = 0$ exactly if a is α-invariant.

3.1 Theorem. *Let G be a compact connected Lie group, and fix a norm on $\mathfrak{g}$. Let α be an ergodic action of G on a unital C^*-algebra A. Let $\mathcal{L} = A^1$ and $L(a) = \|da\|$, and let ρ denote the corresponding metric on the state space S. Then the ρ-topology on S agrees with the weak-$*$ topology.*

Proof. Choose an inner-product on $\mathfrak{g}$. Its corresponding norm is equivalent to the given norm, and so by the Comparison Lemma 1.10 it suffices to deal with the norm from the inner-p roduct. We can left-translate this inner-product over G to obtain a left-invariant Riemannian metric on G, and then a corresponding left-invariant ordinary metric on G. We let $\ell(x)$ denote the corresponding distance from x to e. Then ℓ is a continuous length function on G satisfying conditions 2.1 [G, Ro2].

Then the elements of $\mathcal{L} = A^1$ are Lipschitz for ℓ. This essentially just involves the following standard argument [G, Ro2], which we include for the reader's convenience. Let $a \in A^1$ and let c be a smooth path in G from e to a point $x \in G$. Then ϕ, defined by $\phi(t) = \alpha_{c(t)}(a)$, is differentiable, and so we have

$$\|\alpha_x(a) - a\| = \| \int \phi'(t) dt \| \leq \int \|\alpha_{c(t)}(d_{c'(t)} a)\| dt \leq \|da\| \int \|c'(t)\| dt \ .$$

But the last integral is just the length of c. Thus from the definition of the ordinary metric on G, with its length function ℓ, we obtain

$$\|\alpha_x(a) - a\| \leq \|da\| \ell(x) \ .$$

(Actually, the above argument works for any norm on $\mathfrak{g}$.) Then if we let $\mathcal{L}_0$ and L_0 be defined just in terms of ℓ as in the previous section, we see that $\mathcal{L} \subseteq \mathcal{L}_0$ and $L_0 \leq L$. Thus we are exactly in position to apply the Comparison Lemma 1.10 to obtain the desired conclusion. □

We remark that Weaver (theorem 24 of [Wv1]) in effect proved for this setting the total boundedness of $\mathcal{B}_1$ for the particular case of non-commutative 2-tori, by different methods.

4. Metrics from Dirac operators

Suppose again that G is a compact connected Lie group, and that α is an ergodic action of G on a unital C^*-algebra A. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{g}'$ denote its vector-space dual. Fix any inner-product on $\mathfrak{g}'$. We will denote it by g, or by $\langle\ ,\ \rangle_g$, to distinguish it from the Hilbert space inner-products which will arise.

With this data we can define a spectral triple [C1, C2, C3] for A. For simplicity of exposition we will not include gradings and real structure, and we will oversimplify our treatment of spinors, since the details are not essential for our purposes. But with more care they can be included. (See, e.g. [V, VB].) We proceed as follows. Let $C = \mathrm{Clif}(\mathfrak{g}', -g)$ be the complex Clifford C^*-algebra over $\mathfrak{g}'$ for $-g$. Thus each $\omega \in \mathfrak{g}'$ determines a skew-adjoint element of C such that

$$\omega^2 = -\langle \omega, \omega \rangle_g \; 1_C \ .$$

Depending on whether $\mathfrak{g}$ is even or odd dimensional, C will be a full matrix algebra, or the direct sum of two such. We let $\mathcal{S}$ be the Hilbert space of a finite-dimensional faithful representation of C (the "spinors").

Let A^∞ denote the space of smooth elements of A. (We could just as well use the A^1 of the previous section. We use A^∞ here for variety. It is still a dense $*$-subalgebra [BR].) Let $W = A^\infty \otimes \mathcal{S}$, viewed as a free right A^∞-module. From the Hilbert-space inner-product on $\mathcal{S}$ we obtain an A^∞-valued inner-product on W. Let η be as in the previous section, viewed as a faithful state on A. Combined with the A-valued inner product on W, it gives an ordinary inner-product on W. We will denote the Hilbert space completion by $L^2(W, \eta)$.

Now A^∞ and C have evident commuting left actions on W. These are easily seen to give $*$-representations of A and C on $L^2(W, \eta)$, which we denote by λ and c respectively.

We define the Dirac operator, D, on $L^2(W, \eta)$ in the usual way. Its domain will be W, and it is defined as the composition of operators

$$W \xrightarrow{d} \mathfrak{g}' \otimes W \xrightarrow{i} C \otimes W \xrightarrow{c} W \ .$$

Here d is the operator which takes $b \in A^\infty$ to $db \in \mathfrak{g}' \otimes A^\infty$, defined by $db(X) = d_X(b)$, which we then extend to W so that it takes $b \otimes s$ to $db \otimes s$. The operator i just comes from the canonical inclusion of $\mathfrak{g}'$ into C. The operator c just comes from applying the representation of C on $\mathcal{S}$, and so on W.

It is easily seen that D is a symmetric operator on $L^2(W, \eta)$. It will not be important for us to verify that D is essentially self-adjoint, and that its closure has compact resolvant.

Let $\{e_j\}$ denote an orthonormal basis for $\mathfrak{g}'$, and let $\{E_j\}$ denote the dual basis for $\mathfrak{g}$. Then in terms of these bases we have

$$D(b \otimes s) = \sum \alpha_{E_j}(b) \otimes c(e_j)s \ .$$

When we use this to compute $[D, \lambda_a]$ for $a \in A^\infty$, a straightforward calculation shows that we obtain

$$[D, \lambda_a](b \otimes s) = \sum (\alpha_{E_j}(a) \otimes c(e_j))(b \otimes s) \ .$$

That is,

$$[D, \lambda_a] = \sum \alpha_{E_j}(a) \otimes e_j \ , \tag{4.1}$$

acting on $L^2(W, \eta)$ through the representations λ and c. It is clear from (4.1) that $[D, \lambda_a]$ is bounded for the operator norm from $L^2(W, \eta)$.

We can now set $\mathcal{L} = A^\infty$, and

$$L(a) = \|[D, \lambda_a]\| \ .$$

It is clear that $L(1_A) = 0$. To proceed further we compare L with the semi-norm of the last section. If we view $\mathfrak{g}'$ as contained in the C^*-algebra C, we have $e_j^2 = -1$ and $e_j^* = -e_j$ for each j. In particular, $\|e_j\| = 1$. From (4.1) it is then easy to see that there is a constant, K, such that

$$L(a) \leq K\|da\|$$

for all $a \in \mathcal{L}$, where $\|da\|$ is as in the previous section, for the inner-product dual to that on $\mathfrak{g}'$. However, what we need is an inequality in the reverse direction so that we will be able to apply the Comparison Lemma 1.10.

For this purpose, consider any element $t = \sum b_j \otimes e_j$ in $A \otimes C$, with the e_j as above. Let $f_j = ie_j$, so that $f_j^* = f_j$, $f_j^2 = 1$, and $f_j f_k = -f_k f_j$ for $j \neq k$. Let $p_j = (1 + f_j)/2$ and $q_j = 1 - p_j = (1 - f_j)/2$, both being self-adjoint projections. Then $p_j f_k = f_k q_j$ for $j \neq k$. Consequently $p_j f_k p_j = 0 = q_j f_k q_j$ for $j \neq k$. Thus

$$(1 \otimes p_j)t(1 \otimes p_j) = b_j \otimes p_j e_j p_j = b_j \otimes ip_j$$

and

$$(1 \otimes q_j)t(1 \otimes q_j) = -b_j \otimes iq_j \ .$$

Since at least one of p_j and q_j must be non-zero, it becomes clear that $\|t\| \geq \|b_j\|$ for each j. When we apply this to (4.1) we see that

$$L(a) \geq \|\alpha_{E_j}(a)\|$$

for each j. Consequently, for a suitable constant k we have

$$L(a) \geq k\|da\| \ ,$$

where again $\|da\|$ is as in the previous section. On applying Proposition 1.11, Theorem 3.1, and the Comparison Lemma 1.10, we obtain the proof of:

4.2 Theorem. *Let α be an ergodic action of the compact connected Lie group G with Lie algebra $\mathfrak{g}$ on the unital C^*-algebra A. Pick any inner-product on the dual, $\mathfrak{g}'$, of $\mathfrak{g}$. Let D denote the corresponding Dirac operator, as defined above. Let $\mathcal{L} = A^\infty$, and let L be defined by*

$$L(a) = \|[D, a]\|$$

for $a \in A$. Let ρ be the corresponding metric on S. Then the ρ-topology on S agrees with the weak-$*$ topology.

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