On the Classification of Simple Inductive Limit C^*-Algebras, I: The Reduction Theorem

Dedicated to Professor Ronald G. Douglas on the occasion of his sixtieth birthday

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Abstract. Suppose that

$$A = \lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$$

is a simple C^*-algebra, where $X_{n,i}$ are compact metrizable spaces of uniformly bounded dimensions (this restriction can be relaxed to a condition of very slow dimension growth). It is proved in this article that A can be written as an inductive limit of direct sums of matrix algebras over certain special 3-dimensional spaces. As a consequence it is shown that this class of inductive limit C^*-algebras is classified by the Elliott invariant — consisting of the ordered K-group and the tracial state space — in a subsequent paper joint with G. Elliott and L. Li (Part II of this series). (Note that the C^*-algebras in this class do not enjoy the real rank zero property.)

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0 Introduction

In this article and the subsequent article [EGL], we will classify all the unital simple C^*-algebras A, which can be written as the inductive limit of a sequence

$$\bigoplus_{i=1}^{t_1} P_{1,i}M_{[1,i]}(C(X_{1,i}))P_{1,i} \xrightarrow{\phi_{1,2}} \bigoplus_{i=1}^{t_2} P_{2,i}M_{[2,i]}(C(X_{2,i}))P_{2,i} \xrightarrow{\phi_{2,3}} \cdots,$$

where $X_{n,i}$ are compact metrizable spaces with $\sup\{\dim X_{n,i}\}_{n,i} < +\infty$, $[n,i]$ and t_n are positive integers, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections. The invariant consists of the ordered K-group and the space of traces on the algebra. The main result in the present article is that a C^*-algebra A as above can be written in another way as an inductive limit so that all the spaces $X_{n,i}$ appearing are certain special simplicial complexes of dimension at most three. Then, in [EGL], the classification theorem will be proved by assuming the C^*-algebras are such special inductive limits.

In the special case that the groups $K_*(C(X_{n,i}))$ are torsion free, the C^*-algebra A can be written as an inductive limit of direct sums of matrix algebras over $C(S^1)$(i.e., one can replace $X_{n,i}$ by S^1). Combining this result with [Ell2], without [EGL]–the part II of this series—, we can still obtain the classification theorem for this special case, which is a generalization of the result of Li for the case that $\dim(X_{n,i}) = 1$ (see [Li1-3]).

The theory of C^*-algebras can be regarded as noncommutative topology, and has broad applications in different areas of mathematics and physics (e.g., the study of foliated spaces, manifolds with group actions; see [Con]).

One extreme class of C^*-algebras is the class of commutative C^*-algebras, which corresponds to the category of ordinary locally compact Hausdorff topological spaces. The other extreme, which is of great importance, is the class of simple C^*-algebras, which must be considered to be highly noncommutative. For example, the (reduced) foliation C^*-algebra of a foliated space is simple if and only if every leaf is dense in the total space; the cross product C^*-algebra, for a $\mathbb{Z}$ action on a space X, is simple if and only if the action is minimal.

Even though the commutative C^*-algebras and the simple C^*-algebras are opposite extremes, remarkably, many (unital or nonunital) simple C^*-algebras (including the foliation C^*-algebra of a Kronecker foliation, see [EE]) have been proved to be inductive limits of direct sums of matrix algebras over commutative C^*-algebras, i.e., to be of the form $\lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$. (Note that the only commutative C^*-algebras, or matrix algebras over commutative C^*-algebras, which are simple are the very trivial ones, $\mathbb{C}$ or $M_k(\mathbb{C})$.) In general, it is a conjecture that any stably finite, simple, separable, amenable C^*-algebra is an inductive limit of subalgebras of matrix algebras over commutative C^*-algebras. This conjecture would be analogous to the result of Connes and Haagerup that any amenable von Neumann algebra is generated by an upward directed family of sub von Neumann algebras of type I.

The sweeping classification project of G. Elliott is aimed at the complete classification of simple, separable, amenable C^*-algebras in terms of a certain simple invariant, as we mentioned above, consisting of the ordered K-group and the space of traces on the algebra. Naturally, the class of inductive limit C^*-algebras $A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$, considered in this article, is an essential ingredient of the project. Following Blackadar [Bl1], we will call such inductive limit algebras AH algebras.

The study of AH algebras has its roots in the theory of AF algebras (see [Br] and [Ell4]). But the modern classification theory of AH algebras was inspired by the seminal paper [Bl3] of B. Blackadar and was initiated by Elliott in [Ell5].

The real rank of a C^*-algebra is the noncommutative counterpart of the dimension of a topological space. Until recently, the only known possibilities for the real rank of a simple C^*-algebra were zero or one. It was proved in [DNNP] that any simple AH algebra

$$A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$$

has real rank either zero or one, provided that $\sup\{\dim X_{n,i}\}_{n,i} < +\infty$. (Recently, Villadsen has found a simple C^*-algebra with real rank different from zero and one, see [Vi 2].)

For the case of simple C^*-algebras of real rank zero, the classification is quite successful and satisfactory, even though the problem is still not completely solved. Namely, on one hand, the remarkable result of Kirchberg [Kir] and Phillips [Phi1] completely classified all purely infinite, simple, separable, amenable C^*-algebras with the so called UCT property (see also [R] for an important earlier result). All purely infinite simple C^*-algebras are of real rank zero; see [Zh]. On the other hand, in [EG1-2] Elliott and the author completely classified all the stably finite, simple, real rank zero C^*-algebras which are AH algebras of the form $\varinjlim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ with $\dim(X_{n,i}) \leq 3$. It was proved by Dadarlat and the author that this class includes all simple real rank zero AH algebras with arbitrary but uniformly bounded dimensions for the spaces $X_{n,i}$ (see [D1-2], [G1-4] and [DG]).

In this article, the AH algebras considered are not assumed to have real rank zero. As pointed out above, they must have real rank either zero or one. In fact, in a strong sense, almost all of them have real rank one. The real rank zero C^*-algebras are the very special ones for which the space of traces (one part of the invariant mentioned above) is completely determined by the ordered K-group of the C^*-algebra (the other part of the invariant). Not surprisingly, the lack of the real rank zero property presents new essential difficulties. Presumably, dimension one noncommutative spaces are much richer and more complicated than dimension zero noncommutative spaces. In what follows, we would like to explain one of the main differences between the real rank zero case and the general case in the setting of simple AH algebras.

If $A = \lim\limits_{\rightarrow}(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ is of real rank zero, then Elliott and the author proved a decomposition result (see Theorem 2.21 of [EG2]) which says that $\phi_{n,m}$ (for m large enough) can be approximately decomposed as a sum of two parts, $\phi_1 \oplus \phi_2$; one part, ϕ_1, having a very small support projection, and the other part, ϕ_2, factoring through a finite dimensional algebra.

In §4 of the present paper, we will prove a decomposition theorem which says that, for the simple AH algebra A above (with or without the real rank zero condition), $\phi_{n,m}$ (for m large enough) can be approximately decomposed as a sum of three parts, $\phi_1 \oplus \phi_2 \oplus \phi_3$: the part ϕ_1 having a very small support projection compared with the part ϕ_2; the part ϕ_2 factoring through a finite dimensional algebra; and the third part ϕ_3 factoring through a direct sum of matrix algebras over the interval $[0, 1]$. (Note that, in the case of a real rank zero inductive limit, the part ϕ_3 does not appear. In the general case, though, the part ϕ_3 has a very large support projection compared with the part $\phi_1 \oplus \phi_2$.) With this decomposition theorem, we can often deal with the part $\phi_1 \oplus \phi_2$ by using the techniques developed in the classification of the real rank zero case (see [EG1-2], [G1-4], [D1-2] and [DG]).

This new decomposition theorem is much deeper. It reflects the real rank one (as opposed to real rank zero) property of the simple C^*-algebra. The special case of the decomposition result that the spaces $X_{n,i}$ are already supposed to be one-dimensional spaces is due to L. Li (see [Li3]). The proof for the case of higher dimensional spaces is essentially more difficult. In particular, as preparation, we need to prove certain combinatorial results (see §3) and also the following result (see §2): Any homomorphism from $C(X)$ to $M_k(C(Y))$ can be perturbed to a homomorphism whose maximum spectral multiplicity (for the definition of this terminology, see 1.2.4 below) is not larger than $\dim X + \dim Y$, provided that $X \neq \{pt\}$ and X is path connected.

The special simplicial complexes used in our main reduction theorem are the following spaces: $\{pt\}$, $[0, 1]$, S^1, S^2, $\{T_{II,k}\}_{k=2}^{\infty}$, and $\{T_{III,k}\}_{k=2}^{\infty}$, where the spaces $T_{II,k}$ are two-dimensional connected simplicial complexes with $H^1(T_{II,k}) = 0$ and $H^2(T_{II,k}) = \mathbb{Z}/k$, and the spaces $T_{III,k}$ are three-dimensional connected simplicial complexes with $H^1(T_{III,k}) = 0 = H^2(T_{III,k})$ and $H^3(T_{III,k}) = \mathbb{Z}/k$. (See 4.2 of [EG2] for details.)

The spaces $T_{II,k}$ and $T_{III,k}$ are needed to produce the torsion part of the K-groups of the inductive limit C^*-algebras. Since the algebras $C(T_{II,k}), C(T_{III,k})$, and $C(S^2)$ are not stably generated (see [Lo]), difficulties occur in the construction of homomorphisms from these C^*-algebras, when we prove our main reduction theorem (and the isomorphism theorem in [EGL]). In the case of real rank zero algebras, this difficulty can be avoided by using unsuspended E-theory (see [D1-2] and [G1-4]) combined with a certain uniqueness theorem — Theorem 2.29 of [EG2], which only involves homomorphisms (instead of general completely positive linear maps). Roughly speaking, the trouble is that a completely positive linear $*$-contraction, which is an "almost homomorphism"—a G-δ multiplicative map (see 1.1.2 below for the definition of this concept) for sufficiently large G and sufficiently small δ—, may not be automatically close to a homomorphism. As we mentioned above, after we approximately decompose $\phi_{n,m}$ as $\phi_1 \oplus \phi_2 \oplus \phi_3$, we will deal with the part $\phi_1 \oplus \phi_2$, by using the results and techniques from the real rank zero case, in particular by using Theorem 1.6.9 below—a strengthened version of Theorem 2.29 of [EG2]. Therefore, we will consider the composition of the map $\phi_1 \oplus \phi_2$ and a homomorphism from a matrix algebra over $\{pt\}$, $[0,1]$, S^1, S^2, $\{T_{II,k}\}_{k=2}^{\infty}$, and $\{T_{III,k}\}_{k=2}^{\infty}$, to A_n. We need this composition to be close to a homomorphism, but $\phi_1 \oplus \phi_2$ is not supposed to be close to a homomorphism (it is close to the homomorphism $\phi_{n,m}$ in the case of real rank zero). To overcome the above difficulty, we prove a theorem in §5—a kind of uniqueness theorem, which may be roughly described as follows:

For any $\varepsilon > 0$, positive integer N, and finite set $F \subset A = M_k(C(X))$, where X is one of the spaces $\{pt\}, [0,1], T_{II,k}, T_{III,k}$, and S^2, there are a number $\delta > 0$, a finite set $G \subset A$, and a positive integer L, such that for any two G-δ multiplicative (see 1.1.2 below), completely positive, linear $*$-contractions $\phi, \psi : M_k(C(X)) \to B = M_l(C(Y))$ (where $\dim(Y) \leq N$), if they define the same map on the level of K-theory and also mod-p K-theory (this statement will be made precise in §5), then there are a homomorphism $\lambda : A \to M_L(B)$ with finite dimensional image and a unitary $u \in M_{L+1}(B)$ such that

$$\|(\phi \oplus \lambda)(f) - u(\psi \oplus \lambda)(f)u^*\| < \varepsilon$$

for all $f \in F$.

This result is quite nontrivial, and may be expected to have more general applications. Some similar results appear in the literature (e.g., [EGLP, 3.1.4], [D1, Thm A], [G4, 3.9]). But even for $*$-homomorphisms (which are G-δ multiplicative for any G and δ), all these results (except for contractible spaces) require that the number L, the size of the matrix, depends on the maps ϕ and ψ.

Note that the theorem stated above does not hold if one replaces X by S^1, even if both ϕ and ψ are $*$-homomorphisms. (Fortunately, we do not need the theorem for S^1 in this article, since $C(S^1)$ is stably generated. But on the other hand, the lack of such a theorem for S^1 causes a major difficulty in the formulation and the proof of the uniqueness theorem involving homomorphisms from $C(S^1)$ to $M_k(C(X))$, in [EGL]— part 2 of this series.)

With the above theorem, if a G-δ multiplicative, positive, linear $*$-contraction ϕ and a $*$-homomorphism (name it ψ) define the same map on the level of K-theory and mod-p K-theory, then $\phi\oplus\lambda$ is close to a $*$-homomorphism (e.g., $\mathrm{Ad}u\circ(\psi\oplus\lambda)$) for some $*$-homomorphism $\lambda : A \to M_L(B)$ with finite dimensional image. In particular, the size L of the $*$-homomorphism λ can be controlled. This is essential for the construction of $*$-homomorphisms from $A = M_k(C(X))$, where X is one of $T_{II,k}, T_{III,k}$, and S^2. In particular, once L is fixed, we can construct the decomposition of $\phi_{n,m}$ as $\phi_1 \oplus \phi_2 \oplus \phi_3$, as mentioned above, such that the supporting projection of the part ϕ_2 is larger than the supporting projection of the part ϕ_1 by the amplification of L times. Hence we can prove that, the composition of the map $\phi_1 \oplus \phi_2$ and a homomorphism from a matrix algebra over $T_{II,k}, T_{III,k}$, and S^2 to A_n, is close to a homomorphism (see Theorems 5.32a and 5.32b below for details).

The theorem is also true for a general finite CW complex X, provided that $K_1(C(X))$ is a torsion group.

(Note that for the space S^1 (or the spaces $\{pt\}, [0,1]$), we do not need such a theorem, since any G-δ multiplicative, positive, linear $*$-contraction from $M_k(C(S^1))$ will automatically be close to a $*$-homomorphism if G is sufficiently large and δ is sufficiently small.)

The above mentioned theorem and the decomposition theorem both play important roles in the proof of our main reduction theorem, and also in the proof of the isomorphism theorem in [EGL].

The main results of this article and [EGL] were announced in [G1] and in Elliott's lecture at the International Congress of Mathematicians in Zurich (see [Ell3]). Since then, several classes of simple inductive limit C^*-algebras have been classified (see [EGJS], [JS 1-2], and [Th1]). But all these later results involve only inductive limits of subhomogeneous algebras with 1-dimensional spectra. In particular, the K_0-groups have to be torsion free, since it is impossible to produce the torsion in K_0-group with one-dimensional spectra alone, even with subhomogeneous building blocks.

This article is organized as follows. In §1, we will introduce some notations, collect some known results, prove some preliminary results, and discuss some important preliminary ideas, which will be used in other sections. In particular, in §1.5, we will discuss the general strategy in the proof of the decomposition theorem, of which, the detailed proof will be given in in §2, §3 and §4. In §1.6, we will prove some uniqueness theorem and factorization theorem which are important in the proof of the main theorem. Even though the results in §1.6 are new, most of the methods are modification of known techniques from [EG2], [D2], [G4] and [DG]. In §2, we will prove the result about maximum spectral multiplicities, which will be used in §4 and other papers. In §3, we will prove certain results of a combinatorial nature. In §4, we will combine the results from §2, §3, and the results in [Li2], to prove the decomposition theorem. In §5, we will prove the result mentioned above concerning G-δ multiplicative maps. In §6, we will use §4, §5 and §1.6 to prove our main reduction theorem. Our main result can be generalized from the case of no dimension growth (i.e., $X_{n,i}$

have uniformly bounded dimensions) to the case of very slow dimension growth. Since the proof of this general case is much more tedious and complicated, we will deal with this generalization in [G5], which can be regarded as an appendix to this article.

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1 PREPARATION AND SOME PRELIMINARY IDEAS

We will introduce some conventions, general assumptions, and preliminary results in this section.

1.1 GENERAL ASSUMPTIONS ON INDUCTIVE LIMITS

1.1.1. If A and B are two C^*-algebras, we use $\mathrm{Map}(A, B)$ to denote THE SPACE OF ALL LINEAR, COMPLETELY POSITIVE $*$-CONTRACTIONS from A to B. If both A and B are unital, then $\mathrm{Map}(A, B)_1$ will denote the subset of $\mathrm{Map}(A, B)$ consisting of unital maps. By word "map", we shall mean linear, completely positive $*$-contraction between C^*-algebras, or else we shall mean continuous map between topological spaces, which one will be clear from the context.

BY A HOMOMORPHISM BETWEEN C^*-ALGEBRAS, WILL BE MEANT A $*$-HOMOMORPHISM. Let $\mathrm{Hom}(A, B)$ denote the SPACE OF ALL HOMOMORPHISMS from A to B. Similarly, if both A and B are unital, let $\mathrm{Hom}(A, B)_1$ denote the subset of $\mathrm{Hom}(A, B)$ consisting of unital homomorphisms.

DEFINITION 1.1.2. Let $G \subset A$ be a finite set and $\delta > 0$. We shall say that $\phi \in \mathrm{Map}(A, B)$ is G-δ MULTIPLICATIVE if

$$\|\phi(ab) - \phi(a)\phi(b)\| < \delta$$

for all $a, b \in G$.

Sometimes, we use $\mathrm{Map}_{G-\delta}(A, B)$ to denote all the G-δ multiplicative maps.

1.1.3. In the notation for an inductive system $(A_n, \phi_{n,m})$, we understand that $\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \cdots \circ \phi_{n,n+1}$, where all $\phi_{n,m} : A_n \to A_m$ are homomorphisms.

We shall assume that, for any summand A_n^i in the direct sum $A_n = \bigoplus_{i=1}^{t_n} A_n^i$, necessarily, $\phi_{n,n+1}(\mathbf{1}_{A_n^i}) \neq 0$, since, otherwise, we could simply delete A_n^i from A_n without changing the limit algebra.

1.1.4. If $A_n = \bigoplus_i A_n^i$ and $A_m = \bigoplus_j A_m^j$, we use $\phi_{n,m}^{i,j}$ to denote the partial map of $\phi_{n,m}$ from the i-th block A_n^i of A_n to the j-th block A_m^j of A_m.
In this article, we will assume that all inductive limit C^*-algebras are SIMPLE. That is, the limit algebra has no nontrivial proper closed two sided ideals. We will also assume that every inductive limit C^*-algebra $A = \varinjlim(A_n, \phi_{n,m})$ coming into consideration is different both from $M_k(\mathbb{C})$ (the matrix algebra over $\mathbb{C}$), and from $\mathcal{K}(H)$ (the algebra of all compact operators).
Since $A = \varinjlim(A_n = \bigoplus_i A_n^i, \phi_{n,m})$ is simple, by 5.3.2(b) of [DN], we may assume that $\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i}) \neq 0$ for any blocks A_n^i and A_m^j, where $n < m$.

1.1.5. To avoid certain counter examples (see [V]) of the main result of this article, we will restrict our attention, in this article, to inductive systems satisfying the following VERY SLOW DIMENSION GROWTH CONDITION. This is a strengthened form of the condition of slow dimension growth introduced in [BDR].
If $\varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ is a unital inductive limit system, the very slow dimension growth condition is

$$\lim_{n\to+\infty} \ \max_i \left\{ \frac{(\dim X_{n,i})^3}{\operatorname{rank}(P_{n,i})} \right\} = 0,$$

where $\dim(X_{n,i})$ denotes the (covering) dimension of $X_{n,i}$.
In this article, we will also study non-unital inductive limit algebras. The above formula must then be slightly modified. The very slow dimension growth condition in the non-unital case is that, for any summand
$A_n^i = P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ of a fixed A_n,

$$\lim_{m\to+\infty} \ \max_{i,j} \left\{ \frac{(\dim X_{m,j})^3}{\operatorname{rank}\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})} \right\} = 0,$$

where $\phi_{n,m}^{i,j}$ is the partial map of $\phi_{n,m}$ from A_n^i to A_m^j.
(For a unital inductive limit, the two conditions above are equivalent. Of course, both conditions are only proposed for the simple case.)
If the set $\{\dim X_{n,i}\}$ is bounded, i.e, there is an M such that

$$\dim X_{n,i} \leq M$$

for all n and i, then the inductive system automatically satisfies the very slow dimension growth condition, as we already assume that the limit algebra is not $M_k(\mathbb{C})$ or $\mathcal{K}(H)$.

We will prove our main reduction theorem for the case of uniformly bounded dimensions in this article, since it is significantly simpler than the case of very slow dimension growth. The general case will be discussed in [G5]—an appendix of this article. But the decomposition theorem will be proved for the case of very slow dimension growth.
It must be noted that, without the above assumption on dimension growth, the main theorem of this article does not hold (see [Vi1]). We shall leave the following question open: can the above condition of very slow dimension growth be replaced by the similar (but weaker) condition of slow dimension growth (see [BDR]), in the main theorem of this article?

1.1.6. By 2.3 of [Bl1], in the inductive limit

$$A = \lim_{\rightarrow}(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}),$$

one can always replace the compact metrizable spaces $X_{n,i}$ by finite simplicial complexes. Note that the replacement does not increase the dimensions of the spaces. Therefore, in this article, WE WILL ALWAYS ASSUME THAT ALL THE SPACES $X_{n,i}$ IN A GIVEN INDUCTIVE SYSTEM ARE FINITE SIMPLICIAL COMPLEXES. Also, WE WILL FURTHER ASSUME THAT ALL $X_{n,i}$ ARE PATH CONNECTED. Otherwise, we will separate different components into different direct summands. (Note that a finite simplicial complex has at most finitely many path connected components.)
By SIMPLICIAL COMPLEX we mean finite simplicial complex or polyhedron; see [St].

1.1.7.
(a) We use the notation $\#(\cdot)$ to denote the cardinal number of the set, if the argument is a finite set. Very often, the sets under consideration will be sets with multiplicity, and then we shall also count multiplicity when we use the notation $\#$.
(b) We shall use $a^{\sim k}$ to denote $\underbrace{a, \cdots, a}_{k\ copies}$. For example,

$$\{a^{\sim 3}, b^{\sim 2}\} = \{a, a, a, b, b\}.$$

(c) $\mathrm{int}(\cdot)$ is used to denote the integer part of a real number. We reserve the notation $[\cdot]$ for equivalence classes in possibly different contexts.
(d) For any metric space X, any $x_0 \in X$ and any $c > 0$, let $B_c(x_0) := \{x \in X \mid d(x, x_0) < c\}$ denote the open ball with radius c and centre x_0.
(e) Suppose that A is a C^*-algebra, $B \subset A$ is a subalgebra, $F \subset A$ is a (finite) subset and let $\varepsilon > 0$. If for each element $f \in F$, there is an element $g \in B$ such that $\|f - g\| < \varepsilon$, then we shall say that F is approximately contained in B to within ε, and denote this by $F \subset_\varepsilon B$.

(f) Let X be a compact metric space. For any $\delta > 0$, a finite set $\{x_1, x_2, \cdots x_n\}$ is said to be δ-dense in X, if for any $x \in X$, there is x_i such that $\mathrm{dist}(x, x_i) < \delta$.
(g) We shall use $\bullet$ to denote any possible positive integer. To save notation, $y, y', y'', \cdots$ or $a_1, a_2, \cdots$ may be used for finite sequences if we do not care how many terms are in the sequence. Similarly, $A_1 \cup A_2 \cup \cdots$ or $A_1 \cap A_2 \cap \cdots$ may be used for finite union or finite intersection. If there is a danger of confusion with infinite sequence, union, or intersection, we will write them as $a_1, a_2, \cdots, a_\bullet$, $A_1 \cup A_2 \cup \cdots \cup A_\bullet$, or $A_1 \cap A_2 \cap \cdots A_\bullet$.
(h) For $A = \bigoplus_{i=1}^t M_{k_i}(C(X_i))$, where X_i are path connected simplicial complexes, we use the notation $r(A)$ to denote $\bigoplus_{i=1}^t M_{k_i}(\mathbb{C})$, which could be considered to be a subalgebra of A consisting of t-tuples of constant functions from X_i to $M_{k_i}(\mathbb{C})$ $(i = 1, 2, \cdots, t)$. Fix a base point $x_i^0 \in X_i$ for each X_i, one can define a map $r : A \to r(A)$ by

$$r(f_1, f_2, \cdots, f_t) = (f_1(x_1^0), f_2(x_2^0), \cdots f_t(x_t^0)) \in r(A).$$

(i) For any two projections $p, q \in A$, we use the notation $[p] \leq [q]$ to denote that p is unitarily equivalent to a sub projection of q. And we use $p \sim q$ to denote that p is unitarily equivalent to q.

1.2 Spectrum and spectral variation of a homomorphism

1.2.1. Let Y be a compact metrizable space. Let $P \in M_{k_1}(C(Y))$ be a projection with $\mathrm{rank}(P) = k \leq k_1$. For each y, there is a unitary $u_y \in M_{k_1}(\mathbb{C})$ (depending on y) such that

$$P(y) = u_y \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} u_y^*,$$

where there are k 1's on the diagonal. If the unitary u_y can be chosen to be continuous in y, then P is called a trivial projection.
It is well known that any projection $P \in M_{k_1}(C(Y))$ is locally trivial. That is, for any $y_0 \in Y$, there is an open set $U_{y_0} \ni y_0$, and there is a continuous unitary-valued function

$$u : U_{y_0} \to M_{k_1}(\mathbb{C})$$

such that the above equation holds for $u(y)$ (in place of u_y) for any $y \in U_{y_0}$. If P is trivial, then $PM_{k_1}(C(X))P \cong M_k(C(X))$.

1.2.2. Let X be a compact metrizable space and $\psi : C(X) \to PM_{k_1}(C(Y))P$

be a unital homomorphism. For any given point $y \in Y$, there are points $x_1(y), x_2(y), \cdots, x_k(y) \in X$, and a unitary $U_y \in M_{k_1}(\mathbb{C})$ such that

$$\psi(f)(y) = P(y)U_y \begin{pmatrix} f(x_1(y)) & & & & & \\ & \ddots & & & & \\ & & f(x_k(y)) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U_y^* P(y) \in P(y)M_{k_1}(\mathbb{C})P(y)$$

for all $f \in C(X)$. Equivalently, there are k rank one orthogonal projections $p_1, p_2, \cdots, p_k$ with $\sum_{i=1}^k p_i = P(y)$ and $x_1(y), x_2(y), \cdots, x_k(y) \in X$, such that

$$\psi(f)(y) = \sum_{i=1}^{k} f(x_i(y))p_i, \qquad \forall\, f \in C(X).$$

Let us denote the set $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, counting multiplicities, by $\mathrm{SP}\psi_y$. In other words, if a point is repeated in the diagonal of the above matrix, it is included with the same multiplicity in $\mathrm{SP}\psi_y$. WE SHALL CALL $\mathrm{SP}\psi_y$ THE SPECTRUM OF ψ AT THE POINT y. Let us define the SPECTRUM OF ψ, denoted by $\mathrm{SP}\psi$, to be the closed subset

$$\mathrm{SP}\psi := \overline{\bigcup_{y \in Y} \mathrm{SP}\psi_y} \subseteq X.$$

Alternatively, $\mathrm{SP}\psi$ is the complement of the spectrum of the kernel of ψ, considered as a closed ideal of $C(X)$. The map ψ can be factored as

$$C(X) \xrightarrow{i^*} C(\mathrm{SP}\psi) \xrightarrow{\psi_1} PM_{k_1}(C(Y))P$$

with ψ_1 an injective homomorphism, where i denotes the inclusion $\mathrm{SP}\psi \hookrightarrow X$. Also, if $A = PM_{k_1}(C(Y))P$, then we shall call the space Y the spectrum of the algebra A, and write $\mathrm{SP}A = Y (= \mathrm{SP}(\mathrm{id}))$.

1.2.3. In 1.2.2, if we group together all the repeated points in $\{x_1(y), x_2(y), \cdots, x_k(y)\}$, and sum their corresponding projections, we can write

$$\psi(f)(y) = \sum_{i=1}^{l} f(\lambda_i(y))P_i \qquad\qquad (l \le k),$$

where $\{\lambda_1(y), \lambda_2(y), \cdots, \lambda_l(y)\}$ is equal to $\{x_1(y), x_2(y), \cdots, x_k(y)\}$ as a set, but $\lambda_i(y) \neq \lambda_j(y)$ if $i \neq j$; and each P_i is the sum of the projections corresponding to $\lambda_i(y)$. If $\lambda_i(y)$ has multiplicity m (i.e., it appears m times in $\{x_1(y), x_2(y), \cdots, x_k(y)\}$), then $\mathrm{rank}(P_i) = m$.

DEFINITION 1.2.4. Let ψ, y, and P_i be as above. The MAXIMUM SPECTRAL MULTIPLICITY OF ψ AT THE POINT y is defined to be $\max_i(\mathrm{rank}P_i)$. The MAXIMUM SPECTRAL MULTIPLICITY OF ψ is defined to be the supremum of the maximum spectral multiplicities of ψ at the various points of Y.

The following result is the main theorem in §2, which says that we can make the homomorphism not to have too large spectral multiplicities, up to a small perturbation.

THEOREM 2.1. *Let X and Y be connected simplicial complexes and $X \neq \{pt\}$. Let $l = \dim(X) + \dim(Y)$. For any given finite set $G \subset C(X)$, any $\varepsilon > 0$, and any unital homomorphism $\phi : C(X) \to PM_\bullet(C(Y))P$, where $P \in M_\bullet(C(Y))$ is a projection, there is a unital homomorphism $\phi' : C(X) \to PM_\bullet(C(Y))P$ such that*
(1) $\|\phi(g) - \phi'(g)\| < \varepsilon$ for all $g \in G$;
(2) ϕ' has maximum spectral multiplicity at most l.

1.2.5. Set $P^k(X) = \underbrace{X \times X \times \cdots \times X}_{k} / \sim$, where the equivalence relation $\sim$ is defined by $(x_1, x_2, \cdots, x_k) \sim (x'_1, x'_2, \cdots, x'_k)$ if there is a permutation σ of $\{1, 2, \cdots, k\}$ such that $x_i = x'_{\sigma(i)}$, for each $1 \leq i \leq k$. A metric d on X can be extended to a metric on $P^k(X)$ by

$$d([x_1, x_2, \cdots, x_k], [x'_1, x'_2, \cdots, x'_k]) = \min_{\sigma} \max_{1 \leq i \leq k} d(x_i, x'_{\sigma(i)}),$$

where σ is taken from the set of all permutations, and $[x_1, \cdots, x_k]$ denotes the equivalence class in $P^k(X)$ of $(x_1, \cdots, x_k)$.

1.2.6. Let X be a metric space with metric d. Two k-tuples of (possibly repeating) points $\{x_1, x_2, \cdots, x_k\} \subset X$ and $\{x'_1, x'_2, \cdots, x'_k\} \subset X$ are said to BE PAIRED WITHIN η if there is a permutation σ such that

$$d(x_i, x'_{\sigma(i)}) < \eta, \qquad i = 1, 2, \cdots, k.$$

This is equivalent to the following. If one regards $(x_1, x_2, \cdots, x_k)$ and $(x'_1, x'_2, \cdots, x'_k)$ as two points in P^kX, then

$$d([x_1, x_2, \cdots, x_k], [x'_1, x'_2, \cdots, x'_k]) < \eta.$$

1.2.7. Let $\psi : C(X) \to PM_{k_1}(C(Y))P$ be a unital homomorphism as in 1.2.5. Then

$$\psi^* : y \mapsto \mathrm{SP}\psi_y$$

defines a map $Y \to P^kX$, if one regards $\mathrm{SP}\psi_y$ as an element of P^kX. This map is continuous. In term of this map and the metric d, let us define the spectral variation of ψ:

$$\mathrm{SPV}(\psi) := \text{diameter of the image of } \psi^*.$$

Definition 1.2.8. We shall call the projection P_i in 1.2.3 the spectral projection of ϕ at y with respect to the spectral element $\lambda_i(y)$. If $X_1 \subset X$ is a subset of X, we shall call

$$\sum_{\lambda_i(y)\in X_1} P_i$$

the spectral projection of ϕ at y corresponding to the subset X_1 (or with respect to the subset X_1).
In general, for an open set $U \subset X$, the spectral projection $P(y)$ of ϕ at y corresponding to U does not depend on y continuously. But the following lemma holds.

Lemma 1.2.9. *Let $U \subset X$ be an open subset. Let $\phi : C(X) \to M_\bullet(C(Y))$ be a homomorphism. Suppose that $W \subset Y$ is an open subset such that*

$$SP\phi_y \cap (\overline{U}\backslash U) = \emptyset, \quad \forall y \in W.$$

Then the function

$$y \mapsto \textit{ spectral projection of } \phi \textit{ at } y \textit{ correponding to } U$$

is a continuous function on W. Furthermore, if W is connected then $\#(SP\phi_y \cap U)$ (counting multiplicity) is the same for any $y \in W$, and the map $y \mapsto SP\phi_y \cap U \in P^l X$ is a continuous map on W, where $l = \#(SP\phi_y \cap U)$.

Proof: Let $P(y)$ denote the spectral projection of ϕ at y corresponding to the open set U. Fix $y_0 \in W$. Since $\mathrm{SP}\phi_{y_0}$ is a finite set, there is an open set $U_1 \subset \overline{U_1} \subset U(\subset X)$ such that $\mathrm{SP}\phi_{y_0} \cap U_1 = \mathrm{SP}\phi_{y_0} \cap U(= \mathrm{SP}\phi_{y_0} \cap \overline{U})$, or in other words, $\mathrm{SP}\phi_{y_0} \subset U_1 \cup (X\backslash\overline{U})$. Considering the open set $U_1 \cup (X\backslash\overline{U})$, by the continuity of the function

$$y \mapsto \mathrm{SP}\phi_y \in P^k X,$$

where $k = \mathrm{rank}(\phi(\mathbf{1}))$, there is an open set $W_1 \ni y_0$ such that

$$\mathrm{SP}\phi_y \subset U_1 \cup (X\backslash\overline{U}), \quad \forall y \in W_1. \tag{1}$$

Let $\chi \in C(X)$ be a function satisfying

$$\chi(x) = \begin{cases} 1 & \text{if} \quad x \in \overline{U_1} \\ 0 & \text{if} \quad x \in X\backslash U\ . \end{cases}$$

Then from (1) and the definition of spectral projection it follows that

$$\phi(\chi)(y) = P(y), \ \forall y \in W_1.$$

In particular, $P(y)$ is continuous at y_0.
The additional part of the lemma follows from the continuity of $P(y)$ and the connectedness of W.

$\square$

In the above proof, we used the following fact, a consequence of the continuity of the map $y \mapsto \mathrm{SP}\phi_y$. We state it separately for our future use.

Lemma 1.2.10. *Let X be a finite simplicial complex, $X_1 \subset X$ be a closed subset, and $\phi : C(X) \to M_\bullet(C(Y))$ be a homomorphism. For any $y_0 \in Y$, if $SP\phi_{y_0} \cap X_1 = \emptyset$, then there is an open set $W \ni y_0$ such that $SP\phi_y \cap X_1 = \emptyset$ for any $y \in W$.*
Another equivalent statement is the following. Let $U \subset X$ be an open subset. For any $y_0 \in Y$, if $SP\phi_{y_0} \subset U$, then there is an open set $W \ni y_0$ such that $SP\phi_y \subset U$ for any $y \in W$.

1.2.11. In fact the above lemma is a consequence of the following more general principle: If $\phi : C(X) \to M_\bullet(\mathbb{C})$ is a homomorphism satisfying $\mathrm{SP}\phi \subset U$ for a certain open set U, then for any homomorphism $\psi : C(X) \to M_\bullet(\mathbb{C})$ which is close enough to ϕ, we have $\mathrm{SP}\psi \subset U$. We state it as the following lemma.

Lemma 1.2.12. *Let $F \subset C(X)$ be a finite set of elements which generate $C(X)$ as a C^*-algebra. For any $\varepsilon > 0$, there is a $\delta > 0$ such that if two homomorphisms ϕ, ψ : $C(X) \to M_\bullet(\mathbb{C})$ satisfy*

$$\|\phi(f) - \psi(f)\| < \delta, \qquad \forall f \in F,$$

then $SP\psi$ and $SP\phi$ can be paired within ε. In particular, $SP\psi \subset U$, where U is the open set defined by $U = \{x \in X \mid \exists x' \in SP\phi \text{ with } dist(x, x') < \varepsilon\}$.

1.2.13. For any C^* algebra A (usually we let $A = C(X)$ or $A = PM_k(C(X))P$), any homomorphism $\phi : A \to M_\bullet(C(Y))$, and any closed subset $Y_1 \subset Y$, denote by $\phi|_{Y_1}$ the following composition:

$$A \xrightarrow{\phi} M_\bullet(C(Y)) \overset{\text{restriction}}{\longrightarrow} M_\bullet(C(Y_1)).$$

(As usual, for a subset or subalgebra $A_1 \subset A$, $\phi|_{A_1}$ will be used to denote the restriction of ϕ to A_1. We believe that there will be no danger of confusion as the meaning will be clear from the context.)
The following trivial fact will be used frequently.

Lemma 1.2.14. *Let $Y_1, Y_2 \subset Y$ be two closed subsets. If $\phi_1 : A \to M_k(C(Y_1))$*

and $\phi_2 : A \to M_k(C(Y_2))$ are two homomorphisms with $\phi_1|_{Y_1\cap Y_2} = \phi_2|_{Y_1\cap Y_2}$, then for any $a \in A$, the matrix-valued function $y \mapsto \phi(a)(y)$, where

$$\phi(a)(y) = \begin{cases} \phi_1(a)(y) & \text{if} \quad y \in Y_1 \\ \phi_2(a)(y) & \text{if} \quad y \in Y_2 \end{cases},$$

is a continuous function on $Y_1 \cup Y_2$ (i.e., it is an element of $M_k(C(Y_1 \cup Y_2))$. Furthermore, $a \mapsto \phi(a)$ defines a homomorphism $\phi : A \to M_k(C(Y_1 \cup Y_2))$.

1.2.15. Let X be a compact connected space and let Q be a projection of rank n in $M_N(C(X))$. The WEAK VARIATION OF A FINITE SET $F \subset QM_N(C(X))Q$ is defined by

$$\omega(F) = \sup_{\Pi_1,\Pi_2} \inf_{u\in U(n)} \max_{a\in F} \|u\Pi_1(a)u^* - \Pi_2(a)\|$$

where Π_1, Π_2 run through the set of irreducible representations of $QM_N(C(X))Q$ into $M_n(\mathbb{C})$.
Let X_i be compact connected spaces and $Q_i \in M_{n_i}(C(X_i))$ be projections. For a finite set $F \subset \bigoplus_i Q_iM_{n_i}(C(X_i))Q_i$, define the WEAK VARIATION $\omega(F)$ to be $\max_i \omega(\pi_i(F))$, where $\pi_i : \bigoplus_i Q_iM_{n_i}(C(X_i))Q_i \to Q_iM_{n_i}(C(X_i))Q_i$ is the natural project map onto the i-th block.
The set F is said to be WEAKLY APPROXIMATELY CONSTANT TO WITHIN ε if $\omega(F) < \varepsilon$. The other description of this concept can be found in [EG2, 1.4.11] (see also [D2, 1.3]).

1.2.16. Let $\phi : M_k(C(X)) \to PM_l(C(Y))P$ be a unital homomorphism. Set $\phi(e_{11}) = p$, where e_{11} is the canonical matrix unit corresponding to the upper left corner. Set

$$\phi_1 = \phi|_{e_{11}M_k(C(X))e_{11}} : C(X) \longrightarrow pM_l(C(Y))p.$$

Then $PM_l(C(Y))P$ can be identified with $pM_l(C(Y))p \otimes M_k$ in such a way that

$$\phi = \phi_1 \otimes \mathbf{1}_k.$$

Let us define

$$\mathrm{SP}\phi_y := \mathrm{SP}(\phi_1)_y,$$

$$\mathrm{SP}\phi := \mathrm{SP}\phi_1,$$

$$\mathrm{SPV}(\phi) := \mathrm{SPV}(\phi_1)\ .$$

Suppose that X and Y are connected. Let Q be a projection in $M_k(C(X))$ and $\phi : QM_k(C(X))Q \to PM_l(C(Y))P$ be a unital map. By the Dilation Lemma (2.13 of [EG2]; see Lemma 1.3.1 below), there are an n, a projection $P_1 \in M_n(C(Y))$, and a unital homomorphism

$$\tilde{\phi} : M_k(C(X)) \longrightarrow P_1M_n(C(Y))P_1$$

such that

$$\phi = \tilde{\phi}|_{QM_k(C(X))Q}.$$

(Note that this implies that P is a subprojection of P_1.) We define:

$$\mathrm{SP}\phi_y := \mathrm{SP}\tilde{\phi}_y,$$

$$\mathrm{SP}\phi := \mathrm{SP}\tilde{\phi},$$

$$\mathrm{SPV}(\phi) := \mathrm{SPV}(\tilde{\phi})\ .$$

(Note that these definitions do not depend on the choice of the dilation $\tilde{\phi}$.)
The following lemma was essentially proved in [EG2, 3.27] (the additional part is [EG 1.4.13]).

Lemma 1.2.17. *Let X be a path connected compact metric space. Let $p_0, p_1, p_2, \cdots, p_n \in M_\bullet(C(Y))$ be mutually orthogonal projections such that $rank(p_i) \geq rank(p_0)$, $i = 1, 2, \cdots, n$. Let $\{x_1, x_2, \cdots, x_n\}$ be a $\frac{\delta}{2}$-dense subset of X. If a homomorphism $\phi:\ C(X) \to M_\bullet(C(Y))$ is defined by*

$$\phi(f) = \phi_0(f) \oplus \sum_{i=1}^{n} f(x_i)p_i,$$

where $\phi_0 : C(X) \to p_0 M_\bullet(C(Y))p_0$ is an arbitrary homomorphism, then $SPV(\phi) < \delta$. Consequently, if a finite set $F \subset C(X)$ satisfies the condition that $\|f(x) - f(x')\| < \varepsilon$, for any $f \in F$, whenever $dist(x, x') < \delta$, then $\phi(F)$ is weakly approximately constant to within ε.
(For convenience, we will call such a homomorphism $\psi : C(X) \to M_\bullet(C(Y))$, defined by $\psi(f) = \sum_{i=1}^{n} f(x_i)p_i$, a homomorphism defined by point evaluations on the set $\{x_1, x_2, \cdots, x_n\}$.)

Proof: For any two points $y, y' \in Y$, the sets $\mathrm{SP}\phi_y$ and $\mathrm{SP}\phi_{y'}$ have the following subset in common:

$$\{x_1^{\sim \mathrm{rank}(p_1)}, x_2^{\sim \mathrm{rank}(p_2)}, \cdots, x_n^{\sim \mathrm{rank}(p_n)}\}.$$

The remaining parts of $\mathrm{SP}\phi_y$ and $\mathrm{SP}\phi_{y'}$ are $\mathrm{SP}(\phi_0)_y$ and $\mathrm{SP}(\phi_0)_{y'}$, respectively, which have at most $\mathrm{rank}(p_0)$ elements.
It is easy to prove the following fact. For any $a, b \in X$, the sets $\{a, x_1, x_2, \cdots, x_n\}$ and $\{b, x_1, x_2, \cdots, x_n\}$ can be paired within δ. In fact, by path connectedness of X and $\frac{\delta}{2}$-density of the set $\{x_1, x_2, \cdots, x_n\}$, one can find a sequence

$$a, x_{j_1}, x_{j_2}, \cdots, x_{j_k}, b$$

beginning with a and ending with b such that each pair of consecutive terms has distance smaller than δ. So $\{a, x_{j_1}, \cdots, x_{j_{k-1}}, x_{j_k}\}$ can be paired with $\{x_{j_1}, x_{j_2}, \cdots, x_{j_k}, b\}$ $(= \{b, x_{j_1}, \cdots, x_{j_k}\})$ one by one to within δ. The other parts of the sets are identical, each element can be paired with itself.

Combining the above fact with the condition that $\mathrm{rank}(p_i) \geq \mathrm{rank}(p_0)$ for any i, we know that $\mathrm{SP}\phi_y$ and $\mathrm{SP}\phi_{y'}$ can be paired within δ. That is, $\mathrm{SPV}(\phi) < \delta$. The rest of the lemma is obvious. Namely, for any two points y, y', $\phi(f)(y)$ is approximately unitarily equivalent to $\phi(f)(y')$ to within ε, by the same unitary for all $f \in F$ (see [EG2, 1.4.13]).

$\square$

1.2.18. In the last part of the above lemma, one does not need ϕ_0 to be a homomorphism to guarantee $\phi(F)$ to be weakly approximately constant to within a small number. In fact, the following is true.

Suppose that all the notations are as in 1.2.17 except that the maps $\phi_0 : C(X) \to p_0 M_\bullet(C(Y))p_0$ and $\phi : C(X) \to M_\bullet(C(Y))$ are no longer homomorphisms. Suppose that for any $y \in Y$, there is a homomorphism $\psi_y : C(X) \to p_0(y)M_\bullet(\mathbb{C})p_0(y)$ such that

$$\|\phi_0(f)(y) - \psi_y(f)\| < \varepsilon, \ \forall f \in F.$$

Then the set $\phi(F)$ is weakly approximately constant to within 3ε. One can prove this claim as follows.

For any $y \in Y$, define a homomorphism $\phi_y \to M_\bullet(\mathbb{C})$ by $\phi_y(f) = \psi_y(f) \oplus \sum_{i=1}^{n} f(x_i)p_i$. Then for any two points $y, y' \in Y$, as same as in Lemma 1.2.17, $\mathrm{SP}(\phi_y)$ and $\mathrm{SP}(\phi_{y'})$ can be paired within δ. Therefore, $\phi_y(f)$ is approximately unitarily equivalent to $\phi_{y'}(f)$ to within ε, by the same unitary for all $f \in F$.

On the other hand,

$$\|\phi(f)(y) - \phi_y(f)\| < \varepsilon \quad \text{and} \quad \|\phi(f)(y') - \phi_{y'}(f)\| < \varepsilon, \qquad \forall f \in F.$$

Hence $\phi(f)(y)$ is approximately unitarily equivalent to $\phi(f)(y')$ to within 3ε, by the same unitary for all $f \in F$.

1.2.19. Suppose that $F \subset M_k(C(X))$ is a finite set and $\varepsilon > 0$. Let $F' \subset C(X)$ be the finite set consisting of all entries of elements in F and $\varepsilon' = \frac{\varepsilon}{k}$, where k is the order of the matrix algebra $M_k(C(X))$.

It is well known that, for any $k \times k$ matrix $a = (a_{ij}) \in M_k(B)$ with entries $a_{ij} \in B$, $\|a\| \leq k \max_{ij} \|a_{ij}\|$. This implies the following two facts.

FACT 1. If $\phi_1, \psi_1 \in \mathrm{Map}(C(X), B)$ are (complete positive) linear $*$-contraction (as the notation in 1.1.1) which satisfy

$$\|\phi_1(f) - \psi_1(f)\| < \varepsilon', \ \ \forall f \in F',$$

then $\phi := \phi_1 \otimes \mathrm{id}_k \in \mathrm{Map}(M_k(C(X)), M_k(B))$ and $\psi := \psi_1 \otimes \mathrm{id}_k \in \mathrm{Map}(M_k(C(X)), M_k(B))$ satisfy

$$\|\phi(f) - \psi(f)\| < \varepsilon, \ \ \forall f \in F.$$

FACT 2. Suppose that $\phi_1 \in \mathrm{Map}(C(X), M_\bullet(C(Y)))$ is a (complete positive) linear $*$-contraction. If $\phi_1(F')$ is weakly approximately constant to within ε', then $\phi_1 \otimes \mathrm{id}_k(F)$ is weakly approximately constant to within ε.

Suppose that a homomorphism $\phi_1 \in \mathrm{Hom}(C(X), B)$ has a decomposition described as follows. There exist mutually orthogonal projections $p_1, p_2 \in B$ with $p_1 + p_2 = \mathbf{1}_B$ and $\psi_1 \in \mathrm{Hom}(C(X), p_2Bp_2)$ such that

$$\|\phi_1(f) - p_1\phi_1(f)p_1 \oplus \psi_1(f)\| < \varepsilon', \quad \forall f \in F'.$$

Then there is a decomposition for $\phi := \phi_1 \otimes \mathrm{id}_k$:

$$\|\phi(f) - P_1\phi_1(f)P_1 \oplus \psi(f)\| < \varepsilon, \quad \forall f \in F,$$

where $\psi := \psi_1 \otimes \mathrm{id}_k$ and $P_1 = p_1 \otimes \mathbf{1}_k$.
In particular, if $B = M_\bullet(C(Y))$ and ψ_1 is described by

$$\psi_1(f)(y) = \sum f(\alpha_i(y))q_i(y), \quad \forall f \in C(X),$$

where $\sum q_i = p_2$ and $\alpha_i : Y \to X$ are continuous maps, then ψ can be described by

$$\psi(f)(y) = \sum q_i(y) \otimes f(\alpha_i(y)), \quad \forall f \in M_k(C(X)),$$

regarding $M_k(M_\bullet(C(Y)))$ as $M_\bullet(C(Y)) \otimes M_k$.
If α_i are constant maps, the homomorphism ψ_1 is called a homomorphism defined by point evaluations as in Lemma 1.2.17. In this case, we will also call the above ψ a homomorphism defined by point evaluations.
From the above, we know that to decompose a homomorphism $\phi \in \mathrm{Hom}(M_k(C(X)), M_\bullet(C(Y)))$, one only needs to decompose $\phi_1 := \phi|_{e_{11}M_k(C(X))e_{11}} \in Hom(C(X), \phi(e_{11})M_\bullet(C(Y))\phi(e_{11}))$.

1.3 Full matrix algebras, corners, and the dilation lemma

Some results in this article deal with a corner $QM_N(C(X))Q$ of the matrix algebra $M_N(C(X))$. But using the following lemma and some other techniques, we can reduce the problems to the case of a full matrix algebra $M_N(C(X))$. The following dilation lemma is Lemma 2.13 of [EG2].

Lemma 1.3.1. (cf. Lemma 2.13 of [EG2]) *Let X and Y be any connected finite CW complexes. If $\phi : QM_k(C(X))Q \to PM_n(C(Y))P$ is a unital homomorphism, then there are an n_1, a projection $P_1 \in M_{n_1}(C(Y))$, and a unital homomorphism $\tilde{\phi} : M_k(C(X)) \to P_1M_{n_1}(C(Y))P_1$ with the property that $QM_k(C(X))Q$ and $PM_n(C(Y))P$ can be identified as corner subalgebras of $M_k(C(X))$ and $P_1M_{n_1}(C(Y))P_1$ respectively (i.e., Q and P can be considered to be subprojections of $\mathbf{1}_k$ and P_1, respectively) and, furthermore, in such a way that ϕ is the restriction of $\tilde{\phi}$.*
If $\phi_t : QM_k(C(X))Q \to PM_n(C(Y))P$, $(0 \le t \le 1)$ is a path of unital homomorphisms, then there are $P_1M_{n_1}(C(Y))P_1$ (as above) and a path of unital homomorphisms $\tilde{\phi}_t : M_k(C(X)) \to P_1M_{n_1}(C(Y))P_1$ such that $QM_k(C(X))Q$ and $PM_n(C(Y))P$ are corner subalgebras of $M_k(C(X))$ and $P_1M_{n_1}(C(Y))P_1$ respectively and ϕ_t is the restriction of $\tilde{\phi}_t$.

DEFINITION 1.3.2. Let A be a C^*-algebra. A sub-C^*-algebra $A_1 \subset A$ will be called a LIMIT CORNER SUBALGEBRA of A, if there is a sequence of increasing projections

$$P_1 \leq P_2 \leq \cdots \leq P_n \leq \cdots,$$

such that $A_1 = \overline{\bigcup_{n=1}^{\infty} P_n A P_n}$.
Using Lemma 1.3.1, it is routine to prove the following lemma.

LEMMA 1.3.3. (cf. 4.24 of [EG2]) *For any AH algebra* $A = \lim\limits_{\rightarrow}(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$, *there is an inductive limit* $\tilde{A} = \lim\limits_{\rightarrow}(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{\{n,i\}}(C(X_{n,i})), \tilde{\phi}_{n,m})$ *of full matrix algebras over* $\{X_{n,i}\}$, *such that* A *is isomorphic to a limit corner subalgebra of* $\tilde{A}$. *In particular, each* $P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ *is a corner subalgebra of* $M_{\{n,i\}}(C(X_{n,i}))$ *and* $\phi_{n,m}$ *is the restriction of* $\tilde{\phi}_{n,m}$ *on* $A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$.
Furthermore, A *is stably isomorphic to* $\tilde{A}$.

REMARK 1.3.4. In the above lemma, in general, the homomorphisms $\tilde{\phi}_{n,m}$ cannot be chosen to be unital, even if all the homomorphisms $\phi_{n,m}$ are unital. If A is unital, then A can be chosen to be the cut-down of $\tilde{A}$ by a single projection (rather than a sequence of projections).

REMARK 1.3.5. In Lemma 1.3.3, if A is simple and satisfies the very slow dimension growth condition, then so is $\tilde{A}$. Hence when we consider the nonunital case, we may always assume that A is an inductive limit of direct sums of full matrix algebras over $C(X_{n,i})$ without loss of generality. Since the reduction theorem in this paper will be proved without the assumption of unitality, we may assume that the C^*-algebra A is an inductive limit of full matrix algebras over finite simplicial complexes. But even in this case, we still need to consider the cut-down $PM_l(C(X))P$ of $M_l(C(X))$ in some situations, since the image of a trivial projection may not be trivial.
In the proof of the decomposition theorem in §4, we will NOT assume that A is unital but we will assume that A is the inductive limit of full matrix algebras. Note that a projection in $M_\bullet(C(X))$ corresponds to a complex vector bundle over X. The following result is well known (see Chapter 8 of [Hu]). This result is often useful when we reduce the proof of a result involving the cut-down $PM_l(C(X))P$ to the special case of the full matrix algebra $M_l(C(X))$.

LEMMA 1.3.6. *Let* X *be a connected simplicial complex and* $P \in M_l(C(X))$ *be a non-zero projection. Let* $n = \mathrm{rank}(P) + \dim(X)$ *and* $m = 2\dim(X) + 1$.
Then P *is Murray-von Neumann equivalent to a subprojection of* $\mathbf{1}_n$, *and* $\mathbf{1}_n$ *is Murray-von Neumann equivalent to a subprojection of* $\underbrace{P \oplus P \oplus \cdots P}_{m}$, *where* $\mathbf{1}_n$ *is a trivial projection with rank* n. *Therefore,* $PM_l(C(X))P$ *can be identified*

as a corner subalgebra of $M_n(C(X))$, and $M_n(C(X))$ can be identified as a corner subalgebra of $M_m(PM_l(C(X))P)$.

1.4 Topological preliminaries

In this subsection, we will introduce some notations and results in the topology of simplicial complexes. We will also introduce a well known method for the construction of cross sections of a fibre bundle. The content of this subsection may be found in [St], [Hu] and [Wh].

1.4.1. Let X be a connected simplicial complex. Endow X with a metric d as follows.
For each n-simplex Δ, one can identify Δ with an n-simplex in $\mathbb{R}^n$ whose edges are of length 1 (of course the identification should preserve the affine structure of the simplices). (Such a simplex is the convex hull of $n+1$ points $\{x_0, x_1, \cdots, x_n\}$ in $\mathbb{R}^n$ with $\mathrm{dist}(x_i, x_j) = 1$ for any $i \neq j \in \{0, 1, \cdots n\}$.) Such an identification gives rise to a unique metric on Δ. The restriction of metric d of X to Δ is defined to be the above metric for any simplex $\Delta \subset X$. For any two points $x, y \in X$, $d(x, y)$ is defined to be the length of the shortest path connecting x and y. (The length is measured in individual simplexes, by breaking the path into small pieces.)
If X is not connected, denote by L the maximum of the diameters of all the connected components. Define $d(x, y) = L + 1$, if x and y are in different components. (Recall that all the simplicial complexes in this article are supposed to be finite.)

1.4.2. For a simplex Δ, by $\partial\Delta$, we denote the boundary of the simplex Δ, which is the union of all proper faces of Δ. Note that if Δ is a single point—zero dimensional simplex, then $\partial\Delta = \emptyset$. Obviously, $\dim(\partial\Delta) = \dim(\Delta) - 1$. (We use the standard convention that the dimension of the empty space is -1.) By interior(Δ), we denote $\Delta\backslash\partial\Delta$. Let X be a simplicial complex. Obviously, for each $x \in X$, there is a unique simplex Δ such that $x \in \mathrm{interior}(\Delta)$, which is the simplex Δ of lowest dimension with the condition that $x \in \Delta$. (Here we use the fact that if two different simplices of the same dimension intersect, then the intersection is a simplex of lower dimension.)
For any simplex Δ, define

$$\mathrm{Star}(\Delta) = \bigcup\{\mathrm{interior}(\Delta') \mid \Delta' \cap \Delta \neq \emptyset\}.$$

Then $\mathrm{Star}(\Delta)$ is an open set which covers Δ.
We will use the following two open covers of the simplicial complex X.
(a) For any vertex $x \in X$, let

$$W_x = \mathrm{Star}(\{x\}) (= \bigcup\{interior(\Delta) \mid x \in \Delta\}).$$

Obviously $\{W_x\}_{x\in\mathrm{Vertex}(X)}$ is an open cover parameterized by vertices of X.

In this open cover, the intersection $W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_k}$ is nonempty if and only if $x_1, x_2, \cdots, x_k$ span a simplex of X.
(b) We denote the original simplicial structure of X by σ. Introduce a barycentric subdivision (X, τ) of (X, σ).
Then for each simplex Δ of (X, σ) (before subdivision), there is exactly one point $C_\Delta \in \mathrm{Vertex}(X, \tau)$—the barycenter of Δ, such that $C_\Delta \in \mathrm{interior}(\Delta)$. (Here interior$(\Delta)$ is clearly defined by referring Δ as a simplex of (X, σ).)
Define

$$U_\Delta = \mathrm{Star}_{(X,\tau)}(\{C_\Delta\}).$$

As in (a), $\{U_\Delta \mid \Delta \text{ is a simplex of } (X, \sigma)\}$ is an open cover. In fact, $U_\Delta \supset \mathrm{interior}(\Delta)$. This open cover is parameterized by simplices of (X, σ) (also by vertices of (X, τ), since there is a one to one correspondence between the vertices of (X, τ) and the simplices of (X, σ)).
This cover satisfies the following condition: The intersection $U_{\Delta_1} \cap U_{\Delta_2} \cap \cdots \cap U_{\Delta_k}$ is nonempty if and only if one can reorder the simplices, such that

$$\Delta_1 \subset \Delta_2 \subset \cdots \Delta_n.$$

One can verify that the open cover in (b) is a refinement of the open cover in (a).

1.4.3. It is well known that a simplicial complex X is locally contractible. That is, for any point $x \in X$ and an open neighborhood $U \ni x$, there is an open neighborhood $W \ni x$ with $W \subset U$ such that W can be contracted to a single point inside U. (One can prove this fact directly, using the metric in 1.4.1.)
One can endow different metrics on X, but all the metrics are required to induce the same topology as the one in 1.4.1.
Using the local contractibility and the compactness of X, one can prove the following fact.
For any simplicial complex X with a metric d (may be different from the metric in 1.4.1), there are $\delta_{X,d} > 0$ and a nondecreasing function $\rho : (0, \delta_{X,d}] \to \mathbb{R}^+$ such that the following are true.
(1) $\lim_{\delta \to 0^+} \rho(\delta) = 0$, and
(2) for any $\delta \in (0, \delta_{X,d}]$ and $x_0 \in X$, the ball $B_\delta(x_0)$ with radius δ and centre x_0 (see 1.1.7 (d) for the notation) can be contracted into a single point within the ball $B_{\rho(\delta)}(x_0)$. I.e., there is a continuous map $\alpha : B_\delta(x_0) \times [0,1] \to B_{\rho(\delta)}(x_0)$ such that
(i) $\alpha(x, 0) = x$ for any $x \in B_\delta(x_0)$,
(ii) $\alpha(x, 1) = x_0$ for any $x \in B_\delta(x_0)$.
The following lemma is a consequence of the above fact.

Lemma 1.4.4. *For any simplicial complex X with metric d, there are $\delta_{X,d} > 0$ and a nondecreasing function $\rho : (0, \delta_{X,d}] \to \mathbb{R}^+$ such that the following are true.*

(1) $\lim_{\delta\to 0^+}\rho(\delta)=0$, *and*
(2) for any ball $B_\delta(x_0)$ *with radius* $\delta\le\delta_{X,d}$, *any simplex* Δ *(not assumed to be a simplex in* X*), and any continuous map* $f:\partial\Delta\to B_\delta(x_0)$, *there is a continuous map* $g:\Delta\to B_{\rho(\delta)}(x_0)$ *such that* $g(y)=f(y)$ *for any* $y\in\partial\Delta$.

Proof: The simplex Δ can be identified with $\partial\Delta\times[0,1]/\partial\Delta\times\{1\}$ in such a way that $\partial\Delta$ is identified with $\partial\Delta\times\{0\}$. Define the map g by

$$g(y,t)=\alpha(f(y),t)\in B_{\rho(\delta)}(x_0),\forall y\in\partial\Delta,t\in[0,1],$$

where α is the map in 1.4.3.

$\square$

1.4.5. The following is a well known result in differential topology: Suppose that M is an m-dimensional smooth manifold, $N\subset M$ is an n-dimensional submanifold. If Y is an l-dimensional simplicial complex with $l<m-n$, then for any continuous map $f:Y\to M$ and any $\varepsilon>0$, there is a continuous map $g:Y\to M$ such that
(i) $g(Y)\cap N=\emptyset$ and
(ii) $\mathrm{dist}(g(y),f(y))<\varepsilon$, for any $y\in Y$.
There is an analogous result in the case of a simplicial complex M and a sub-complex N. Instead of the assumption that M is an m-dimensional smooth manifold, let us suppose that M has the PROPERTY $D(m)$: for each $x\in M$, there is a contractible open neighborhood $U_x\ni x$ such that $U_x\setminus\{x\}$ is $(m-2)$-connected, i.e.,

$$\pi_i(U_x\setminus\{x\})=0\quad\text{for any}\quad i\in\{0,1,\cdots m-2\}.$$

(We use the following convention: by $\pi_0(X)=0$, it will be meant that X is a path connected nonempty space.)
Note that $\mathbb{R}^m\backslash\{0\}$ is $(m-2)$-connected. Therefore, any m-dimensional manifold has property $D(m)$.
The following result is the relative version of Theorem 5.4.16 of [St] (see page 111 of [St]), which also holds according to the top of page 112 of [St].

Proposition 1.4.6. Suppose that M is a simplicial complex with property $D(m)$, and $N\subset M$ is a sub-simplicial complex. Suppose that Y is a simplicial complex of dimension $l<m-\dim(N)$, and suppose that $Y_1\subset Y$ is a sub-simplicial complex. Suppose that $f:Y\to M$ is a continuous map such that $f(Y_1)\cap N=\emptyset$. For any $\varepsilon>0$, there is a continuous map $f_1:Y\to M$ such that
(i) $f_1|_{Y_1}=f|_{Y_1}$,
(ii) $f_1(Y)\cap N=\emptyset$, and
(iii) $d(f(y),f_1(y))<\varepsilon$ for any $y\in Y$.

1.4.7. Let X,F be two simplicial complexes.

Let $\Gamma \subset Homeo(F)$ be a subgroup of the group of homeomorphisms of the space F.
Let us recall the definition of fibre bundle. A FIBRE BUNDLE OVER X WITH FIBRE F AND STRUCTURE GROUP Γ, is a simplicial complex M with a continuous surjection $p : M \to X$ such that the following is true. There is an open cover $\mathcal{U}$ of X, and associated to each $U \in \mathcal{U}$, there is a homeomorphism

$$t_U : p^{-1}(U) \to U \times F$$

(called a local trivialization of the bundle) such that
(1) Each t_U takes the fibre of $p^{-1}(U)$ at $x \in U$ to the fibre of $U \times F$ at the same point x—a trivialization of the restriction $p^{-1}(U)$ of the fibre bundle to U, i.e., the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "t_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

is commutative, where p_1 denotes the project map from the product $U \times F$ to the first factor U, and
(2) The given local trivializations differ fibre-wise only by homeomorphisms in the structure group Γ: for any $U, V \in \mathcal{U}$ and $x \in U \cap V$,

$$t_U \circ t_V^{-1}|_{\{x\}\times F} \in \Gamma \subset Homeo(F).$$

Furthermore, we will also suppose that THE METRIC d OF F IS INVARIANT UNDER THE ACTION OF ANY ELEMENT $g \in \Gamma$, i.e., $d(g(x), g(y)) = d(x, y)$ for any $x, y \in F$. We will see, the fibre bundles constructed in §2, satisfy this condition.
(Note that, if F is the vector space $\mathbb{R}^n$ with Euclidean metric d, then d is invariant under the action of $O(n) \subset Homeo(F)$, but not invariant under the action of $Gl(n) \subset Homeo(F)$.)
A subset $F_1 \subset F$ is called Γ-INVARIANT if for any $g \in \Gamma$, $g(F_1) \subset F_1$.

1.4.8. A CROSS SECTION of fibre bundle $p : M \to X$ is a continuous map $f : X \to M$ such that

$$(p \circ f)(x) = x \quad \text{for any} \quad x \in X.$$

The following Theorem is a consequence of Proposition 1.4.6. The proof is a standard argument often used in the construction of cross sections for fibre bundles (see [Wh]). (In the literature such an argument is often taken for granted.) We give it here for the convenience of the reader.

Theorem 1.4.9. *Suppose that $p : M \to X$ is a fibre bundle with fibre F. Suppose that $F_1 \subset F$ is a Γ-invariant sub-simplicial complex of F. Suppose that F has the property $D(m)$ and $\dim(X) < m - \dim(F_1)$. Then for any cross section $s : X \to M$ and any $\varepsilon > 0$, there is a cross section $s_1 : X \to M$ such that the following two statements are true:*
(1) $(t_U \circ s_1)(x) \notin \{x\} \times F_1$ for any $x \in U \in \mathcal{U}$, where $\mathcal{U}$ is the open cover of X and $\{t_U\}_{U\in\mathcal{U}}$ is the local trivialization of the fibre bundle. That is, $s_1(X)$ avoids F_1 in each fibre;
(2) $d(s_1(x), s(x)) < \varepsilon$ for any $x \in X$, where the distance is taken in the fibre F.

Proof: If the fibre bundle is trivial, then the cross sections of the bundle can be identified with maps from X to F. The conclusion follows immediately from 1.4.6.
For the general case, we will use the local trivializations.
For each open set $U \in \mathcal{U}$, using the trivialization

$$t_U : p^{-1}(U) \to U \times F,$$

each cross section f on U induces a continuous map $\tilde{f}_U : U \to F$ by

$$\tilde{f}_U(x) = p_2(t_U(f(x))),$$

where $p_2 : U \times F \to F$ is the projection onto the second factor.
Suppose that δ_F is as in 1.4.3 (see 1.4.4 also). That is, there is a nondecreasing function $\rho : (0, \delta_F] \to (0, \infty)$ such that $\lim_{\delta\to 0^+} \rho(\delta) = 0$ and such that any δ-ball $B_\delta(x)$ can be contracted to a single point within $B_{\rho(\delta)}(x)$.
Let $\dim(X) = n$. Choose a finite sequence of positive numbers

$$\varepsilon_n > \varepsilon_{n-1} > \varepsilon_{n-2} > \cdots > \varepsilon_1 > \varepsilon_0 > 0$$

as follows. Set $\varepsilon_n = \min\{\delta_F, \varepsilon\}$. Then choose ε_{n-1} to satisfy

$$\rho(3\varepsilon_{n-1}) < \frac{1}{3}\varepsilon_n \quad \text{and} \quad 3\varepsilon_{n-1} < \frac{1}{3}\varepsilon_n.$$

Once ε_l is defined, then choose ε_{l-1} to satisfy

$$\rho(3\varepsilon_{l-1}) < \frac{1}{3}\varepsilon_l \;\; \text{and} \;\; 3\varepsilon_{l-1} < \frac{1}{3}\varepsilon_l.$$

Repeat this procedure until we choose ε_0 to satisfy

$$\rho(3\varepsilon_0) < \frac{1}{3}\varepsilon_1 \;\; \text{and} \;\; 3\varepsilon_0 < \frac{1}{3}\varepsilon_1.$$

Let us refine the given simplicial complex structure on X in such a way that each simplex Δ is covered by an open set $U \in \mathcal{U}$ and that for any simplex Δ

and an open set $U \in \mathcal{U}$ which covers Δ, the map $\tilde{s}_U|_\Delta : \Delta \to F$, induced by the cross section s, satisfies

$$\text{diameter}(\tilde{s}_U(\Delta)) < \varepsilon_0.$$

(Since the metric on F is invariant under the action of any element in Γ, the above inequality holds or not does not depend on the choice of the open set U which covers Δ. In what follows, we will use this fact many times without saying so.)

We will apply Proposition 1.4.6 to each simplex of X from the lowest dimension to the highest dimension.

For any $l \in \{0, 1, \cdots, n\}$, let us denote the l-skeleton of X by $X^{(l)}$. So $X^{(n)} = X$, and $X^{(0)}$ is the set of vertices of X.

Step 1. Fix a vertex $x \in X^{(0)}$, and suppose that $x \in U \in \mathcal{U}$. Applying Proposition 1.4.6 to $\{x\}$ (in place of Y with $Y_1 = \emptyset$) and F (in place of M with $N = F_1$), there exists $\tilde{s}^0(x) \in F \setminus F_1$ such that

$$d(\tilde{s}^0(x), \tilde{s}_U(x)) < \varepsilon_0.$$

Any choice of $\tilde{s}^0(x)$ gives a cross section s^0 on $\{x\}$ by

$$s^0(x) = t_U^{-1}(x, \tilde{s}^0(x)),$$

where $(x, \tilde{s}^0(x)) \in \{x\} \times F \subset U \times F$. Defining s^0 on all vertices, we obtain a cross section s^0 on $X^{(0)}$ such that

$$d(\tilde{s}^0_U(x), \tilde{s}_U(x)) < \varepsilon_0$$

for each $x \in U \cap X^{(0)}$.

Step 2. Suppose that for $l < n = \dim(X)$, there is a cross section $s^l : X^{(l)} \to M$ such that for any $U \in \mathcal{U}$ and any $x \in X^{(l)} \cap U$, we have $\tilde{s}^l_U(x) \notin F_1$, and

$$(*) \qquad d(\tilde{s}^l_U(x), \tilde{s}_U(x)) < \varepsilon_l.$$

Let us define a cross section $s^{l+1} : X^{(l+1)} \to M$ as follows. We will work one by one on each $(l+1)$-simplex Δ.

First, we shall simply extend the cross section $s^l|_{\partial\Delta}$ to a cross section on Δ (see Substep 2.1 below). Then, apply Proposition 1.4.6 to perturb the cross section $s^l|_\Delta$ to avoid F_1 in each fibre (see Substep 2.2 below). Again, since Proposition 1.4.6 is only for maps (not for cross sections), we will use $\tilde{s}^l_U|\partial\Delta : \partial\Delta \to F$ to replace $s^l|\partial\Delta$, as in Step 1.

Substep 2.1. Let Δ be an $(l+1)$-simplex. Suppose that $\Delta \subset U \in \mathcal{U}$. Then $\tilde{s}^l_U|_{\partial\Delta} : \partial\Delta \to F$ is a continuous map. Since $(*)$ holds for any $x \in \partial\Delta$, and since

$$\text{diameter}(\tilde{s}_U(\Delta)) < \varepsilon_0,$$

we have

$$\text{diameter}(\tilde{s}^l_U(\partial\Delta)) < \varepsilon_l + \varepsilon_l + \varepsilon_0.$$

Let $\delta = \varepsilon_l + \varepsilon_l + \varepsilon_0 < \delta_F$. Then there is a $y \in F$ such that $\tilde{s}^l_U(\partial\Delta) \subset B_\delta(y)$. Since $\rho(\delta) \le \rho(3\varepsilon_l) < \frac{1}{3}\varepsilon_{l+1}$, by Lemma 1.4.4, $\tilde{s}^l_U : \partial\Delta \to F$ can be extended to a map (still denoted by $\tilde{s}^l_U$)

$$\tilde{s}^l_U : \Delta \to F,$$

such that $\tilde{s}^l_U(\Delta) \subset B_{\frac{1}{3}\varepsilon_{l+1}}(y)$. Consequently, the extended map $\tilde{s}^l_U$ also satisfies that diameter$(\tilde{s}^l_U(\Delta)) < \frac{2}{3}\varepsilon_{l+1}$.
Substep 2.2. Note that $\tilde{s}^l_U(x) \notin F_1$ for any $x \in \partial\Delta$. Applying Proposition 1.4.6 to Δ (in place of Y with subcomplex $Y_1 = \partial\Delta$) and to F (in the place of M with subcomplex $N = F_1$), we obtain a continuous map

$$\tilde{s}^{l+1} : \Delta \to F$$

such that
(1) $\tilde{s}^{l+1}(x) \notin F_1$ for any $x \in \Delta$,
(2) $d(\tilde{s}^{l+1}(x), \tilde{s}^l_U(x)) < \varepsilon_0$, for any $x \in \Delta$, and
(3) $\tilde{s}^{l+1}|\partial\Delta = \tilde{s}^l_U|\partial\Delta$.
The map $\tilde{s}^{l+1}$ defines a cross section s^{l+1} by

$$s^{l+1}(x) = t_U^{-1}(x, \tilde{s}^{l+1}(x)).$$

After working out all the $(l+1)$-simplices, we obtain a cross section s^{l+1} on $X^{(l+1)}$—it is a continuous cross section because it is continuous on each $(l+1)$-simplex and $s^{l+1}|\partial\Delta = s^l|\partial\Delta$ from (3) above.
Recall that diameter$(\tilde{s}_U(\Delta)) < \varepsilon_0$ and diameter$(\tilde{s}^l_U(\Delta)) < \frac{2}{3}\varepsilon_{l+1}$. Combining these facts with $(*)$, we have

$$(**) \qquad d(\tilde{s}^l_U(x), \tilde{s}_U(x)) < \varepsilon_l + \frac{2}{3}\varepsilon_{l+1} + \varepsilon_0$$

for any $x \in \Delta$. Combining $(**)$ and (2) above, we have

$$d(\tilde{s}^{l+1}_U(x), \tilde{s}_U(x)) < \varepsilon_l + \frac{2}{3}\varepsilon_{l+1} + 2\varepsilon_0 < \varepsilon_{l+1}$$

for any $x \in X^{l+1} \cap U$. This is $(*)$ for $l+1$ (in place of l).
Step 3. By mathematical induction, we can define s^l for each $l = 0, 1, \cdots n$ as the above. Let $s_1 = s^n$ to finish the proof.

$\square$

The following relative version of the theorem is also true.

Corollary 1.4.10. *Suppose that $p : M \to X$ is a fibre bundle with fibre F and F_1 is a Γ-invariant sub-simplicial complex of F. Suppose that F has the property $D(m)$ and that $\dim(X) < m - \dim(F_1)$. Suppose that $X_1 \subset X$ is a sub-simplicial complex. Suppose that the cross section $s : X \to M$ satisfies that $(t_U \circ s)(x) \notin \{x\} \times F_1$ for any $U \in \mathcal{U}$ and any $x \in X_1 \cap U$, where $\mathcal{U}$ and t_U*

are as in the definition of fibre bundle in 1.4.7. Then for any $\varepsilon > 0$, there is a cross section $s_1 : X \to M$ such that the following three statements are true:
(1) $(t_U \circ s_1)(x) \notin \{x\} \times F_1$, for any $x \in U$ and $U \in \mathcal{U}$;
(2) $d(s_1(x), s(x)) < \varepsilon$ for any $x \in X$, where the distance is taken inside the fibre F;
(3) $s_1|_{X_1} = s|_{X_1}$.

Proof: In the proof of Theorem 1.4.9, we have essentially proved this relative version. If fact, in Step 2, we proved that a cross section on a simplex Δ can be constructed within arbitrarily small distance of the original cross section such that
(1) it avoids F_1 in each fibre, and
(2) it agrees with the original cross section on $\partial\Delta$, provided that the original cross section avoids F_1 on $\partial\Delta$.
This is a local version of the Corollary. To prove the Corollary, one only needs to apply this local version, repeatedly, to the simplices Δ with $\Delta\backslash\partial\Delta \subset X\backslash X_1$, from the lowest dimension to the highest dimension.

$\square$

1.5 About the decomposition theorem

In this subsection, we will briefly discuss the main ideas in the proof of the decomposition theorem stated in §4. Mainly, we will review the ideas in the proofs of special cases already in the literature (see especially [EG2, Theorem 2.21]), point out the additional difficulties in our new setting, and discuss how to overcome these difficulties. This subsection could be skipped without any logical gap, but we do not encourage the reader to do so, except for the expert in the classification theory. By reading this subsection, the reader will get the overall picture of the proof. In particular, how §2, §3, and the results of [Li2] fit into the picture. We will also discuss some ideas in the proof of the combinatorial results of §3. This subsection may also be helpful for understanding the corresponding parts of [EG2], [Li3], and (perhaps) other papers. Even though the discussion in this subsection is sketched, the proof of Lemma 1.5.4 and Propositions 1.5.7 and 1.5.7' are complete. We will begin our discussion with some very elementary facts.

1.5.1. Let A and B be unital C^*-algebras, and $\phi : A \to B$, a unital homomorphism. If $P \in B$ is a projection which commutes with the image of ϕ, i.e., such that

$$P\phi(a) - \phi(a)P = 0, \ \forall a \in A,$$

then $\phi(a)$ can be decomposed into two mutually orthogonal parts $\phi(a)P = P\phi(a)P$ and $\phi(a)(\mathbf{1} - P) = (\mathbf{1} - P)\phi(a)(\mathbf{1} - P)$:

$$\phi(a) = P\phi(a)P + (\mathbf{1} - P)\phi(a)(\mathbf{1} - P).$$

1.5.2. In 1.5.1, let us consider the case that $A = C(X)$. Let $F \subset C(X)$ be a finite set. Let unital homomorphism $\phi : C(X) \to B$ and projection $P \in B$ be as in 1.5.1. Furthermore, suppose that there is a point $x_0 \in X$ such that $P\phi(f)P = \phi(f)P$ is approximately equal to $f(x_0)P$ to within ε on F:

$$\|\phi(f)P - f(x_0)P\| < \varepsilon, \quad \forall f \in F.$$

Then

$$\|\phi(f) - (\mathbf{1} - P)\phi(f)(\mathbf{1} - P) \oplus f(x_0)P\| < \varepsilon, \quad \forall f \in F.$$

More generally, if there are mutually orthogonal projections $P_1, P_2, \cdots, P_n \in B$, which commute with $\phi(C(X))$, and points $x_1, x_2, \cdots, x_n \in X$ such that

$$(*) \qquad \|\phi(f)P_i - f(x_i)P_i\| < \varepsilon, \quad \forall f \in F,\ i = 1, 2, \cdots n,$$

then

$$(**) \qquad \|\phi(f) - (\mathbf{1} - \sum_{i=1}^{n} P_i)\phi(f)(\mathbf{1} - \sum_{i=1}^{n} P_i) \oplus \sum_{i=1}^{n} f(x_i)P_i\| < \varepsilon,\ \forall f \in F.$$

Here, we used the following fact: the norm of the summation of a set of mutually orthogonal elements in a C^*-algebra is the maximum of the norms of all individual elements in the set. In this paper, this fact will be used many times without saying so.

Example 1.5.3. Let $F \subset C(X)$ be a finite set, and $\varepsilon > 0$. Choose $\eta > 0$ such that if $\mathrm{dist}(x, x') < \eta$, then $|f(x) - f(x')| < \varepsilon$ for any $f \in F$.
Let $x_1, x_2, \cdots, x_n \in X$ be distinct points, and $U_1 \ni x_1, U_2 \ni x_2, \cdots, U_n \ni x_n$ be mutually disjoint open neighborhoods with $U_i \subset B_\eta(x_i)$ $(= \{x \in X \mid \mathrm{dist}(x, x_i) < \eta\})$.
Consider the case that $B = M_\bullet(\mathbb{C})$ and let $\phi : C(X) \to M_\bullet(\mathbb{C})$ be a homomorphism. If $P_i,\ i = 1, 2, \cdots, n$ are the spectral projections corresponding to the open sets U_i (see Definition 1.2.4), then the projections P_i commute with $\phi(C(X))$ and satisfy (*) in 1.5.2. Therefore, the decomposition

$$(**) \qquad \|\phi(f) - (\mathbf{1} - \sum_{i=1}^{n} P_i)\phi(f)(\mathbf{1} - \sum_{i=1}^{n} P_i) \oplus \sum_{i=1}^{n} f(x_i)P_i\| < \varepsilon,$$

holds for all $f \in F$.
We remark that if $\#(\mathrm{SP}\phi \cap U_i)$ (counting multiplicities) is large, then, in the decomposition, $\mathrm{rank}(P_i)(= \#(\mathrm{SP}\phi \cap U_i))$ is large.
In the setting of 1.5.2, not only is (**) true for the original projections $P_1, P_2, \cdots, P_n$, but also it is true for any subprojections $p_1 \leq P_1, p_2 \leq P_2, \cdots, p_n \leq P_n$, with ε replaced by 3ε. Namely, the following lemma holds.

LEMMA 1.5.4. *Let X be a compact metrizable space, and write $A = C(X)$. Let $F \subset A$ be a finite set. Let B be a unital C^* -algebra, and $\phi : A \to B$ be a homomorphism. Let $\varepsilon > 0$. Suppose that there are mutually orthogonal projections $P_1, P_2, \cdots, P_n$ in B and points $x_1, x_2, \cdots, x_n$ in X such that $P_i\phi(f) = \phi(f)P_i \quad (i = 1, 2, \cdots, n)$ for any $f \in C(X)$ and such that*

$$(*) \qquad \|\phi(f)P_i - f(x_i)P_i\| < \varepsilon \quad (i = 1, 2, \cdots, n) \quad \text{for any } f \in F.$$

If $p_1, p_2, \cdots, p_n$ are subprojections of $P_1, P_2, \cdots, P_n$ respectively, then

$$\|\phi(f) - (\mathbf{1} - \sum_{i=1}^{n} p_i)\phi(f)(\mathbf{1} - \sum_{i=1}^{n} p_i) \oplus \sum_{i=1}^{n} f(x_i)p_i\| < 3\varepsilon,$$

for any $f \in F$.
(Notice that the condition that the projections P_i commute with $\phi(f)$ does not by itself imply that the p_i almost commute with $\phi(f)$, but this does follows if (*) holds.)
Different versions of this lemma have appeared in a number of papers (especially, [Cu], [GL], [EGLP]).

Proof: The proof is a straightforward calculation.
One verifies directly that

$$\|(\sum P_i)\phi(f)(\sum P_i) - \sum f(x_i)P_i\| < \varepsilon, \quad \forall f \in F.$$

Hence on multiplying by $\mathbf{1} - \sum p_i$ and $\sum p_i$ (one on each side),

$$\|(\mathbf{1} - \sum p_i)\phi(f)(\sum p_i)\| < \varepsilon, \text{ and } \|(\sum p_i)\phi(f)(\mathbf{1} - \sum p_i)\| < \varepsilon, \ \forall f \in F;$$

on multiplying by $\sum p_i$ on both sides,

$$\|(\sum p_i)\phi(f)(\sum p_i) - \sum f(x_i)p_i\| < \varepsilon, \quad \forall f \in F.$$

The desired conclusion follows from identity

$$\phi(f) = ((\mathbf{1} - \sum p_i) + \sum p_i)\phi(f)((\mathbf{1} - \sum p_i) + \sum p_i).$$

□

REMARK 1.5.5. One may wonder why we need the decomposition given in the preceding lemma. In fact, the decomposition (**) of 1.5.2, with the original projections, has a better estimation. Why do we need to use subprojections? The reason is as follows.
Suppose that the C^*-algebra $A = C(X)$, the finite set $F \subset A$, the points $x_1, x_2, \cdots, x_n \in X$, and the open sets $U_1 \ni x_1, U_2 \ni x_2, \cdots, U_n \ni x_n$ are as in 1.5.3. Let us consider the case $B = M_\bullet(C(Y))$ (instead of $M_\bullet(\mathbb{C})$ in 1.5.3), where Y is a simplicial complex.

Let $\phi: C(X) \to M_\bullet(C(Y))$ be a unital homomorphism. As in 1.5.3, let $P_i(y)$ denote the spectral projection of $\phi|_y$ corresponding to the open set U_i (see 1.2.8). Then for each $y \in Y$, we have the inequality (**) above,

$$\|\phi(f)(y) - (\mathbf{1} - \sum_{i=1}^{n} P_i(y))\phi(f)(y)(\mathbf{1} - \sum_{i=1}^{n} P_i(y)) \oplus \sum_{i=1}^{n} f(x_i)P_i(y)\| < \varepsilon, \ \forall f \in F.$$

Unfortunately, $P_i(y)$ does not in general depend continuously on y, and so this estimation does not give rise to a decomposition for ϕ globally.
On the other hand, one can construct a globally defined continuous projection $p_i(y)$ which is a subprojection of $P_i(y)$ at each point y, and is such that $\mathrm{rank}(p_i)$ is not much smaller than $\min_{y\in Y} \mathrm{rank}(P_i(y))$ (more precisely, $\mathrm{rank}(p_i) \geq \min_{y\in Y} \mathrm{rank}(P_i(y)) - \dim(Y)$), by using the continuous selection theorem of [DNNP] as 1.5.6 below.
Once this is done , then for each $y \in Y$, applying the lemma, we have

$$\|\phi(f)(y) - (\mathbf{1} - \sum_{i=1}^{n} p_i(y))\phi(f)(y)(\mathbf{1} - \sum_{i=1}^{n} p_i(y)) \oplus \sum_{i=1}^{n} f(x_i)p_i(y)\| < 3\varepsilon, \ \forall f \in F.$$

Since the projections $p_i(y)$ depend continuously on y, they define elements $p_i \in B$. We can then rewrite the preceding estimate as

$$\|\phi(f) - (\mathbf{1} - \sum_{i=1}^{n} p_i)\phi(f)(\mathbf{1} - \sum_{i=1}^{n} p_i) \oplus \sum_{i=1}^{n} f(x_i)p_i\| < 3\varepsilon, \ \forall f \in F.$$

1.5.6. We would like to discuss how to construct the projections p_i referred to in 1.5.5, using the selection theorem [DNNP 3.2].
To guarantee p_i to have a large rank, we should assume that $P_i(y)$ has a large rank at every point y. So let us assume that for some positive integer k_i and for every point $y \in Y$,

$$\#(\mathrm{SP}\phi_y \cap U_i) \geq k_i,$$

equivalently, $\mathrm{rank}(P_i(y)) \geq k_i$.
For the sake of simplicity, let us fix i and write U for U_i ($U \subset X$), P for P_i, k for k_i, and p for the desired projection p_i. So for every point $y \in Y$,

$$\#(\mathrm{SP}\phi_y \cap U) \geq k,$$

equivalently, $\mathrm{rank}P(y) \geq k$. Let us construct a projection $p(y)$, depending continuously on y, such that $\mathrm{rank}p(y) \geq k - \dim(Y)$ and $p(y) \leq P(y)$ for each $y \in Y$.
For each fixed $y_0 \in Y$, since $\mathrm{SP}\phi_{y_0} \cap U$ is a finite set, one can choose an open set $U' \subset \overline{U'} \subset U$ such that $\mathrm{SP}\phi_{y_0} \cap U' = \mathrm{SP}\phi_{y_0} \cap U$. In particular, $\mathrm{SP}\phi_{y_0} \cap (\overline{U'} \backslash U') = \emptyset$. By Lemma 1.2.10, there is a connected open set $W \ni y_0$ in Y such that

$$\mathrm{SP}\phi_y \cap (\overline{U'} \backslash U') = \emptyset, \ \forall y \in W.$$

Let $P^W(y)$ be the spectral projection of ϕ_y corresponding to open set U'. By Lemma 1.2.9, this depends continuously on y, and so defines a continuous projection-valued function

$$P^W : W \to \text{ projections of } M_\bullet(\mathbb{C}).$$

Furthermore, $P^W(y) \leq P(y)$ for any $y \in Y$ and, for each y in the (connected) subset W,

$$\operatorname{rank}(P^W(y)) = \#(\mathrm{SP}\phi_{y_0} \cap U') = \#(\mathrm{SP}\phi_{y_0} \cap U) \geq k.$$

Once we have the above locally defined continuous projection-valued functions $P^W(y)$, the existence of a globally defined continuous projection-valued function $p(y)$ follows from the following result.

Proposition ([DNNP 3.2]). *Let Y be a simplicial complex, and let k be a positive integer. Suppose that $\mathcal{W}$ is an open covering of Y such that for each $W \in \mathcal{W}$, there is a continuous projection-valued map $P^W : W \to M_\bullet(\mathbb{C})$ satisfying*

$$rank P^W(y) \geq k \qquad \text{for all } \; y \in W.$$

Then there is a continuous projection-valued map $p : Y \to M_\bullet(\mathbb{C})$ such that for each $y \in Y$,

$$rank \;\; p(y) \geq k - dim(Y), and$$

$$p(y) \leq \bigvee\{P^W(y); \quad W \in \mathcal{W}, y \in W\}.$$

Let $p(y)$ be as given in the preceding proposition with respect to P^W as defined above. Then as $P^W(y) \leq P(y)$ for each W, $p(y) \leq P(y)$ also holds .
Recall, we write U for U_i, P for P_i and p for p_i. So, we obtain a projection p_i such that $p_i(y)$ is a subprojection of $P_i(y)$ for every y. Since $P_i(y)$, $i = 1, 2, \cdots, n$, are the spectral projections corresponding to U_i, $i = 1, 2, \cdots n$, which are mutually disjoint, the projections $P_i(y)$, $i = 1, 2, \cdots, n$, are mutually orthogonal, and so are the projections p_i, $i = 1, 2, \cdots, n$. Combining this construction with Lemma 1.5.4, we have the following result.

Proposition 1.5.7. *Let X be a simplicial complex, and $F \subset C(X)$ a finite subset. Suppose that $\varepsilon > 0$ and $\eta > 0$ are as in 1.5.3, i.e., such that if $dist(x, x') < \eta$, then $|f(x) - f(x')| < \varepsilon$ for any $f \in F$.*
Suppose that $U_1, U_2, \cdots, U_n$ are disjoint open neighborhoods of (distinct) points $x_1, x_2, \cdots, x_n \in X$, respectively, such that $U_i \subset B_\eta(x_i)$ for all $1 \leq i \leq n$. Suppose that $\phi : C(X) \to M_\bullet(C(Y))$ is a unital homomorphism, where Y is a simplicial complex, such that

$$\#(SP\phi_y \cap U_i) \geq k_i \qquad \text{for } 1 \leq i \leq n, \text{ and for all } y \in Y.$$

Then there are mutually orthogonal projections $p_1, p_2, \cdots, p_n \in M_\bullet(C(Y))$ *with* $rank(p_i) \geq k_i - dim(Y)$ *such that*

$$\|\phi(f) - p_0\phi(f)p_0 \oplus \sum_{i=1}^{n} f(x_i)p_i\| < 3\varepsilon \qquad \textit{for all } f \in F,$$

where $p_0 = \mathbf{1} - \sum p_i$. *Consequently,*

$$rank(p_0) \leq (\#(SP\phi_y) - \sum_{i=1}^{n} k_i) + n \cdot dim(Y).$$

(Note that $\#(SP\phi_y)$ *is the order of the matrix algebra* $M_\bullet(C(Y))$*.)*
(In fact, the above is also true if one replaces $M_\bullet(C(Y))$ by $PM_\bullet(C(Y))P$, with the exact same proof.)

1.5.8. Proposition 1.5.7 is implicitly contained in the proof of the main decomposition theorem—Theorem 2.21 of [EG2], and explicitly stated as Theorem 2.3 of [Li3], for the case of dimension one.
To use 1.5.7 to decompose a partial map $\phi^{i,j}_{m,m'} : M_{[m,i]}(C(X_{m,i})) \to M_{[m',j]}(C(X_{m',j}))$ of the connecting homomorphism $\phi_{m,m'} : A_m \to A_{m'}$ in the inductive system $(A_m, \phi_{m,m'})$, we only need to write

$$\phi^{i,j}_{m,m'} = \phi \otimes \mathrm{id}_{[m,i]},$$

(see 1.2.9), and then decompose ϕ (cf. 1.2.19). In [EG2], we proved that such a map ϕ (for m' large enough) satisfies the condition in Proposition 1.5.7, if the inductive limit is of real rank zero. More precisely, we constructed mutually disjoint open sets $U_1, U_2, \cdots, U_n$, with small diameter, such that $\sum_{i=1}^{n} k_i$ is very large compared with $(\#(\mathrm{SP}\phi_y) - \sum_{i=1}^{n} k_i)$, where $k_i = \min_{y\in Y} \#(\mathrm{SP}\phi_y \cap U_i)$. (See the open sets W_i in the proof of Theorem 2.21 of [EG2].) Therefore, in the above decomposition, the part $\sum_{i=1}^{n} f(x_i)p_i$, which has rank at least $(\sum_{i=1}^{n} k_i) - n \cdot \dim(Y)$, has much larger size than the size of the part $p_0\phi(f)p_0$, which has rank at most $(\#(\mathrm{SP}\phi_y) - \sum_{i=1}^{n} k_i) + n \cdot \dim(Y)$, if $n \cdot \dim(Y)$ is very small compared with $\#(\mathrm{SP}\phi_y)$. (Notice that if ϕ is not unital, then $p_0 = \phi(\mathbf{1}) - \sum_{i=1}^{n} p_i$ and $\#(\mathrm{SP}\phi_y) = \mathrm{rank}\phi(\mathbf{1})$.) (We should mention that $n \cdot \dim(Y)$ is automatically small from the construction. This is a kind of technical detail, to which the reader should not pay much attention now. The number n depends only on η above, but $\#(\mathrm{SP}\phi_y)$ could be very large as m' (for $\phi_{m,m'}$) is large. In particular, it could be much larger than $\dim(Y)$ (note that $Y = X_{m',j}$), if the inductive limit has slow dimension growth.)
The above construction is not trivial. It depends heavily on the real rank zero property and Su's result concerning spectral variation (see [Su]).
What was proved by this construction in [EG2] is the decomposition theorem for the real rank zero case, as mentioned in the introduction.

1.5.9. For the case of a non real rank zero inductive system, we can not construct the mutually disjoint open set $\{U_i\}$ as described in 1.5.8. Notice that in the decomposition described in 1.5.8, the major part ψ defined by $\psi(f) := \sum_{i=1}^n f(x_i)p_i$ has property that

$$\mathrm{SP}\psi_y = \{x_1^{\sim \mathrm{rank}(p_1)}, x_2^{\sim \mathrm{rank}(p_2)}, \cdots, x_n^{\sim \mathrm{rank}(p_n)}\}.$$

That is, the spectrum consists of several fixed points $\{x_i\}_{i=1}^n (\subset X)$ with multiplicities. This kind of decomposition depends on the real rank zero property. For the decomposition of the simple inductive limit algebras, we are forced to allow the major part to have variable spectrum— $\mathrm{SP}\psi_y$ varies when y varies. The following results can be proved exactly the same as the way Proposition 1.5.7 is proved (see 1.5.6 and 1.5.4). (Proposition 1.5.7 is a special case of the following result by taking $\alpha_i(y) = x_i$, the constant maps, and $U_i(y) = U_i \ni x_i$, the fixed open sets.)

Proposition 1.5.7'. *Let X be a simplicial complex, and $F \subset C(X)$, a finite subset. Suppose that $\varepsilon > 0$ and $\eta > 0$ are as in 1.5.3, i.e., such that if dist$(x, x') < \eta$, then $|f(x) - f(x')| < \varepsilon$ for any $f \in F$.*
Suppose that $\alpha_1, \alpha_2, \cdots \alpha_n : Y \to X$ are continuous maps from a simplicial complex Y to X. Suppose that $U_1(y), U_2(y), \cdots, U_n(y)$ are mutually disjoint open sets satisfying $U_i \subset B_\eta(\alpha_i(y))$ and satisfying the following continuity condition:
For any $y_0 \in Y$ and closed set $F \subset U_i(y_0)$, there is an open set $W \ni y_0$ such that $F \subset U_i(y)$ for any $y \in W$.
Suppose that $\phi : C(X) \to M_\bullet(C(Y))$ is a unital homomorphism such that

$$\#(SP\phi_y \bigcap U_i(y)) \geq k_i, \qquad \forall i \in \{1, 2, \cdots, n\},\ y \in Y.$$

Then there are mutually orthogonal projections $p_1, p_2, \cdots, p_n \in M_\bullet(C(Y))$ with rank$(p_i) \geq k_i -$ dim(Y) such that

$$\|\phi(f)(y) - p_0(y)\phi(f)(y)p_0(y) \oplus \sum_{i=1}^n f(\alpha_i(y))p_i(y)\| < 3\varepsilon, \qquad \forall f \in F,$$

where $p_0 = \mathbf{1} - \sum p_i$. Consequently,

$$rank(p_0) \leq (\#(SP\phi_y) - \sum_{i=1}^n k_i) + n\ dim(Y).$$

(It is easy to see that the proof of Proposition 1.5.7 (see 1.5.6) can be generalized to this case. Notice that the above continuity condition for $U_i(y)$ assures that $\overline{U'} \subset U_i(y)$ for any $y \in W$, where U' is described in 1.5.6 corresponding to y_0 and $U = U_i(y_0)$. Then it will assure that $P^W(y) \leq P(y)$, where $P^W(y)$ is the locally defined continuous projection-valued function described in 1.5.6.)

1.5.10. In the above proposition, if one can choose the maps $\alpha_i : Y \to X$ factoring through the interval $[0,1]$—the best possible 1-dimensional space—as

$$\alpha_i : Y \xrightarrow{a_i} [0,1] \xrightarrow{b} X,$$

then the part ψ, defined by $\psi(f)(y) = \sum_{i=1}^{n} f(\alpha_i(y))p_i(y)$, factors through $C[0,1]$ as

$$C(X) \xrightarrow{b^*} C([0,1]) \xrightarrow{\psi'} (\bigoplus_{i=1}^{n} p_i)M_\bullet(C(Y))(\bigoplus_{i=1}^{n} p_i),$$

where b^* is induced by $b : [0,1] \to X$ and ψ' is defined by

$$\psi'(f)(y) = \sum_{i=1}^{n} f(a_i(y))p_i(y).$$

In particular, if

$$\sum_{i=1}^{n} k_i >> (\#(\mathrm{SP}\phi_y) - \sum_{i=1}^{n} k_i), \tag{1}$$

where $k_i = \min_{y\in Y} \#(\mathrm{SP}\phi_y \cap U_i)$, then we obtain a decomposition with major part factoring through the interval algebras—direct sum of matrix algebras over interval.

1.5.11. The ideal approach for obtaining a decomposition of $\phi_{m,m'}$ with major part factoring through an interval algebra, is to reduce it to the setting of Proposition 1.5.7', that is, to construct continuous maps $\{\alpha_i\}$ (factoring through interval) and the mutually disjoint open sets $\{U_i\}$ as described in 1.5.7', such that (1) in 1.5.10 holds for homomorphism ϕ induced from the partial connecting homomorphism $\phi^{i,j}_{m,m'}$ described in 1.5.8.
Unfortunately, it seems impossible to realize such a construction globally.
A consequence of the property of α_i described in Proposition 1.5.7', is the following property of α_i, called property (Pairing):
Property (Pairing): For each $y \in Y$, there is a subset of $\mathrm{SP}\phi_y$, which can be paired with

$$\{\alpha_1(y)^{\sim k_1}, \alpha_2(y)^{\sim k_2}, \cdots, \alpha_n(y)^{\sim k_n}\}$$

to within η, counting multiplicities. (See 1.1.7 (b) for the notation $x^{\sim k}$.)
Even though one can not construct the continuous maps $\{\alpha_i\}$ (factoring through interval $[0,1]$ and open sets $\{U_i\}$ to satisfy the conditions in Proposition 1.5.7' together with the condition (1) in 1.5.10, for the connecting homomorphisms in the simple inductive limit, Li constructed the maps $\{\alpha_i\}$ to satisfy the above weaker property (Pairing) and (1) in 1.5.10.
In fact, Li proves the following lemma.

Lemma. *Suppose that* $\lim(A_m, \phi_{m,m'})$ *is a simple AH-inductive limit with slow dimension growth and with injective connecting homomorphisms. For any* $\eta > 0$*, and* A_m*, there are a* $\delta > 0$ *and an integer* $N > m$*, such that for any* $m' > N$*,* $SP(\phi^{i,j}_{m,m'})_y$ *can be paired with*

$$\Theta(y) = \{\alpha_1(y)^{\sim T_1}, \alpha_2(y)^{\sim T_2}, \cdots, \alpha_L(y)^{\sim T_L}\}$$

to within η*, counting multiplicities, for certain continuous maps* $\alpha_i: Y \to X$ *factoring through [0,1], where* $X = X_{m,i}$ *and* $Y = X_{m',j}$*. Furthermore, if we denote* $\frac{rank(\phi^{i,j}_{m,m'}(\mathbf{1}_{A^i_m}))}{rank(\mathbf{1}_{A^i_m})}$ *by* K*, then* $T_i \geq int(\delta K)$ *and therefore* $L < \frac{2}{\delta}$*, provided* $\delta K \geq 2$*, where* $int(\delta K)$ *is the integer part of* δK *as in 1.1.7.*
(This lemma is Theorem 2.19 (see also Remark 2.21) of [Li2]. The additional part about the size of T_i could be obtained by inspecting the proof of the theorem (see 2.16 and 2.18 of [Li2]).)
(Note that when we write $\phi^{i,j}_{m,m'} = \phi \otimes \mathrm{id}_{[m,i]}$, we have $\mathrm{SP}(\phi^{i,j}_{m,m'})_y = \mathrm{SP}\phi_y$.)
In particular, the condition (1) in 1.5.10 holds, since the right hand side is zero. Therefore the following theorem will be useful for our setting, which is the first theorem in §4.

Theorem 4.1. *Let* X *be a connected finite simplicial complex, and* $F \subset C(X)$ *be a finite set which generates* $C(X)$*. For any* $\varepsilon > 0$*, there is an* $\eta > 0$ *such that the following statement is true.*
Suppose that a unital homomorphism $\phi : C(X) \to PM_\bullet(C(Y))P$ *(*$rank(P) = K$*) (where* Y *is a finite simplicial complex) satisfies the following condition: There are* L *continuous maps*

$$a_1, a_2, \cdots, a_L : Y \longrightarrow X$$

such that for each $y \in Y$*,* $SP\phi_y$ *and* $\Theta(y)$ *can be paired within* η*, where*

$$\Theta(y) = \{a_1(y)^{\sim T_1}, a_2(y)^{\sim T_2}, \cdots, a_L(y)^{\sim T_L}\}$$

and $T_1, T_2, \cdots, T_L$ *are positive integers with*

$$T_1 + T_2 + \cdots + T_L = K = rank(P).$$

Let $T = 2^L(\dim X + \dim Y)^3$*. It follows that there are* L *mutually orthogonal projections* $p_1, p_2, \cdots, p_L \in PM_\bullet(C(Y))P$ *such that*
(i) $\|\phi(f)(y) - p_0(y)\phi(f)(y)p_0(y) \oplus \sum_{i=1}^L f(a_i(y))p_i(y)\| < \varepsilon$ *for any* $f \in F$ *and* $y \in Y$*, where* $p_0 = P - \sum_{i=1}^L p_i$*;*
(ii) $\|p_0(y)\phi(f)(y) - \phi(f)(y)p_0(y)\| < \varepsilon$ *for any* $f \in F$ *and* $y \in Y$*;*
(iii) $rank(p_i) \geq T_i - T$ *for* $1 \leq i \leq L$*, and hence* $rank(p_0) \leq LT$*.*
(In the above, η can be chosen to be any number satisfying that if $\mathrm{dist}(x, x') < 2\eta$, then $|f(x) - f(x')| < \frac{\varepsilon}{3}$, $\forall f \in F$.)
(Note that we can not make the number T in the above theorem as small as $\dim(Y)$, as in Proposition 1.5.7 or 1.5.7', for some technical difficulties. This

is the reason that we are forced to use the stronger condition of very slow dimension growth instead of slow dimension growth in our main decomposition theorem.)

1.5.12. The proof of the above theorem is much more difficult than that of Proposition 1.5.7 or 1.5.7'. In particular, Theorem 2.1 (see §1.2 above), and results in §3 are only for the purpose of proving the above theorem. With these results in hand, the proof of Theorem 4.1 will be given in 4.2–4.19. Then the main decomposition theorem described in the introduction—Theorems 4.35 and 4.37—will be proved based on Theorem 4.1 and the results from [Li2].
We would like to explain the difficulties, and how Theorem 2.1 and §3 will be used to over come the difficulties.
Now, our notations are as in Theorem 4.1 above.
Fix i. Let $U_i(y) = \{x \in X \mid \mathrm{dist}(x, a_i(y)) < \eta\}$, then from the condition of Theorem 4.1, we have

$$\#(\mathrm{SP}\phi_y \cap U_i(y)) \geq T_i.$$

Let $P_i(y)$ be the spectral projection corresponding to the open set $U_i(y)$. This is not a continuously defined projection. But using the same procedure in 1.5.6 (see 1.5.7 and 1.5.7'), one can construct a globally defined projection $p_i(y)$ such that $p_i(y) \leq P_i(y)$, and $\mathrm{rank}(p_i(y)) \geq T_i - \dim(Y)$.
But unfortunately, those $p_i(y)$ are not mutually orthogonal, since $U_i(y)$ are not mutually disjoint, and therefore $P_i(y)$ are not mutually orthogonal.

1.5.13. If we assume that the maximum spectral multiplicity of ϕ is at most Ω, then for each $y \in Y$, we can divide the set $\mathrm{SP}\phi_y$ (with multiplicity) into L mutually disjoint subsets $E_1, E_2, \cdots, E_L$ such that, for each $\lambda \in E_i$, $\mathrm{dist}(\lambda, a_i(y)) \leq \eta,\ i = 1, 2, \cdots, L$, and such that

$$T_i - \Omega < \#(E_i) < T_i + \Omega,\ i = 1, 2, \cdots, L,$$

counting multiplicity. By $\{E_i\}$ being mutually disjoint, we mean that if an element $\lambda \in \mathrm{SP}\phi_y$ has multiplicity k, then we put the entire k copies of λ into one of E_i, without separating them. (In the above, $\phi,\ a_i,\ T_i$, and L are all from Theorem 4.1.)
(Note that if we require that $\#(E_i) = T_i$, then we can not guarantee $\{E_i\}$ are mutually disjoint, because of spectral multiplicity.)
Then we can construct mutually disjoint open sets $U_1(y), U_2(y), \cdots, U_L(y)$ such that $E_i \subset U_i(y)$ and $U_i(y) \subset B_\eta(a_i(y))$. We can further assume that these open sets have mutually disjoint closures. That is $\overline{U_i(y)} \cap \overline{U_j(y)} = \emptyset$, for $i \neq j,\ i, j \in \{1, 2, \cdots, L\}$.
(The open sets from such construction usually do not satisfy the continuity condition in Proposition 1.5.7', so we can not apply Proposition 1.5.7'. We need to check the proof of it (e.g. the argument in 1.5.6) against our new setting.)

For each $y_0 \in Y$, there is an open set $W(y_0) \ni y_0$ such that

$$\mathrm{SP}\phi_y \subset U_1(y_0) \cup U_2(y_0) \cup \cdots \cup U_L(y_0),\ \forall y \in W(y_0).$$

As in 1.5.6, one can construct the mutually orthogonal locally defined continuous projection-valued functions

$$P_i^{W(y_0)}:\ W(y_0) \to \text{projection of } M_\bullet(\mathbb{C}),\ i = 1, 2, \cdots, L,$$

where $P_i^{W(y_0)}(y)$ $(y \in W(y_0))$ are the spectral projections of ϕ_y corresponding to open sets $U_i(y_0)$ (or $\mathrm{SP}\phi_y \cap U_i(y_0)$). Furthermore, $\mathrm{rank} P_i^{W(y_0)} = \#(E_i) > T_i - \Omega$.

(Note that we do not need to introduce the smaller open set U' as in 1.5.6, because it is automatically true that $\mathrm{SP}\phi_y \cap (\overline{U_i(y_0)} \backslash U_i(y_0)) = \emptyset$, as $\{\overline{U_i(y_0)}\}_{i=1}^L$ are mutually disjoint and $\mathrm{SP}\phi_y \subset U_1(y_0) \cup U_2(y_0) \cup \cdots \cup U_L(y_0)$.)

Theorem 2.1 guarantees that Ω is controlled, it is at most $\dim(X) + \dim(Y)$, which will be very small compared with T_i, in our future application.

1.5.14. There is a finite subcover $\mathcal{W} = \{W(y_j)\}_j$ of the open cover $\{W(y)\}_{y \in Y}$ of Y.

We can use the selection theorem [DNNP 3.2] (see 1.5.6 above) to construct global defined continuous projection valued functions $p_i(y)$, $i = 1, 2, \cdots, L$, of ranks at least $T_i - \Omega - \dim(Y)$, such that

$$(*) \qquad p_i(y) \le \bigvee \{P_i^{W(y_j)}(y) \mid y \in W(y_j) \text{ and } W(y_j) \in \mathcal{W}\}.$$

For any $i_1 \ne i_2 \in \{1, 2, \cdots, L\}$, $W(y_j) \in \mathcal{W}$, and $y \in W(y_j)$, we have $P_{i_1}^{W(y_j)}(y) \perp P_{i_2}^{W(y_j)}(y)$.

Unfortunately, when $y \in W(y_{j_1}) \cap W(y_{j_2})$, we do not have

$$P_{i_1}^{W(y_{j_1})}(y) \perp P_{i_2}^{W(y_{j_2})}(y).$$

Therefore, one can not conclude that $p_{i_1}(y) \perp p_{i_2}(y)$ from the above (*).

(Notice that, in the above, $P_i^{W(y_0)}(y)$ is the spectral projection of ϕ_y with respect to the open set $U_i(y_0)$ (not $U_i(y)$), and in general, $U_{i_1}(y_{j_1}) \cap U_{i_2}(y_{j_2}) \ne \emptyset$ if $j_1 \ne j_2$. In Propositions 1.5.7 and 1.5.7', we do not have such problem, since $P_i^W(y)$ is the spectral projection of U', which is an open subset of U_i (does not depend on y) in the case of Proposition 1.5.7, or which is an open subset of $U_i(y_0) \cap U_i(y)$ in the case of Proposition 1.5.7'; see 1.5.6 and the explanation after Proposition 1.5.7' for more details.)

1.5.15. For any $W \in \mathcal{W}$ and $y \in W$, define $Q_i^W(y)$, $i = 1, 2, \cdots, L$, to be the spectral projections of ϕ at point y with respect to the open sets $\bigcap_{\{j:\ W(y_j) \cap W \ne \emptyset\}} U_i(y_j), i = 1, 2, \cdots, L$. These are subprojections of $P_i^W(y)$.

The advantage of using these projections is the following fact. For any $y \in W(y_{j_1}) \cap W(y_{j_2})$, we DO have

$$Q_{i_1}^{W(y_{j_1})}(y) \perp Q_{i_2}^{W(y_{j_2})}(y)$$

for any $i_1 \neq i_2 \in \{1, 2, \cdots, L\}$, because $U_{i_1}(y_{j_1}) \cap U_{i_2}(y_{j_1}) = \emptyset$, $\bigcap_{\{j:\ W(y_j)\cap W(y_{j_1})\neq\emptyset\}} U_{i_1}(y_j) \subset U_{i_1}(y_{j_1})$ and $\bigcap_{\{j:\ W(y_j)\cap W(y_{j_2})\neq\emptyset\}} U_{i_2}(y_j) \subset U_{i_2}(y_{j_1})$ (the second inclusion follows from $W(y_{j_1}) \cap W(y_{j_2}) \neq \emptyset$).
Then we apply the selection theorem to Q_i^W (instead of P_i^W) to find globally defined continuous projection-valued functions $p_i(y)$ such that

$$p_i(y) \leq \bigvee \{Q_i^{W(y_j)}(y) \mid y \in W(y_j) \text{ and } W(y_j) \in \mathcal{W}\},$$

and such that

$$\operatorname{rank}(p_i(y)) \geq \min_{y\in W\in\mathcal{W}} \{\operatorname{rank}(Q_i^W(y))\} - \dim(Y).$$

(The readers may notice that $Q_i^W(y)$ are not continuous on W, so one can not apply the selection theorem directly. But one can introduce the open subsets U' as in 1.5.6 for $\bigcap_{\{j:\ W(y_j)\cap W\neq\emptyset\}} U_i(y_j)$ (instead of U_i). We omit the details.)
This time, $p_{i_1}(y) \perp p_{i_2}(y)$ for any $i_1 \neq i_2 \in \{1, 2, \cdots, L\}$.
To guarantee $\operatorname{rank}(p_i(y))$ to be large—not too much smaller than T_i, $\min_{y\in W\in\mathcal{W}}\{\operatorname{rank}(Q_i^W(y))\}$ must be large.

1.5.16. Fixed $y_0 \in Y$ with $W(y_0) \in \mathcal{W}$. Recall from the definitions of $U_i(y_0)$ and $W(y_0)$ (see 1.5.13),

$$\mathrm{SP}\phi_y = \big(\mathrm{SP}\phi_y \cap U_1(y_0)\big) \cup \big(\mathrm{SP}\phi_y \cap U_2(y_0)\big) \cup \cdots \cup \big(\mathrm{SP}\phi_y \cap U_L(y_0)\big), \forall y \in W(y_0).$$

Define $E_i^{W(y_0)}(y) := \mathrm{SP}\phi_y \cap U_i(y_0)$. ($E_i^{W(y_0)}(y_0)$ is the set E_i in 1.5.13, with y_0 in place of y.) Then for each $y \in W \in \mathcal{W}$, $\{E_i^W(y)\}_{i=1}^L$ is a division of $\mathrm{SP}\phi_y$ (in the terminology in §3, it will be called a grouping of $\mathrm{SP}\phi_y$).
From 1.5.15,

$$\begin{aligned}\operatorname{rank}(Q_i^W(y)) &= \#\left(\mathrm{SP}\phi_y \cap \bigcap_{\{j:\ W(y_j)\cap W\neq\emptyset\}} U_i(y_j)\right) \\ &= \#\Big(\bigcap_{\{j:\ W(y_j)\cap W\neq\emptyset\}} E_i^{W(y_j)}(y)\Big).\end{aligned}$$

Roughly speaking, for our construction in 1.5.15 to work, we need the following condition.

Condition: For each $y \in Y$, the number $\#(\bigcap_{\{W:\ y\in W\in\mathcal{W}\}} E_i^W(y))$ is large—not too much smaller than T_i.

(This condition is a little weaker than $Q_i^W(y)$ to be large. But we are going to use some special open cover so that the above weaker condition will be enough. We are not going to discuss details here, and the reader does not need to pay much attention.)
But in 1.5.15, we only have $\#(E_i^W(y)) > T_i - \Omega$. To obtain the above condition, we need the combinatorial results in §3. We are going to discuss it now.

1.5.17. For an intersection to be large, it will be certainly natural to require the number of sets involved in the intersection be as small as possible. As in the setting of 1.5.16, we should require that for any $y \in Y$, the number of sets in $\mathcal{W}$ which cover y—$\#\{W \mid y \in W \in \mathcal{W}\}$— is not too large.
From the definition of covering dimension, we know that for any n-dimensional compact metrizable space Y and any finite cover $\mathcal{U}$ of Y, there is a refined cover $\mathcal{U}_1$ of $\mathcal{U}$ such that for any point $y \in Y$, there are at most $n+1$ open sets in $\mathcal{U}_1$ to cover the point y. In particular, for a simplicial complex Y, the construct of such open cover is given in 1.4.2 (a).
Let $\{W_y\}_{y \in \text{Vertex}(Y)}$ be the open cover of Y given in 1.4.2(a). Recall that, for any vertices $y_0, y_1, \cdots, y_k$, the intersection $\bigcap_{i=1}^k W_{y_i}$ is nonempty if and only if $y_0, y_1, \cdots, y_k$ share one simplex.
For any finite open cover, there is a finite open cover of the above form (for some refined simplicial complex structure), refining the given open cover.
Without loss of generality, we can assume the open cover $\mathcal{W} = \{W(y_j)\}$ in 1.5.14 is of the above form. Hence $\{y_j\}$ are vertices of the simplicial complex Y and $W(y_j)$ are the open sets W_{y_j} defined above. Then the condition in 1.5.16 becomes the following.
For any simplex Δ of Y with vertices $y_0, y_1, \cdots, y_k$,

$$\#\left(E_i^{W(y_0)}(y) \cap E_i^{W(y_1)}(y) \cap \cdots \cap E_i^{W(y_k)}(y)\right) \geq T_i - C$$

for any $y \in W(y_0) \cap W(y_1) \cap \cdots \cap W(y_k)$, where C is not too large. (In the proof of Theorem 4.1, the number C will be chosen to be $2^L\Omega(1 + \dim(Y)(\dim(Y) + 1))$, where Ω is the maximum spectral multiplicity which is bounded by $\dim(X) + \dim(Y)$, by Theorem 2.1.)

1.5.18. To make the discussion simpler, we suppose that the homomorphism ϕ has distinct spectrum at any point $y \in Y$. That is, the maximum spectral multiplicity of ϕ is one. (Of course, in the proof of Theorem 4.1 in §4, we will not make this assumption.)
If the simplicial structure is sufficiently refined, by the distinct property of the spectrum, we can assume the following holds: For any simplex Δ with $Z = \bigcup_{y \in \text{Vertex}(\Delta)} W(y)(\supset \Delta)$, there are continuous maps

$$\lambda_1, \lambda_2, \cdots, \lambda_K : Z \to X,$$

where $K = \text{rank}(P)$ as in Theorem 4.1, such that

$$\text{SP}\phi_y = \{\lambda_1(y), \lambda_2(y), \cdots, \lambda_K(y)\}, \ \forall y \in Y.$$

Then for any $y \in \text{Vertex}(\Delta)$, the division (or grouping) $E_i^{W(y)}(y)$ of $\text{SP}\phi_y$ gives rise to a division (or grouping) $E_i(y)$ of $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}$. In the case of distinct spectrum, the condition in 1.5.13 concerning $\#(E_i)$ is $\#(E_i(y)) = T_i$. The condition at the end of 1.5.17 is

$$\#\left(\bigcap_{y\in\text{Vertex}(\Delta)} E_i(y)\right) \geq T_i - C.$$

This is of course not true in general, unless we make some special arrangement. But with the following lemma, we can always subdivide (or refine) the simplicial complex and introduce the groupings of $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}$ for all newly introduced vertices, to make the above true for any simplex of new simplicial structure (after subdivision).

The following formal definition of grouping is in 3.2 of §3.

Definition. Let $E = \{1, 2, \cdots, K\}$ be an index set. Let $T_1, T_2, \cdots, T_L$ be non negative integers with

$$T_1 + T_2 + \cdots + T_L = K.$$

A grouping of E of type $(T_1, T_2, \cdots, T_L)$ is a set of L mutually disjoint index sets $E_1, E_2, \cdots, E_L$ with

$$E = E_1 \cup E_2 \cup \cdots \cup E_L,$$

and $\#(E_j) = T_j$ for each $1 \leq j \leq L$.

Lemma. *Suppose that (Δ, σ) is a simplex, where σ is the standard simplicial structure of the simplex Δ. Suppose that for each vertex $x \in$ Vertex(Δ, σ), there is a grouping $E_1(x), E_2(x), \cdots, E_L(x)$ of E of type $(T_1, T_2, \cdots, T_L)$.*

It follows that there is a subdivision (Δ, τ) of (Δ, σ), and there is an extension of the definition of the groupings of E for Vertex(Δ, σ) to the groupings of E (of type $(T_1, T_2, \cdots, T_L)$) for Vertex(Δ, τ) ($\supset$ Vertex(Δ, σ)) such that:

(1) For each newly introduced vertex $x \in$ Vertex(X, τ),

$$E_j(x) \subset \bigcup_{y\in \mathit{Vertex}(\Delta,\sigma)} E_j(y), \qquad j = 1, 2, \cdots, L.$$

(2) For any simplex Δ_1 of (X, τ) (after subdivision),

$$\#(\bigcap_{x\in \mathit{Vertex}(\Delta_1)} E_j(x)) \geq T_j - \frac{\dim(\Delta)(\dim(\Delta)+1)}{2}, \qquad j = 1, 2, \cdots, L.$$

1.5.19. Condition (1) above is important for the following reason. In 1.5.13, when we define E_i as a subset of $\text{SP}\phi_y$, we require that $\text{dist}(\lambda, a_i(y)) < \eta$ for any $\lambda \in E_i$. This condition guarantees that the projection P_i^W in 1.5.13 (or Q_i^W in 1.5.16) satisfies that $\phi(f)(y)P_i^W(y)$ (or $\phi(f)(y)Q_i^W(y)$) is approximately

equal to $f(a_i(y))P_i^W(y)$ (or $f(a_i(y))Q_i^W(y)$) to within ε, which is the condition (*) in Lemma 1.5.4.
We consider the grouping of E as the grouping of the spectral functions $\{\lambda_1, \lambda_2, \cdots, \lambda_K\}$ in 1.5.18. Then the condition (1) in the lemma implies the following fact. If for any vertex $y_0 \in \mathrm{Vertex}(\Delta, \sigma)$ and any element $k \in E_i(y_0)$, we have $\mathrm{dist}(\lambda_k(y), a_i(y)) < \eta$, $\forall y \in \Delta$, then for any newly introduced vertex $y_1 \in \mathrm{Vertex}(\Delta, \tau)$ and any element $k' \in E_i(y_1)$ (here $E_i(y_1)$ is the set E_i for the newly introduced grouping for the vertex y_1), we still have $\mathrm{dist}(\lambda_{k'}(y), a_i(y)) < \eta$, $\forall y \in \Delta$.

1.5.20. In fact, in the proof of Theorem 4.1, we need the relative version of the result: Suppose that there are a subdivision $(\partial\Delta, \tau')$ of the boundary $(\partial\Delta, \sigma)$ and groupings for all vertices in $\mathrm{Vertex}(\partial\Delta, \tau')$ ($\supset \mathrm{Vertex}(\Delta, \sigma)$), such that the above (1) holds for any vertex in $\mathrm{Vertex}(\partial\Delta, \tau')$ and such that the above (2) holds for any simplex Δ_1 of $(\partial\Delta, \tau')$ with $\dim(\partial\Delta)$ in place of $\dim(\Delta)$. Then there is a subdivision (Δ, τ) of (Δ, σ), and groupings for all vertices in $\mathrm{Vertex}(\Delta, \tau)$ such that the above (1) and (2) hold and in addition, the following holds: the restriction of (Δ, τ) onto the boundary $\partial\Delta$ is $(\partial\Delta, \tau')$ and the grouping associated to any vertex in $\mathrm{Vertex}(\partial\Delta, \tau)(= \mathrm{Vertex}(\partial\Delta, \tau'))$ is the same as the old one.
In §3, we will prove the above relative version. In fact, to prove the absolute version will automatically force us to prove the stronger one—the relative version.
Another complication comes from the multiplicities, since we can not assume the spectrum to be distinct. In this case, even the definition of grouping needs to be modified.

1.5.21. To give the readers some feeling about the lemma in 1.5.18, we shall discuss the special case of $\dim(\Delta) = 1$. That is, $\Delta = [0, 1]$, the interval.
In this case, we have two groupings for the end points, $\{E_1(0), E_2(0), \cdots E_L(0)\}$ and $\{E_1(1), E_2(1), \cdots E_L(1)\}$ of E of type $(T_1, T_2, \cdots, T_L)$. Then we need to introduce a sequence of points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1,$$

(this give rise to a subdivision of $\Delta = [0, 1]$) and define groupings $\{E_1(t_j), E_2(t_j), \cdots E_L(t_j)\}$ for $j = 1, 2, \cdots n-1$ such that conditions (1) and (2) in the lemma holds.
The condition (1) in the lemma means

$$E_i(t_j) \subset E_i(0) \cup E_i(1),\ \forall i \in \{1, 2, \cdots L\},\ j \in \{1, 2, \cdots, n-1\}.$$

The condition (2) in the lemma means

$$\#(E_i(t_j) \cap E_i(t_{j+1})) \geq T_i - 1,$$

i.e., for any i, and any pair of adjacent points t_j, t_{j+1}, the set $E_i(t_{j+1})$ differs from the set $E_i(t_j)$ by at most one element.

Let us discuss how to make the above (2) hold for E_1. Suppose that $E_1(0)$ and $E_1(1)$ are different (otherwise, we do not need to do anything for them). We can modify $E_1(0)$ to obtain $E_1(t_1)$ as follows. Taking one element λ in $E_1(1)\backslash E_1(0)$ to replace an element μ in $E_1(0)\backslash E_1(1)$, and define it to be $E_1(t_1)$. So $E_1(t_1)$ contains λ but not μ. Since $\lambda \notin E_1(0)$, it must be in some $E_i(0)$, $i > 1$. In the set $E_i(0)$, after we take out λ and put it into $E_0(t_1)$, E_i has one element less than it should have, so we can put μ in it, and call it $E_i(t_1)$. For $j \neq 1$ or i, $E_j(t_1)$ should be the same as $E_j(0)$. In such a way, we construct the grouping for t_1, which satisfies the above (2) for the pair $0, t_1$. Furthermore, compare to $\{E_i(0)\}_i$, the new grouping $\{E_i(t_1)\}_i$ is one step closer to the grouping $\{E_i(1)\}_i$. Repeating the above construction (e.g., for t_1 in place of 0) we can construct $E_i(t_2)$ and so on. Finally, we will reach the grouping at the other end point $1 \in [0,1]$.
If one does not require the condition (1), the above is the complete proof of the lemma for the one-dimensional case.

1.5.22. Since we require the condition (1), when we add an element $\lambda \in E_1(1)\backslash E_1(0)$ into $E_1(0)$ to define $E_1(t_1)$ (as in 1.5.21), we shall carefully choose the element $\mu \in E_1(0)\backslash E_1(1)$ to be replaced by λ. In §3, we shall prove the following assertion: For any $\lambda \in E_1(1)\backslash E_1(0)$, there is $\mu \in E_1(0)\backslash E_1(1)$ to satisfy the following condition: let $F = (E_1(0)\backslash\{\mu\}) \cup \{\lambda\}$; the set $E\backslash F$ can be grouped into $E'_2, E'_3, \cdots, E'_L$ ($\#(E'_i) = T_i$), in such a way that

$$E_i \subset E_i(0) \cup E_i(1).$$

(Such an element μ is the element we should choose.) This is Lemma 3.9 with $E_i(0) \cup E_i(1)$ in place of H_i.
With the above ideas in mind, it should not be difficult (hopefully) to read the first part of §3, which does not involve multiplicity. The main step of §3 is contained in the proof of Lemma 3.11.

1.5.23. In the case with multiplicity, there are two possible ways of proceeding.
1. Define a grouping of

$$E = \{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_k^{\sim w_k}\}$$

to be a (set theoretical) partition of E as a disjoint union of L sets $E = E_1 \cup E_2 \cup \cdots E_L$. Using this definition, we have to allow that, at different vertices, the groupings may be of different types. That is, $\#(E_i)$ may be different for different vertices. (One can compare with $T_i - \Omega < \#(E_i) < T_i + \Omega$ in 1.5.13.)
2. Define a grouping of

$$E = \{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_k^{\sim w_k}\}$$

to be a collection of L subsets $E_1, E_2, \cdots, E_L$ with

$$E_j = \{\lambda_1^{\sim p_1^j}, \lambda_2^{\sim p_2^j}, \cdots, \lambda_k^{\sim p_k^j}\},$$

where $0 \leq p_i^j \leq w_j$, such that

$$\sum_{j=1}^{L} p_i^j = w_i, \qquad \text{for each } i = 1, 2, \cdots, k.$$

The grouping is called to be of type $(T_1, T_2, \cdots, T_L)$ if

$$\#(E_j) = \sum_{i=1}^{k} p_i^j = T_j, \qquad \text{for each } j = 1, 2, \cdots, L.$$

In this way, as will be seen, all the groupings corresponding to vertices of a simplicial complex (as in the proof of Theorem 4.1), may be chosen to be of the same type. But the conclusion (2) in the lemma should be modified. Instead of $\#(\bigcap_{x \in \mathrm{Vertex}(\Delta_1)} E_j(x)),\ j = 1, 2, \cdots, L$ to be big, we require $\#(\bigcap_{x \in \mathrm{Vertex}(\Delta_1)} \overset{\circ}{E}_j(x)),\ j = 1, 2, \cdots, L$, to be big, where for any set F with multiplicity, $\overset{\circ}{F}$ is the subset of F consisting of all such elements λ_i that $\{\lambda_i^{\sim w_i}\}$ are entirely inside F. (See 3.22 for detailed definition of $\overset{\circ}{F}$.)
In fact, either approaches can be carried out for our purpose. It turns out that the second approach is shorter and more elegant. Therefore we shall take this approach.
Even though, in our approach, it is allowed to separate some set $\{\lambda_i^{\sim w_i}\}$ into different groups E_i of the grouping, we should still group as many whole sets $\{\lambda_j^{\sim w_j}\}$ of the index set $\{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_n^{\sim w_n}\}$ as possible into the same set E_i of $\{E_1, E_2, \cdots, E_L\}$. Assumption 3.27 and Lemma 3.28 are for this purpose. (One needs to pay special attention to the definition and properties of G_I in 3.25.) Except this idea, all other parts of the proof are the same as the case of multiplicity one.

1.5.24. Once we have the combinatorial results in §3, and the explanations in 1.5.1–1.5.19, it will not be hard to understand the proof of Theorem 4.1, though there are some other small techniques, which will be clearly explained in the proof (see 4.2–4.19).

1.5.25. Combining Theorem 4.1 and the result of [Li2] (see the lemma stated in 1.5.11), we can obtain a decomposition $\phi_1 \oplus \psi$ of $\phi_{m,m'}$ (for m' large enough) such that the major part ψ factors through an interval algebra.
But to deal with the part ϕ_1, we should add to it, a relatively large (comparing with ϕ_1) homomorphism ϕ_2, which factors through a finite dimensional C^*-algebra—or which is defined by certain point evaluations (on a δ-dense subset of $X_{n,i}$ for some small number δ).
In [Li3], Li deals with this problem by another decomposition, taking such a homomorphism out of the part ψ. (She only proved the one dimensional case.) We take a different approach. Going back to the construction of the maps α_i in [Li2] (see the lemma inside 1.5.11 above), we can choose sufficiently many

of them to be constant maps (see Lemma 4.33 in §4 below). Therefore ψ automatically has such a part defined by point evaluations.

We believe that our approach is easier to understand than Li's approach, though the spirit is the same. Furthermore, our decomposition is a quantitative version (see Theorem 4.35), and is stronger than Li's theorem even in the case of one dimensional spaces. (This will be important in [EGL].)

We shall not use any result from [Li3]. But we encourage the reader to read the short article [Li2], on which our proof heavily depends.

1.6 Some uniqueness theorems and a factorization theorem

First, this subsection contains some uniqueness theorems. In general, a uniqueness theorem states that, under certain conditions, two maps $\phi, \psi : A \to B$ (homomorphisms or completely positive linear contractions between C^*-algebras A and B) are approximately unitarily equivalent to each other to within a given small number ε on a given finite set $F \subset A$, that is, there is a unitary $u \in B$ such that

$$\|\phi(f) - u\psi(f)u^*\| < \varepsilon, \quad \forall f \in F.$$

This subsection also contains a factorization theorem, which says that, there is a homomorphism (in the class of the so called unital simple embeddings) between matrix algebras over (perhaps higher dimensional) spaces, which must approximately factor through a sum of matrix algebras over the special spaces $\{pt\}, [0,1], S^1, \{T_{II,k}\}_{k=2}^{\infty}, \{T_{III,k}\}_{k=2}^{\infty}$, and S^2, by means of almost multiplicative maps.

We put these two kinds of results together into one subsection, since the proofs of them have some similarity. Also, in the proof of the factorization theorem, we use some uniqueness theorems of this same subsection.

Most of the results are modifications of some results in the literature [EG2], [D1-2], [G4] and [DG] (see [Phi], [GL], [Lin1-2] and [EGLP] also). One of the main theorems (Theorem 1.6.9) is a generalization of Theorem 2.29 of [EG2]—the main uniqueness theorem in the classification of real rank zero AH algebras. The proof given here is shorter than the proof given in [EG2]. Another main theorem—Theorem 1.6.26 (see also Corollary 1.6.29) is a refinement of Lemma 2.2 of [D2] (see also Lemma 3.13 and 3.14 of [G4]). Both Theorem 1.6.9 and Corollary 1.6.29 are important in the proof of our main results in §6.

The following well known result (see [Lo]) will be used frequently.

Lemma 1.6.1. *Suppose that $A = \bigoplus_{i=1}^{t} M_{k_i}(C(X_i))$, where $X_i = \{pt\}, [0,1]$, or S^1. For any finite set $F \subset A$ and any number $\varepsilon > 0$, there is a finite set $G \subset A$ and there is a number $\delta > 0$ such that if C is a C^*-algebra and $\phi \in Map(A, C)$ is a G-δ multiplicative map, then there is a homomorphism $\phi' \in Hom(A, C)$ satisfying*

$$\|\phi(f) - \phi'(f)\| < \varepsilon, \quad \forall f \in F.$$

The following result is essentially contained in [EG2] (see [EG2, 2.11]) and is stated as Theorem 1.2 of [D1] (see also [G4, 3.2 and 3.8]).

LEMMA 1.6.2. ([D1, 1.2]) *Let X be a finite simplicial complex. For any finite subset $F \subset C(X)$ and any number $\varepsilon > 0$, there are a positive integer L, a unital homomorphism $\tau : C(X) \to M_L(C(X))$, and a unital homomorphism $\mu : C(X) \to M_{L+1}(C(X))$ with finite dimensional image such that*

$$\|diag(f, \tau(f)) - \mu(f)\| < \varepsilon, \quad \forall f \in F.$$

By the argument in 1.2.19, in the above lemma, the algebra $C(X)$ can be replaced by $M_n(C(X))$.

LEMMA 1.6.3. *Let X be a finite simplicial complex and $A = M_n(C(X))$. For any finite subset $F \subset A$ and any number $\varepsilon > 0$, there are a positive integer L, a unital homomorphism $\tau : A \to M_L(A)(= M_{nL}(C(X)))$, and a unital homomorphism $\mu : A \to M_{L+1}(A)$ with finite dimensional image such that*

$$\|diag(f, \tau(f)) - \mu(f)\| < \varepsilon, \quad \forall f \in F.$$

REMARK 1.6.4. In general, a unital homomorphism $\lambda : C(X) \to B$ with finite dimensional image is always of the form:

$$\lambda(f) = \sum f(x_i)p_i, \quad \forall f \in F,$$

where $\{x_i\}$ is a finite subset of X, and $\{p_i\} \subset B$ is a set of mutually orthogonal projections with $\sum p_i = \mathbf{1}_B$. A homomorphism $\lambda : A = M_n(C(X)) \to B$ with finite dimensional image is of the form

$$\lambda(f) = \sum p_i \otimes f(x_i), \quad \forall f \in M_n(C(X))$$

for a certain identification of $\lambda(\mathbf{1}_A)B\lambda(\mathbf{1}_A) \cong (\lambda(e_{11})B\lambda(e_{11})) \otimes M_n(\mathbb{C})$, where $\{p_i\}$ is a set of mutually orthogonal projections in $\lambda(e_{11})B\lambda(e_{11})$.
The following Lemma is essentially proved in [D1, Lemma 1.4] (see [G4, Theorem 3.9] also), using the idea from [Phi] and [GL].

LEMMA 1.6.5. *Let X be a finite simplicial complex and $A = M_n(C(X))$. For a finite set $F \subset M_n(C(X))$, a positive number $\varepsilon > 0$ and a positive integer N, there are a finite set $G \subset M_n(C(X))$, a positive number $\delta > 0$ and a positive integer L, such that the following is true.*
For any unital C^*-algebra B, any $N+1$ completely positive G-δ multiplicative linear $*$-contraction $\phi_0, \phi_1, \cdots \phi_N \in Map_{G-\delta}(A, B)$, there are a homomorphism $\lambda \in Hom(A, M_L(B))$ with finite dimensional image and a unitary $u \in M_{L+1}(B)$ such that

$$\|diag(\phi_0(f), \lambda(f)) - u diag(\phi_N(f), \lambda(f))u^*\| < \varepsilon + \omega, \quad \forall f \in F,$$

where

$$\omega = \max_{f\in F} \max_{0\le j\le N-1} \|\phi_j(f) - \phi_{j+1}(f)\|.$$

Proof: If we allow the number L to depend on the maps $\{\phi_j\}$, then this is Lemma 1.4 of [D1]. In fact, in the proof of [D1, Lemma 1.4], the author proves this stronger version of the lemma. We will not repeat the entire proof in [D1], instead, we will only repeat the construction of G, δ in [D1] and at the same time choose the number L.
Apply Lemma 1.6.3 to $F \subset A$, $\frac{\varepsilon}{4} > 0$ to obtain the integer L_1, $\tau : A \to M_{L_1}(A)$ and $\mu : A \to M_{L_1+1}(A)$ as in Lemma 1.6.3. Then $D := \mu(A)$ is a finite dimensional C^*-subalgebra of $M_{L_1+1}(A)$. By Lemma 1.6.1, there are a finite set $F_1 \subset D(\subset M_{L_1+1}(A))$ and a positive number $\delta_1 > 0$ such that if C is any C^*-algebra and $\psi \in \mathrm{Map}(D, C)$ is any F_1-δ_1 multiplicative map, then there is a homomorphism $\psi' \in \mathrm{Hom}(D, C)$ such that

$$\|\psi'(f) - \psi(f)\| < \frac{\varepsilon}{4}, \quad \forall f \in \mu(F)\ (\subset D).$$

From 1.2.19, there exist a finite set $G \subset A$ and a positive number $\delta > 0$ such that if $\phi \in \mathrm{Map}(A, B)$ is G-δ multiplicative, then $\phi \otimes \mathrm{id}_{L_1+1} \in \mathrm{Map}(M_{L_1+1}(A), M_{L_1+1}(B))$ is F_1-δ_1 multiplicative.
Let $L := N(L_1 + 1)$. The proof of [D1, Lemma 1.4] proves that such G, δ and L are as desired. (Notice that the size of the homomorphism η on line 9 of page 122 of [D1] is the number L above.)

$\square$

In Lemma 1.6.5, if we further assume that $F \subset A$ is weakly approximately constant to within ε, then one can replace the homomorphism λ in Lemma 1.6.5 by an arbitrary homomorphism with finite dimensional image of sufficiently large size (with ε replaced by 5ε). One can even use two different homomorphisms (provided that the images of the matrix unit e_{11} under these two different homomorphisms are unitarily equivalent) for ϕ_0 and ϕ_N, i.e., instead of $\mathrm{diag}(\phi_0, \lambda)$ and $\mathrm{diag}(\phi_N, \lambda)$, one can use $\mathrm{diag}(\phi_0, \lambda_1)$ and $\mathrm{diag}(\phi_N, \lambda_2)$ in the estimation. Namely, we can prove the following result.

Corollary 1.6.6. *Let X be a finite simplicial complex and $A = M_n(C(X))$. Suppose that $\varepsilon > 0$ and that a finite set $F \subset M_n(C(X))$ is weakly approximately constant to within ε. Suppose that N is a positive integer. Then there are a finite set $G \subset M_n(C(X))$, a positive number $\delta > 0$, and a positive integer L such that the following is true.*
For any unital C^*-algebra B and projection $p \in B$, any $N+1$ completely positive G-δ multiplicative linear $*$-contractions $\phi_0, \phi_1, \cdots, \phi_N \in Map_{G-\delta}(A, pBp)$, any $\lambda^1, \lambda^2 \in Hom(A, (1-p)B(1-p))$ with finite dimensional images and with $\lambda^1(e_{11}) \sim \lambda^2(e_{11})$ (see 1.1.7(i)) and $[\lambda^1(e_{11})] \ge L \cdot [p]$, there is a unitary $u \in B$ such that

$$\|diag(\phi_0(f), \lambda^1(f)) - u\,diag(\phi_N(f), \lambda^2(f))u^*\| < 5\varepsilon + \omega, \quad \forall f \in F,$$

where

$$\omega = \max_{f \in F} \max_{0 \le j \le N-1} \|\phi_j(f) - \phi_{j+1}(f)\|.$$

Proof: Suppose that L and $\lambda : A \to M_L(pBp)$ are as in Lemma 1.6.5 for the G-δ multiplicative maps $\phi_0, \phi_1, \cdots \phi_N \in \mathrm{Map}(A, pBp)$.
From 1.6.4, λ is of the following form

$$\lambda(f) = \sum_{i=1}^{s} p_i \otimes f(x_i), \quad \forall f \in M_n(C(X))$$

for a certain identification of $\lambda(\mathbf{1}_A)B\lambda(\mathbf{1}_A) \cong (\lambda(e_{11})B\lambda(e_{11})) \otimes M_n(\mathbb{C})$, where $p_i,\ i = 1, 2, \cdots, s$, are mutually orthogonal projections with $\sum_{i=1}^{s} p_i = \lambda(e_{11}) \in M_L(pBp)$, and $\{x_i\} \subset X$.
Fix a base point $x_0 \in X$. Since F is weakly approximately constant to within ε, for each i, $\{f(x_i)\}_{f\in F}$ is approximately unitarily equivalent to $\{f(x_0)\}_{f\in F}$ to within ε, one by one by the same unitary. I.e., for each i, there is a unitary $v \in M_n(\mathbb{C})$ such that $\|vf(x_i)v^* - f(x_0)\| < \varepsilon$, for any $f \in F$.
Define $new\lambda$ by $new\lambda(f) = \sum_{i=1}^{s} p_i \otimes f(x_0) = E \otimes f(x_0)$, where $E := \lambda(e_{11}) = \sum_{i=1}^{s} p_i$. Then $new\lambda$ is approximately unitarily equivalent to the old λ to within ε on F. Therefore, with this $new\lambda$, we still have

$$\|\mathrm{diag}(\phi_0(f), \lambda(f)) - u_1 \mathrm{diag}(\phi_N(f), \lambda(f))u_1\| < 3\varepsilon + \omega, \quad \forall f \in F, \tag{1}$$

for some unitary $u_1 \in M_{L+1}(pBp)$.
Since $\lambda^1(e_{11}) \sim \lambda^2(e_{11})$, without loss of generality, we can assume that $\lambda^1|_{M_n(\mathbb{C})} = \lambda^2|_{M_n(\mathbb{C})}$, where $M_n(\mathbb{C}) \subset M_n(C(X))\ (= A)$. In particular, $\lambda^1(\mathbf{1}_A) = \lambda^2(\mathbf{1}_A)$ and $\lambda^1(e_{11}) = \lambda^2(e_{11})$. Denote $\lambda^1(e_{11})$ by E'. Similar to the case of λ, we can assume that

$$\lambda^1(f) = \sum_{i=1}^{s_1} q_i^1 \otimes f(x_i^1), \quad \forall f \in M_n(C(X)),$$

$$\lambda^2(f) = \sum_{i=1}^{s_2} q_i^2 \otimes f(x_i^2), \quad \forall f \in M_n(C(X))$$

for a certain identification of $\lambda^1(\mathbf{1}_A)B\lambda^1(\mathbf{1}_A) \cong (E'BE') \otimes M_n(\mathbb{C})$, where $\{q_i^1\}$ and $\{q_i^2\}$ are two sets of mutually orthogonal projections with $\sum_{i=1}^{s_1} q_i^1 = \sum_{i=1}^{s_2} q_i^2 = E' \in (1-p)B(1-p)$, and $\{x_i^1\}, \{x_i^2\} \subset X$.
Define $\tilde{\lambda} : A \to \lambda^1(\mathbf{1}_A)B\lambda^1(\mathbf{1}_A)$ by

$$\tilde{\lambda}(f) = E' \otimes f(x_0), \quad \forall f \in F.$$

Similar to the argument for λ, both λ^1 and λ^2 are approximately unitarily equivalent to $\tilde{\lambda}$ to within ε on F.

Since $[E] \leq L\cdot[p] \leq [E']$ $(= [\lambda^1(e_{11})])$, there is a sub-projection $E_1 \leq E'$ which is unitarily equivalent to E.
Write $\tilde{\lambda} = \mu_1 \oplus \mu_2$, where $\mu_1(f) = E_1 \otimes f(x_0)$ and $\mu_2(f) = (E' - E_1) \otimes f(x_0)$. Then μ_1 is unitarily equivalent to λ (strictly speaking, $new\lambda$). From (1), we have

$$\|\mathrm{diag}(\phi_0(f), \mu_1(f)) - u_2\mathrm{diag}(\phi_N(f), \mu_1(f))u_2^*\| < 3\varepsilon + \omega, \quad \forall f \in F$$

for a unitary $u_2 \in (p \oplus E_1 \otimes \mathbf{1}_n)B(p \oplus E_1 \otimes \mathbf{1}_n)$. Notice that $E_1 \otimes \mathbf{1}_n \leq E' \otimes \mathbf{1}_n = \lambda^1(\mathbf{1}_A) \leq (1-p)$.
Therefore,

$$\|\mathrm{diag}(\phi_0(f), \tilde{\lambda}(f)) - u_3\mathrm{diag}(\phi_N(f), \tilde{\lambda}(f))u_3^*\| < 3\varepsilon + \omega, \quad \forall f \in F,$$

where $u_3 := u_2 \oplus ((E' - E_1) \otimes \mathbf{1}_n) \in (p \oplus (E' \otimes \mathbf{1}_n))B(p \oplus (E' \otimes \mathbf{1}_n))$.
We already know that both λ^1 and λ^2 are approximately unitarily equivalent to $\tilde{\lambda}$ on F to within ε, so we have

$$\|\mathrm{diag}(\phi_0(f), \lambda^1(f)) - u\mathrm{diag}(\phi_N(f), \lambda^2(f))u^*\| < 5\varepsilon + \omega, \quad \forall f \in F,$$

for a unitary $u \in B$.

$\square$

Lemma 1.6.7. *Suppose that $A = M_k(C(X))$, and $F \subset A$ is weakly approximately constant to within ε. Suppose that A_1 is a C^*-algebra, and two homomorphisms ϕ and $\psi \in Hom(A, A_1)$ are homotopic to each other. There are a finite set $G \subset A_1$, a number $\delta > 0$, and a positive integer $L > 0$ such that the following is true.*
If B is a unital C^*-algebra, $p \in B$ is a projection, $\lambda_0 \in Map(A_1, pBp)$ is G-δ multiplicative, $\lambda_1 \in Hom(A_1, (\mathbf{1}-p)B(\mathbf{1}-p))$ is a homomorphism with finite dimensional image satisfying $[(\lambda_1 \circ \phi)(e_{11})] \geq L\cdot[p]$, and $\lambda \in Map(A_1, B)$ is defined by $\lambda = \lambda_0 \oplus \lambda_1$, then there is a unitary $u \in B$ such that

$$\|(\lambda \circ \phi)(f) - u(\lambda \circ \psi)(f)u^*\| < 6\varepsilon, \quad \forall f \in F.$$

Proof: Since ϕ is homotopic to ψ. There is a continuous path of homomorphisms $\phi_t, 0 \leq t \leq 1$, such that $\phi_0 = \phi$ and $\phi_1 = \psi$. Choose $0 = t_0 < t_1 < \cdots t_{N-1} < t_N = 1$ such that

$$\|\phi_{t_{j+1}}(f) - \phi_{t_j}(f)\| < \varepsilon, \quad \forall j \in \{0, 1, \cdots, N-1\} \text{ and } \forall f \in F.$$

Applying Corollary 1.6.6 to ε, $F \subset A$ (which is weakly approximately constant to within ε), and the number N from the above, there are $G_1 \subset A$ and $\delta > 0$ and L as in the Corollary 1.6.6.
The set $G := \bigcup_{j=0}^{N} \phi_{t_j}(G_1) \subset A_1$, $\delta > 0$ and number L are as desired.

Suppose that λ_0, λ_1 are the maps satisfying the conditions described in the lemma for G, δ, and L as chosen above. Choosing the sequence of G_1-δ multiplicative maps in Corollary 1.6.6 to be $\lambda_0 \circ \phi_{t_0} (= \lambda_0 \circ \phi), \lambda_0 \circ \phi_{t_1}, \cdots, \lambda_0 \circ \phi_{t_N} (= \lambda_0 \circ \psi)$, and the homomorphisms λ^1 and λ^2 (with finite dimensional images) to be $\lambda^1 = \lambda_1 \circ \phi$ and $\lambda^2 = \lambda_1 \circ \psi$, and using that $\lambda = \lambda_0 \oplus \lambda_1$, we have

$$\|(\lambda \circ \phi)(f) - u(\lambda \circ \psi)(f)u^*\| < 5\varepsilon + \omega, \quad \forall f \in F,$$

for a certain unitary $u \in B$, where

$$\omega = \max_{f \in F} \max_{0 \le j \le N-1} \|(\lambda_0 \circ \phi_{t_{j+1}})(f) - (\lambda_0 \circ \phi_{t_j})(f)\| < \varepsilon,$$

since λ_0 is a contraction—norm decreasing map. So the Lemma follows. (Note that if λ_0 is G-δ multiplicative, then $\lambda_0 \circ \phi_{t_j}$ is G_1-δ multiplicative. Also note that we have the condition that $[\lambda_1 \circ \phi(e_{11})] \geq L \cdot [p]$. Another condition $\lambda_1 \circ \phi(e_{11}) \sim \lambda_1 \circ \psi(e_{11})$ follows from the condition $\phi \sim_h \psi$.)

$\square$

The author is indebted to Professor G. Elliott for pointing out the proof of the following result to him.

Lemma 1.6.8. *Suppose that C is a unital C^*-algebra, and $D \subset C$ is a finite dimensional C^*-subalgebra. For any finite set $F \subset C$ and any positive number $\varepsilon > 0$, there are a finite set $G \subset C$ and a number $\delta > 0$ such that if B is a unital C^*-algebra, $\lambda \in Map(C, B)$ is G-δ multiplicative, then there is a $\lambda' \in Map(C, B)$ satisfying the following conditions.*
1. $\lambda'|_D$ is a homomorphism.
2. $\|\lambda'(f) - \lambda(f)\| < \varepsilon, \quad \forall f \in F$.

Proof: Without loss of generality, we assume that $\|f\| \leq 1$ for all $f \in F$.
By Kasparov's version of Stinespring Dilation Theorem, for the completely positive linear $*$-contraction $\lambda : C \to B$, there is a homomorphism $\phi : C \to M(B \otimes \mathcal{K})$ such that $\lambda(f) = p\phi(f)p$ for all $f \in F$, where $\mathcal{K}$ is the algebra of all compact operators on an infinite dimensional separable Hilbert space, $M(B \otimes \mathcal{K})$ is the multiplier algebra of $B \otimes \mathcal{K}$, and $p = \mathbf{1}_B \otimes e_{11} \in B \otimes \mathcal{K} \subset M(B \otimes \mathcal{K})$.
For the above λ and ϕ, it is straight forward to check that, for any fixed element $a \in C$, if $\|\lambda(a \cdot a^*) - \lambda(a)\lambda(a^*)\| < \delta$, then $\|p\phi(a)(1-p) \cdot (p\phi(a)(1-p))^*\| < \delta$. Therefore, if we choose the finite set G to satisfy that $G = G^*$, then the G-δ-multiplicativity of the map λ implies the following property of the dilation ϕ and the cutting down projection p:

$$(*) \qquad \|\phi(a) - (p\phi(a)p + (\mathbf{1} - p)\phi(a)(\mathbf{1} - p))\| < 2\sqrt{\delta}, \quad \forall a \in G,$$

where $\mathbf{1}$ is the unit of $M(B \otimes \mathcal{K})$.
By a well known perturbation technique (see [Gli] and [Br]), we have the following: If G contains all matrix units e_{ij} of each block of D and δ is small enough,

then the above condition (*) implies that there is a unitary $u \in M(B\otimes\mathcal{K})$ with $\|u-\mathbf{1}\|<\frac{\varepsilon}{2}$, such that

$$u\phi(D)u^* \subset pM(B\otimes\mathcal{K})p \oplus (\mathbf{1}-p)M(B\otimes\mathcal{K})(\mathbf{1}-p).$$

(One can obtain the above assertion by applying Lemma III.3.2 of [Da] (or even stronger result of [Ch]) with $\phi(D)$ and $pM(B\otimes\mathcal{K})p\oplus(\mathbf{1}-p)M(B\otimes\mathcal{K})(\mathbf{1}-p)$ in place of $\mathcal{U}$ and $\mathcal{B}$ in [Da, III.3.2], respectively.)
The map $\lambda' : C \to B$, defined by $\lambda'(f) = pu\phi(f)u^*p$, is as desired.

□

The following result can be considered to be a generalization of Theorem 2.29 of [EG 2].

Theorem 1.6.9. *Suppose that $A = \bigoplus_{i=1}^{s} M_{k_i}(C(X_i))$ and $F \subset A$ is weakly approximately constant to within ε. Suppose that C is a C^*-algebra, the homomorphisms ϕ and $\psi \in Hom(A,C)$ are homotopic to each other. There are a finite set $G \subset C$, a number $\delta > 0$, and a positive integer $L > 0$ such that the following is true.*
If B is a unital C^*-algebra, $p \in B$ is a projection, $\lambda_0 \in Map(C, pBp)$ is G-δ multiplicative, $\lambda_1 \in Hom(C, (\mathbf{1}-p)B(\mathbf{1}-p))$ is a homomorphism with finite dimensional image satisfying that for each $i \in \{1,2,\cdots,s\}$, $[(\lambda_1\circ\phi)(e_{11}^i)] \geq L\cdot[p]$, where e_{11}^i is the matrix unit (of upper left corner) of the i-th block, $M_{k_i}(C(X_i))$, of A, then there is a unitary $u \in B$ such that

$$\|(\lambda\circ\phi)(f) - u(\lambda\circ\psi)(f)u^*\| < 8\varepsilon, \quad \forall f \in F,$$

where $\lambda \in Map(A_1, B)$ is defined by $\lambda = \lambda_0 \oplus \lambda_1$.

Proof: Let ϕ_t be the homotopy between ϕ and ψ. It is well known that there is a unitary path $u_t \in C$ such that

$$\phi_t(\mathbf{1}_{A^i}) = u_t\phi_0(\mathbf{1}_{A^i})u_t^*,$$

for all blocks $A^i = M_{k_i}(C(X_i))$. Therefore, without loss of generality, we assume that $\phi(\mathbf{1}_{A^i}) = \psi(\mathbf{1}_{A^i})$, and that $\phi|_{A^i}$ is homotopic to $\psi|_{A^i}$ within the corner $\phi(\mathbf{1}_{A^i})C\phi(\mathbf{1}_{A^i})$.
Apply Lemma 1.6.7 to $\phi|_{A^i}$, $\psi|_{A^i}$ and $\pi_i(F)$, where π_i is the quotient map from A to A^i, to obtain G_1 ($\subset C$), δ_1 and L as G, δ and L in Lemma 1.6.7. For convenience, without loss of generality, we assume that $\|g\| \leq 1$ for all $g \in G_1$. Let $E_i = \phi(\mathbf{1}_{A^i})$. Consider the finite dimensional subalgebra $D := \mathbb{C}\cdot E_1 \oplus \mathbb{C}\cdot E_2 \oplus \cdots \oplus \mathbb{C}\cdot E_s \subset C$. Applying Lemma 1.6.8, there are $G \subset C$ with $G \supset G_1$ and $\delta > 0$ with $\delta < \frac{\delta_1}{3}$ such that if $\lambda_0 \in \mathrm{Map}(C, pBp)$ is G-δ multiplicative, then there is another map $\lambda_0' \in \mathrm{Map}(C, pBp)$ satisfying the following conditions.
1. The restriction $\lambda_0'|_D$ is a homomorphism.
2. $\|\lambda_0'(f) - \lambda_0(f)\| < \min(\frac{\delta_1}{3}, \varepsilon), \quad \forall f \in G_1 \cup \phi(F) \cup \psi(F)$.

As a consequence we also have
3. λ_0' is G_1-δ_1 multiplicative.
The condition 1 above yields that $\{\lambda_0'(E_i)\}_{i=1}^s$ are mutually orthogonal projections.
Such G, δ and L are as desired.
Suppose that λ_0 and λ_0' are as above. Set $\lambda' = \lambda_0' \oplus \lambda_1$. From Lemma 1.6.7 and the ways G_1, δ_1 and L are chosen, there are unitaries $u_i \in \lambda'(E_i)B\lambda'(E_i)$ such that

$$\|(\lambda' \circ \phi|_{A^i})(f) - u_i(\lambda' \circ \psi|_{A^i})(f)u_i^*\| < 6\varepsilon, \quad \forall f \in F_i.$$

Then the unitary $u = \bigoplus_i u_i \oplus (\mathbf{1} - \sum_i \lambda'(E_i))$ satisfies

$$\|(\lambda' \circ \phi)(f) - u(\lambda' \circ \psi)(f)u^*\| < 6\varepsilon, \quad \forall f \in F.$$

Hence

$$\|(\lambda \circ \phi)(f) - u(\lambda \circ \psi)(f)u^*\| < 6\varepsilon + 2\varepsilon = 8\varepsilon, \quad \forall f \in F.$$

$\square$

Remark 1.6.10. The version of Theorem 2.29 of [EG2] with A being a direct sum of full matrix algebras is a direct consequence of the above theorem and Corollary 2.24 of [EG2] (see [EG2, Theorem 2.21] also). In order to obtain the general version of Theorem 2.29 of [EG2], one needs to apply the dilation lemma [EG2, 2.13] and Lemma 1.6.8 above. (The number 8ε should be changed to $5 \cdot 8\varepsilon = 40\varepsilon$ which is still better than 70ε in [EG2].)
The following lemma is a direct consequence of Lemma 1.6.5.

Lemma 1.6.11. *Let X be a finite simplicial complex and $A = C(X)$. Let $F \subset A$ be a finite set and $\varepsilon > 0$. There are a finite set $G \subset A$ and a number $\delta > 0$ with the following property.*
If B is a unital C^*-algebra, $\phi_t : A \to B$, $0 \le t \le 1$ is a continuous path of G-δ multiplicative maps (i.e., $\phi_t \in Map_{G-\delta}(A, B)$), then there are a positive integer L, a homomorphism $\lambda : A \to M_L(B)$ with finite dimensional image, and a unitary $u \in M_{L+1}(B)$ such that

$$\|(\phi_0 \oplus \lambda)(f) - u(\phi_1 \oplus \lambda)(f)u^*\| < \varepsilon, \quad \forall f \in F.$$

The proof of the following corollary has some similarity to the proof of Corollary 1.6.6. Such method will be used frequently.

Corollary 1.6.12. *Let X be a finite simplicial complex and $A = C(X)$. Let $F \subset A$ be a finite set and $\varepsilon > 0$. There are a finite set $G \subset A$ and a number $\delta > 0$ with the following property.*
If B is a unital C^*-algebra, $p \in B$ is a projection, $\phi_t : A \to pBp$, $0 \le t \le 1$ is a continuous path of G-δ multiplicative maps , then there are a positive integer L

and a number $\eta > 0$ such that for any η-dense subset $\{x_1, x_2, \cdots, x_\bullet\} \subset X$, any set of mutually orthogonal projections $\{p_1, p_2, \cdots, p_\bullet\} \subset B \otimes \mathcal{K}$ with $[p_i] \geq L \cdot [p]$ and $p_i \perp p$, we have

$$\|\phi_0(f) \oplus \sum_{i=1}^{\bullet} f(x_i)p_i - u(\phi_1(f) \oplus \sum_{i=1}^{\bullet} f(x_i)p_i)u^*\| < \varepsilon, \quad \forall f \in F,$$

for a certain unitary $u \in (p \oplus p_1 \oplus \cdots p_\bullet)(B \otimes \mathcal{K})(p \oplus p_1 \oplus \cdots p_\bullet)$.

Proof: For the finite set $F \subset A$, choose η small enough such that if $\operatorname{dist}(x, x') < \eta$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{3}$ for all $f \in F$. Apply Lemma 1.6.11 to F and $\frac{\varepsilon}{3}$ to obtain G and δ. For the path $\phi_t : A \to pBp$, there exist a positive integer L, a homomorphism $\lambda : A \to M_L(pBp)$, and a unitary $u_1 \in M_{L+1}(pBp)$ as in Lemma 1.6.11. That is

$$\|(\phi_0 \oplus \lambda)(f) - u_1(\phi_1 \oplus \lambda)(f)u_1^*\| < \frac{\varepsilon}{3}, \quad \forall f \in F.$$

From 1.6.4, λ is of the form

$$\lambda(f) = \sum_{i=1}^{l} f(y_i)q_i,$$

where $\{y_1, y_2, \cdots, y_l\} \subset X$, and $\{q_1, q_2, \cdots, q_l\} \subset M_L(pBp)$ is a set of mutually orthogonal projections.
Since $\{x_1, x_2, \cdots, x_\bullet\}$ is an η-dense subset of X, we can divide the set $\{y_1, y_2, \cdots, y_l\}$ into a disjoint union of subsets $X_1 \cup X_2 \cup \cdots \cup X_\bullet$ (some X_i may be empty) such that $\operatorname{dist}(y, x_i) < \eta$ for any $y \in X_i$. Set $p_i' := \sum_{y_j \in X_i} q_j$ and define $\lambda' : C(X) \to M_L(pBp)$ by $\lambda'(f) = \sum_{i=1}^{\bullet} f(x_i)p_i'$. (Note that for some i, p_i' might be 0.) Then from the way η is chosen, we have

$$\|\lambda'(f) - \lambda(f)\| < \frac{\varepsilon}{3}, \quad \forall f \in F.$$

Therefore,

$$\|(\phi_0 \oplus \lambda')(f) - u_1(\phi_1 \oplus \lambda')(f)u_1^*\| < \varepsilon, \quad \forall f \in F.$$

Our corollary follows from the fact $[p_i'] \leq L \cdot [p] \leq p_i$.

$\square$

If X does not contain any isolated point, then in the above corollary, we can change the condition $[p_i] \geq L \cdot [p]$ to a weaker condition $[p_i] \geq [p]$, by choosing η smaller. (Roughly speaking, this is true because η could be chosen so small that if $\{x_i\}$ is η-dense, then there are at least L points of x_i in the η'-neighborhood of any point in X for a pre-given small number η'. If X is a space of single point, then this is not true.) Therefore, the number L does not appear in the following corollary.

COROLLARY 1.6.13. *Let X be a finite simplicial complex without any single*

point components, $A = C(X)$. Let $F \subset A$ be a finite set and $\varepsilon > 0$. There are a finite set $G \subset A$ and a number $\delta > 0$ with the following property.
If B is a unital C^*-algebra, $p \in B$ is a projection, $\phi_t : A \to pBp$, $0 \le t \le 1$, is a continuous path of G-δ multiplicative maps , then there is a number $\eta > 0$ such that for any η-dense subset $\{x_1, x_2, \cdots, x_\bullet\} \subset X$, any set of mutually orthogonal projections $\{p_1, p_2, \cdots, p_\bullet\} \subset B \otimes \mathcal{K}$ with $[p_i] \ge [p]$ and $p_i \perp p$, we have

$$\|\phi_0(f) \oplus \sum_{i=1}^{\bullet} f(x_i)p_i - u(\phi_1(f) \oplus \sum_{i=1}^{\bullet} f(x_i)p_i)u^*\| < \varepsilon, \quad \forall f \in F$$

for a certain unitary $u \in (p \oplus p_1 \oplus \cdots p_\bullet)(B \otimes \mathcal{K})(p \oplus p_1 \oplus \cdots p_\bullet)$.

Proof: Let L and η_1 (in place of η) be as in Corollary 1.6.12. Let η_2 be the minimum of the diameters of path connected components of X, which is positive since X has no single point component. And let η_3 be a positive number such that if $\mathrm{dist}(x, x') < \eta_3$, then $\|f(x) - f(x')\| < \varepsilon$.
Define $\eta' = \min(\eta_1, \eta_2, \eta_3)$. Let $\eta = \frac{\eta'}{8L}$.
Suppose that $X' = \{x_1, x_2, \cdots, x_\bullet\}$ is an η-dense finite subset of X. Choose a η'-dense subset $\{x_{k_1}, x_{k_2}, \cdots, x_{k_l}\} \subset X'$ such that $\mathrm{dist}(x_{k_i}, x_{k_j}) \ge \frac{\eta'}{2}$ if $i \ne j$. (Such subset exists. It could be chosen to be a maximum subset of X' such that the distance of any two points in the set is at least $\frac{\eta'}{2}$. Then the η'-density follows from the maximality.) It is easy to see that there is a partition of X' as $X' = X_1 \cup X_2 \cup \cdots \cup X_l$ such that

$$X' \cap B_{\frac{\eta'}{4}}(x_{k_i}) \subset X_i \subset X' \cap B_{\eta'}(x_{k_i}).$$

Since X' is η-dense and $\eta = \frac{\eta'}{8L}$,

$$\#(X_i) \ge \#(X' \cap B_{\frac{\eta'}{4}}(x_{k_i})) \ge L.$$

(Here we also use the fact that the connected component of x_{k_i} in X has diameter at least η'.)
Let p_j, $j = 1, 2, \cdots \bullet$, be the projections as in the corollary. Define $q_i = \sum_{x_j \in X_i} p_j$, $i = 1, 2, \cdots l$. Then from $[p_j] \ge [p]$ and $\#(X_i) \ge L$, we have, $[q_i] \ge L \cdot [p]$.
Our corollary (with 3ε in place of ε) follows from an application of Corollary 1.6.12 to $\{x_{k_i}\}_{i=1}^l$ and $\{q_i\}_{i=1}^l$, and the estimation

$$\|\sum_{i=1}^{\bullet} f(x_i)p_i - \sum_{i=1}^{l} f(x_{k_i})q_i\| < \varepsilon, \quad \forall f \in F.$$

(The above estimation is a consequence of the way η_3 is chosen and the fact that $X_i \subset B_{\eta'}(x_{k_i})$ with $\eta' < \eta_3$.)

$\square$

The following lemma is proved by applying Lemma 1.6.8.

Lemma 1.6.14. *Let $A = \bigoplus A^k = \bigoplus_{k=1}^{l} M_{s(k)}(C(X_k))$, where X_k are connected simplicial complexes and $\{s(k)\}$ are positive integers. For any finite set $G' \subset A$, any number $\delta' > 0$, any finite sets $G_1^k \subset C(X_k)$ and any numbers $\delta_1^k > 0$, $k = 1, 2, \cdots l$, there are a finite set $G \subset A$ and a number $\delta > 0$ such that if $\phi \in Map(A, B)$ is G-δ multiplicative, then there is a map $\phi' \in Map(A, B)$ satisfying the following conditions.*
(1) ϕ' is G'-δ' multiplicative;
(2) $\|\phi'(g) - \phi(g)\| < \delta'$ for all $g \in G$;
(3) $\{\phi'(\mathbf{1}_{A^k})\}_{k=1}^{l}$ are mutually orthogonal projections in B and $\phi'(e_{11}^k) \in B$ are subprojections of $\phi'(\mathbf{1}_{A^k}) \in B$. And if each $\phi_1^k \in Map(C(X_k), B)$ is the restriction of ϕ' on $e_{11}^k M_{s(k)}(C(X_k))e_{11}^k \cong C(X_k)$, then one can identify $\phi'(\mathbf{1}_{A^k})B\phi'(\mathbf{1}_{A^k})$ with $\phi'(e_{11}^k)B\phi'(e_{11}^k) \otimes M_{s(k)}$ such that

$$\phi' = \bigoplus_{k=1}^{l} \phi_1^k \otimes id_{s(k)}.$$

Furthermore, ϕ_1^k is G_1^k-δ_1^k Multiplicative.

Proof: The part of G_1^k-δ_1^k multiplicativity of ϕ_1^k follows from the G'-δ' multiplicativity of ϕ' if we enlarge the set G' and reduce the number δ' so that $G' \supset \{g \cdot e_{11}^k \mid g \in G_1^k\}$ and $\delta' < \delta_1^k$. Also we can assume that G' contains $\{e_{ij}^k\}$—the set of all matrix units.
By Lemma 1.6.8, without loss of generality, we assume that the restriction $\phi|_{\bigoplus_{k=1}^{l} M_{s(k)}(\mathbb{C})}$ is a homomorphism.
Let $\phi_1^k = \phi|_{e_{11}^k A e_{11}^k} \in \mathrm{Map}(C(X_k), \phi(e_{11}^k)B\phi(e_{11}^k))$. Then $\phi' := \bigoplus_{k=1}^{l} \phi_1^k \otimes \mathrm{id}_{s(k)}$ is defined by

$$\phi'(f) = \sum_{i,j} \phi(e_{i1}^k)\phi(f_{ij} \cdot e_{11}^k)\phi(e_{1j}^k),$$

where $f = (f_{ij})_{s(k)\times s(k)} \in A^k$.
Note that for the above $f \in A^k$, one can write $f = \sum_{i,j} e_{i1}^k \cdot (f_{ij} \cdot e_{11}^k) \cdot e_{1j}^k$. Obviously, if we choose G to be the set of all the elements which can be expressed as products of at most ten elements from the set G', and if we choose δ small enough, then (1) and (2) will hold for ϕ'. (Notice that G' contains all the matrix units.)

□

The following result follows from Lemma 1.6.14 and Corollary 1.6.12 (see also 1.2.19).

Corollary 1.6.15. *Let $A = \bigoplus_{k=1}^{l} M_{s(k)}(C(X_k))$, where X_k are connected finite simplicial complexes and $s(k)$ are positive integers. Let $F \subset A$ be a finite set and $\varepsilon > 0$. There are a finite set $G \subset A$ and a number $\delta > 0$ with the following property.*

If B is a unital C^*-algebra, $p \in B$ is a projection, $\phi_t : A \to pBp$, $0 \le t \le 1$ is a continuous path of G-δ multiplicative maps , then there are a positive integer L, and $\eta > 0$ such that for a homomorphism $\lambda : A \to B \otimes \mathcal{K}$ (with finite dimensional image), there is a unitary $u \in B$ satisfying:

$$\|\phi_0(f) \oplus \lambda(f) - u(\phi_1(f) \oplus \lambda(f))u^*\| < \varepsilon, \quad \forall f \in F,$$

provided that λ is of the following form: there are an η-dense subset $\{x_1, x_2, \cdots, x_\bullet\} \subset \coprod_{k=1}^{l} X_k \quad (= SP(A))$, and a set of mutually orthogonal projections $\{p_1, p_2, \cdots, p_\bullet\} \subset \lambda(\bigoplus_k e_{11}^k)(B \otimes \mathcal{K})\lambda(\bigoplus_k e_{11}^k)$ with $[p_i] \ge L \cdot [p]$, such that

$$\lambda(f) = \sum_{i=1}^{\bullet} p_i \otimes f(x_i), \quad \forall f \in A$$

under the identification $\lambda(\mathbf{1}_{A^k})B\lambda(\mathbf{1}_{A^k}) \cong (\lambda(e_{11}^k)B\lambda(e_{11}^k)) \otimes M_{s(k)}(\mathbb{C})$.

Proof: One can apply Lemma 1.6.14 to $\phi_t \in \mathrm{Map}(A, pBp[0,1])$ to reduce the problem to the case of $A = C(X_k)$ which is Corollary 1.6.12. (Here $pBp[0,1]$ is defined to be the C^*-algebra of continuous pBp valued functions on $[0,1]$.) $\square$

For convenience, we introduce the following definitions.

DEFINITION 1.6.16. A homomorphism $\phi : A = \bigoplus_{i=1}^{n} M_{k_i}(C(X_i)) \to B = \bigoplus_{j=1}^{n'} M_{l_j}(C(Y_j))$ is called m-LARGE if for each partial map $\phi^{ij} : A^i = M_{k_i}(C(X_i)) \to B^j = M_{l_j}(C(Y_j))$ of ϕ,

$$\mathrm{rank}(\phi^{ij}(\mathbf{1}_{A^i}) \ge m \cdot \mathrm{rank}(\mathbf{1}_{A^i})(= m \cdot k_i).$$

DEFINITION 1.6.17. Let X be a connected finite simplicial complex, $A = M_k(C(X))$. A unital *-monomorphism $\phi : A \to M_l(A)$ is called a (UNITAL) SIMPLE EMBEDDING if it is homotopic to the homomorphism $\mathrm{id} \oplus \lambda$, where $\lambda : A \to M_{l-1}(A)$ is defined by

$$\lambda(f) = \mathrm{diag}(\underbrace{f(x_0), f(x_0), \cdots, f(x_0)}_{l-1}),$$

for a fixed base point $x_0 \in X$.
Let $A = \bigoplus_{i=1}^{n} M_{k_i}(C(X_i))$, where X_i are connected finite simplicial complexes. A unital *-monomorphism $\phi : A \to M_l(A)$ is called a (unital) simple embedding, if ϕ is of the form $\phi = \oplus\phi^i$ defined by

$$\phi(f_1, f_2, \cdots, f_n) = (\phi^1(f_1), \phi^2(f_2), \cdots, \phi^n(f_n)),$$

where the homomorphisms $\phi^i : A^i(= M_{k_i}(C(X_i))) \to M_l(A^i)$ are unital simple embeddings.

1.6.18. For each connected finite simplicial complex X, there is a three dimensional connected simplicial complex $Y = Y_1 \vee Y_2 \vee \cdots \vee Y_\bullet$ such that $K^*(X) = K^*(Y)$, where Y_i are the following special spaces: $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$ and S^2.
The space $[0,1]$ could be avoided in the construction of Y. But we would like to use the space $[0,1]$ for the following special case: If $K^0(X) = \mathbb{Z}$ and $K^1(X) = 0$ (e.g., X is contractible) and X is not the space of a single point, then we choose $Y = [0,1]$. When X is the space of a single point, choose $Y = \{pt\}$.
The following result is Lemma 2.1 of [D2] (see Lemma 3.13 and Lemma 3.14 of [G4] also).

Lemma 1.6.19. ([D2, 2.1]) *Let $B_1 = \bigoplus_{j=1}^s M_{k(j)}(C(Y_j))$, where Y_j are the following spaces: $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$, and S^2. Let X be a connected finite simplicial complex, let Y be the three dimensional space defined in 1.6.18 with $K^*(X) = K^*(Y)$, and let $A = M_N(C(X))$.*
Let $\alpha_1 : B_1 \to A$ be a homomorphism. For any finite sets $G \subset B_1$ and $F \subset A$, and any number $\delta > 0$, there exists a diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2 \end{array},$$

where
$A' = M_L(A), B_2 = M_S(C(Y))$;
ψ is a homomorphism, α_2 is a unital homomorphism, and ϕ is a unital simple embedding (see 1.6.17);
$\beta \in Map(A, B_2)$ is F-δ multiplicative.
Moreover there exist homotopies $\Psi \in Map(B_1, B_2[0,1])$ and $\Phi \in Map(A, A'[0,1])$ such that Ψ is G-δ multiplicative, Φ is F-δ multiplicative, and

$$\Psi|_1 = \psi, \quad \Psi|_0 = \beta \circ \alpha_1, \quad \Phi|_0 = \alpha_2 \circ \beta \quad \textit{and} \quad \Phi|_1 = \phi.$$

(In the application of this lemma, it is important to require that ϕ is a unital simple embedding. This requirement means that ϕ defines the same element in $kk(X,X)$ (connective KK theory) as the identity map id : $A \to A$. Roughly speaking, this lemma (and Theorem 6.26 below) means that an “identity map” could factor through matrix algebras over Y — a special space of dimension three.)

Proof: If one assumes that $\alpha_1 : B_1 \to A$ is m-large (see 1.6.16) for a number $m > 4\dim(X)$, then this lemma becomes Lemma 2.1 of [D2]. We make use of this special case to prove the general case as below.

Define a unital simple embedding $\lambda: A \to M_m(A)$ ($m > 4\dim(Y)$) by

$$\lambda(f) = \operatorname{diag}(f, \underbrace{f(x_0), f(x_0), \cdots, f(x_0)}_{m-1}).$$

Then $\alpha_1' = \lambda \circ \alpha_1$ is m-large. Apply Lemma 2.1 of [D2]— the special case of the lemma to α_1', $\lambda(F) \subset M_m(A)$ and $G \subset B_1$ to obtain the following diagram

$$\begin{array}{ccc} M_m(A) & \xrightarrow{\phi'} & A' \\ \Big\uparrow{\scriptstyle \alpha_1'} & \overset{\beta'}{\searrow} & \Big\uparrow{\scriptstyle \alpha_2'} \\ B_1 & \xrightarrow{\psi'} & B_2 \end{array}$$

with homotopy paths Ψ' and Φ' with properties described in the lemma for the homomorphism α_1', finite sets $\lambda(F) \subset M_m(A)$ and $G \subset B_1$.
Define $\beta = \beta' \circ \lambda$, $\phi = \phi' \circ \lambda$, $\alpha_2 = \alpha_2'$, $\psi = \psi'$, $\Psi = \Psi'$ and $\Phi = \Phi' \circ \lambda$. Then we have the desired diagram with the desired properties.

$\square$

Remark 1.6.20. From the construction of ϕ and α_2 in the proof of [D2, Lemma 2.1], we know that ϕ and α_2 take trivial projections to trivial projections. But ψ may not take trivial projections to trivial projections unless α_1 does.

1.6.21. Let X and Y be path connected finite simplicial complexes, and $C = M_k(C(Y)), D = M_l(C(X))$. Let $x_0 \in X$ and $y_0 \in Y$ be fixed base points. Then from Lemma 3.14 of [EG2], we have the following: any homomorphism $\phi \in \operatorname{Hom}(C, D)$ is homotopy equivalent to a homomorphism $\phi' \in \operatorname{Hom}(C, D)$ (within $\operatorname{Hom}(C, D)$) such that $\phi'(C^0) \subset D^0$, where C^0, D^0, are the ideals of C and D, respectively, which consist of matrix valued functions vanishing on the base points (see 1.1.7(h)). In other words, there is a unitary $U \in M_l(\mathbb{C})$ such that

$$\phi'(f)(x_0) = U \begin{pmatrix} f(y_0) & & & & & \\ & \ddots & & & & \\ & & f(y_0) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} U^* \in M_l(\mathbb{C}) \quad \forall f \in C.$$

Notice that if a homomorphism α_2 is as desired in Lemma 1.6.19, then any homomorphism, which is homotopic to α_2, is also as desired. Therefore, in Lemma 1.6.19, we can require that the homomorphism $\alpha_2 : B_2 (= M_S(C(Y)) \to A' (= M_L(C(X)))$ is of the above form for certain base points $y_0 \in Y$ and $x_0 \in X$.

In the following, let us explain why we can also choose the homomorphism α_2 to be injective.
If X is the space of a single point, then Y is also the space of a single point by our choice. And therefore, α_2 is injective, since B_2 is simple.
If the connected simplicial complex X is not a single point (i.e., $\dim(X) \geq 1$), then it can be proved that there is a continuous surjective map $g : X \to Y$, using the standard idea of Peano curve. In fact, one can assume that the map g is homotopy trivial—one can make it factor through an interval.
On the other hand, by Theorem 6.4.4 of [DN], if $L \geq 3S(\dim(X)+1)$, then for the unital homomorphism $\alpha_2 : M_S(C(Y)) \to M_L(C(X))$, there is a homomorphism $\alpha' : M_S(C(Y)) \to M_{L-S}(C(X))$ such that α_2 is homotopic to the homomorphism defined by

$$f \mapsto \mathrm{diag}(\alpha'(f), f(y_0)),$$

Then α_2 is homotopic to $\mathrm{diag}(\alpha', g^*)$ defined by

$$f \mapsto \mathrm{diag}(\alpha'(f), f \circ g),$$

since g is homotopy trivial. So we can replace α_2 by $\mathrm{diag}(\alpha', g^*)$ which is injective, since g is surjective.
Similarly, if $\mathrm{SP}(B_2) = Y$ is not a single point space (i.e., X is not the space of a single point), then the homomorphism $\psi : B_1 \to B_2$ could be chosen to be injective with in the same homotopy class of $\mathrm{Hom}(B_1, B_2)$, provided that $\alpha_1(\mathbf{1}_{B_1^i}) \neq 0$, for each block B_1^i of B_1 (later on, we will always assume α_1 satisfies this condition, since otherwise this block can be deleted from B_1).

Lemma 1.6.22. *Let $B = M_k(C(Y))$, $A = M_l(C(X))$. Suppose that a unital homomorphism $\alpha : B \to A$ satisfies $\alpha(B^0) \subset A^0$, and takes any trivial projections of B to trivial projections of A, where $B^0 = M_k(C_0(Y)) := \{f \in M_k(C(Y)) \mid f(y_0) = 0\}$, and $A^0 = M_l(C_0(X)) := \{f \in M_l(C(X)) \mid f(x_0) = 0\}$, for some fixed base points $y_0 \in Y$ and $x_0 \in X$. Let $\beta_0 : B \to M_n(B)$ and $\beta_1 : A \to M_n(A)$ be unital homomorphisms defined by*

$$\beta_0(f)(y) = diag(f(y), f(y_0), \cdots, f(y_0)), \quad \forall f \in B,$$

and

$$\beta_1(f)(x) = diag(f(x), f(x_0), \cdots, f(x_0)), \quad \forall f \in A.$$

Then the following diagram commutes up to unitary equivalence.

$$\begin{array}{ccc} A & \xrightarrow{\beta_1} & M_n(A) \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha\otimes id_n} \\ B & \xrightarrow{\beta_0} & M_n(B). \end{array}$$

I.e., there is a unitary $u \in M_n(A)$ such that

$$\beta_1 \circ \alpha = Adu \circ (\alpha \otimes id_n) \circ \beta_0.$$

Proof: $\beta_1 \circ \alpha$ is defined by:

$$f \mapsto \alpha(f) \mapsto \mathrm{diag}(\alpha(f), \alpha(f)(x_0), \cdots, \alpha(f)(x_0)),$$

and $(\alpha \otimes \mathrm{id}_n) \circ \beta_0$ is defined by:

$$f \mapsto \mathrm{diag}(f, f(y_0), \cdots, f(y_0)) \mapsto \mathrm{diag}(\alpha(f), \alpha(f(y_0)), \cdots, \alpha(f(y_0))),$$

where $\alpha(f(y_0))$ denotes the result of α acting on the constant function $g = f(y_0)$.
On the other hand, from $\alpha(B^0) \subset A^0$, we get

$$\alpha(f)(x_0) \ \sim_u \ diag(\underbrace{f(y_0), \cdots, f(y_0)}_{\frac{l}{k}}),$$

and from the fact that α takes trivial projections to trivial projections, we get

$$diag(\underbrace{f(y_0), \cdots, f(y_0)}_{\frac{l}{k}}) \ \sim_u \ \alpha(f(y_0)),$$

where the symbol $\sim_u$ means to be unitarily equivalent.

$\square$

The following result is from [EG2] (see 5.10, 5.11 of [EG2]).

Lemma 1.6.23. *Let $Y = Y_1 \vee Y_2 \vee \cdots \vee Y_m$. If n is large enough, then any unital homomorphism $\beta : M_k(C(Y)) \to M_{nk}(C(Y))$ is homotopic to a homomorphism $\beta' : M_k(C(Y)) \to M_{nk}(C(Y))$, which factors through $\bigoplus_{i=1}^m M_{k_i}(C(Y_i))$, for certain integers $\{k_i\}$, as*

$$\beta' : \ M_k(C(Y)) \xrightarrow{\beta_1} \bigoplus_{i=1}^m M_{k_i}(C(Y_i)) \xrightarrow{\beta_2} M_{nk}(C(Y)).$$

Furthermore, β_1 and β_2 above can be chosen to be injective.

Proof: If $k = 1$, then the lemma is Lemma 5.11 of [EG2]. (Notice that, we choose both spaces X and Y in Lemma 5.11 of [EG2] to be the above space Y. In addition, the spaces X_i in Lemma 5.11 of [EG2] could be chosen to be spaces Y_i in our case, according to 5.10 of [EG2].)
For the general case, one writes β as $b \otimes \mathrm{id}_k$, where $b = \beta|_{e_{11}M_k(C(Y))e_{11}} : C(Y) \to \beta(e_{11})M_{nk}(C(Y))\beta(e_{11})$, then apply Lemma 5.11 of [EG2] to b.
Furthermore, one can make β_1 and β_2 injective in the same way as in the end of 1.6.21. (Or one observes that the maps β_1 and β_2 constructed in Lemma 5.11 of [EG2] are already injective for our case.)

$\square$

Combining Lemmas 1.6.19, 1.6.21, 1.6.22 and 1.6.23, we have the following Lemma:

Lemma 1.6.24. *Let* $B_1 = \bigoplus_{j=1}^{s} M_{k(j)}(C(Y_j))$, *where* Y_j *are spaces:* $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, $\{T_{III,k}\}_{k=2}^{\infty}$, *and* S^2. *Let* X *be a connected finite simplicial complex and let* $A = M_N(C(X))$.
Let $\alpha_1 : B_1 \to A$ *be a homomorphism with* $\alpha_1(\mathbf{1}_{B_1^i}) \neq 0$ *for each block* B_1^i *of* B_1. *For any finite sets* $G \subset B_1$ *and* $F \subset A$, *and any number* $\delta > 0$, *there exists a diagram*

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2 \end{array},$$

where
$A' = M_L(A)$, *and* B_2 *is a direct sum of matrix algebras over the spaces:* $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, $\{T_{III,k}\}_{k=2}^{\infty}$, *and* S^2;
ψ *is a homomorphism,* α_2 *is a unital injective homomorphism, and* ϕ *is a unital simple embedding (see 1.6.17).*
$\beta \in Map(A, B_2)$ *is* F-δ *multiplicative.*
Moreover, there exist homotopies $\Psi \in Map(B_1, B_2[0,1])$ *and* $\Phi \in Map(A, A'[0,1])$ *such that* Ψ *is* G-δ *multiplicative,* Φ *is* F-δ *multiplicative, and*

$$\Psi|_1 = \psi, \quad \Psi|_0 = \beta \circ \alpha_1, \quad \Phi|_0 = \alpha_2 \circ \beta \quad \text{and} \quad \Phi|_1 = \phi.$$

Furthermore, if X *is not the space of a single point, then at least one of the blocks of* B_2 *has spectrum different from the space of single point and* ψ *can be chosen to be injective.*

Proof: Let

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2 \end{array},$$

be the diagram described in Lemma 1.6.19 with homotopies Φ and Ψ. Let n be the integer obtained by applying Lemma 1.6.23 to $B_2 = M_k(C(Y))$. Then apply 1.6.22 to $\alpha_2 : B_2 \to A'$ to obtain a diagram

$$\begin{array}{ccc} A' & \xrightarrow{\beta_1} & M_n(A') \\ {\scriptstyle \alpha_2}\uparrow & & \uparrow{\scriptstyle \alpha_2 \otimes \mathrm{id}_n} \\ B_2 & \xrightarrow{\beta_0} & M_n(B_2). \end{array}$$

which commutes up to homotopy. (Here we have the condition that α_2 takes trivial projections to trivial projections from Remark 1.6.20. Also, α_2 is homotopic to a homomorphism which takes B^0 to A'^0.)

Furthermore, from Lemma 1.6.23, β_0 is homotopic to a homomorphism β_0' factoring through a C^*-algebra $newB_2$ which is a direct sum of matrix algebras over spaces $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, $\{T_{III,k}\}_{k=2}^{\infty}$, and S^2. Now it is routine to finish the construction of the diagram. We omit the details.

$\square$

1.6.25. Our next task is to add a homomorphism $\lambda : A \to M_n(B_2)$ into the diagram in Lemma 1.6.24 to obtain diagrams,

$$\begin{array}{ccc} A & & \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta\oplus\lambda} & \\ B_1 & \xrightarrow{\psi\oplus(\lambda\circ\alpha_1)} & M_{n+1}(B_2) \end{array}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\phi\oplus((\alpha_2\otimes id_n)\circ\lambda)} & M_{n+1}(A') \\ & \searrow^{\beta\oplus\lambda} & \uparrow{\scriptstyle \alpha_2\otimes id_{n+1}} \\ & & M_{n+1}(B_2) \end{array}$$

which are almost commutative up to unitary equivalence, using Corollary 1.6.15.

To do so, we make the following assumption.

Assumption: $\alpha_1 : B_1 \to A$ is injective.

Let $G_1 \subset B_1$ and $F_1 \subset A$ be any finite sets, and $\varepsilon > 0$. We will make the above diagrams approximately commute on G_1 and F_1, respectively, to within ε, up to unitary equivalence.

Apply Corollary 1.6.15 to $G_1 \subset B_1$ (in place of $F \subset A$) and $\varepsilon > 0$, to obtain $G \subset B_1$ and δ_1 (in place of the set G and the number δ, respectively in Corollary 1.6.15). Similarly, apply Corollary 1.6.15 to $F_1 \subset A$ (in place of $F \subset A$) and $\varepsilon > 0$ to obtain $F \subset A$ and δ_2 (in place of the set G and the number δ in Corollary 1.6.15).

Let $\delta = \min(\delta_1, \delta_2)$.

Suppose that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

is the one constructed in Lemma 1.6.24 with homotopy path $\Psi \in \mathrm{Map}(B_1, B_2[0,1])$ between $\beta \circ \alpha_1$ and ψ, and homotopy path $\Phi \in \mathrm{Map}(A, A'[0,1])$ between $\alpha_2 \circ \beta$ and ϕ, corresponding to the sets $G \subset B_1$, $F \subset A$ and the number $\delta > 0$.

Regarding the homotopy path Ψ as the homotopy path ϕ_t in Corollary 1.6.15, we can obtain η_1, L_1 as the numbers η and L in Corollary 1.6,15. Similarly, replacing the above Ψ by Φ, we obtain η_2, L_2.

Notice that the injectivity of α_1 implies that, for each block B_1^j of B_1, $\mathrm{SP}(\alpha_1^j) = Y_j (= \mathrm{SP}(B_1^j))$. Therefore there is an η_2-dense subset $\{x_1, x_2, \cdots x_m\}$ of X such that $\bigcup_{i=1}^m \mathrm{SP}\alpha_1^j|_{x_i}$ is η_1-dense in Y_j for each $j \in \{1, 2, \cdots, s\}$.
Define $\lambda_1 : A (= M_N(C(X))) \to M_{mN}(B_2)$ by

$$\lambda_1(f) = \mathrm{diag}(\mathbf{1}_{B_2} \otimes f(x_1), \mathbf{1}_{B_2} \otimes f(x_2), \cdots, \mathbf{1}_{B_2} \otimes f(x_m)).$$

Then $\lambda_1 \circ \alpha_1 : B_1 \to M_{mN}(B_2)$ is a homomorphism defined by the point evaluations on the η_1-dense subset $\bigcup_{j=1}^s \bigcup_{i=1}^m \mathrm{SP}\alpha_1^j|_{x_i} \subset \mathrm{SP}B_1$. Also $(\alpha_2 \otimes \mathrm{id}_{mN}) \circ \lambda_1 : A \to M_{mN}(A')$ is defined by point evaluations on the η_2-dense subset $\{x_j\}_{j=1}^m \subset X$.
Let $L = \max(L_1, L_2)$ and $n = mNL$. Define $\lambda : A \to M_n(B_2) = M_L(M_{mN}(B_2))$ by $\lambda = \mathrm{diag}(\underbrace{\lambda_1, \lambda_1, \cdots, \lambda_1}_{L})$.
Then, obviously, $\lambda \circ \alpha_1 : B_1 \to M_n(B_2)$ satisfies the condition for λ in Corollary 1.6.15, for the homotopy Ψ, positive integer L_1, and $\eta_1 > 0$. And so does $(\alpha_2 \otimes \mathrm{id}_n) \circ \lambda : A \to M_n(A')$ for Φ, L_2 and η_2.
Therefore, there are unitaries $u_1 \in M_{n+1}(B_2)$ and $u_2 \in M_{n+1}(A')$ such that

$$\|((\beta \oplus \lambda) \circ \alpha_1)(f) - u_1(\psi \oplus (\lambda \circ \alpha_1))(f)u_1^*\| < \varepsilon, \quad \forall f \in G_1,$$

$$\|(\phi \oplus ((\alpha_2 \otimes \mathrm{id}_n) \circ \lambda))(f) - u_2((\alpha_2 \otimes id_{n+1}) \circ (\beta \oplus \lambda))(f)u_2^*\| < \varepsilon, \quad \forall f \in F_1.$$

In the diagram in Lemma 1.6.24, if we replace B_2 by $M_{n+1}(B_2)$, A' by $M_{n+1}(A')$, ψ by $\mathrm{Ad}u_1 \circ (\psi \oplus (\lambda \circ \alpha_1))$, β by $\beta \oplus \lambda$, α_2 by $\mathrm{Ad}u_2 \circ (\alpha_2 \otimes id_{n+1})$, and finally ϕ by $\phi \oplus ((\alpha_2 \otimes \mathrm{id}_n) \circ \lambda)$, then we have the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] & A' \\ B_1 \arrow[u, "\alpha_1"] \arrow[r, "\psi"] & B_2 \arrow[u, "\alpha_2"'] \arrow[from=1-1, to=2-2, "\beta"] \end{tikzcd}$$

for which, the lower left triangle is approximately commutative on G_1 to within ε and the upper right triangle is approximately commutative on F_1 to within ε. Since G_1 and F_1 are arbitrary finite subsets, we proved the following main factorization result.

Theorem 1.6.26. *Let $B_1 = \bigoplus_{j=1}^s M_{k(j)}(C(Y_j))$, where Y_j are spaces: $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$, and S^2. Let X be a connected finite simplicial complex and let $A = M_N(C(X))$.*
Let $\alpha_1 : B_1 \to A$ be an injective homomorphism. For any finite sets $G \subset B_1$ and $F \subset A$, and for any numbers $\varepsilon > 0$ and $\delta > 0$ there exists a diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] & A' \\ B_1 \arrow[u, "\alpha_1"] \arrow[r, "\psi"] & B_2 , \arrow[u, "\alpha_2"'] \arrow[from=1-1, to=2-2, "\beta"] \end{tikzcd}$$

where
$A' = M_L(A)$, *and* B_2 *is a direct sum of matrix algebras over the spaces:* $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^{\infty}$, $\{T_{III,k}\}_{k=2}^{\infty}$, *and* S^2*;*
ψ *is an injective homomorphism,* α_2 *is a unital injective homomorphism, and* ϕ *is a unital simple embedding (see 1.6.17).*
$\beta \in Map(A, B_2)$ *is* F*-*δ *multiplicative.*
Moreover

$$\|\psi(f) - (\beta \circ \alpha_1)(f)\| < \varepsilon, \quad \forall f \in G;$$

$$\|\phi(f) - (\alpha_2 \circ \beta)(f)\| < \varepsilon, \quad \forall f \in F.$$

Corollary 1.6.27. *Theorem 1.6.26 still holds if one replaces the injectivity condition of* α_1 *by the following condition:*
For each block B_1^j *of* B_1*, either* α_1^j *is injective or* $\alpha_1^j(B_1^j)$ *is a finite dimensional subalgebra of* A*.*
(Of course, one still needs to assume that $\alpha_1(\mathbf{1}_{B_1^j}) \neq 0$ *for each block* B_1^j *of* B_1 *and that at least one block of* B_2 *has spectrum different from the space of single point (equivalently,* $X \neq \{pt\}$*) , if he wants the homomorphism* ψ *to be injective.*
If one does not assume the above condition, he could still get the following dichotomy condition for ψ*: For each block* B_1^j *of* B_1 *and* B_2^k *of* B_2*, either* $\psi^{j,k}$ *is injective or* $\psi^{j,k}$ *has a finite dimensional image.)*

Proof: Write $B_1 = B' \oplus B''$ such that α_1 is injective on B' and $\alpha_1(B'') \subset A$ is of finite dimension.
Consider the finite dimensional algebra

$$D := \bigoplus_{B_1^i \subset B'} (\alpha_1(\mathbf{1}_{B_1^i}) \cdot \mathbb{C}) \bigoplus \alpha_1(B'') \quad \subset \ A.$$

By Lemma 1.6.8, if $\beta : A \to B_2$ is sufficiently multiplicative, then β is close to such a map β' that the restriction $\beta'|_D$ is a homomorphism. β' can be connected to the original β by a linear path. If the original map β is sufficiently multiplicative, then the connecting path, regarded as a map from A to $B_2[0,1]$, is F-δ multiplicative for any pre-given finite set $F \subset A$ and number $\delta > 0$. Therefore, with out loss of generality, we assume that $\beta|_D$ is a homomorphism for the original map β in 1.6.24.
By Lemma 1.6.8 again, if $\Psi : \ B_1 \to B_2[0,1]$ is sufficiently multiplicative, then Ψ is close to a map Ψ' such that $\Psi'|_{r(B_1)}$ is a homomorphism, where $r(B_1)$ is defined in 1.1.7(h). Note that $\Psi|_1 = \psi$ is a homomorphism and $(\Psi|_0)|_{r(B_1)} = \beta|_D \circ (\alpha_1|_{r(B_1)})$ is also a homomorphism. From the proof of Lemma 1.6.8, we can see that the above Ψ' can be chosen such that $\Psi'|_1 = \Psi|_1$ and $\Psi'|_0 = \Psi|_0$. Therefore, without loss of generality, we can assume that the homotopy path Ψ in Lemma 1.6.24 satisfies that $\Psi|_{r(B_1)}$ is a homomorphism.

Up to a unitary equivalence, we can further assume that $\Psi_t(\mathbf{1}_{B_1^i}) = \Psi_{t'}(\mathbf{1}_{B_1^i})$ for any $t, t' \in [0,1]$ and any block B_1^i of B_1.
One can repeat the procedure in 1.6.25 to construct the homomorphism $\lambda: A \to M_n(A)$, defined by point evaluations on an η_2-dense subset $\{x_1, x_2, \cdots, x_m\} \subset X$, to satisfy the condition that $\lambda \circ \alpha_1^j$ is defined by point evaluations on an η_1-dense subset $\bigcup_{i=1}^m \mathrm{SP}\alpha_1^j|_{x_i} \subset \mathrm{SP}B_1^j$ of sufficiently large size, for each block B_1^j of the part B'. As in 1.6.25, we can define newβ to be $\beta \oplus \lambda$. At the same time, ϕ and α_2 can also be defined as in 1.6.25. To define ψ, we need to consider two cases. For the blocks B_1^j in B', ψ can be defined as in 1.6.25, since $\lambda \circ \alpha_1^j$ is defined by point evaluations on an η_1-dense subset (of sufficiently large size). For the blocks $B_1^j \subset B''$, we define ψ to be $(\beta \oplus \lambda) \circ \alpha_1^j = (\text{new}\beta) \circ \alpha_1^j$. (Note that $\beta|_{\alpha_1(B'')}$ is a homomorphism.)

□

Remark 1.6.28. Once the diagram in Theorem 1.6.26 (or Corollary 1.6.27) exists for $A' = M_L(A)$, then for any $L' > L$, one can construct a diagram with the same property as in the theorem or the corollary with $A' = M_{L'}(A)$. This is easily seen from the following.
Let $r(A) = M_N(\mathbb{C})$ and $r: A \to r(A)$ be as in 1.1.7(h). Let newB_2 = old$B_2 \oplus r(A)$, newβ = old$\beta \oplus r$, newϕ = $\mathrm{diag}(\text{old}\phi, \underbrace{i \circ r, \cdots, i \circ r}_{L'-L})$, new$\psi$ = old$\psi \oplus (r \circ \alpha_1)$, and new$\alpha_2$ = old$\alpha_2 \oplus \mathrm{diag}(\underbrace{i, \cdots, i}_{L'-L})$, where $i: r(A) \to A \subset M_{L'-L}(A) \subset M_{L'}(A)$ is the inclusion (note that $r(A)$ is a subalgebra of A as in 1.1.7(h)), and oldB_2, oldβ, oldϕ and oldα_2 are B_2, β, ϕ and α_2, respectively, from Lemma 1.6.26 or Corollary 1.6.27.

Corollary 1.6.29. *Let $B_1 = \bigoplus_{j=1}^s M_{k(j)}(C(Y_j))$, where Y_j are spaces: $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$, and S^2. Let $A = \bigoplus_{j=1}^t M_{l(j)}(C(X_j))$, where X_j are connected finite simplicial complex.*
Let $\alpha_1: B_1 \to A$ be a homomorphism satisfying the following condition:
For each pair of blocks B_1^i of B_1 and A^j of A, either the partial map $\alpha_1^{i,j}$ is injective or $\alpha_1^{i,j}(B_1^i)$ is a finite dimensional subalgebra of A^j.
For any finite sets $G \subset B_1$ and $F \subset A$, and for any numbers $\varepsilon > 0$ and $\delta > 0$, there exists a diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2 \end{array}$$

where
$A' = M_L(A)$, and B_2 is a direct sum of matrix algebras over the spaces: $\{pt\}$, $[0,1]$, S^1, $\{T_{II,k}\}_{k=2}^\infty$, $\{T_{III,k}\}_{k=2}^\infty$, and S^2;

ψ is a homomorphism, α_2 is a unital injective homomorphism, and ϕ is a unital simple embedding (see 1.6.17).
$\beta \in Map(A, B_2)$ is F-δ multiplicative.
Moreover,

$$\|\psi(f) - (\beta \circ \alpha_1)(f)\| < \varepsilon, \quad \forall f \in G;$$

$$\|\phi(f) - (\alpha_2 \circ \beta)(f)\| < \varepsilon, \quad \forall f \in F.$$

If we further assume that α_1 satisfies the condition that $\alpha_1^{i,j}(\mathbf{1}_{B_1^i}) \neq 0 \in A^j$ for any partial map $\alpha_1^{i,j} : B_1^i \to A^j$ of α_1, then either the homomorphism ψ could be chosen to be injective, or the spectra of all blocks of B_2 could be chosen to be the spaces of a single point.

Proof: We can construct the diagram for each block A^j of A, then put them together in the obvious way. Using Remark 1.6.28, we can assume for each block A^j, $A'_j = M_L(A_j)$ for the same L.

$\square$

The following is Lemma 4.6 of [G4] (see Lemma 1.2 of [D2]).

LEMMA 1.6.30. *Let $A = \bigoplus_{i=1}^{t} M_{l(i)}(C(X_i))$, where X_i are connected finite simplicial complexes. Let $A' = M_L(A)$. Let the algebra $r(A)$ and the homomorphism $r : A \to r(A)$ be as in 1.1.7(h). Let B be a direct sum of matrix algebras over finite simplicial complexes of dimension at most m. Let $\phi : A \to A'$ be a unital simple embedding (see Definition 1.6.17). For any (not necessarily unital) $(m \cdot L)$-large homomorphism $\phi' : A \to B$, there is a homomorphism $\lambda : A' \oplus r(A) \to B$ such that ϕ' is homotopic to $\lambda \circ (\phi \oplus r)$.*
Furthermore, λ could be chosen to satisfy the condition that for any block B^j with $SP(B^j) \neq \{pt\}$, the partial map $\lambda^{\cdot,j} : A' \oplus r(A) \to B^j$ of λ is injective as remarked in 1.6.21.
(This lemma will be applied in conjunction with Lemma 1.6.26 or Corollary 1.6.29. From here, one can see the importance of the requirement that ϕ is a unital simple embedding.)

REMARK 1.6.31. In order to apply Lemma 1.6.30 later, we would like to do one more modification for Corollary 1.6.29. Let $r : A \to r(A)$ be as in 1.1.7(h). Then the diagram in Corollary 1.6.29 could be modified to the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi \oplus r} & A' \oplus r(A) \\ \uparrow{\scriptstyle \alpha_1} & \searrow^{\beta \oplus r} & \uparrow{\scriptstyle \alpha_2 \oplus id} \\ B_1 & \xrightarrow{\psi \oplus (r \circ \alpha_1)} & B_2 \oplus r(A) \end{array}$$

which satisfies that $\beta \oplus r$ is F-δ multiplicative and

$$\|(\psi \oplus (r \circ \alpha_1))(f) - ((\beta \oplus r) \circ \alpha_1)(f)\| < \varepsilon, \quad \forall f \in G;$$

$$\|(\phi \oplus r)(f) - ((\alpha_2 \oplus \mathrm{id}) \circ (\beta \oplus r))(f)\| < \varepsilon, \quad \forall f \in F.$$

In the application of 1.6.29 and 1.6.30 in the proof of our main reduction theorem, we will still denote $B_2 \oplus r(A)$ by B_2, $\beta \oplus r$ by β, $\psi \oplus (r \circ \alpha_1)$ by ψ, and $\alpha_2 \oplus \mathrm{id}$ by α_2. So the diagram is

$$\begin{array}{ccc} A & \xrightarrow{\phi \oplus r} & A' \oplus r(A) \\ \Big\uparrow{\scriptstyle \alpha_1} & \searrow^{\beta} & \Big\uparrow{\scriptstyle \alpha_2} \\ B_1 & \xrightarrow{\psi} & B_2\ . \end{array}$$

2 Spectral Multiplicity

In this section, we will show how to perturb a homomorphism $\phi : C(X) \to PM_k(C(Y))P$ in such a way that the resulting homomorphism does not have large spectral multiplicities (see 1.2.4). Namely, the following result will be proved.

Theorem 2.1. *Let X and Y be connected simplicial complexes with $X \neq \{pt\}$. Set $\dim(X) + \dim(Y) = l$. For any given finite set $G \subset C(X)$, any $\varepsilon > 0$, and any unital homomorphism $\phi : C(X) \to PM_\bullet(C(Y))P$, where $P \in M_\bullet(C(Y))$ is a projection, there is a unital homomorphism $\phi' : C(X) \to PM_\bullet(C(Y))P$ such that*
(1) $\|\phi(g) - \phi'(g)\| < \varepsilon$ for all $g \in G$, and
(2) ϕ' has maximum spectral multiplicity at most l.

2.2. Let k be a positive integer. Let $\mathrm{Hom}(C(X), M_k(\mathbb{C}))_1 = F^k X$. The space $F^k X$ is compact and metrizable. We can endow the space $F^k X$ with a fixed metric d as below.
Choose a finite set $\{f_i\}_{i=1}^n \subset C(X)$ which generates $C(X)$ as a C^*-algebra (e.g. one can embed X into $\mathbb{R}^n$, then choose $\{f_i\}$ to be the coordinate functions). For any $\phi, \psi \in F^k X$ which, by definition, are unital homomorphisms from $C(X)$ to $M_k(\mathbb{C})$, define

$$d(\phi, \psi) = \sum_{i=1}^{n} \|\phi(f_i) - \psi(f_i)\|.$$

Without loss of generality, we can assume that the above finite set $\{f_i\}_{i=1}^n \subset G$. On the other hand, $F^k X$ is a finite simplicial complex (see [DN], [Se] and [Bl1]).

2.3. Let $k = \mathrm{rank}(P)$, where P is the projection in Theorem 2.1.
For any fixed y, there is a unitary $u_y \in M_\bullet(\mathbb{C})$ such that $P(y) = u_y diag(\mathbf{1}_k, 0) u_y^*$ (as in 1.2.1). Using this unitary, one can identify $P(y)M_\bullet(\mathbb{C})P(y)$ with $M_k(\mathbb{C})$ by sending $a \in P(y)M_\bullet(\mathbb{C})P(y)$ to the element in $M_k(\mathbb{C})$ corresponding to the upper left corner of $u_y^* a u_y$. (Notice that for any $a \in P(y)M_\bullet(\mathbb{C})P(y)$, the matrix

$$u_y^* a u_y = u_y^* P(y) a P(y) u_y = u_y^* P(y) u_y u_y^* a u_y u_y^* P(y) u_y$$

$$= diag(\mathbf{1}_k, 0)u_y^* a u_y diag(\mathbf{1}_k, 0)$$

has the form

$$\begin{pmatrix} (*)_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}.)$$

In this way, for any $y \in Y$, the space $\mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$ can be identified with F^kX.
Consider the disjoint union

$$\bigcup_{y\in Y} \mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$$

as a subspace of $\mathrm{Hom}(C(X), M_\bullet(\mathbb{C}))\times Y$ with the induced topology. Using the above identification we can define a locally trivial fibre bundle

$$\begin{array}{c} \bigcup_{y\in Y} \mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1 \\ \downarrow \pi \\ Y \end{array}$$

with fibre F^kX, as shown below, where π is the natural map sending any element in the set $\mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$ to the point y.
For simplicity, write $E_P := \bigcup_{y\in Y} \mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$.
For any point $y_0 \in Y$, there are an open set $U \ni y$, and a continuous unitary valued function $u : U \to M_\bullet(\mathbb{C})$ such that $P(y) = u(y)diag(\mathbf{1}_k, 0)u^*(y)$. (See 1.2.1.) Let $R : M_\bullet(\mathbb{C}) \to M_k(\mathbb{C})$ be the map taking any element in $M_\bullet(\mathbb{C})$ to the $k\times k$ upper left corner of the element. Let the trivialization $t_U : \pi^{-1}(U) \to U \times F^kX$ be defined as follows. For any $\phi \in \mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1 \subset \pi^{-1}(U)$, where $y \in U$, define $t_u(\phi) = (y, \psi)$, where $\psi \in F^kX$ is defined by

$$\psi(f) = R(u^*(y)\phi(f)u(y)) \quad \text{for any} \quad f \in C(X).$$

(Again, $u^*(y)\phi(f)u(y)$ is of the form $\begin{pmatrix} (*)_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}$.)
Since the set Y is compact, there is a finite cover $\mathcal{U} = \{U\}$ of Y with the above trivialization for each U. This defines a fibre bundle $\pi : E_P \to Y$.
(See §1.4 for the definition and other materials of fibre bundle.)

2.4. In the above fibre bundle, the structure group $\Gamma \subset Homeo(F^kX)$ could be chosen to be the collection of all $\gamma \in Homeo(F^kX)$ of the form: there is a unitary $u \in M_k(\mathbb{C})$ such that

$$\gamma(\phi)(f) = u^*\phi(f)u \quad \text{for any} \quad \phi \in F^kX \text{ and } f \in C(X).$$

One can see this as follows.

Suppose that U and V are two open sets in $\mathcal{U}$, and t_U and t_V are trivializations, as in 2.3, defined by unitary valued functions $u : U \to M_\bullet(\mathbb{C})$ and $v : V \to M_\bullet(\mathbb{C})$, respectively.
For any point $y \in U \cap V$, the map $t_U \circ t_V^{-1} : F^kX \to F^kX$, can be computed as below.
For any $\phi \in F^kX$, define $\tilde{\phi} : C(X) \to M_\bullet(\mathbb{C})$ by

$$\tilde{\phi}(f) = \begin{pmatrix} \phi(f)_{k\times k} & 0 \\ 0 & 0 \end{pmatrix}, \forall f \in C(X).$$

Then

$$t_V^{-1}(\phi)(f) = v(y)\tilde{\phi}(f)v^*(y) \in P(y)M_\bullet(\mathbb{C})P(y),$$

and

$$t_U \circ t_V^{-1}(\phi)(f) = R(u^*(y)v(y)\tilde{\phi}(f)v^*(y)u(y)).$$

Notice that

$$u(y)diag(\mathbf{1}_k, 0)u^*(y) = P(y) = v(y)diag(\mathbf{1}_k, 0)v^*(y).$$

It follows that $v^*(y)u(y)$ commutes with $diag(\mathbf{1}_k, 0)$. This implies that this matrix has the form

$$\begin{pmatrix} (w_1)_{k\times k} & 0 \\ 0 & w_2 \end{pmatrix},$$

where both w_1 and w_2 are unitaries. This shows that

$$t_U \circ t_V^{-1}(\phi)(f) = w_1^*\phi(f)w_1, \quad \forall \phi \in F^kX, f \in C(X).$$

In other words, $t_U \circ t_V^{-1} \in \Gamma$.
Obviously, $\mathrm{Hom}(C(X), PM_\bullet(C(Y))P)_1$ can be regarded as a collection of continuous cross sections of the bundle $\pi : E_P \to Y$.
Since for any elements $a, b \in M_k(\mathbb{C})$ and unitary $u \in M_k(\mathbb{C})$,

$$\|uau^* - ubu^*\| = \|a - b\|,$$

it is easy to see that THE METRIC d ON F^kX defined in 2.2 is INVARIANT UNDER THE ACTION OF ANY ELEMENT IN Γ in the sense of 1.4.6.

2.5. There is a natural map

$$\theta : F^kX \longrightarrow P^kX$$

defined as follows. For any $\phi \in F^kX$ given by $\phi : C(X) \to M_k(\mathbb{C})$, define

$$\theta(\phi) = \mathrm{SP}(\phi) \in P^kX,$$

counting multiplicities. (See 1.2.5 and 1.2.7.)
For each point $x = [x_1, x_2, \cdots, x_k] \in P^kX$, if the element x_i appears μ_i times in x for $i = 1, 2, \cdots, k$, then the MAXIMUM MULTIPLICITY of x is defined to be

the maximum of $\mu_1, \mu_2, \cdots, \mu_k$. The maximum multiplicity of a point $\phi \in F^k X$ is defined to be the maximum multiplicity of $\theta(\phi) \in P^k X$, which agrees with the maximum multiplicity of homomorphism $\phi : C(X) \to M_k(\mathbb{C})$ defined in 1.2.4.

The homomorphism $\phi \in \mathrm{Hom}(C(X), PM_\bullet(C(Y))P)_1$ corresponds to a continuous cross section $f : Y \to E_P$. This correspondence is one to one. For any cross section $f : Y \to E_P$, any point $y \in Y$, the maximum multiplicity of $f(y)$ is understood to be that obtained by regarding $f(y)$ as an element in $F^k X$ by an identification of $\mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$ with $F^k X$. Note that the maximum multiplicity of an element $\phi \in F^k X$ is invariant under the action of any element of Γ.

2.6. It is easy to see that for any finite set $G \subset C(X)$ and $\varepsilon > 0$, there is an $\varepsilon' > 0$ such that if $d(\phi_y, \phi'_y) < \varepsilon'$ for any $y \in Y$, then $\|\phi(g) - \phi'(g)\| < \varepsilon$ for any $g \in G$, where $\phi_y, \phi'_y \in F^k X$ are determined by an identification of $\mathrm{Hom}(C(X), P(y)M_\bullet(\mathbb{C})P(y))_1$ with $F^k X$, as above. (Again the choice of the identification is not important, because the metric d is invariant under the action of any element in Γ.)

Before proving Theorem 2.1, we prove the following weak version of Theorem 2.1.

Lemma 2.7. *Let X and Y be as in Theorem 2.1. Let $k > l = \dim(X)+\dim(Y)$. For any given finite set $G \subset C(X)$, any $\varepsilon > 0$, and any unital homomorphism $\phi : C(X) \to PM_\bullet(C(Y))P$, where $P \in M_\bullet(C(Y))$ is a projection with rank$(P) = k$, there is a unital homomorphism $\phi' : C(X) \to PM_\bullet(C(Y))P$ such that*

(1) $\|\phi(g) - \phi'(g)\| < \varepsilon$ for all $g \in G$, and

(2) ϕ' has maximum spectral multiplicity at most $k - 1$.

Comparing with Theorem 2.1, in the above result, we allow the maximum spectral multiplicity of the resulting homomorphism to be larger than l— only require it to be smaller than $k = \mathrm{rank}(P)$—the maximum possible multiplicity. Since we assume that all the generators f_i of $C(X)$ are inside the set G, Lemma 2.7 is equivalent to the following theorem.

Lemma 2.8. *Suppose that X, Y, and P are as in Theorem 2.1 and that rank$(P) = k > \dim(X) + \dim(Y)$. For any $\varepsilon > 0$ and any cross section $f : Y \to E_P$, there is a cross section $f' : Y \to E_P$ such that*

(1) $d(f(y), f'(y)) < \varepsilon$ for all $y \in Y$, and

(2) $f'(y)$ has multiplicity at most $k - 1$ for all $y \in Y$.

To prove our main theorem of this section—Theorem 2.1, we need the following result. The proof of this result will be given after the proof of Theorem 2.1.

Theorem 2.9. *Suppose that X is a connected simplicial complex and $X \neq \{pt\}$. For any $\varepsilon > 0$ and any $x \in F^m X$, there is a contractible open neighbor-*

hood $U_x \ni x$ with $U_x \subset B_\varepsilon(x) \subset F^m X$ such that

$$\pi_i(U_x \backslash \{x\}) = 0$$

for any $0 \le i \le m-2$. In other words, $F^m X$ has property $D(m)$ as in 1.4.3.
We will use Theorem 2.9 and Corollary 1.4.10 (see also Theorem 1.4.9) to prove the following relative version of Lemma 2.8 (which gives rise to Lemma 2.8, by taking $Y_1 = \emptyset$).

Lemma 2.10. *Let X, Y, and P be as in Theorem 2.1, and $Y_1 \subset Y$ be a sub simplicial complex. Suppose that rank$(P) = k > \dim(X) + \dim(Y)$. Suppose that a cross section $f : Y \to E_P$ satisfies the condition that $f(y)$ has multiplicity at most $k-1$, for any $y \in Y_1$. It follows that for any $\varepsilon > 0$, there is a cross section $f' : Y \to E_P$ such that*
(1) $d(f(y), f'(y)) < \varepsilon$ for all $y \in Y$, and
(2) $f'(y)$ has multiplicity at most $k-1$ for all $y \in Y$.
(3) $f'(y) = f(y)$, for any $y \in Y_1$.

Proof: Let $F_1 \subset F^k X$ denote the subset of all elements of maximum multiplicity equal to k. In other words, a homomorphism in F_1 has one dimensional range. Obviously, F_1 is the set of all homomorphisms $\phi \in Hom(C(X), M_k(\mathbb{C}))_1$ which are of the form

$$\phi(f) = \begin{pmatrix} f(x) & & & \\ & f(x) & & \\ & & \ddots & \\ & & & f(x) \end{pmatrix}$$

for a certain point $x \in X$. Hence F_1 is homeomorphic to X, and $\dim(F_1) = \dim(X)$.
As mentioned in 2.5, the maximum multiplicity of an element of $F^k X$ is invariant under the action of Γ. So F_1 is an invariant subset under the action of Γ.
The conclusion of the Lemma 2.10 follows from Corollary 1.4.10 with $E_P \to Y$, $F^k X$, F_1, Y_1 and k in place of $M \to X$, F, F_1 , X_1 and m in Corollary 1.4.10, respectively. (Note that, from Theorem 2.9, $F^k X$ has property $D(k)$.)

$\square$

The above lemma is equivalent to the following lemma (we stated it with projection Q instead of P to emphasis that we may use projections other than P—we will use subprojections of P).

Lemma 2.11. *Let $Y_2 \subset Y_1 \subset Y$ be sub-simplicial complexes of Y. Let $Q \in M_\bullet(C(Y_1))$ be a projection with rank $m > l = \dim(X) + \dim(Y)$. For any given finite set $G \subset C(X)$, any $\varepsilon > 0$, and any unital homomorphism $\psi : C(X) \to QM_\bullet(C(Y_1))Q$ with the property that for any $y \in Y_2$, the multiplicity of ψ at y*

is at most $m-1$*, there is a unital homomorphism* $\psi' : C(X) \to QM_\bullet(C(Y_1))Q$ *such that*
(1) $\|\psi(g)(y) - \psi'(g)(y)\| < \varepsilon$ *for all* $g \in G$ *and* $y \in Y_1$*,*
(2) ψ' *has spectral multiplicity at most* $m-1$*,*
(3) $\psi'|_{Y_2} = \psi|_{Y_2}$*.*
In the proof of Theorem 2.1, we will not use Theorem 2.9 or Lemma 2.10 directly. We will use Lemma 2.11 instead. (So we do not need anything from fibre bundles in the rest of the proof of Theorem 2.1.)

2.12 Sketch of the idea of the proof of Theorem 2.1. Note that the proof of Lemma 2.10 can not be used to prove Theorem 2.1 (or the fibre bundle version of Theorem 2.1) in a straightforward way. For example, if we let $F_1 \subset F^kX$ be the subset of all elements with maximum multiplicity at least $l+1$ (instead of k), then $\dim(F_1)$ may be very large— much larger than $\dim(X)$. In fact, $\dim(F_1)$ also depends on k and l.
In Lemma 2.11 (or Lemma 2.10), we have already perturbed the homomorphism to avoid the largest possibility of maximum multiplicity—k. Next, we will perturb it again to avoid the next largest possibility of maximum multiplicity—$k-1$. We will continue the procedure in this way.
In general, suppose that the homomorphism $\phi : C(X) \to PM_\bullet(C(Y))P$ has maximum multiplicity m, with $l < m \le k$, we will prove that ϕ can be approximated arbitrarily well by another homomorphism ϕ' with maximum multiplicity at most $m-1$. Once this is done, Theorem 2.1 follows from a reverse induction argument beginning with $m = k > l$. (Note that for the case $k \le l$, we have nothing to prove.)
To do the above, we need to work simplex by simplex. In fact, on each small simplex, the homomorphism ϕ can be decomposed into a direct sum of several homomorphisms $\bigoplus_i \phi_i$, such that the projections $\phi_i(\mathbf{1})$ has rank at most m. Then we can apply Lemma 2.11 to each ϕ_i to avoid maximum multiplicity m. With these ideas in mind, it will not be difficult for the reader to construct the proof of Theorem 2.1. The complete detail will be contained in the next few lemmas, in particular, see the proof of Lemma 2.16.

Lemma 2.13. *Suppose that* P, X *and* Y *are as in Theorem 2.1. For any* $\varepsilon > 0$ *and any positive integer* d*, there is a* $\delta > 0$ *such that if* $\Delta \subset Y$ *is a simplex of dimension* d*, if* P *is regarded as a projection in* $M_\bullet(C(\Delta))$*, and if* $\psi : C(X) \to PM_\bullet(C(\partial\Delta))P$ *is a homomorphism such that*

$$\|\psi(g)(y) - \psi(g)(y')\| < \delta, \quad \forall g \in G, y, y' \in \partial\Delta,$$

then there is a homomorphism $\psi' : C(X) \to PM_\bullet(C(\Delta))P$ *such that*
(1) $\|\psi'(g)(y) - \psi'(g)(y')\| < \varepsilon, \quad \forall g \in G, y, y' \in \Delta$*, and*
(2) $\psi'|_{\partial\Delta} = \psi$*.*

Proof: Note that $P|_\Delta$ is a trivial projection, since a simplex Δ is contractible.

So $PM_\bullet(C(\Delta))P \cong M_k(C(\Delta))$, where $k = \mathrm{rank}(P)$. The lemma follows from the fact that

$$F^k X = \mathrm{Hom}(C(X), M_k(\mathbb{C}))_1$$

is a simplicial complex (see [DN] and [Bl]), which is locally contractible (see 1.4.2 and 1.4.3).

$\square$

We need the following lemma, which is obviously true.

Lemma 2.14. *Suppose that $\phi : C(X) \to PM_\bullet(C(Y))P$ has maximum spectral multiplicity at most m. Then there exist $\eta > 0$ and $\delta > 0$ such that the following statement holds.*
For any subset $Z \subset Y$ with diameter$(Z) < \eta$, and homomorphism $\psi : C(X) \to PM_\bullet(C(Z))P$ with the property that

$$\|\psi(g)(z) - \phi(g)(z)\| < \delta, \quad \forall z \in Z, g \in G,$$

there is a decomposition of ψ such as described below.
There are open sets $O_1, O_2, \cdots, O_t \subset X$, with mutually disjoint closures (i.e., $\bar{O}_i \cap \bar{O}_j = \emptyset$, $\forall i \neq j$), and there are mutually orthogonal projections $Q_1, Q_2, \cdots, Q_t \in M_\bullet(C(Z))$ and homomorphisms $\psi_i : C(X) \to Q_i M_\bullet(C(Z))Q_i$ such that
1. $\psi = \sum_{i=1}^t \psi_i$
2. $P(z) = \sum_{i=1}^t Q_i(z), \forall z \in Z$,
3. rank$(Q_i) \leq m$, and
4. $SP\psi_i \subset O_i$ for all i.

Proof: One can prove it using the following fact. Suppose that $\mathrm{SP}\phi \subset \cup O_i$. If ψ is close enough to ϕ, then $\mathrm{SP}\psi \subset \cup O_i$ and $\#(\mathrm{SP}\phi \cap O_i) = \#(\mathrm{SP}\psi \cap O_i)$ (see 1.2.12).
Q_i in the lemma should be chosen to be the spectral projections of ψ corresponding to the open sets O_i (see 1.2.9).
(Notice that Y is compact and that $G \subset C(X)$ contains a set of the generators.)

$\square$

Lemma 2.15. *Suppose that $\phi : C(X) \to PM_\bullet(C(Y))P$ has maximum spectral multiplicity at most m. Then there exist $\eta > 0$ and $\delta > 0$ such that the following statement holds.*
For any $\varepsilon > 0$, any simplex $\Delta \subset Y$ (of any simplicial decomposition of Y) with diameter$(\Delta) < \eta$, and any homomorphism $\psi : C(X) \to PM_\bullet(C(\Delta))P$ with the following properties:
(i) $\|\psi(g)(z) - \phi(g)(z)\| < \delta$, $\forall z \in \Delta, g \in G$, and
(ii) $\psi|_{\partial\Delta}$ has maximum multiplicity at most $m-1$,
there exists a homomorphism $\psi' : C(X) \to PM_\bullet(C(\Delta))P$ such that
(1) $\|\psi(g)(y) - \psi'(g)(y)\| < \varepsilon$ for all $g \in G$ and $y \in \Delta$;
(2) ψ' has spectral multiplicity at most $m-1$;

(3) $\psi'|_{\partial\Delta} = \psi|_{\partial\Delta}$.

Proof: . Suppose that η and δ are as in Lemma 2.14. If ψ is as described in this lemma, then one can obtain the decomposition $\psi = \sum_{i=1}^{t} \psi_i$ of ψ as in Lemma 2.14.

Then we only need to apply Lemma 2.11 to each map ψ_i to obtain $\psi_i' : C(X) \to Q_i M_\bullet(C(Z))Q_i$ to satisfy the conclusion of Lemma 2.11 with ψ_i, Δ, $\partial\Delta$, and Q_i in place of ψ, Y_1, Y_2, and Q, respectively.

If ε is small enough, then $\mathrm{SP}\psi_i' \subset O_i$, where the open sets O_i are from Lemma 2.14. Hence the sum $\psi' = \sum_{i=1}^{t} \psi_i'$ is as desired.

□

Lemma 2.16. *Suppose that $\phi : C(X) \to PM_\bullet(C(Y))P$ has maximum spectral multiplicity at most $m > l = \dim(X) + \dim(Y)$. For any simplicial subcomplex $Y_1 \subset Y$, with respect to any simplicial decomposition of Y, and any $\varepsilon > 0$, there is a homomorphism $\phi' : C(X) \to PM_\bullet(C(Y_1))P$ with multiplicity at most $m-1$ such that*

$$\|\phi'(g)(y) - \phi(g)(y)\| < \varepsilon, \quad \forall g \in G, y \in Y_1.$$

(In particular, the above is true for $Y_1 = Y$.)

Proof: We will prove the lemma by induction on $\dim(Y_1)$.

If $\dim(Y_1) = 0$, the lemma follows from the fact that, for a connected simplicial complex X with $X \neq \{pt\}$, the subset of homomorphisms with distinct spectrum (maximum spectral multiplicity one) is dense in $\mathrm{Hom}(C(X), M_k(\mathbb{C}))_1$.

Suppose that the lemma is true for any simplicial subcomplex of dimension n, with respect to any simplicial decomposition.

Let $Y_1 \subset Y$ be a simplicial complex of dimension $n+1 \leq \dim Y$, with respect to some simplicial decomposition of Y.

Let $\varepsilon > 0$.

Let δ_1, η_1 be the δ and η of Lemma 2.15.

Apply Lemma 2.13 with $n+1$ in place of d, and $\frac{1}{4}\min(\varepsilon, \delta_1)$ in place of ε, to find δ_2 as δ in the lemma.

Choose $\eta_2 > 0$ such that if $\mathrm{dist}(y, y') < \eta_2$, then

$$(*) \qquad \|\phi(g)(y) - \phi(g)(y')\| < \frac{1}{4}\min(\varepsilon, \delta_1, \delta_2), \forall g \in G.$$

Endow Y_1 with a simplicial complex structure such that $\mathrm{diameter}(\Delta) < \min(\eta_1, \eta_2)$ for any simplex Δ of Y_1. Let $Y' \subset Y_1$ be the n-skeleton of Y_1 with respect to the simplicial structure.

From the inductive assumption, there is a homomorphism $\phi_1 : C(X) \to PM_\bullet(C(Y'))P$, with multiplicity at most $m-1$, such that

$$(**) \qquad \|\phi_1(g)(y) - \phi(g)(y)\| < \frac{1}{4}\min(\varepsilon, \delta_1, \delta_2), \quad \forall g \in G \text{ and } y \in Y'.(**)$$

Consider a fixed simplex $\Delta \subset Y_1$ of top dimension (i.e., $\dim(\Delta) = n+1$). Let us extend $\phi_1|_{\partial\Delta}$ to Δ (notice that $\partial\Delta \subset Y'$).
For any two points $y, y' \in \partial\Delta$, applying (*) to the pair of points y and y', applying (**) to the points y and y' separately, and combining all these three inequalities together, we get

$$\|\phi_1(g)(y) - \phi_1(g)(y')\| < \frac{3}{4}\min(\varepsilon, \delta_1, \delta_2) < \delta_2, \quad \forall g \in G.$$

By Lemma 2.13 and the way δ_2 is chosen, there is a homomorphism (let us still denote it by ϕ_1) $\phi_1 : C(X) \to PM_\bullet(C(\Delta))P$, which extends the original $\phi_1|_{\partial\Delta}$, such that

$$(***) \qquad \|\phi_1(g)(y) - \phi_1(g)(y')\| < \frac{1}{4}\min(\varepsilon, \delta_1), \quad \forall g \in G \quad \text{and} \quad y, y' \in \Delta.$$

For any point $y \in \Delta$, choose a point $y' \in \partial\Delta$. Applying both (*) and (***) to the pair (y, y'), applying (**) to the point $y' \in \partial\Delta \subset Y'$ and combining all these three inequalities together, we get

$$(****) \qquad \|\phi_1(g)(y) - \phi(g)(y)\| < \frac{3}{4}\min(\varepsilon, \delta_1), \forall g \in G \text{ and } y \in \Delta.$$

Since δ_1 and η_1 are chosen as in Lemma 2.15, and diameter$(\Delta) < \eta_1$, it follows from $(* * **)$ and Lemma 2.15 that there is a homomorphism $\phi' : C(X) \to PM_\bullet(C(\Delta))P$ such that
(1) $\|\phi'(g)(y) - \phi_1(g)(y)\| < \frac{1}{4}\varepsilon, \quad \forall g \in G$ and $y \in \Delta$.
(2) ϕ' has spectral multiplicity at most $m-1$.
(3) $\phi'|_{\partial\Delta} = \phi_1|_{\partial\Delta}$.
Combining (1) above with (****), yields

$$\|\phi'(g)(y) - \phi(g)(y)\| < \frac{3}{4}\min(\varepsilon, \delta_1) + \frac{1}{4}\varepsilon \le \varepsilon, \forall g \in G \text{ and } y \in \Delta.$$

Carry out the above construction independently for each simplex Δ. Since the definition of ϕ' on $\partial\Delta$ is as same as ϕ_1, the definitions of ϕ' on different simplices are agree on their intersection. By Lemma 1.2.14, this yields a homomorphism over the whole set Y_1. The lemma follows.

□

Obviously, Theorem 2.1 follows from Lemma 2.16 by reverse induction argument beginning with $m = k$. (Note that we only need Lemma 2.16 for the case $Y_1 = Y$.)
Now we are going to prove Theorem 2.9, which is the only missing part in the proof of Theorem 2.1. The proof is somewhat similar to the proof of Theorem 6.4.2 of [DN]. It will therefore be convenient to recall some of the terminology and notation of [DN]. (It will be important to consider a certain method of decomposing the space F^kX.)

2.17. Recall from 6.17 of [DN] (cf. 1.2.4 above) that there is a map λ :

$X^k \times U(k) \to F^k X$, defined as follows. If $u \in U(k)$ and $(x_1, x_2, \cdots, x_k) \in X^k$, then

$$(\lambda(x_1, x_2, \cdots, x_k, u))(f) = u \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \ddots & \\ & & & f(x_k) \end{pmatrix} u^*$$

for any $f \in C(X)$. Since λ is surjective, $F^k X$ can be regarded as a quotient space $X^k \times U(k)$. Therefore, for convenience, a point in $F^k X$ will be written as

$$[x_1, x_2, \cdots, x_k, u]$$

which means $\lambda(x_1, x_2, \cdots, x_k, u)$.
With the above notation, it is easy to see that, if X is path connected and is not a single point, then any element in $F^k X$ can be approximated arbitrarily well by elements in $F^k X$ with distinct spectra.

2.18. If $X_1 \subset X$ is a subset, then define $F^k X_1$ to be the subset of $F^k X$ consisting of those homomorphisms $\phi \in \mathrm{Hom}(C(X), M_k(\mathbb{C}))$ with $\mathrm{SP}(\phi) \subset X_1$ as a set. Obviously, if X_1 is open (closed resp.), then $F^k X_1$ is open (closed resp.).
If $X_1, X_2, \cdots, X_i$ are disjoint subspaces of X, and $k_1, k_2, \cdots, k_i$ are nonnegative integers with

$$k_1 + k_2 + \cdots + k_i = k,$$

then define $F^{(k_1, k_2, \cdots, k_i)}(X_1, X_2, \cdots, X_i)$ to be the subset of $F^k X$ consisting of all ϕ with

$$\#(\mathrm{SP}(\phi) \cap X_i) = k_i$$

counting multiplicity.
Usually when we use the above notation, we suppose that $\bar{X}_{i_1} \cap \bar{X}_{i_2} = \emptyset$ if $i_1 \neq i_2$, where $\bar{X}_i$ is the closure of X_i. In this case,

$$F^k(X_1 \cup X_2 \cup \cdots \cup X_i) = \coprod_{k_1 + k_2 + \cdots + k_i = k} F^{(k_1, k_2, \cdots, k_i)}(X_1, X_2, \cdots, X_i)$$

is a disjoint union of separate components.

2.19. For each i-tuple $(k_1, k_2, \cdots, k_i)$ with

$$k_1 + k_2 + \cdots + k_i = k,$$

one can define $G_{(k_1, k_2, \cdots, k_i)}(\mathbb{C}^k)$ to be the collection of i-tuples $(p_1, p_2, \cdots, p_i)$ of orthogonal projections $p_j \in M_k(\mathbb{C})$ with $\mathrm{rank}(p_j) = k_j$ $(1 \leq j \leq i)$ and $\sum_{j=1}^{i} p_j = \mathbf{1} \in M_k(\mathbb{C})$. Note that if $i = 2$, $G_{(k_1, k_2)}(\mathbb{C}^k)$ is the ordinary complex Grassmannian manifold $G_{k_1}(\mathbb{C}^k) = G_{k_2}(\mathbb{C}^k)$.

For each fixed i-tuple $(k_1, k_2, \cdots, k_i)$, there is a locally trivial fibre bundle

$$\begin{array}{ccc} F^{k_1}(X_1) \times F^{k_2}(X_2) \times \cdots \times F^{k_i}(X_i) & \longrightarrow & F^{(k_1,k_2,\cdots,k_i)}(X_1, X_2, \cdots, X_i) \\ & & \downarrow \\ & & G_{(k_1,k_2,\cdots,k_i)}(\mathbb{C}^k). \end{array}$$

2.20. For certain purposes, it is more convenient to use CW complexes (instead of simplicial complexes).

For the terminology used below, see [Wh].

Suppose that (X, A) is a relative CW complex pair. If X is path connected, then (X, A) is zero connected CW complex pair, no matter A is connected or not. In particular, (X, A) is homotopy equivalent to (X_1, A), where X_1 is obtained from A by attaching finitely many cells of dimension ≥ 1 (see Theorem 2.6 of Chapter five of [Wh]). This can not be done if one only uses simplicial complex pair. (Note, we always assume our CW complexes to be finite CW complexes without saying so.)

For a relative CW complex pair (X, A), define $F^m_A X \subset F^m X$ to be the subspace consisting of those elements $x \in F^m X$, with

$$\mathrm{SP}(x) \cap A \neq \emptyset.$$

(This is different from the set $F^m A$ (defined in 2.18) which consists of elements $x \in F^m X$ such that $\mathrm{SP}(x) \subset A$.)

LEMMA 2.21. *Suppose that (X, A) is a relative CW complex pair. Suppose that X is obtained from A by attaching cells of dimension at least 1. It follows that the inclusion*

$$F^m_A X \hookrightarrow F^m X$$

is $m-1$ equivalent, i.e., $i_* : \pi_j(F^m_A X) \to \pi_j(F^m X)$ is an isomorphism for any $0 \leq j \leq m-2$ and a surjection for $j = m-1$, where i_* is induced by the inclusion map.

The proof of this lemma is divided into two steps.

LEMMA 2.22. *Lemma 2.21 is true if X is obtained from A by attaching several cells of dimension 1.*

Proof: Let $X = A \cup e_1 \cup e_2 \cup \cdots \cup e_t$, where $e_1, e_2, \cdots, e_t$ are 1-cells with $\partial e_i \subset A$. Then $F^m X \backslash F^m_A X$ consists of those points whose spectra are contained in

$$\overset{\circ}{e}_1 \cup \overset{\circ}{e}_2 \cup \cdots \cup \overset{\circ}{e}_t,$$

where each $\overset{\circ}{e}_j = e_j \backslash \partial e_j$ is homeomorphic to $(0, 1)$.

In other words,

$$F^m X \backslash F^m_A X \quad = \coprod_{k_1+k_2+\cdots+k_t=m} F^{(k_1,k_2,\cdots,k_t)}(\overset{\circ}{e}_1, \overset{\circ}{e}_2, \cdots, \overset{\circ}{e}_t).$$

For each fixed t-tuple $(k_1, k_2, \cdots, k_t)$ with $\sum k_i = m$, the space $F^{(k_1,k_2,\cdots,k_t)}(\overset{\circ}{e}_1, \overset{\circ}{e}_2, \cdots, \overset{\circ}{e}_t)$ is a smooth manifold. To see this, we can consider the fibre bundle

$$\begin{array}{ccc} F^{k_1}(\overset{\circ}{e}_1) \times F^{k_2}(\overset{\circ}{e}_2) \times \cdots \times F^{k_t}(\overset{\circ}{e}_t) & \longrightarrow & F^{(k_1,k_2,\cdots,k_t)}(\overset{\circ}{e}_1, \overset{\circ}{e}_2, \cdots, \overset{\circ}{e}_t) \\ & & \downarrow \\ & & G_{(k_1,k_2,\cdots,k_t)}(\mathbb{C}) \end{array}$$

introduced in 2.19. Evidently, the fibre of the bundle is

$$F^{k_1}(\overset{\circ}{e}_1) \times F^{k_2}(\overset{\circ}{e}_2) \times \cdots \times F^{k_t}(\overset{\circ}{e}_t) \cong \mathbb{R}^{k_1^2} \times \mathbb{R}^{k_2^2} \times \cdots \times \mathbb{R}^{k_t^2}.$$

Note that the above fibre bundle has an obvious cross section (see [DN]). Therefore, the fibre bundle can be regarded as a smooth vector bundle with the vector space $\mathbb{R}^{k_1^2+k_2^2+\cdots+k_t^2}$ as the fibre. The zero section of the bundle has codimension

$$k_1^2 + k_2^2 + \cdots + k_t^2 \geq k_1 + k_2 + \cdots + k_t = m.$$

By a standard argument from topology, using the transversality theorem, the lemma can be proved. (See [DN, 6.3.4] for details.)

$\square$

2.23. The next step is to prove Lemma 2.21 by induction, starting with 2.22. Since the proof is a complete repetition of the proof of Theorem 6.4.2 of [DN], we omit the detail—only point out how to define the collection $W(n, r)$ in our setting, and several small modifications.
Let $W(1, 0) = \{A\}$—the set with single element: the space A. For $r > 0$, $W(n, r)$ is the class of all finite CW complexes, each of which is obtained by attaching at most one cell of dimension n to a space in $W(n, r-1)$. Let

$$W(n+1, 0) = \bigcup_{r=0}^{+\infty} W(n, r).$$

Lemma 2.22 says that $F_A^m X \to F^m X$ is $m-1$ equivalent if $X \in W(2, 0)$. In applying the argument in [DN, 6.4.2], $F_A^m X$ is in place of $F^k(X)$ and $F^m X$ is in place of $F^{k+1}(X)$. All the other parts of the proof follow from [DN]. The only thing needs mentioning is that the inclusion

$$F_A^m(X \backslash \alpha_I) \hookrightarrow F_A^m X$$

is 1-equivalent, where α_I is a set of finitely many points inside one of the n-cells of X, and $n \geq 2$. To prove this statement, one needs to prove that any continuous map from S^1 to $F_A^m X$ can be perturbed to a map from S^1 to $F_A^m(X \backslash \alpha_I)$. To do this, he can first perturb a map to a piecewise linear map for which the image will be one dimensional. And the resulting map can be easily perturbed again to a map whose spectrum avoids any given finite set of points in any cell of dimension at least 2.

$\square$

2.24. Let X be a simplicial complex, and $x_0 \in X$ be a vertex.
Define X' to be the sub-simplicial complex consisting of all the simplices Δ (and their faces) with $\Delta \ni x_0$. Then X' is contractible.
We also use x_0 to denote the point in $F^m X$ defined by

$$\phi(f) = f(x_0) \cdot \mathbf{1}_m \in M_m(\mathbb{C})$$

for each $f \in C(X)$.
We can easily prove the following claim: The map $F^m X' \backslash \{x_0\} \hookrightarrow F^m X'$ is $(m-1)$ equivalent.
To see this, let

$$A = \cup\{\Delta \mid \Delta \text{ is a simplex of } X' \text{ and } x_0 \notin \Delta\}.$$

Then A is a sub-simplicial complex of X'. A may not be connected, but (X', A) is 0-connected. In the notation of 1.4.2,

$$X' = \overline{\mathrm{Star}(\{x_0\})} \quad \text{and} \quad A = \overline{\mathrm{Star}(\{x_0\})} \backslash \mathrm{Star}(\{x_0\}).$$

It is obvious that $A \hookrightarrow X' \backslash \{x_0\}$ is a homotopy equivalence. Therefore, $F^m_A X' \hookrightarrow (F^m X') \backslash \{x_0\}$ is a homotopy equivalence. By Lemma 2.21, the claim holds. In particular,

$$\pi_i(F^m X' \backslash \{x_0\}) = 0$$

for any $0 \le i \le m-2$, since $F^m X'$ is contractible. Equivalently,

$$\pi_i(F^m(X' \backslash A) \backslash \{x_0\}) = 0$$

for any $0 \le i \le m-2$. Note that $X' \backslash A$ is an open neighborhood of $x_0 \in X$, which is the interior of X'.

2.25. Proof of Theorem 2.9. Suppose that

$$\mathrm{SP}(x) = \{ \underbrace{\lambda_1, \lambda_1, \cdots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \lambda_2, \cdots, \lambda_2}_{k_2}, \cdots, \underbrace{\lambda_i, \lambda_i, \cdots, \lambda_i}_{k_i} \},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_i \in X$ are distinct points and $k_1 + k_2 + \cdots + k_i = m$. Choose mutually disjoint open sets $U_1 \ni \lambda_1, U_2 \ni \lambda_2, \cdots, U_i \ni \lambda_i$, in X. Then there is a locally trivial fibre bundle

$$\begin{array}{ccc} F^{k_1}(U_1) \times F^{k_2}(U_2) \times \cdots \times F^{k_i}(U_i) & \longrightarrow & F^{(k_1,k_2,\cdots,k_i)}(U_1, U_2, \cdots, U_i) \\ & & \downarrow \\ & & G_{(k_1,k_2,\cdots,k_i)}(\mathbb{C}). \end{array}$$

Note that $G_{(k_1,k_2,\cdots,k_i)}(\mathbb{C}) = U(m)/(U(k_1)\times U(k_2)\times\cdots\times U(k_i))$ is a smooth manifold of dimension $t := m^2 - \sum_{j=1}^{i} k_j^2$. There is a small contractible neighborhood $U_x \subset B_\varepsilon(x)$ which is homeomorphic to the space

$$F^{k_1}(X_1)\times F^{k_2}(X_2)\times\cdots F^{k_i}(X_i)\times\mathbb{R}^t,$$

where $X_1, X_2, \cdots, X_i$ are mutually disjoint open subsets of X. The space X_i can be chosen so that $\bar{X}_i$ is the simplicial complex X' as in 2.24 corresponding to vertex λ_i, with respect to some simplicial decomposition of X.

The following fact is well known in topology. Suppose that X and Y are connected CW complexes with base points x_0, y_0 respectively. If $X\backslash\{x_0\}$ is l_1-connected and $X\backslash\{y_0\}$ is l_2-connected, then $(X\times Y)\backslash\{x_0,y_0\}$ is (l_1+l_2+2)-connected.

Combining this fact with 2.24, we conclude that $U_x\backslash\{x\}$ is

$$\sum_{j=1}^{i} k_j - 2 + t = m - 2 + t$$

connected. This ends the proof.

□

3 Combinatorial Results

In this section, we will prove certain results of a combinatorial nature, for the preparation of the proof of the Decomposition Theorem—Theorem 4.1 of the next section.

We will need the results in the case that certain multiplicities are general—not just equal to one. For the sake of clarity, we will first state and prove the results in the special case of multiplicity one. We will then consider the general case.

3.1. Suppose that X is a simplicial complex. Let σ denote the simplicial complex structure of X—which tells what the simplices of X are, and what the faces of each simplex of X are. In this section, we will use (X,σ) to denote the simplicial complex X with simplicial structure σ, to emphasize that we may endow the same space X with different simplicial complex structures. In this section, we will reserve the notation, $\sigma, \tau, \sigma_1, \tau_1, \cdots$, etc., for simplicial complex structures.

Recall that, if Δ is a simplex, its boundary is denoted by $\partial\Delta$. For example, if $\dim(\Delta) = 0$, i.e., $\Delta = \{pt\}$, the set consisting of a single point, then $\partial\Delta = \emptyset$; if $\dim(\Delta) = 1$, i.e., Δ is an interval, then $\partial\Delta$ is the set consisting of the two extreme points of the interval. Let us also consider the set $\Delta\backslash\partial\Delta$, and denote it by interior(Δ).

If (X,σ) is a simplicial complex, then for any point $x\in X$, there is a unique simplex Δ such that $x\in$ interior(Δ).

As usual, if each simplex of (X,σ_1) is a union of certain simplices of (X,σ_2), then we shall call σ_2 a subdivision of σ_1. This is equivalent to the property that any simplex of (X,σ_2) is contained in a simplex of (X,σ_1).
The notation Vertex(X,σ) (respectively, Vertex(Δ)) will be used to denote the set of vertices of (X,σ) (or of the simplex Δ).

DEFINITION 3.2. Let $E=\{1,2,\cdots,K\}$ be an index set. (The index set E can be any set with exactly K elements.) Let $K_1,K_2,\cdots,K_m$ be non negative integers with

$$K_1+K_2+\cdots+K_m=K.$$

A GROUPING OF E OF TYPE $(K_1,K_2,\cdots,K_m)$ is a collection of m mutually disjoint index sets $E_1,E_2,\cdots,E_m$ with

$$E=E_1\cup E_2\cup\cdots\cup E_m,$$

and $\#(E_j)=K_j$ for each $1\le j\le m$. (Cf. 1.5.18.)
Usually, we will keep the tuple $(K_1,K_2,\cdots,K_m)$ fixed and just call the collection $E_1,E_2,\cdots,E_m$ a grouping of E (without mentioning the type).
(Most of the time, $K_1,K_2,\cdots K_m$ will be positive integers, i.e., nonzero. But for convenience, we allow some numbers $K_i=0$, and then the corresponding sets E_i should be the empty set.)

3.3. Let (X,σ) be a simplicial complex. Suppose that, associated to each vertex $x\in X$, there is a grouping $E_1(x),E_2(x),\cdots,E_m(x)$ of E of type $(K_1,K_2,\cdots,K_m)$. (In our application in the proof of Theorem 4.1, the index set E will be the spectrum of a homomorphism at the given vertex, see 1.5.13, 1.5.17–1.5.22.)
Suppose that these groupings for all the vertices are chosen arbitrarily. Then, in general, for a simplex Δ with vertices $x_0,x_1,\cdots,x_n$, the intersections

$$\bigcap_{x\in \mathrm{Vertex}(\Delta)} E_j(x)=E_j(x_0)\cap E_j(x_1)\cap\cdots\cap E_j(x_n),\ \ j=1,2,\cdots,m,$$

may have very few elements—the sets $E_j(x_0),E_j(x_1),\cdots,E_j(x_n)$ may be very different.
The purpose of this section is to introduce a subdivision (X,τ) of (X,σ), and to associate to each new vertex of (X,τ) a grouping to make the following true: For any simplex Δ of (X,τ) (after the subdivision), for each j, the number of elements in the intersection

$$\bigcap_{x\in \mathrm{Vertex}(\Delta)} E_j(x)$$

is not much less than the number of elements in each individual set $E_j(x)$ (note that $\#(E_j(x))=K_j$ for each x); in other words, the groupings of adjacent vertices (after subdivision) should be almost as the same as each other.

First we will state the following lemma (the proof will be given in 3.15). Later on, we will need a relative version of the lemma.
(See 1.5.17 to 1.5.23 for the explanations of the role of this lemma in §4. To visualize the following lemma, see 1.5.21 for the explanation of the one dimensional case.)

Lemma 3.4. *Let (X,σ) be a simplicial complex consisting of a single simplex X and its faces. Suppose that associated to each $x \in Vertex(X,\sigma)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E (of type $(K_1, K_2, \cdots, K_m)$).*
It follows that there is a subdivision (X,τ) of (X,σ), and associated to each new vertex $x \in Vertex(X,\tau)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E (of type $(K_1, K_2, \cdots, K_m)$), (for any old vertex of (X,σ), the grouping should not be changed), such that the following hold.
For each newly introduced vertex $x \in Vertex(X,\tau)$,

$$(1) \qquad \bigcap_{y \in \, Vertex(X,\sigma)} E_j(y) \subset E_j(x), \qquad j = 1, 2, \cdots, m,$$

and

$$(2) \qquad E_j(x) \subset \bigcup_{y \in \, Vertex(X,\sigma)} E_j(y), \qquad j = 1, 2, \cdots, m.$$

For any simplex Δ of (X,τ) (after subdivision),

$$(3) \qquad \#\Big(\bigcap_{x \in \, Vertex(\Delta)} E_j(x)\Big) \geq K_j - \frac{n(n+1)}{2}, \qquad j = 1, 2, \cdots, m,$$

where $n = \dim X$.
(When we apply this lemma in §4, the simplex X will be a simplex of a simplicial complex Y, and $K_j >> (\dim Y)^3$; from this it follows that

$$\#\Big(\bigcap_{x \in \mathrm{Vertex}(\Delta)} E_j(x)\Big) \geq K_j - \frac{n(n+1)}{2} >> (\dim Y)^3, \qquad j = 1, 2, \cdots, m.)$$

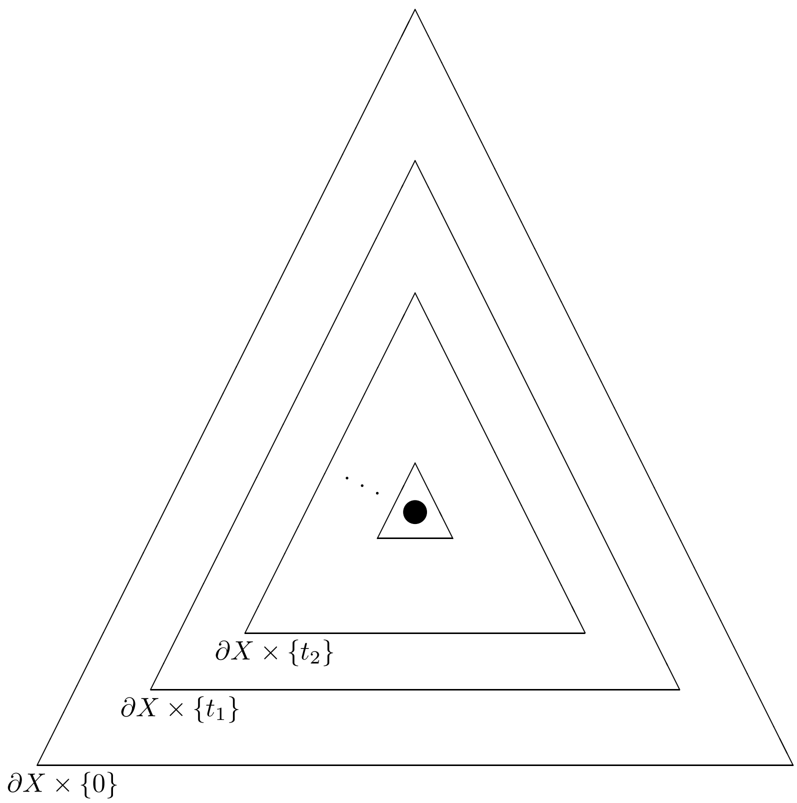
Remark 3.5. The inclusion $E_j(x) \subset \bigcup_{y \in \mathrm{Vertex}(X,\sigma)} E_j(y)$ in the condition (2) of Lemma 3.4 is important for our application in §4 (see 1.5.19 for the explanation). We will put this inclusion into a more general context, 3.7. So we will only discuss the condition (1) in this remark.
The inclusion $\bigcap_{y \in \mathrm{Vertex}(X,\sigma)} E_j(y) \subset E_j(x)$ in the condition (1) above, will not be used in our application in §4. But taking this inclusion as a part of the conclusion will make the induction argument easier in the proof.
We would like to point out that the weak version of the above lemma without requiring the inclusion in (1) will automatically imply the above stronger version. (This can be seen from the proof of Corollary 3.14 below.)

3.6. Strategy and logistics of the proof of Lemma 3.4

We shall prove it by induction. So we assume that the lemma is true for simplex of dimension at most $n-1$, and prove the case that $\dim(X) = n$.

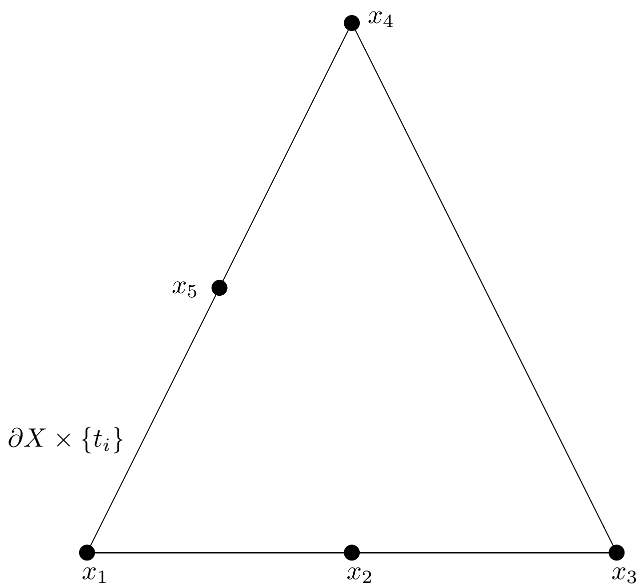
First, we introduce a new vertex, which is the barycenter of X, and introduce a model grouping $E_1^{model}, E_2^{model}, \cdots, E_m^{model}$ of E for the new vertex.

We shall view X as many layers similar to the boundary ∂X: $\partial X \times \{0\}, \partial X \times \{t_1\}, \cdots,$ and the top layer $\partial X \times \{1\}$ is identified into a single point which is the barycenter, where $t_1, t_2, \cdots,$ is a finite sequence of increasing numbers between 0 and 1 (the number of terms in this sequence depending in a certain sense on the distance between the giving groupings at the vertices and the model grouping at the barycenter). See the picture below.



We will introduce a subdivision of each layer $\partial X \times \{t_i\}$ (identifying the layer as a set with ∂X and thereby endow it with a simplicial complex structure), and a grouping for each vertex on this layer. The general principle we shall follow is: the higher the layer is, the closer the groupings are to the model grouping. We should, gradually, change the groupings from each layer to the next higher layer.

Let us explain it for the case $\dim(X) = 2$ and $\dim(\partial X) = 1$. Fix a t_i, and suppose that we have the simplicial structure and groupings for all vertices on $\partial X \times \{t_i\}$. Let us using the following picture to show the vertices of $\partial X \times \{t_i\}$.



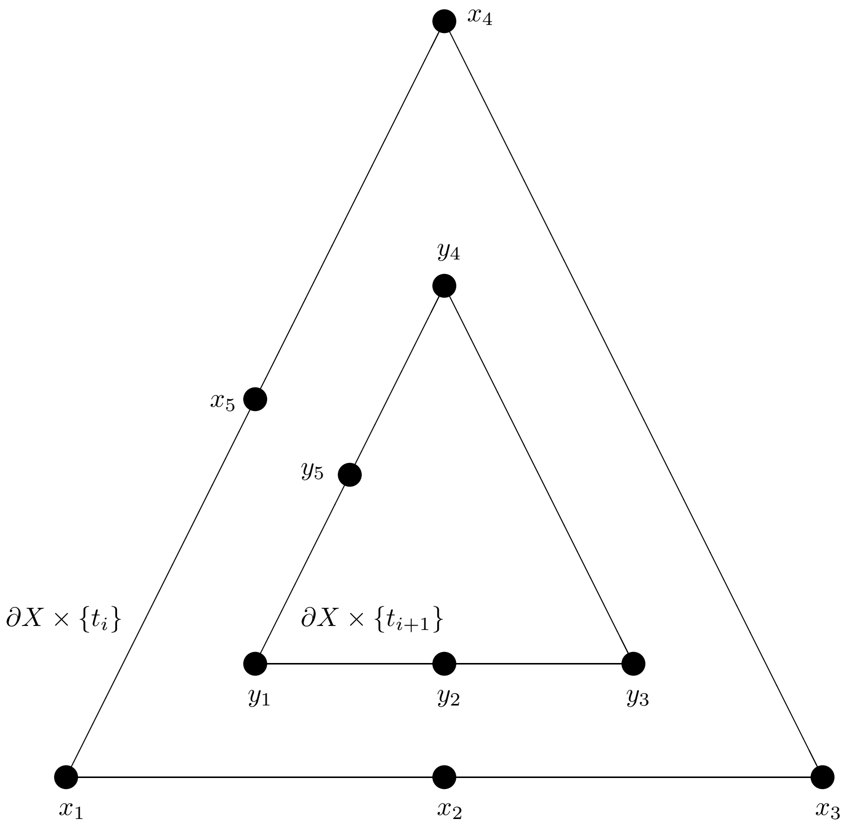
(I.e., there are five 1-dimensional simplices $[x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_5]$ and $[x_5, x_1]$.)

Let us assume that the condition (3) holds for any simplex of $\partial X \times \{t_i\}$ with $\dim(X)$ replaced by $\dim(\partial X)$. (We will also discuss the condition (1) below, but not the condition (2).)

We shall construct simplicial structure and groupings on $\partial X \times \{t_{i+1}\}$. To begin with, let us provisionally define the simplicial structure on $\partial X \times \{t_{i+1}\}$ to be as the same as that on $\partial X \times \{t_i\}$, as in the picture on the next page.

Fix an element $\lambda \in E_1^{model}$ such that $\lambda \notin \bigcap_{k=1}^{5} E_1(x_k)$. (If such an element does not exist, then the groupings are already good for E_1. In other words, $E_1(x_k)$ contains and therefore equals E_1^{model} for every k. Then we should go on to E_2 or other parts.)

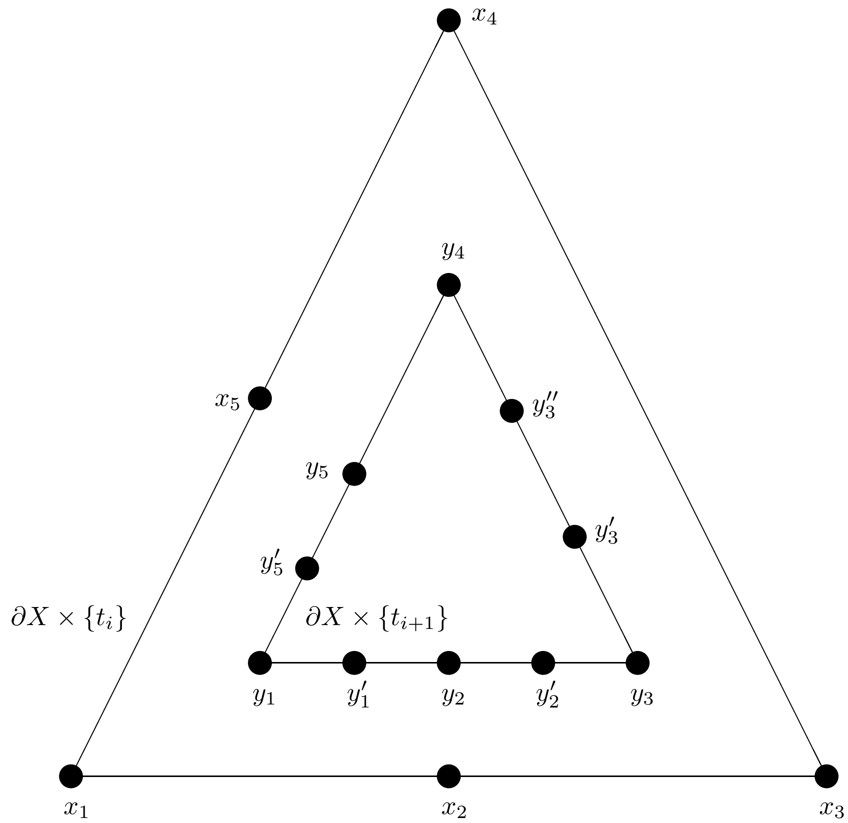
The grouping on the vertex $y_j, j = 1, \cdots, 5$ will be taken to be either the grouping on the corresponding vertex x_j, if $E_1(x_j) \ni \lambda$, or the grouping on the corresponding vertex x_j, with a certain element of $E_1(x_j) \backslash E_1^{model}$ replaced by $\lambda \in E_1^{model}$ if $E_1(x_j) \not\ni \lambda$. Lemma 3.9 below tells which element should be chosen to be replaced. Of course the other part $E_t,\ t > 1$ of the grouping must also be slightly modified. Lemma 3.9 also guarantees that such modification exists. Subsections 3.7 and 3.8 give the definition used in 3.9. (This consideration are all in order to ensure the condition (2).)



Now, $\bigcap_{k=1}^{5} E_1(y_k)$ contains one more element of E_1^{model} than $\bigcap_{k=1}^{5} E_1(x_k)$, namely, λ. So for E_1, the groupings on $\partial X \times \{t_{i+1}\}$ are (globally) closer to the model grouping than that on $\partial X \times \{t_i\}$.

But the groupings on $\partial X \times \{t_{i+1}\}$ may not satisfy the condition (3) with $\dim(X)$ replaced by $\dim(\partial X)$, as the groupings on $\partial X \times \{t_i\}$ do .

By the induction assumption, applied to each individual simplex of $\partial X \times \{t_{i+1}\}$ (with the provisional simplicial structure), we can introduce a subdivision for $\partial X \times \{t_{i+1}\}$ and groupings for the new vertices to make the condition (3), with $\dim(X)$ replaced by $\dim(\partial X)$, hold for $\partial X \times \{t_{i+1}\}$. The picture now looks like



In this picture, y_1', y_2', y_3', y_3'', and y_5' are the new vertices introduced in the subdivision.

(Of course this picture only shows a special case.)

It goes without saying that we wish to ensure the condition (1) (and also the condition (2)) for the groupings associated to the new vertices inside each provisional simplex of $\partial X \times \{t_{i+1}\}$. In the other words, when we introduce the groupings for a new vertex inside a fixed provisional simplex of $\partial X \times \{t_{i+1}\}$ (e.g., y_2' inside $[y_2, y_3]$), for each k we should keep the intersection of the sets E_k over vertices of this simplex (e.g., $E_k(y_2) \cap E_k(y_3)$) inside the set E_k for the new vertex (e.g., inside $E_k(y_2')$). This is the condition (1) for this provisional simplex. The condition (1) for all the individual simplices implies that after the subdivision, the intersection over the whole layer $\bigcap_{y \in \mathrm{Vertex}(X \times \{t_{i+1}\})} E_1(y)$ is equal to the intersection over the vertices of provisional simplicial structure $\bigcap_{k=1}^{5} E_1(y_k)$, and therefore still contains one more element of E_1^{model} than $\bigcap_{k=1}^{5} E_1(x_k)$ (namely , λ).

One may notice that the subset $\partial X \times [t_i, t_{i+1}]$ is not automatically a simplicial complex. We shall use Lemma 3.10 below to decompose it into a simplicial complex.

Because we do not change much from the grouping of x_j to the grouping of y_j and because we make (1) true when introduce groupings for new vertices y'_j, y''_j, etc., the groupings for any simplex inside $\partial X \times [t_i, t_{i+1}]$ will satisfy the condition (3) (of course with $\dim(X)$ NOT replaced by $\dim(\partial X)$).

Finally, let us mention that, we carry out the above construction separately for E_1, E_2, etc. Once this has been done for E_1, the same method can be used for E_2. The condition (1) will guarantee that when we work on E_2, we will not affect the condition (3) for E_1, which was supposed to be already satisfied. (The details will be contained in the proof of Lemma 3.11)

As we mentioned above, when we construct $E_1(y_j)$ from $E_1(x_j)$, we need to replace one element of $E_1(x_j)\backslash E_1^{model}$ by the element $\lambda \in E_1^{model}\backslash E_1(x_j)$. If we choose an arbitrary element $\mu \in E_1(x_j)\backslash E_1^{model}$ to be replaced by λ to define $E_1(y_j)$, then in general, $E_1(y_j)$ may not be extended to a grouping satisfying the condition (2), in other words, there may not exist a grouping $E_1, E_2, \cdots, E_m$ of E of type $(K_1, K_2, \cdots, K_m)$ such that $E_1 = E_1(y_j)$ and

$$E_k \subset \bigcup_{x \in \mathrm{Vertex}(X)} E_k(x), \quad k = 1, 2, \cdots, m.$$

So we need to give a condition to ensure that a subset $E_1 \subset E$ can be extended to a grouping satisfying condition (2). This will be discussed in 3.7 and 3.8. (See condition $(**)$ in 3.8.)

The proof of Lemma 3.4 will be given in 3.7 to 3.16.

3.7. We will put the inclusion $E_j(x) \subset \bigcup_{y \in \mathrm{Vertex}(X,\sigma)} E_j(y)$ in the condition (2) of Lemma 3.4, into a more general form, as follows. (In fact, we will use this more general form in our application.)

Suppose that $H_1, H_2, \cdots, H_m$ are (not necessarily disjoint) subsets of E, satisfying the following condition (called Condition $(*)$). For each subset $I \subset \{1, 2, \cdots, m\}$,

$$\#(\bigcup_{i \in I} H_i) \geq \sum_{i \in I} K_i \ . \tag{$*$}$$

It follows obviously that $H_1 \cup H_2 \cup \cdots \cup H_m = E$, since $\#(H_1 \cup H_2 \cup \cdots \cup H_m) \geq \sum_{i=1}^m K_i = \#(E)$.

From the Marriage Lemma of [HV] (or the Pairing Lemma in [Su]), the condition $(*)$ is a necessary and sufficient condition for the existence of a grouping $E_1, E_2, \cdots, E_m$ of E of type $(K_1, K_2, \cdots, K_m)$ with the condition $E_i \subset H_i$.

(Recall that, the Marriage Lemma of [HV] is stated as follows.

Suppose that there are two groups of K boys and K girls. Suppose that the following condition holds:

For any subset of K_1 girls ($K_1 = 1, 2, \cdots, K$), there are at least K_1 boys, each of them knows at least one girl from this subset.
Then there is a way to arrange marriage between them such that each boy marries one of the girls he knows.
Our claim above is a special case of this Marriage Lemma. One can see this as follows. Suppose that the K girls are from m different clubs, and the i-th club has exactly K_i girls. Number the boys by $1, 2, \cdots, K$. Let us define the relation consisting of a boy knowing a girl as follows. If $j \in H_i$, then the boy j knows all the girls in the i-th club. Otherwise, he does not know any girl in the i-th club. (Notice that the boy j could be in different H_i, so he could know girls from different clubs.) Obviously the condition (*) becomes the above condition in the Marriage Lemma. So if the condition $(*)$ holds, then there is a way to arrange the marriage as in the lemma. One can define E_i to be the set of boys each of whom marries a girl from the i-th club. Obviously $E_i \subset H_i$. This proves the sufficiency part of the condition. The necessary part is trivial.)
If we let $H_i = \bigcup_{y \in \mathrm{Vertex}(X,\sigma)} E_i(y)$, then the inclusion in (2) of Lemma 3.4 becomes $E_i(x) \subset H_i$ for each $x \in \mathrm{Vertex}(X, \sigma)$.
For any subset $I \subset \{1, 2, \cdots, m\}$, let

$$H_I = \bigcup_{i \in I} H_i \ .$$

3.8. We say that a subset $E_1 \subset H_1$, of K_1 elements, satisfies Condition $(**)$ if for any $I \subset \{2, 3, \cdots, m\}$,

$$(**) \qquad \#(H_I \backslash E_1) \geq \sum_{i \in I} K_i \ .$$

(Caution: $1 \notin I$.) Again, from the Marriage Lemma, $E_1 \subset H_1$ satisfies $(**)$ if and only if E_1 can be extended to a grouping $E_1, E_2, \cdots, E_m$ of E of type $(K_1, K_2, \cdots, K_m)$ such that $E_i \subset H_i$.

Lemma 3.9. *Suppose that E_1, F_1 $(\subset H_1)$ are two subsets satisfying $(**)$. If $\lambda \in F_1 \backslash E_1$, then there is a $\mu \in E_1 \backslash F_1$ such that*

$$E_1' = (E_1 \backslash \{\mu\}) \cup \{\lambda\},$$

satisfies $(**)$.

Proof: Let $G = E_1 \cup \{\lambda\}$. Since E_1 satisfies $(**)$, necessarily,

$$\#(H_I \backslash G) \geq \sum_{i \in I} K_i - 1 \ ,$$

for all subsets $I \subset \{2, 3, \cdots, m\}$.
Let $\tilde{H}_i = H_i \backslash G$, $i \in \{2, 3, \cdots, m\}$. And let $\tilde{H}_I = \cup_{i \in I} \tilde{H}_i$ for any $I \subset \{2, 3, \cdots, m\}$. The above inequality becomes $\#(\tilde{H}_I) \geq \sum_{i \in I} K_i - 1$.

Let I_0 be a minimum subset of $\{2,3,\cdots,m\}$ such that

$$\#(\tilde{H}_{I_0}) = \sum_{i\in I_0} K_i - 1 \ .$$

Note that such set I_0 exists, since if $I = \{2,3,\cdots,m\}$,

$$\#(\tilde{H}_I) = \sum_{i\in I} K_i - 1 \ .$$

Using the fact that $\#(H_{I_0}\backslash F_1) \geq \sum_{i\in I_0} K_i$, we can prove that

$$E_1 \cap H_{I_0} \not\subset F_1.$$

If it is not true, then $G\cap H_{I_0} \subset F_1$, since $\lambda \in F_1$ and $G = E_1 \cup \{\lambda\}$. And therefore,

$$\#(\tilde{H}_{I_0}) = \#(H_{I_0}\backslash G) \geq \#(H_{I_0}\backslash F_1) \geq \sum_{i\in I_0} K_i \ ,$$

which contradicts with the above equation.
Choose any element $\mu \in (E_1 \cap H_{I_0})\backslash F_1$; we will prove that μ is as desired in the lemma. I.e., the set

$$E_1' = (E_1\backslash\{\mu\}) \cup \{\lambda\} = G\backslash\{\mu\}$$

satisfies $(**)$. That is, for any $J \subset \{2,3,\cdots,m\}$, $\#(H_J\backslash E_1') \geq \sum_{i\in J} K_i$.
The proof is divided into three cases.
(i) *The case that* $J\cap I_0 = \emptyset$. By the relations

$$\tilde{H}_{I_0\cup J} = (\tilde{H}_J\backslash \tilde{H}_{I_0}) \cup \tilde{H}_{I_0} \qquad\qquad \text{(disjoint union)}$$

and

$$\#(\tilde{H}_{I_0\cup J}) \ \geq \sum_{i\in I_0\cup J} K_i - 1 \ ,$$

combined with the definition of I_0, one knows that

$$(a) \qquad \#(\tilde{H}_J\backslash \tilde{H}_{I_0}) \ \geq \sum_{i\in J} K_i \ ,$$

which is stronger than the condition

$$\#(H_J\backslash E_1') \ \geq \sum_{i\in J} K_i \ .$$

(ii) *The case that* $J \subset I_0$. Obviously, for $J = I_0$, we have

$$\#(H_{I_0}\backslash E_1') \ = \#(H_{I_0}\backslash G) + 1 = \sum_{i\in I_0} K_i \ ,$$

since $E_1' = G\backslash\{\mu\}$ and $\mu \in H_{I_0} \cap G$.
So we can suppose that $J \subsetneq I_0$.

By the minimality of I_0, we know that

$$\#(H_J\backslash G) \geq \sum_{i\in J} K_i \ .$$

Therefore,

$$\#(H_J\backslash E_1') \ \geq \sum_{i\in J} K_i \ .$$

(iii) *The general case.* Let $J_0 = J \cap I_0$, and $J_1 = J\backslash J_0$. Then

$$(H_J\backslash E_1') \supset (\tilde{H}_{J_1}\backslash \tilde{H}_{I_0}) \cup (H_{J_0}\backslash E_1'),$$

where the right hand side is a disjoint union since $J_0 \subset I_0$.
Evidently, this case follows from (a) above and case (ii).

□

The following lemma is perhaps well known.

LEMMA 3.10. *Let (X, σ_0) be a simplicial complex and (X, σ_1) be a subdivision of (X, σ_0). It follows that there is a simplicial structure σ of $X \times [0,1]$ such that*
(1) all vertices of $(X \times [0,1], \sigma)$ are on three subsets $X \times \{0\}$, $X \times \{\frac{1}{2}\}$, and $X \times \{1\}$;
(2) $(X \times [0,1], \sigma)|_{X\times\{0\}} = (X, \sigma_0)$, and $(X \times [0,1], \sigma)|_{X\times\{1\}} = (X, \sigma_1)$;
(3) For a simplex Δ of $(X \times [0,1], \sigma)$, there is a simplex Δ_0 of (X, σ_0) (caution: we do not use (X, σ_1)) such that

$$\Delta \subset \Delta_0 \times [0,1],$$

as a subset.

Proof: We prove it by induction on $\dim(X)$.
If X is 0-dimensional simplicial complex which consists of finitely many points, the conclusion is obvious, since $X \times [0,1]$ is finitely many disjoint intervals. (Note that, at this case, necessarily, $(X, \sigma_0) = (X, \sigma_1)$.) For us to visualize the general case later on, we introduce a new vertex $(x, \frac{1}{2}) \in X \times \{\frac{1}{2}\}$ for each $x \in X$. That is, we divide the interval $\{x\} \times [0,1]$ into two simplices $\{x\} \times [0, \frac{1}{2}]$ and $\{x\} \times [\frac{1}{2}, 1]$.
As the induction assumption, let us assume that the lemma is true for any n-dimensional complex. Let $\dim(X) = n + 1$.
Let $X^{(n)}$ be the n-skeleton of (X, σ_0) (we use σ_0 not σ_1 here). By the induction assumption, there is a simplicial structure σ' of $X^{(n)} \times [0,1]$ such that
(1) all vertices of $(X^{(n)} \times [0,1], \sigma')$ are on three subsets $X^{(n)} \times \{0\}$, $X^{(n)} \times \{\frac{1}{2}\}$, and $X^{(n)} \times \{1\}$;

(2) $(X^{(n)} \times [0,1], \sigma')|_{X^{(n)} \times \{0\}} = (X^{(n)}, \sigma_0|_{X^{(n)}})$, and
$(X^{(n)} \times [0,1], \sigma')|_{X^{(n)} \times \{1\}} = (X^{(n)}, \sigma_1|_{X^{(n)}})$;

(3) For an simplex Δ of $(X^{(n)} \times [0,1], \sigma')$, there is a simplex Δ_0 of $(X^{(n)}, \sigma_0)$ such that

$$\Delta \subset \Delta_0 \times [0,1],$$

as a subset.

Let us introduce the simplicial structure σ on $X \times [0,1]$ such that $(X \times [0,1], \sigma)|_{X^{(n)} \times [0,1]} = (X^{(n)} \times [0,1], \sigma')$, $(X \times [0,1], \sigma)|_{X \times \{0\}} = (X, \sigma_0)$, and $(X \times [0,1], \sigma)|_{X \times \{1\}} = (X, \sigma_1)$.

Consider each $\Delta \times [0,1]$ for any $(n+1)$-simplex Δ of (X, σ_0) (again, we use σ_0 not σ_1). From the above, we already have the simplicial structure on the boundary

$$\partial(\Delta \times [0,1]) = (\Delta \times \{0\}) \cup (\partial\Delta \times [0,1]) \cup (\Delta \times \{1\}).$$

Namely, on $\Delta \times \{0\}$, we use σ_0; on $\partial\Delta \times [0,1]$, we use σ'; and on $\Delta \times \{1\}$, we use σ_1.

Let c be the barycenter of Δ, introduce a new vertex $C = (c, \frac{1}{2}) \in X \times \{\frac{1}{2}\}$. The simplices of σ on $\Delta \times [0,1]$ are of the following forms.

(i) C itself is a zero dimensional simplex;

(ii) Any simplex of the boundary $\partial(\Delta \times [0,1])$ is a simplex for σ on $\Delta \times [0,1]$; and

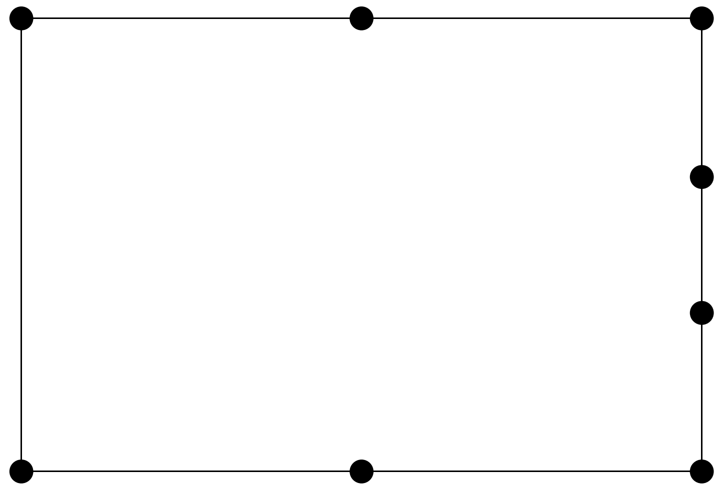
(iii) For any simplex Δ' of the boundary $\partial(\Delta \times [0,1])$, the convex hull of $\Delta' \cup \{C\}$ is a simplex of dimension $(\dim(\Delta') + 1)$ for σ on $\Delta \times [0,1]$.

Define such simplicial structure for each $(n+1)$-simplex separately, and put them together give rise to a simplicial structure of $X \times [0,1]$, which obviously satisfies the conditions (1), (2), and (3).

(Note that the simplicial structure on $\partial\Delta \times [0,1]$ is as the same as σ', therefore the simplicial structure on $\Delta \times [0,1]$ and on $\Delta_1 \times [0,1]$ for different $(n+1)$-dimensional simplices Δ and Δ_1 are compatible on the intersection $(\Delta \cap \Delta_1) \times [0,1]$.)

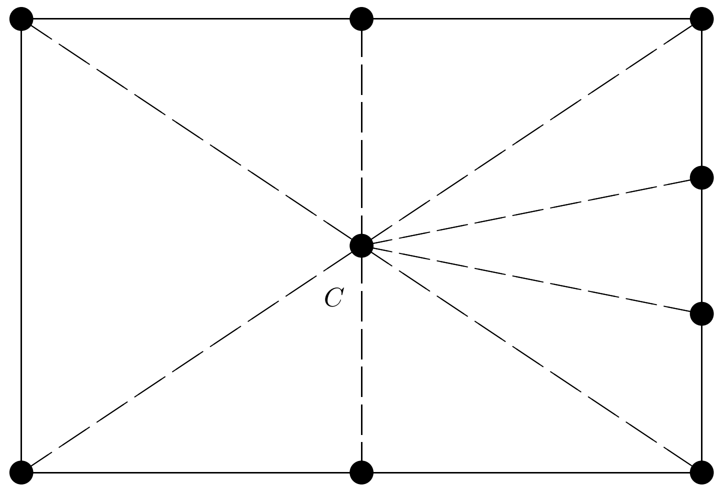
The following pictures may help the reader to visualize the construction. They are pictures only for the case $n = 0$, $\dim(\Delta) = 1$, and $\dim(\Delta \times [0,1]) = 2$.

Suppose the simplicial structure for the boundary $\partial(\Delta \times [0,1])$ is as follows. (The dots represent vertices.)



$\Delta \times \{0\}$ $\Delta \times \{1\}$

Then the simplicial structure on $\Delta \times [0,1]$ will be described by the following picture.



$\Delta \times \{0\}$ $\Delta \times \{\frac{1}{2}\}$ $\Delta \times \{1\}$

$\square$

The following lemma presents the main technical step of this section.

Lemma 3.11. *Suppose that $\{H_1, H_2, \cdots, H_m\}$ satisfies the condition $(*)$. Suppose that (X,σ) is a simplicial complex consisting of a single simplex Δ_0 and its faces. Let $(Y,\sigma) = (\partial\Delta_0, \sigma)$, and (Y,τ) be a subdivision of (Y,σ). Suppose that it is assigned, for each vertex $x \in (Y,\tau)$, a set $E_1(x) \subset H_1$ which satisfies the condition $(**)$. Furthermore, suppose that for any simplex Δ of (Y,τ),*

$$\#\left(\bigcap_{y\in\, Vertex(\Delta)} E_1(y)\right) \geq K_1 - \frac{\dim Y(\dim Y+1)}{2}\ .$$

It follows that there are a subdivision $(X,\tilde{\tau})$ of (X,σ) and an assignment, for each vertex $x \in Vertex(X,\tilde{\tau})$, a set $E_1(x) \subset H_1$, satisfying condition $(**)$, with the following conditions.
(1) $(X,\tilde{\tau})|_Y = (Y,\tau)$, and for each vertex $y \in Vertex(Y,\tau)$, the assignment $E_1(y)$ is as same as the original one.
(2) For any $x \in (X,\tilde{\tau})$,

$$E_1(x) \supset \bigcap_{y\in\, Vertex(Y,\tau)} E_1(y)\ .$$

(3) For any simplex Δ of $(X,\tilde{\tau})$,

$$\#\left(\bigcap_{x\in\, Vertex(\Delta)} E_1(x)\right) \geq K_1 - \frac{\dim X(\dim X+1)}{2}\ .$$

Proof: The Lemma is proved by induction on the dimension of the simplex. If $dim(\Delta_0) = 0$, then $\Delta_0 = \{pt\}$, a set of single point, and $\partial\Delta_0 = \emptyset$. Obviously, the lemma holds by choosing any $E_1(pt) \subseteq H_1$ of K_1 element to satisfy $(**)$. Let us prove the 1-dimensional case. Logically, this part could be skipped. But the proof of this case will be easier to visualize which can be used to understand the general case.
Suppose that $dim(\Delta_0) = 1$. Δ_0 is a line segment $[0,1]$. Divide $[0, 1]$ into several subintervals by

$$0 = t_0^0 < t_1^0 < t_2^0 < \cdots < t_{a-1}^0 < t_a^0 = \frac{1}{2} = t_a^1 < t_{a-1}^1 < \cdots < t_2^1 < t_1^1 < t_0^1 = 1.$$

(The natural number a is to be determined later.) The points $\{t_j^i\}_{i=0,1;j=1,2\cdots,a}$ will be the new vertices of $(\Delta_0, \tilde{\tau})$. (Note that t_a^0 is the same vertex as t_a^1.) Choose a model $E_1^{model} \subset H_1$ to satisfy $(**)$ and

$$E_1^{model} \supset E_1(t_0^0) \cap E_1(t_0^1).$$

In fact, one can choose E_1^{model} to be either $E_1(t_0^0)$ or $E_0(t_0^1)$. (Note that $t_0^0 = 0$ and $t_0^1 = 1$ are vertices of (Δ_0, σ).)
Let $G = E_1(t_0^0) \cap E_1(t_0^1) \cap E_1^{model}$. Without loss of generality, we can assume that there is $\lambda \in E_1^{model} \backslash G$. Otherwise, $E_1^{model} = G = E_1(t_0^0) = E_1(t_0^1)$, and the conclusion already holds before introducing any subdivision.
By Lemma 3.9, if $\lambda \notin E_1(t_0^i)$ $(i = 0, 1)$, then there is a $\mu \in E_1(t_0^i) \backslash E_1^{model}$ such that $E_1(t_0^i) \cup \{\lambda\} \backslash \{\mu\}$ satisfies $(**)$. Define

$$E_1(t_1^i) = \begin{cases} E_1(t_0^i) & \text{if } \lambda \in E_1(t_0^i) \\ (E_1(t_0^i) \cup \{\lambda\}) \backslash \{\mu\} & \text{if } \lambda \notin E_1(t_0^i) \ . \end{cases}$$

Then $E_1(t_1^i) \supset G \cup \{\lambda\}$. Therefore,

$$E_1(t_1^0) \cap E_1(t_1^1) \cap E_1^{model} \underset{\neq}{\supset} E_1(t_0^0) \cap E_1(t_0^1) \cap E_1^{model}.$$

Suppose that we already have the definitions of $E_1(t_i^0)$ and $E_1(t_i^1)$, we can define $E_1(t_{i+1}^0)$ and $E_1(t_{i+1}^1)$ exactly the same as above (i in place of 0, and $i+1$ in place of 1), and obtain

$$E_1(t_{i+1}^0) \cap E_1(t_{i+1}^1) \cap E_1^{model} \underset{\neq}{\supset} E_1(t_i^0) \cap E_1(t_i^1) \cap E_1^{model}.$$

Carrying out this procedure for at most finitely many times, we will reach $E_1(t_{a-1}^0) \cap E_1(t_{a-1}^1) \cap E_1^{model} = E_1^{model}$. Then define $E_1(t_a^i) = E_1^{model}$. (Note that $t_a^1 = t_a^0 = \frac{1}{2}$.)
For $i = 0, 1; j = 0, 1, 2, \cdots, a-1$,

$$\#(E_1(t_j^i) \cap E_1(t_{j+1}^i)) \geq K_1 - 1 = K_1 - \frac{dim(\Delta_0)(dim(\Delta_0)+1)}{2},$$

since we take out at most one point from $E_1(t_j^i)$ to define $E_1(t_{j+1}^i)$. This proves that the lemma holds for $n = 1$.
(Let us point out that for one dimensional case, the proof could be simpler. We choose the above proof to present some idea for the general case below.)
Suppose that the lemma is true for any simplex of dimension $\leq n-1$. We will prove it for $\dim(X) = n$. (One should compare to the explanation in 3.6.)
Step 1. Identify Δ_0 with $\partial\Delta_0 \times [0,1]/\partial\Delta_0 \times \{1\}$. Regard $\partial\Delta_0$ as $\partial\Delta_0 \times \{0\} \subset \partial\Delta_0 \times [0,1]$. Note that $\partial\Delta_0 \times \{1\}$ is identified as a single point which is the center of Δ_0, and is NOT a vertex of (Δ_0, σ).
Choose $0 = t_0 < t_1 < \cdots < t_a = 1$. (The natural number a is to be determined later.)
We will first introduce some new vertices (for the subdivision $(X, \tilde{\tau})$) on $\partial\Delta_0 \times \{t_1\}, \partial\Delta_0 \times \{t_2\}, \cdots, \partial\Delta_0 \times \{t_a\}$, and define E_1 for those vertices.
Later on (in Step 4), we will consider each $\partial\Delta \times [t_i, t_{i+1}]$ to be $X \times [0,1]$ in Lemma 3.10, and introduce new vertices on $\partial\Delta \times \{\frac{t_i + t_{i+1}}{2}\}$ (in place of $X \times \{\frac{1}{2}\}$). (We need to do this, because $\partial\Delta \times [t_i, t_{i+1}]$ is not automatically a simplicial complex.)

Choose a model $E_1^{model} \subset H_1$ to satisfy $(**)$. We also require that

$$E_1^{model} \supset \bigcap_{y \in \mathrm{Vertex}(Y,\tau)} E_1(y) \ .$$

(One can choose E_1^{model} to be $E_1(y)$ for any vertex $y \in \mathrm{Vertex}(Y,\tau)$.)
Define $E_1\{(x,1)\} = E_1^{model}$. (Note that $\{(x,1)\} \subset \partial\Delta_0 \times [0,1]$ is identified to a single point, the center of Δ_0.)
The construction will be carried out in Step 2, 3 and 4. The procedure can be outlined as follows. If we already have the construction of simplicial structure $\tilde{\tau}$ for $\partial\Delta_0 \times \{t_{i-1}\}$ and the definition of E_1 on all vertices in $\mathrm{Vertex}(\partial\Delta_0 \times \{t_{i-1}\})$, then, to define the simplicial structure on $\partial\Delta_0 \times [t_{i-1}, t_i]$ (in particular, to introduce vertices on $\partial\Delta_0 \times \{t_i\}$), and to define E_1 on the newly introduced vertices (on $\partial\Delta_0 \times \{t_i\}$), we will only use the simplicial structure and the definition of E_1 on $\partial\Delta_0 \times \{t_{i-1}\}$.
In this procedure, if there is a vertex $x \in \mathrm{Vertex}(\partial\Delta_0 \times \{t_{i-1}\}, \tilde{\tau})$ such that

$$E_1(x) \neq E_1^{model} \ ,$$

then we will require that

$$(a) \qquad \bigcap_{x \in \mathrm{Vertex}(\partial\Delta_0 \times \{t_i\}, \tilde{\tau})} E_1(x) \cap E_1^{model} \underset{\neq}{\supset} \bigcap_{x \in \mathrm{Vertex}(\partial\Delta_0 \times \{t_{i-1}\}, \tilde{\tau})} E_1(x) \cap E_1^{model} \ .$$

(That is, the sets E_1's on $\partial\Delta_0 \times \{t_i\}$ are globally closer to E_1^{model} than those on $\partial\Delta_0 \times \{t_{i-1}\}$.) Finally, within finitely many steps, we will reach that, for certain $i-1$, and for all vertices $x \in \mathrm{Vertex}(\partial\Delta_0 \times \{t_{i-1}\}, \tilde{\tau})$,

$$E_1(x) = E_1^{model} \ .$$

Then we choose $t_i = t_a = 1$, and choose any simplicial structure on $\partial\Delta_0 \times [t_{a-1}, 1]/\partial\Delta_0 \times \{1\}$ with vertex set to be $\mathrm{Vertex}(\partial\Delta_0 \times \{t_{a-1}\}) \cup \partial\Delta_0 \times \{1\}$. Recall that the set $\partial\Delta_0 \times \{1\}$ is identified as a single point with $E_1(\partial\Delta_0 \times \{1\}) = E_1^{model}$.)
Furthermore, in this procedure, we not only make (3) true for any simplex in $\partial\Delta_0 \times [t_{i-1}, t_i]$, but also make the following stronger statement true for any simplex Δ lies on $\partial\Delta_0 \times \{t_i\}$:

$$(b) \qquad \#\left(\bigcap_{y \in \mathrm{Vertex}(\Delta)} E_1(y) \right) \ \geq \ K_1 - \frac{(n-1)n}{2} \ .$$

(Note that $n-1 = \dim(\partial\Delta_0 \times \{t_i\}) = \dim(\partial\Delta_0)$.) This condition has to be satisfied for the construction of the next step by induction.
Step 2. We will do all the above construction only for $\partial\Delta_0 \times [t_0, t_1]$. For the other part of the construction, one uses induction argument with aid of (b) (i.e., let t_{i-1} play the role of t_0, and t_i play the role of t_1.)

Let $\{(y_1, t_0), (y_2, t_0), \cdots, (y_p, t_0)\}$ be the vertices of $\partial\Delta_0 \times \{t_0\} = Y$. There is a simplicial complex structure on $\partial\Delta_0 \times \{t_1\}$, which is exactly the same as that of $(\partial\Delta_0 \times \{t_0\}, \tilde{\tau})$, since both $\partial\Delta_0 \times \{t_1\}$ and $\partial\Delta_0 \times \{t_0\}$ can be regarded as $\partial\Delta_0$. We call such simplicial complex $\tilde{\tau}_{pre}$. Therefore, each point (y_i, t_1) $(1 \leq i \leq p)$ is a vertex of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$. We will introduce more vertices later.
Let $G = E_1(y_1, t_0) \cap E_1(y_2, t_0) \cap \cdots \cap E_1(y_p, t_0) \cap E_1^{model}$. If $G = E_1^{model}$, then $E_1(y_i, t_0) = E_1^{model}$ for each $1 \leq i \leq p$ and the construction is done. So we assume

$$G \neq E_1^{model}.$$

Choose $\lambda \in E_1^{model} \backslash G$. When we define $E_1(x)$ for any vertex $x \in \partial\Delta_0 \times \{t_1\}$, it is always required that

$$E_1(x) \supset G \cup \{\lambda\}.$$

Therefore, (a) holds for the pair $\{t_0, t_1\}$.
For each point (y_i, t_0), if $\lambda \notin E_1(y_i, t_0)$, by Lemma 3.9, there is $\mu \in E_1(y_i, t_0) \backslash E_1^{model}$ such that $(E_1(y_i, t_0) \cup \{\lambda\}) \backslash \{\mu\}$ satisfies $(**)$. Define

$$E_1(y_i, t_1) = \begin{cases} E_1(y_i, t_0) & \text{if } \lambda \in E_1(y_i, t_0) \\ (E_1(y_i, t_0) \cup \{\lambda\}) \backslash \{\mu\} & \text{if } \lambda \notin E_1(y_i, t_0) \end{cases}.$$

In this way, obviously, $E_1(x) \supset G \cup \{\lambda\}$ for each vertex $x = (y_i, t_1) \in \mathrm{Vertex}(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$.
Step 3. Note that the definition of E_1 on $\mathrm{Vertex}(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ may not satisfies (b). Therefore we can not use the simplicial structure $\tilde{\tau}_{pre}$ and the definition of E_1 on $\mathrm{Vertex}(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ to construct simplicial structure and the definition of E_1 for $\partial\Delta_0 \times \{t_2\}$. We need to introduce a subdivision for $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ and the definitions of E_1 for new vertices to make (b) true. (This step is not needed in the one dimensional case, since for any zero dimensional simplex (which is a point), (b) automatically holds.)
Apply the induction assumption to each simplex of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ with the above definition of E_1 on $\mathrm{Vertex}(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$, from the simplices of the lowest dimension (dimension 1) to the simplices of the highest dimension (dimension $n-1$). (Note that each such simplex has dimension at most $n-1$.) One should begin with each 1-simplex (with boundary being two points — two 0-simplices), then each 2-simplex, and so on.
First, let e be any 1-simplex of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ with boundary $\partial e = \{v_0, v_1\}$. Obviously, the condition of Lemma 3.11 automatically holds for simplex e in place of Δ_0 and ∂e in place of $\partial\Delta_0$, since ∂e is zero-dimensional. By the induction assumption, there is a subdivision $(e, \tilde{\tau})$ of $(e, \tilde{\tau}_{pre})$ and the definition of E_1 for each vertex of $(e, \tilde{\tau})$ such that
(1) The definition of E_1 on the original vertices $\{v_0, v_1\}$ are the same as before.
(2) For any $x \in \mathrm{Vertex}(e, \tilde{\tau})$,

$$E_1(x) \supset E_1(v_0) \cap E_1(v_1).$$

(3) For any simplex e' of $(e,\tilde{\tau})$ (a line segment of e)

$$\bigcap_{x\in \mathrm{Vertex}(e')} E_1(x) \geq K_1 - \frac{dim(e)(dim(e)+1)}{2}.$$

After we have done the above procedure for each 1-simplex, we can do it for each 2-simplex, since we already have simplicial structure and the definition of E_1 for the boundary of any 2-simplex as required in the condition of Lemma 3.11.

Going through this way, finally, one obtains a subdivision $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau})$ of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ and the definition of E_1 for each newly introduced vertex, such that the following two statements hold.

1. For each old simplex Δ of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau}_{pre})$ and any new vertex $x \in \Delta$,

$$(c) \qquad E_1(x) \supset \bigcap_{y\in \mathrm{Vertex}(\Delta,\tilde{\tau}_{pre})} E_1(y)\ .$$

2. If Δ is a simplex of $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau})$, then

$$\#\left(\bigcap_{y\in \mathrm{Vertex}(\Delta)} E_1(y)\right) \quad \geq \quad K_1 - \frac{\dim Y(\dim Y+1)}{2} = K_1 - \frac{(n-1)n}{2}\ .$$

(This is the requirement (b) in Step 1.)

The first statement is the induction assumption of validity of (2) and the second statement is the induction assumption of validity of (3).

Step 4. In this step, we will apply Lemma 3.10 to define the simplicial structure $\tilde{\tau}$ on $\partial\Delta_0 \times [t_0,t_1]$ and the definitions of E_1 on all vertices. Note that we already have simplicial structure $\tilde{\tau}$ on $\partial\Delta_0 \times \{t_0\}$ and on $\partial\Delta_0 \times \{t_1\}$. Furthermore, $\tilde{\tau}|_{\partial\Delta_0\times\{t_1\}}$ is a subdivision of $\tilde{\tau}|_{\partial\Delta_0\times\{t_0\}}$ if we regard both $\partial\Delta_0 \times \{t_0\}$ and $\partial\Delta_0 \times \{t_1\}$ as $\partial\Delta_0$. Apply Lemma 3.10 (with $\partial\Delta_0$ in place of X) to obtain the simplicial structure on $\partial\Delta_0 \times [t_0,t_1]$ (we only need to introduce new vertices on $\partial\Delta_0 \times \{\frac{t_0+t_1}{2}\}$).

For each new vertex $(u, \frac{t_0+t_1}{2}) \in \partial\Delta_0 \times \{\frac{t_0+t_1}{2}\}$, consider $(u,t_0) \in \partial\Delta_0 \times \{t_0\}$. From 3.1, there is a unique simplex Δ of $(\partial\Delta_0 \times \{t_0\}, \tilde{\tau})$ such that $(u,t_0) \in$ interior(Δ). Choose any vertex x of Δ and define $E_1(u, \frac{t_0+t_1}{2}) = E_1(x)$.

So we have the simplicial structure $\tilde{\tau}$ on $\partial\Delta_0 \times [t_0,t_1]$ and the definition of $E_1(x)$ for each $x \in \mathrm{Vertex}(\partial\Delta_0 \times [t_0,t_1])$. We need to verify the condition (3). Let Δ be any simplex of $(\partial\Delta_0 \times \{t_0\}, \tilde{\tau})$ with vertices $\{(u_0,t_0),(u_1,t_0),\cdots,(u_i,t_0)\}$. Then

$$\#(E_1(u_0,t_0)\cap E_1(u_1,t_0)\cap\cdots\cap E_1(u_i,t_0)) \geq K_1 - \frac{\dim Y(\dim Y+1)}{2}\ .$$

Let $G_1 = E_1(u_0, t_0) \cap E_1(u_1, t_0) \cap \cdots \cap E_1(u_i, t_0)$. From the above definition of E_1 for vertices of $\partial\Delta_0 \times \{\frac{t_0+t_1}{2}\}$, we know that if $(u, t_0) \in \Delta$ and $(u, \frac{t_0+t_1}{2}) \in \mathrm{Vertex}(\partial\Delta_0 \times [0,1], \tilde{\tau})$, then

$$E_1(u, \frac{t_0+t_1}{2}) \supset G_1. \tag{d}$$

Since each $E_1(u_j, t_1)$ is either $E_1(u_j, t_0)$ or is obtained by replacing one element of $E_1(u_j, t_0)$ by λ, we have

$$\begin{aligned} &\#(G_1 \cap E_1(u_0, t_1) \cap E_1(u_1, t_1) \cap \cdots \cap E_1(u_i, t_1)) \\ &\quad \geq K_1 - \tfrac{\dim Y(\dim Y+1)}{2} - (i+1) \\ &\quad \geq K_1 - \tfrac{(n-1)n}{2} - n \\ &\quad = K_1 - \tfrac{n(n+1)}{2}\ . \end{aligned} \tag{e}$$

(Note that there are $i+1$ $(\leq n)$ sets of $\{E_1(u_j, t_1)\}_{j=0}^{i}$, and, therefore, at most $i+1$ points were taken out from G_1.)
Recall that $\tilde{\tau}$ on $\Delta_0 \times \{t_1\}$ is the subdivision of $\tilde{\tau}_{pre}$. By (c) of Step 3, we have

$$\begin{aligned} \bigcap_{x \in \mathrm{Vertex}(\Delta \times \{t_1\}, \tilde{\tau})} E_1(x) &\supset \bigcap_{y \in \mathrm{Vertex}(\Delta \times \{t_1\}, \tilde{\tau}_{pre})} E_1(y) \\ &= E_1(u_0, t_1) \cap E_1(u_1, t_1) \cap \cdots \cap E_1(u_i, t_1). \end{aligned}$$

(Note that (c) implies that the above "$\supset$" holds if the left hand side of "$\supset$" is replaced by $E_1(x)$ for any $x \in \mathrm{Vertex}(\Delta \times \{t_1\}, \tilde{\tau})$, so it also holds for the intersection of these $E_1(x)$. In fact, the above "$\supset$" can be replaced by "$=$".)
Then combining it with (d), we have

$$\bigcap_{x \in \mathrm{Vertex}(\Delta \times [t_0, t_1], \tilde{\tau})} E_1(x) = G_1 \cap E_1(u_0, t_1) \cap E_1(u_1, t_1) \cap \cdots \cap E_1(u_i, t_1)$$

which has at least $K_1 - \frac{n(n+1)}{2}$ elements by (e). Combining this fact with (3) of Lemma 3.10, we know that the desired condition (3) holds for any simplex of $(\partial\Delta_0 \times [t_0, t_1], \tilde{\tau})$.
Evidently, (2) holds from the construction.
Since (b) holds for $\partial\Delta_0 \times \{t_1\}$, one can continue this procedure. This ends the proof.

$\square$

Corollary 3.12. *Suppose that $\{H_1, H_2, \cdots, H_m\}$ satisfies the condition (*). Suppose that (X, σ) is a simplicial complex consisting of a single simplex and its faces. Suppose that there is assigned, for each vertex $x \in (X, \sigma)$, a set $E_1(x) \subset H_1$ which satisfies the condition (**).*
It follows that there are a subdivision (X, τ) of (X, σ) and an assignment, for each new vertex $x \in$ Vertex(X, τ), a set $E_1(x) \subset H_1$, satisfying condition (**),

with the following conditions. (The definition of E_1 for the old vertex should not be changed.)
(1) For any $x \in$ Vertex(X,τ),

$$E_1(x) \supset \bigcap_{y \in \, Vertex(X,\sigma)} E_1(y) \ .$$

(2) For any simplex Δ of (X,τ),

$$\#\left(\bigcap_{x \in \, Vertex(\Delta)} E_1(x) \right) \geq K_1 - \frac{\dim X(\dim X + 1)}{2} \ .$$

Proof: To prove this corollary, one needs to apply Lemma 3.11 to simplices from the lowest dimension (e.g. dimension one simplex whose boundary consists two vertices of (X,σ)) to the highest dimension (e.g the simplex X itself with boundary ∂X). Each time, we only work on a single simplex Δ of (X,σ). And when we work on Δ, we should assume that we already have the subdivision and the definition of E_1 on the boundary $\partial\Delta$ to satisfy the condition in Lemma 3.11 with $\dim(\partial\Delta)$ in place of $\dim(Y)$.

$\square$

Corollary 3.13. *Let (X,σ) be a simplicial complex consisting of a single simplex X and all its faces. Suppose that associated to each $x \in$ Vertex(X,σ), there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E.*
It follows that there is a subdivision (X,τ) of (X,σ), and associated to each new vertex $x \in$ Vertex(Δ,τ), there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E (for any old vertex of (Δ,σ), the grouping should not be changed), such that the following hold.
For each newly introduced vertex $x \in$ Vertex(X,τ),

$$E_j(x) \subset \bigcup_{y \in \, Vertex(X,\sigma)} E_j(y), \qquad j = 1, 2, \cdots, m. \tag{2}$$

For any simplex Δ of (X,τ) (after subdivision),

$$\#(\bigcap_{x \in \, Vertex(\Delta)} E_1(x)) \geq K_1 - \frac{n(n+1)}{2} \tag{3}$$

where $n = \dim X$.
(In this corollary, we do not require the condition (1) in Lemma 3.4. This will be done in the next corollary.)

Proof: Set $\bigcup_{y \in \mathrm{Vertex}(X,\sigma)} E_j(y) := H_j, \quad j = 1, 2, \cdots, m.$ Then

$H_1, H_2, \cdots, H_m$ satisfy condition $(*)$, and for each $x \in \text{Vertex}(X,\sigma)$, $E_1(x) \subset H_1$ satisfies $(**)$.
Applying Corollary 3.12, we obtain the subdivision (X,τ) and the definition of $E_1(x)$, for each new vertex, to satisfy condition $(**)$, and (1) and (2) in the Corollary 3.12.
For each new vertex x, since $E_1(x)$ satisfies $(**)$, we can extend it to a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ such that $E_i(x) \subset H_i$. Therefore this grouping satisfy the condition (2) of our corollary.
The condition (3) follows from the condition (2) of Corollary 3.12. Thus the corollary is proved.

$\square$

Corollary 3.14. *Let (X,σ) be a simplicial complex consisting of a single simplex X and all its faces. Suppose that associated to each $x \in \mathit{Vertex}(X,\sigma)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E.*
It follows that there is a subdivision (X,τ) of (X,σ), and associated to each new vertex $x \in \mathit{Vertex}(\Delta,\tau)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E (for any old vertex of (Δ,σ), the grouping should not be changed), such that the following hold.
For each newly introduced vertex $x \in \mathit{Vertex}(X,\tau)$,

$$(1) \qquad \bigcap_{y \in \mathit{Vertex}(X,\sigma)} E_j(y) \subset E_j(x), \qquad j = 1,2,\cdots,m,$$

and

$$(2) \qquad E_j(x) \subset \bigcup_{y \in \mathit{Vertex}(X,\sigma)} E_j(y), \qquad j = 1,2,\cdots,m.$$

For any simplex Δ of (X,τ) (after subdivision),

$$(3) \qquad \#\Big(\bigcap_{x \in \mathit{Vertex}(\Delta)} E_1(x)\Big) \geq K_1 - \frac{n(n+1)}{2}.$$

where $n = \dim X$.
(Comparing this Corollary to Lemma 3.4, the only difference is that we require (3) holds only for E_1 in the corollary.)

Proof: The only difference between this corollary and Corollary 3.13 is that we require condition (1) holds. To make (1) hold, we need to do the following. Reserve all the subsets $\bigcap_{y\in\text{Vertex}(X,\sigma)} E_j(y)$, $j = 1,2,\cdots,m$, which are supposed to be in $E_j(x)$ (if we want the condition (1) to hold), for any newly introduced vertex x; group the rest of the elements of E (using Corollary 3.13); and finally put $\bigcap_{y\in\text{Vertex}(X,\sigma)} E_j(y)$ into each $E_j(x)$. The details are as follows.
Set $\bigcap_{y\in\text{Vertex}(X,\sigma)} E_j(y) := D_j$, $j = 1,2,\cdots,m$. Then D_j, $j = 1,2,\cdots,m$ are mutually disjoint. To see this, we fix a $y \in \text{Vertex}(X,\sigma)$, and notice

that $D_j \subset E_j(y)$, and $E_j(y)$, $j = 1, 2, \cdots, m$ are mutually disjoint, from the definition of grouping. Similarly, if $j_1 \neq j_2$, then $D_{j_1} \cap E_{j_2}(y) = \emptyset$, for any $y \in \mathrm{Vertex}(X, \sigma)$.
Consider $E' = E \backslash (\cup_j D_j)$ and the m-tuple

$$(K_1', K_2', \cdots, K_m') = (K_1 - \#(D_1), K_2 - \#(D_2), \cdots, K_m - \#(D_m)).$$

For any $y \in \mathrm{Vertex}(X, \sigma)$, the grouping $E_1(y), E_2(y), \cdots, E_m(y)$ of E of type $(K_1, K_2, \cdots, K_m)$ induces a grouping $E_1'(y), E_2'(y), \cdots, E_m'(y)$ of E' of type $(K_1', K_2', \cdots, K_m')$, by setting $E_j'(y) = E_j(y) \backslash D_j$.
Apply Corollary 3.13 to the simplex (X, σ) and those groupings of E', to obtain a subdivision (X, τ) and groupings $E_1'(x), E_2'(x), \cdots, E_m'(x)$ of E' for all newly introduced vertices $x \in \mathrm{Vertex}(X, \tau)$ such that the following hold.
For each newly introduced vertex $x \in \mathrm{Vertex}(X, \tau)$,

$$(2') \qquad E_j'(x) \subset \bigcup_{y \in \mathrm{Vertex}(X,\sigma)} E_j'(y), \qquad j = 1, 2, \cdots, m.$$

For any simplex Δ of (X, τ) (after subdivision),

$$(3') \qquad \#\Big(\bigcap_{x \in \mathrm{Vertex}(\Delta)} E_1'(x)\Big) \geq K_1' - \frac{n(n+1)}{2},$$

where $n = \dim X$.
Finally, let $E_j(x) = E_j' \cup D_j$ for any $x \in \mathrm{Vertex}(X, \tau)$. Then the desired condition (1) of the corollary means $D_j \subset E_j(x)$, which is true from the definition. Also the conditions (2) and (3) of the corollary follows from (2′) and (3′).

$\square$

Corollary 3.15. *Suppose that (X, σ) is a simplicial complex. Suppose that for each vertex $x \in \mathrm{Vertex}(X, \sigma)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E.*
It follows that there is a subdivision (X, τ) of (X, σ), and there is an extension of the definition of the groupings of E for $\mathrm{Vertex}(X, \sigma)$ to the groupings of E for $\mathrm{Vertex}(X, \tau) \supset \mathrm{Vertex}(X, \sigma)$ such that the following properties hold.
For each newly introduced vertex $x \in \mathrm{Vertex}(X, \tau)$, if $x \in \Delta$, where Δ is a simplex of (X, σ)(before subdivision), then

$$(1) \qquad \bigcap_{y \in \mathrm{Vertex}(\Delta,\sigma)} E_j(y) \subset E_j(x), \qquad j = 1, 2, \cdots, m,$$

and

$$(2) \qquad E_j(x) \subset \bigcup_{y \in \mathrm{Vertex}(\Delta,\sigma)} E_j(y), \qquad j = 1, 2, \cdots, m.$$

For any simplex Δ_1 of (X,τ) (after subdivision),

$$\#\Big(\bigcap_{x\in \mathit{Vertex}(\Delta_1)} E_1(x)\Big) \geq K_1 - \frac{n(n+1)}{2}, \tag{3}$$

where $n = \dim X$.
(The above (1) and (2) imply that for any $x \in \mathrm{Vertex}(X,\tau)$,

$$\bigcap_{y\in \mathrm{Vertex}(X,\sigma)} E_j(y) \subset E_j(x) \subset \bigcup_{y\in \mathrm{Vertex}(X,\sigma)} E_j(y), \qquad j = 1,2,\cdots,m.)$$

Proof: The proof is exactly the same as that of Corollaries 3.13 and 3.14. In fact, in the proof of Corollary 3.13, we were working simplex by simplex from the lowest dimension to the highest dimension. As same as Corollary 3.13, when we work on simplex Δ, we should suppose that, we have already done with $\partial\Delta$. The only difference is the following. We should choose the sets H_i, D_i differently according to the simplex we are working on. For simplex Δ, choose $H_i = \bigcup_{y\in \mathrm{Vertex}(\Delta)} E_i(y),\ D_i = \bigcap_{y\in \mathrm{Vertex}(\Delta)} E_i(y),\ i = 1,2\cdots,m.$ $\square$

Lemma 3.4 is a special case of the following theorem.

Theorem 3.16. *Suppose that (X,σ) is a simplicial complex. Suppose that for each vertex $x \in \mathit{Vertex}(X,\sigma)$, there is a grouping $E_1(x), E_2(x), \cdots, E_m(x)$ of E.*
It follows that there is a subdivision (X,τ) of (X,σ), and there is an extension of the definition of the groupings of E for $\mathit{Vertex}(X,\sigma)$ to the groupings of E for $\mathit{Vertex}(X,\tau) \supset \mathit{Vertex}(X,\sigma)$ such that the following properties hold.
For each newly introduced vertex $x \in \mathit{Vertex}(X,\tau)$, if $x \in \Delta$, where Δ is a simplex of (X,σ) (before subdivision), then

$$\bigcap_{y\in \mathit{Vertex}(\Delta,\sigma)} E_j(y) \subset E_j(x), \qquad j = 1,2,\cdots,m, \tag{1}$$

and

$$E_j(x) \subset \bigcup_{y\in \mathit{Vertex}(\Delta,\sigma)} E_j(y), \qquad j = 1,2,\cdots,m. \tag{2}$$

For any simplex Δ_1 of (X,τ) (after subdivision),

$$\#\Big(\bigcap_{x\in \mathit{Vertex}(\Delta_1)} E_j(x)\Big) \geq K_j - \frac{n(n+1)}{2}, \quad j = 1,2,\cdots,m, \tag{3}$$

where $n = \dim X$.

Proof: We will apply Corollary 3.15 to prove our theorem. First we can apply

Corollary 3.15 to E_1 and (X,σ) to make the condition (3) of the theorem hold for E_1 and any simplex of the subdivision, and also the conditions (1) and (2) of the theorem hold. We call the simplicial structure after this step, τ_1.
Then we apply Corollary 3.15 to E_2 (in place of E_1) and (X,τ_1) (in place of (X,σ)). We call this new subdivision τ_2. Now (3) for E_2 holds for any simplex of new subdivision τ_2. Furthermore (1) and (2) of Corollary 3.15 hold for (X,τ_1) as the simplicial structure before subdivision (i.e., in place of (X,σ)) and (X,τ_2) as the subdivision (i.e., in place of (X,τ)).
The important point is, (3) for E_1 holds for any simplex Δ_2 of (X,τ_2), because (3) for E_1 holds for the simplex Δ_1 of (X,τ_1) which supports Δ_2 (i.e., $\Delta_1\supset\Delta_2$), and because (1) holds for τ_1 (in place of σ) and τ_2 (in place of τ). So, now (3) holds for both E_1 and E_2.
(It is also obvious that (1) for σ and τ_2 (in place of τ) follows from (1) for σ and τ_1 (in place of τ), together with (1) for τ_1 (in place of σ) and τ_2 (in place of τ). The same thing also holds for (2).)
Repeating this procedure, we can define τ_3, τ_4, and so on, until τ_m. Then (1), (2), (3) hold for σ and τ_m and any $E_j,\ j=1,2,\cdots,m$. Let $\tau=\tau_m$.

$\square$

Remark 3.17. Let us remark that, in the proof of Lemma 3.11 when we construct the sets E_1, simplex by simplex for (X,σ), it is impossible to obtain

$$\#\left(\bigcap_{x\in\mathrm{Vertex}(\Delta')} E_1(x)\right) \quad\geq\quad K_1-\frac{\dim(\Delta')(\dim(\Delta')+1)}{2}\ ,$$

for each simplex Δ' of subdivision (X,τ) of (X,σ). (Explained below.)
But from the proof of Corollary 3.12, we can make the following hold,

$$\#\left(\bigcap_{x\in\mathrm{Vertex}(\Delta')} E_1(x)\right) \quad\geq\quad K_1-\frac{\dim(\Delta)(\dim(\Delta)+1)}{2}\ ,$$

where Δ is any simplex of (X,σ) which support Δ' (i.e., $\Delta'\subset\Delta$ as spaces). In other words,

$$\#\left(\bigcap_{x\in\mathrm{Vertex}(\Delta')} E_1(x)\right) \quad\geq\quad K_1-\frac{l(l+1)}{2}\ ,$$

if Δ' is a subspace of l-skeleton $X^{(l)}$ of (X,σ).
In the induction construction from dimension not larger than $n-1$ to dimension n (see the proof of Lemma 3.11), in particular, from $(\partial\Delta_0\times\{t_0\},\tilde\tau)$ to $(\partial\Delta_0\times[t_0,t_1],\tilde\tau)$, for any simplex Δ inside one of $(\partial\Delta_0\times\{t_0\},\tilde\tau)$ and $(\partial\Delta_0\times\{t_1\},\tilde\tau)$, we do have

$$\bigcap_{x\in\mathrm{Vertex}(\Delta)} E_1(x)\geq K_1-\frac{(n-1)n}{2},$$

from our construction (see condition (b) in the proof of Lemma 3.11). But for simplices Δ which are not completely sitting inside one of $(\partial\Delta_0 \times \{t_0\}, \tilde{\tau})$ and $(\partial\Delta_0 \times \{t_1\}, \tilde{\tau})$, we do NOT have

$$\bigcap_{x\in \mathrm{Vertex}(\Delta)} E_1(x) \geq K_1 - \frac{(n-1)n}{2},$$

even if we assume $dim(\Delta) \leq n-1$.

For the application we have in mind, we need the following strengthened form of Theorem 3.16 (in fact, we will need the version of the following result which allow multiplicities; see Theorem 3.32).

Theorem 3.18. *Let (X,σ) be a simplicial complex and $Y = X^{(l)}$ be the l-skeleton of X. Suppose that there is a subdivision (Y,τ) of (Y,σ) and a grouping for each vertex of (Y,τ) (and (X,σ)), such that*
(a) if Δ is a simplex of (Y,τ), then

$$\#\left(\bigcap_{y\in \mathit{Vertex}(\Delta,\tau)} E_j(y)\right) \quad \geq \quad K_j - \frac{l(l+1)}{2}\ . \qquad\qquad j=1,2,\cdots,m;$$

(b) if Δ is a simplex of $(Y,\sigma) \subset (X,\sigma)$, and $y\in\Delta$ is a vertex of (Y,τ), then

$$\bigcap_{x\in \mathit{Vertex}(\Delta,\sigma)} E_j(x) \subset E_j(y) \subset \bigcup_{x\in \mathit{Vertex}(\Delta,\sigma)} E_j(x), \qquad\qquad j=1,2,\cdots,m.$$

It follows that there is a subdivision $(X,\tilde{\tau})$ of (X,σ) and groupings for all the vertices, such that
(1) $(X,\tilde{\tau})|_Y = (Y,\tau)$, and groupings on Vertex(Y,τ) are the same as the old ones.
(2) if Δ is a simplex of (X,σ), and $x_1\in\Delta$ is a newly introduced vertex of $(X,\tilde{\tau})$, then

$$\bigcap_{x\in \mathit{Vertex}(\Delta,\sigma)} E_j(x) \subset E_j(x_1) \subset \bigcup_{x\in \mathit{Vertex}(\Delta,\sigma)} E_j(x), \qquad\qquad j=1,2,\cdots,m;$$

(3) for each simplex Δ of $(X,\tilde{\tau})$, if Δ is inside the l'-skeleton $(X,\sigma)^{(l')}$ $(l'\geq l)$ of (X,σ), then

$$\#\left(\bigcap_{x\in \mathit{Vertex}(\Delta,\tilde{\tau})} E_j(x)\right) \quad \geq \quad K_j - \frac{l'(l'+1)}{2}\ . \qquad\qquad j=1,2,\cdots,m.$$

Proof: If one does not require (1) (i.e., if it is allowed to introduce more vertices into (Y,τ)), then the theorem is Theorem 3.16 (see 3.17 also).

Recall, in the proof of Theorem 3.16, we first constructed a subdivision (X, τ_1) and the groupings to make the above (3) hold for E_1. Then based on (X, τ_1), we constructed a new subdivision (X, τ_2) and groupings to make the above (3) hold also for E_2, and so on. If we use the same procedure to prove Theorem 3.18, we will encounter a difficulty in the second step. We have no problem for the first step, since we can begin with what we already have on $(X^{(l)}, \tau)$ and work on each of the simplexes of dimension larger than l (see Lemma 3.11, the proof of Corollary 3.12 and Remark 3.17). But for the second step, the condition (a) may not hold for l-skeleton $(X, \tau_1)^{(l)}$ of (X, τ_1). So we need to start with the simplex of the lowest dimension, which forced us to introduce vertices on $(Y, \tau) = (X^{(l)}, \tau)$.
The following small trick can be used to avoid the difficulty mentioned above. Consider simplex Δ. Suppose that the subdivision $(\partial\Delta, \tilde{\tau})$ and the groupings for those vertices are chosen. Identify Δ with $\partial\Delta \times [0,1]/\partial\Delta \times \{1\}$ as in the proof of Lemma 3.11. Choose a point $t_0 \in (0,1)$, and write

$$\Delta = \partial\Delta \times [0, t_0] \cup (\partial\Delta \times [t_0, 1]/\partial\Delta \times \{1\}).$$

Substitute Δ by $\Delta^{sub} = \partial\Delta \times [t_0, 1]/\partial\Delta \times \{1\}$. The simplicial structure $\tilde{\tau}_{pre}$ and the groupings on $\partial\Delta^{sub} = \partial\Delta \times \{t_0\}$ should be endowed the same as $\tilde{\tau}$ and the groupings on $\partial\Delta = \partial\Delta \times \{0\}$. Then apply Theorem 3.16 to Δ^{sub}. One may introduce new vertices on $(\partial\Delta \times \{t_0\}, \tilde{\tau}_{pre})$, but no new vertices are introduced on $\partial\Delta = \partial\Delta \times \{0\}$. Finally, for the part $\partial\Delta \times [0, t_0]$, same as in the Step 4 of the proof of Lemma 3.11, we apply Lemma 3.10 to make this part a simplicial complex, in which we do not introduce any new vertices on $\partial\Delta \times \{0\}$.

$\square$

3.19. For convenience, define

$$E_j(\Delta) = \bigcap_{x \in \mathrm{Vertex}(\Delta)} E_j(x), \qquad j = 1, 2, \cdots, m$$

for each simplex Δ of $(X, \tilde{\tau})$. Then (3) of 3.18 becomes

$$\#(E_j(\Delta)) \quad \geq \quad K_j - \frac{l'(l'+1)}{2} \ ,$$

if Δ is in the l'-skeleton of (X, σ) $(l' \geq l)$.

3.20. We need a different version of Theorem 3.18 which allows multiplicity. Let $w_1, w_2, \cdots, w_k$ be a k-tuple of positive integers. Let

$$E = \{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_k^{\sim w_k}\}$$

be an index set with multiplicity and $\lambda_i \neq \lambda_j$ if $i \neq j$. (See 1.1.7 (b) for the notation $\lambda^{\sim w}$.) Let $w_1 + w_2 + \cdots + w_k = K$. (3.2 is a special case with each $w_i =$

1.) Let $K_1, K_2, \cdots, K_m$ be non negative integers with $K_1+K_2+\cdots+K_m = K$. Suppose that

$$E_j = \{\lambda_1^{\sim p_1^j}, \lambda_2^{\sim p_2^j}, \cdots, \lambda_k^{\sim p_k^j}\}, \qquad j = 1, 2, \cdots, m,$$

where p_i^j are nonnegative integers. If $\{E_1, E_2, \cdots, E_m\}$ satisfies

$$\sum_{i=1}^{k} p_i^j = K_j \qquad \text{for each } j = 1, 2, \cdots, m, \quad \text{and}$$

$$\sum_{j=1}^{m} p_i^j = w_i \qquad \text{for each } i = 1, 2, \cdots, k,$$

then we call $\{E_1, E_2, \cdots, E_m\}$ a grouping of E of type $(K_1, K_2, \cdots, K_m)$, or just a grouping of E.

3.21. It is convenient to use the notations of union, intersection etc. for the sets with multiplicity. A is called a subset of E if A is of the form

$$\{\lambda_1^{\sim t_1}, \lambda_2^{\sim t_2}, \cdots, \lambda_k^{\sim t_k}\}$$

with $0 \le t_i \le w_i$, for each $1 \le i \le k$. Note that if all $t_i = 0$, then A is called the empty set. If $t_i = w_i$, then $A = E$. Let B be another subset of E of form $\{\lambda_1^{\sim s_1}, \lambda_2^{\sim s_2}, \cdots, \lambda_k^{\sim s_k}\}$. A is a subset of B (denoted by $A \subset B$) if $t_i \le s_i$ for all i. In general, define the union, intersection, and difference of two subsets A and B of E as follows.

$$A \cup B = \{\lambda_1^{\sim \max(t_1, s_1)}, \lambda_2^{\sim \max(t_2, s_2)}, \cdots, \lambda_k^{\sim \max(t_k, s_k)}\}$$

$$A \cap B = \{\lambda_1^{\sim \min(t_1, s_1)}, \lambda_2^{\sim \min(t_2, s_2)}, \cdots, \lambda_k^{\sim \min(t_k, s_k)}\}$$

$$A \backslash B = \{\lambda_1^{\sim \max(0, t_1 - s_1)}, \lambda_2^{\sim \max(0, t_2 - s_2)}, \cdots, \lambda_k^{\sim \max(0, t_k - s_k)}\} .$$

(The definitions of union and intersection can be easily generalized to finitely many subsets of E.)
Warning 1: $B \cap (A \backslash B)$ may be a nonempty set.
Warning 2: The assumption that $\{E_1, E_2, \cdots E_m\}$ is a grouping of E does NOT imply that $\cup_{i=1}^m E_i = E$ or that $E_i \cap E_{i'} = \emptyset$ for $i \ne i'$. (See 3.20.)
But $A \backslash B = A \backslash (A \cap B)$ still holds.

3.22. Let (X, σ) be a simplicial complex. Suppose that each vertex $x \in X$ is associated with a grouping $\{E_1, E_2, \cdots, E_m\}$, satisfying

$$\#(E_j) = \sum_{i=1}^{k} p_i^j = K_j.$$

(Recall that, for the notation of #, we count multiplicity.)
In Theorem 3.18, we introduced subdivisions (X,τ) of (X,σ), and groupings for newly introduced vertices to make $\#(E_j(\Delta))$ large, for all simplices Δ of (X,τ). In order to prove the decomposition theorem in the next section, we need a stronger result, since the multiplicity of the spectrum of the homomorphism is involved. (See §2. We can not always perturb the map to have distinct spectrum, like the one dimensional case.) Fortunately, this stronger result can be proved in the same way as that for Theorem 3.18, with a few modifications. For any subset $F=\{\lambda_1^{\sim u_1},\lambda_2^{\sim u_2},\cdots,\lambda_k^{\sim u_k}\}\subset E$, define

$$\mathring{F}=\{\lambda_1^{\sim v_1},\lambda_2^{\sim v_2},\cdots,\lambda_k^{\sim v_k}\},$$

where

$$v_i=\begin{cases} u_i & \text{if } u_i=w_i\\ 0 & \text{if } u_i<w_i\ .\end{cases}$$

That is, $\mathring{F}$ is the set of all those elements λ_i, which are entirely inside F. Evidently,

$$\mathring{E}_j(\Delta)=\bigcap_{x\in \mathrm{Vertex}(\Delta)}\mathring{E}_j(x).$$

Instead of the condition that $E_j(\Delta)$ is large (see 3.18 and 3.19), we need to make $\mathring{E}_j(\Delta)$ large for any simplex Δ of $(X,\tilde{\tau})$. For this purpose, $\mathring{E}_j(x)$ should be large for each vertex of (X,σ) at the beginning.

3.23. For each set $F=\{\lambda_1^{\sim u_1},\lambda_2^{\sim u_2},\cdots,\lambda_k^{\sim u_k}\}\subset E$, define

$$\bar{F}=\{\lambda_1^{\sim v_1},\lambda_2^{\sim v_2},\cdots,\lambda_k^{\sim v_k}\}$$

where

$$v_i=\begin{cases} w_i & \text{if } \ u_i>0\\ 0 & \text{if } \ u_i=0\ .\end{cases}$$

Obviously,

$$\mathring{F}\subset F\subset \bar{F}.$$

3.24. Let $H_1,H_2,\cdots,H_m$ (not necessarily disjoint) be finite subsets of E satisfying condition $(*)$ in 3.7, and $E=H_1\cup H_2\cup\cdots\cup H_m$. Suppose that

$$\mathring{H}_i=H_i=\bar{H}_i \qquad \text{for each } i=1,2,\cdots,m.$$

In what follows, we will require that

$$E_j\subset H_j, \qquad j=1,2,\cdots,m$$

(comparing with the condition (2) in Theorem 3.18).

3.25. For each subset $I \subset \{1, 2, \cdots, m\}$, define

$$H_I = \bigcup_{j\in I} H_j.$$

Let $G'_I = \cap_{j\in I} H_j$. Then define

$$G_I = G'_I \backslash \bigcup_{J \underset{\neq}{\supset} I} G'_J \ .$$

Another way to define G_I is by

$$G_I = \{\lambda \in E \mid \lambda \in H_i \text{ if and only if } i \in I\}.$$

(Note that G_I may be an empty set for some I.) Obviously,

$$\mathring{G}_I = G_I = \bar{G}_I \subset \mathring{H}_I = H_I = \bar{H}_I.$$

If $I \cap J = \emptyset$, then $H_I \cap G_J = \emptyset$.
Furthermore, for any $\lambda \in E$, there is a unique set I (defined by $I = \{i \mid \lambda \in H_i\}$) such that $\lambda \in G_I$. Hence E is a disjoint union of

$$\{G_I, \emptyset \neq I \subset \{1, 2, \cdots, m\}\}.$$

Similarly, we have

$$H_I = \bigcup_{J\cap I\neq\emptyset} G_J.$$

3.26. Under the above partition G_I of E, two elements $\lambda, \mu \in E$ are in the same part, if and only if the following is true. For any $i = 1, 2, \cdots, m$, either H_i contains both λ and μ, or H_i contains none of λ and μ.
For any $E_1, E'_1 \subset H_1$, if $\#(E_1 \cap G_I) = \#(E'_1 \cap G_I)$ for any $I \subset \{1, 2, \cdots, m\}$, then from the end of 3.25,

$$\#(E_1 \cap H_I) = \sum_{J\cap I\neq\emptyset} \#(E_1 \cap G_J) = \sum_{J\cap I\neq\emptyset} \#(E'_1 \cap G_J) = \#(E'_1 \cap H_I)$$

for any $I \subset \{1, 2, \cdots, m\}$. Hence $\#(H_I \backslash E_1) = \#(H_I \backslash E'_1)$. At this circumstance, either both of E_1 and E'_1 satisfy $(**)$ in 3.8, or both of them do not satisfy $(**)$ in 3.8.
Note that $H_i = \cup_{I\ni i} G_I$. A grouping $\{E_1, E_2, \cdots E_m\}$ satisfies $E_i \subset H_i$ $(i = 1, 2, \cdots m)$ if and only if for any $i \notin I$, $E_i \cap G_I = \emptyset$.
For the rest of the section, let $\Omega \geq \max(w_1, w_2, \cdots, w_m)$ be a fixed number, where $w_1, w_2, \cdots, w_m$ are the multiplicities in E. Note that for our application, sometimes, we have to allow Ω to be larger than the maximum multiplicity.

Assumption 3.27. For each grouping $\{E_1(x), E_2(x), \cdots, E_m(x)\}$, we always assume that

$$\#(\mathring{E}_j(x)) \geq \#(E_j(x)) - M\Omega = K_j - M\Omega, \qquad j = 1, 2, \cdots, m,$$

where $M = 2^m - 1$. We not only require each initial grouping for (X, σ) to satisfy the above assumption, but also require any new groupings for vertices of (X, τ) to satisfy the assumption.
Since $M = 2^m - 1$, there are totally M non-empty subsets $I \subset \{1, 2, \cdots, m\}$. If the grouping $\{E_1, E_2, \cdots, E_m\}$ satisfies

$$\#(\mathring{E}_j \cap G_I) \geq \#(E_j \cap G_I) - \Omega$$

for all $j = 1, 2, \cdots, m$ and for all $I \subset \{1, 2, \cdots, m\}$, then it also satisfies Assumption 3.27.

Lemma 3.28. *If $\{E_1, E_2, \cdots, E_m\}$ is a grouping of E with $E_i \subset H_i$, then there is a grouping $\{E_1', E_2', \cdots, E_m'\}$ of E satisfying Assumption 3.27, and*

$$E_i' \subset H_i, \quad E_i' \supset \mathring{E}_i \qquad \textit{for all } \; i = 1, 2, \cdots, m.$$

Proof: The proof is straight forward. Consider $E_1', E_2', \cdots, E_m'$ to be m boxes with no element at the beginning, and put each element of E into one of the boxes, following the procedures described below.
Step 1. Put all the elements of $\mathring{E}_i$ into box E_i' for each $i = 1, 2, \cdots, m$. (Thus $E_i' \supset \mathring{E}_i$.)
Step 2. Fix $I \subset \{1, 2, \cdots, m\}$. For the set E_1', if there is a $\lambda_i \in G_I \backslash (E_1' \cup E_2' \cup \cdots \cup E_m')$ such that

$$\#(E_1' \cap G_I) + w_i \leq \#(E_1 \cap G_I),$$

where w_i is the multiplicity of λ_i in E, then put the entire set $\{\lambda_i^{\sim w_i}\}$ into E_1'. (Note that if $1 \notin I$, then $E_1 \cap G_I = \emptyset$. Hence for I, we need not do anything for E_1.) Repeat this procedure until no such i exists. Thus, so far,

$$\#(\mathring{E}_1' \cap G_I) = \#(E_1' \cap G_I) \geq \#(E_1 \cap G_I) - (\Omega - 1).$$

For the same I above, repeat the above construction for the set E_2', then E_3', etc.
After this step has been completed for each I, (it is done for each set I separately) we have the following:

$$\#(\mathring{E}_j' \cap G_I) = \#(E_j' \cap G_I) \geq \#(E_j \cap G_I) - (\Omega - 1).$$

Step 3. Put what left for each G_I from the previous steps, arbitrarily into the boxes

$$E_1', E_2', \cdots, E_m'$$

to make the following condition hold:

$$\#(E_j' \cap G_I) = \#(E_j \cap G_I).$$

From the end of 3.26, $E_i' \subset H_i$ is a consequence of the above equation. (Note that $E_i \subset H_i$.) Evidently, $\{E_1', E_2', \cdots, E_m'\}$ is as desired.

$\square$

3.29. A set $E_1 (\subset H_1)$ of K_1 elements is said to satisfy the condition $(***)$, if there is a grouping $(E_1, E_2, \cdots, E_m)$ of E (of type $(K_1, K_2, \cdots K_m)$), $E_i \subset H_i$, satisfying Assumption 3.27.
Obviously, $(***)$ implies $(**)$.
The following corollary is a direct consequence of Lemma 3.28.

Corollary 3.30. *For any set E_1 $(\subset H_1)$ satisfying $(**)$, there is a set E_1' $(\subset H_1)$ satisfying $(***)$ such that*

$$E_1' \supset \mathring{E}_1.$$

Proof: Since E_1 satisfies $(**)$, we can extend E_1 to a grouping $\{E_1, E_2, \cdots, E_m\}$ of E such that $E_i \subset H_i$ for each i. By Lemma 3.28, there is a grouping $\{E_1', E_2', \cdots, E_m'\}$ satisfying Assumption 3.27, and $E_i' \subset H_i$ for each i. This is condition $(***)$ for E_1'.

$\square$

Lemma 3.31. *Let E_1 and F_1 be two sets satisfying condition $(***)$. Suppose that there is a $\lambda \in E$ such that*

$$\{\lambda^{\sim w}\} \subset \mathring{F}_1 \backslash \mathring{E}_1,$$

where w is the multiplicity of λ in E. Then there are (perhaps repeating) elements $\mu_1, \mu_2, \cdots, \mu_t \in E_1 \backslash \mathring{F}_1$, where $t = w - \#(\{\lambda^{\sim w}\} \cap E_1)$, such that

$$E_1' = (E_1 \cup \{\lambda^{\sim w}\}) \backslash \{\mu_1, \mu_2, \cdots, \mu_t\}$$

satisfies $(**)$ and

$$\begin{aligned} \#((\mathring{E}_1' \cap \mathring{E}_1) \cap G_I) &\geq \#(\mathring{E}_1 \cap G_I) - (w + \Omega) \\ &\geq \#(\mathring{E}_1 \cap G_I) - 2\Omega \end{aligned}$$

for each $I \subset \{1, 2, \cdots, m\}$. As a consequence,

$$\#(\mathring{E}_1' \cap \mathring{E}_1) \geq \#(\mathring{E}_1) - 2M\Omega.$$

Proof: Let $t_1 = \#(\{\lambda^{\sim w}\} \cap E_1)$. Then $t_1 < w$. Applying Lemma 3.9 $t := w - t_1$ times, one can obtain a (possibly repeating) set $T' = \{\nu_1, \nu_2, \cdots, \nu_t\} \subset E_1 \backslash F_1 \subset E_1 \backslash \mathring{F}_1$, such that

$$\tilde{E}_1 = (E_1 \cup \{\lambda^{\sim w}\}) \backslash T'$$

satisfies $(**)$.
From 3.26, if another set $T = \{\mu_1, \mu_2, \cdots, \mu_t\} \subset E_1 \backslash \mathring{F}_1$, satisfies that

$$\#(T \cap G_I) = \#(T' \cap G_I)$$

for each $I \subset \{1, 2, \cdots, m\}$, then $E_1' = (E_1 \cup \{\lambda^{\sim w}\}) \backslash T$ also satisfies $(**)$.
$T \subset E_1 \backslash \mathring{F}_1$ will be constructed to satisfy the following condition. For each $I \subset \{1, 2, \cdots, m\}$,

$$\#(T \cap G_I) = \#(T' \cap G_I),$$

and $(T \cap (E_1 \backslash T)) \cap G_I$ is either empty or $\{\mu_i^{\sim s}\}$ for a certain $\mu_i \in \{\mu_1, \mu_2, \cdots, \mu_t\}$. (Note that $T \cap (E_1 \backslash T)$ may not be empty, since we are dealing with sets with multiplicities.)
To do the above, write

$$(E_1 \backslash \mathring{F}_1) \cap G_I = \{\lambda_{i_1}^{\sim s_1}, \lambda_{i_2}^{\sim s_2}, \cdots, \}.$$

Then put each of the sets $\{\lambda_{i_1}^{\sim s_1}\}, \{\lambda_{i_2}^{\sim s_2}\}, \cdots$, entirely into T one by one until we can not do it without violating the restriction

$$\#(T \cap G_I) \leq \#(T' \cap G_I).$$

Then make T to satisfy $\#(T \cap G_I) = \#(T' \cap G_I)$ by putting part of $\{\lambda_{i_j}^{\sim s_j}\}$ into T if necessary.
Since $\#(T) \leq w$, combining with the above condition for $(T \cap (E_1 \backslash T)) \cap G_I$, after a moment thinking, one can obtain,

$$\#((E_1 \backslash T)^\circ \cap G_I) \geq \#(\mathring{E}_1 \cap G_I) - (w + \Omega).$$

(In fact, $\mathring{E}_1 \backslash (E_1 \backslash T)^\circ \subset \mathring{T} \cup (T \cap (E_1 \backslash T))$, and $(\mathring{T} \cup (T \cap (E_1 \backslash T))) \cap G_I$ has at most $w + \Omega$ elements.)
Hence

$$\#((\mathring{E}_1' \cap \mathring{E}_1) \cap G_I) \geq \#(\mathring{E}_1 \cap G_I) - (w + \Omega).$$

$\square$

The following is the main result of this section. Together with Lemma 3.28, it will be used in §4.

Theorem 3.32. *Let (X,σ) be a simplicial complex, and $Y = X^{(l)}$, the l-skeleton of X. Suppose that (Y,τ) is a subdivision of (Y,σ) and, for each vertex $y \in$ Vertex(Y,τ), there is a grouping $E_1(y), E_2(y), \cdots, E_m(y)$ of E (of type $(K_1, K_2, \cdots, K_m)$). Suppose that the groupings satisfy the following three conditions:*

(a) For each simplex Δ of (Y,τ), and $i = 1,2,\cdots,m$,

$$\#(\mathring{E}_i(\Delta)) \geq K_i - (M\Omega + M\Omega \dim Y \cdot (\dim Y + 1)),$$

where $M = 2^m - 1$.
(b) $E_i(x) \subset H_i,\ i = 1,2,\cdots,m$, for each $x \in$ Vertex(Y,τ).

(c) Each grouping, for a vertex of (Y,τ), satisfies Assumption 3.27.

It follows that there exist a subdivision $(X,\tilde{\tau})$ of (X,σ) and a grouping for each vertex of $(X,\tilde{\tau})$, satisfying the following conditions.
(1) $(X,\tilde{\tau})|_Y = (Y,\tau)$, and each grouping on Vertex(Y,τ) is as same as the old one.
(2) $E_i(x) \subset H_i,\ i = 1,2,\cdots,m$, for each $x \in$ Vertex$(X,\tilde{\tau})$, and if Δ is a simplex of (X,σ) (before the subdivision), and $x \in \Delta$ is a newly introduced vertex of $(X,\tilde{\tau})$, then

$$\mathring{E}_j(x_1) \ \supset \bigcap_{y \in \mathit{Vertex}(\Delta\cap Y,\tau)} \mathring{E}_j(y).$$

(3) For each simplex Δ of $(X,\tilde{\tau})$, if Δ is inside the l'-skeleton $(X,\sigma)^{(l')}$, $(l' > l)$, of (X,σ), then

$$\#(\mathring{E}_j(\Delta)) \geq K_j - (M\Omega + M\Omega l'(l'+1)).$$

(4) Each grouping on Vertex$(X,\tilde{\tau})$ satisfies Assumption 3.27.

Proof: (Sketch) The proof is the same as the one of 3.18 (see 3.11 to 3.18), using Lemma 3.31 to replace Lemma 3.9. The arguments in 3.12 – 3.18 are easily adopted in this new setting. We only give the proof for the part corresponding to 3.11 and sketch the differences for other parts.
As in 3.11, consider only one simplex $X = \Delta_0$ with $Y = \partial\Delta_0$, and only one set $E_1(x)$.
Similar to Step 1 of 3.11, choose E_1^{model} to satisfy condition $(***)$ and

$$\mathring{E}_1^{\,model} \ \supset \bigcap_{x\in \mathrm{Vertex}(\partial\Delta_0,\tau)} \mathring{E}_1(x).$$

Replace (a) of 3.11 by

$$\bigcap_{x\in\mathrm{Vertex}(\partial\Delta\times\{t_i\},\tilde{\tau})} (\mathring{E}_1(x)\cap\mathring{E}_1^{\,model}) \ \underset{\neq}{\supset} \bigcap_{x\in\mathrm{Vertex}(\partial\Delta\times\{t_{i-1}\},\tilde{\tau})} (\mathring{E}_1(x)\cap\mathring{E}_1^{\,model}) \ .$$

Keeping the notations in 3.11, in Step 2, replace G by

$$G = \mathring{E}_1(y_1, t_0) \cap \mathring{E}_1(y_2, t_0) \cap \cdots \cap \mathring{E}_1(y_p, t_0) \cap \mathring{E}_1^{\,model}.$$

If $G = \mathring{E}_1^{\,model}$, then define

$$E_1(y_i, t_1) = E_1^{\,model}, \qquad i = 1, 2, \cdots, p.$$

Suppose that $G \neq \mathring{E}_1^{\,model}$. Choose $\lambda \in \mathring{E}_1^{\,model} \backslash G$. Let w be the multiplicity of λ. Then

$$\{\lambda^{\sim w}\} \subset \mathring{E}_1^{\,model} \backslash G.$$

For each (y_i, t_0), if $\lambda \in \mathring{E}_1(y_i, t_0)$, then define $E_1(y_i, t_1) = E_1(y_i, t_0)$ (as in Step 2 of 3.11). If $\lambda \notin \mathring{E}_1(y_i, t_0)$, apply Lemma 3.31 to obtain E_1' satisfying($**$), $\mathring{E}_1' \supset G \cup \{\lambda^{\sim w}\}$ and

$$(A) \qquad \#(\mathring{E}_1' \cap \mathring{E}_1) \geq \#(\mathring{E}_1) - 2M\Omega.$$

Then we can apply Corollary 3.29 to find E_1'' satisfying $(***)$ and $\mathring{E}_1'' \supset \mathring{E}_1'$. Define

$$E_1(y_i, t_1) = E_1''.$$

Then

$$\mathring{E}_1(y_i, t_1) \supset \mathring{E}_1' \supset G \cup \{\lambda^{\sim w}\}.$$

The arguments in Step 3 and Step 4 of 3.11 can also be employed here. (Of course, at many places (not all places), one needs to replace E_i by $\mathring{E}_i$.) The estimation (e) in Step 4 of 3.11 will be changed to

$$\begin{aligned}
\#\Big(\mathring{E}_1(u_0, t_0) \cap \mathring{E}_1(u_1, t_0) \cap \cdots \cap \mathring{E}_1(u_i, t_0) \cap \mathring{E}_1(u_0, t_1) \cap \mathring{E}_1(u_1, t_1) \cap \cdots \cap \mathring{E}_1(u_i, t_1)\Big)& \\
\geq K_1 - [M\Omega + M\Omega \dim Y \cdot (\dim Y + 1)] - 2M\Omega(i+1)& \\
= K_1 - [M\Omega + M\Omega \cdot (n-1) \cdot n] - 2M\Omega \cdot n& \\
= K_1 - [M\Omega + M\Omega \cdot n \cdot (n+1)].&
\end{aligned}$$

(Here we used the above estimation (A) which is from Lemma 3.31.)
Since $E_1(y_i, t_1)$ satisfies $(***)$, all the other parts (e.g., induction arguments) in 3.11—3.18 can go through easily. In the part corresponding to the proof of Corollary 3.14, the definition of D_i should be changed to

$$D_i = \bigcap_{x \in \mathrm{Vertex}(\Delta)} \mathring{E}_i(x).$$

$\square$

WE REMARK THAT IN §4, WE WILL ONLY USE THE THEOREM OF THE CASE THAT $X = \Delta$, A SINGLE SIMPLEX WITH $Y = \partial\Delta$.

REMARK 3.33. The condition $E_i(x) \subset H_i$ in (2) can be strengthened as

$$E_i(x) \subset \bigcup_{y \in \mathrm{Vertex}(\Delta \cap Y, \tau)} \bar{E}_i(y),$$

where Δ is a simplex of (X, σ) (before the subdivision) such that $x \in \Delta$ is a newly introduced vertex. (The notation is from 3.23.)

4 DECOMPOSITION THEOREMS

In this section, we will prove the decomposition theorems which are needed for the proof of our main Reduction Theorem and the main results in [EGL]. The following Theorem 4.1 is one version of the Decomposition Theorem. After Theorem 4.1 has been proved, we will use [Li 2] to verify that the condition of Theorem 4.1 holds for connecting homomorphisms $\phi_{n,m}$(for each fixed n, m should be large enough), with the maps $a_1, a_2, \cdots a_L$(see below) factoring through interval $[0,1]$ or the single point space $\{pt\}$. In such a way, we can prove our main decomposition theorems (Theorem 4.35 and Theorem 4.37).

THEOREM 4.1. *Let X be a connected finite simplicial complex, and $F \subset C(X)$ be a finite set which generates $C(X)$. For any $\varepsilon > 0$, there is an $\eta > 0$ such that the following statement is true.*
Suppose that a unital homomorphism $\phi : C(X) \to PM_{K'}(C(Y))P$ (rank$(P) = K$) (where Y is a finite simplicial complex) satisfies the following condition: There are L continuous maps

$$a_1, a_2, \cdots, a_L : Y \longrightarrow X$$

such that for each $y \in Y$, $SP\phi_y$ and $\Theta(y)$ can be paired within η, where

$$\Theta(y) = \{a_1(y)^{\sim T_1}, a_2(y)^{\sim T_2}, \cdots, a_L(y)^{\sim T_L}\}$$

and $T_1, T_2, \cdots, T_L$ are positive integers with

$$T_1 + T_2 + \cdots + T_L = K = \mathit{rank}(P).$$

(See 1.1.7(b) for notation $x^{\sim T_i}$.) Let $T = 2^L(\dim X + \dim Y)^3$. It follows that there are L mutually orthogonal projections $p_1, p_2, \cdots, p_L \in PM_{K'}(C(Y))P$ such that
(i) $\|\phi(f)(y) - p_0(y)\phi(f)(y)p_0(y) \oplus \sum_{i=1}^L f(a_i(y))p_i(y)\| < \varepsilon$, for any $f \in F$ and $y \in Y$, where $p_0 = P - \sum_{i=1}^L p_i$;
(ii) $\|p_0(y)\phi(f)(y) - \phi(f)(y)p_0(y)\| < \varepsilon$ for any $f \in F$ and $y \in Y$;
(iii) rank$(p_i) \geq T_i - T$ for $1 \leq i \leq L$, and hence rank$(p_0) \leq LT$.

4.2. In the above theorem, some of the p_i may be zero projections if $T_i \le T$. But when the theorem is applied later in this article, the positive integers T_i are always very large compared with $T = 2^L(\dim X + \dim Y)^3$.
The proof of this theorem will be divided into several small steps. The results in §2 and §3 will be used. In fact, §2 and §3 will only be used in the proof of Theorem 4.1, no other place in this paper or [EGL]. (The results in §2 have some other applications.)

4.3. The theorem is trivial if $X = \{pt\}$, applying Theorem 1.2 of [Hu,Chapter 8]. Without loss of generality, we assume that $X \ne \{pt\}$. By the results in §2, we can assume that ϕ has maximum spectral multiplicity at most $\Omega := \dim X + \dim Y$.

4.4. For any $\varepsilon > 0$, there is an $\eta > 0$ such that for any $x_1, x_2 \in X$, if $\mathrm{dist}(x_1, x_2) < 2\eta$, then

$$|f(x_1) - f(x_2)| < \frac{\varepsilon}{3} \qquad\qquad \text{for all } f \in F.$$

We will prove that this η is as desired.

4.5. Recall, from 1.2.5, for any positive integer n, P^nX is the symmetric product of n-copies of X. Also, any element $\Lambda \in P^KX$ can be considered as a set with multiplicity. So $\mathrm{SP}\phi_y \in P^KX$.
Suppose that $\Lambda_1 \in P^{k_1}X, \Lambda_2 \in P^{k_2}X, \cdots, \Lambda_t \in P^{k_t}X$. Write

$$\Lambda_1 = \{\lambda_1, \lambda_2, \cdots, \lambda_{k_1}\}$$

$$\Lambda_2 = \{\lambda_{k_1+1}, \lambda_{k_1+2}, \cdots, \lambda_{k_1+k_2}\}$$

$$\vdots$$

$$\Lambda_t = \{\lambda_{k_1+\cdots+k_{t-1}+1}, \cdots, \lambda_{k_1+\cdots+k_t}\}$$

as sets with multiplicity. By abusing the notation, we use $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_t\}$ to denote

$$\{\lambda_1, \lambda_2, \cdots, \lambda_{k_1},\ \lambda_{k_1+1},\ \cdots,\ \lambda_{k_1+k_2},\ \cdots\cdots,\ \lambda_{k_1+k_2+\cdots+k_t}\},$$

which defines an element in $P^{k_1+k_2+\cdots+k_t}X$.
(Note that $\{\Lambda_1, \Lambda_2, \cdots, \Lambda_t\} = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_t$, if $\{\Lambda_j\}$ are mutually disjoint. See 3.21 for the definition of unions of sets with multiplicity.)

4.6. For any fixed point $y \in Y$, write

$$\mathrm{SP}\phi_y = \{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_k^{\sim w_k}\}$$

with $\lambda_i \neq \lambda_j$ if $i \neq j$. Note that $w_1 + w_2 + \cdots + w_k = K$. We denote the above k by t_y to indicate that this integer depends on y. Define

$$(a) \qquad \theta(y) = \frac{1}{4} \min_{1 \leq i < j \leq t_y} \mathrm{dist}(\lambda_i, \lambda_j).$$

Then $\theta(y) > 0$ for any $y \in Y$. (Of course, $\theta(y)$, in general, does not depend on y continuously.)
For each $i = 1, 2, \cdots, t_y$, there is an open set $U(y, i) \ni \lambda_i$ such that

$$(b) \qquad \mathit{diameter}(U(y,i)) \leq \min\left(\frac{\eta}{2(\dim Y + 1)}, \theta(y)\right).$$

Then, obviously,

$$(c) \qquad \mathrm{dist}(U(y,i), U(y,j)) \geq 2\theta(y) \qquad \text{if} \qquad i \neq j.$$

Applying Lemma 1.2.10, there is an (connected) open neighborhood $O(y)$ of y such that

$$\mathrm{SP}\phi_{y'} \subset U(y,1) \bigcup U(y,2) \bigcup \cdots \bigcup U(y, t_y)$$

for all $y' \in O(y)$. Define the continuous maps

$$\Lambda_1 : O(y) \longrightarrow P^{w_1} U(y,1) \ (\subset P^{w_1} X)$$

$$\Lambda_2 : O(y) \longrightarrow P^{w_2} U(y,2) \ (\subset P^{w_2} X)$$

$$\vdots$$

$$\Lambda_{t_y} : O(y) \longrightarrow P^{w_{t_y}} U(y,1) \ (\subset P^{w_{t_y}} X)$$

by $\Lambda_i(y') = \mathrm{SP}\phi_{y'} \cap U(y,i)$. Then

$$\mathrm{SP}\phi_{y'} = \{\Lambda_1(y'),\ \Lambda_2(y'),\ \cdots,\ \Lambda_{t_y}(y')\}$$

for each $y' \in O(y)$. Later on, we will use the disjoint open cover

$$U(y,1) \bigcup U(y,2) \bigcup \cdots \bigcup U(y,t_y) \supset \mathrm{SP}\phi_{y'}$$

of $\mathrm{SP}\phi_{y'}$ to decompose $\mathrm{SP}\phi_{y'}$ into a disjoint union of $\mathrm{SP}\phi_{y'} \cap U(y,t)$ and to identify the elements in each set $\mathrm{SP}\phi_{y'} \cap U(y,t)$ as a single element with multiplicity w_t.
We further require that $O(y)$ is so small that

$$(c) \qquad \mathrm{diameter}(a_i(O(y))) \leq \frac{\eta}{2(\dim Y + 1)},$$

where $a_i : Y \to X$ is any one of the continuous maps $a_1, a_2, \cdots, a_L$, appeared in Theorem 4.1.

4.7. Considering the open cover $\{O(y)\}_{y\in Y}$ of Y, where the open sets $O(y)$ are from 4.6, there exists a finite sub-cover

$$\mathcal{O} = \{O_1, O_2, \cdots, O_\bullet\} \subset \{O(y)\}_{y\in Y},$$

of Y.

Without loss of generality, we assume that the simplicial complex structure (Y,σ) of Y satisfies the following condition, because we can always refine it if necessary.

For each simplex Δ of (Y,σ), the closure of

$$\mathrm{Star}(\Delta) := \bigcup_{\Delta'\cap\Delta\neq\emptyset} \mathrm{interior}(\Delta')$$

can be covered by an open set $O_i \in \mathcal{O}$, where

$$\mathrm{interior}(\Delta') = \Delta' \backslash \partial\Delta'.$$

(Note that $\mathrm{Star}(\Delta)$ is an open set, see 1.4.2)

4.8. For each $y \in Y$, in order to construct $p_1(y), p_2(y), \cdots, p_L(y)$, as in Theorem 4.1, we need to split $\mathrm{SP}\phi_y$ into L sets $E_1(y), E_2(y), \cdots, E_L(y)$ such that each set $E_i(y)$ is contained in an open ball of $a_i(y)$ with small radius (smaller than 2η) and that $\#(E_i(y)) = T_i$, $1 \le i \le L$, where a_i and T_i are maps and positive integers appeared in Theorem 4.1. Since $E_1(y), E_2(y), \cdots, E_L(y)$ may have non-empty intersection (because of the multiplicity of the spectrum), we need to introduce certain subsets of them, which are $\mathring{E}_1 \subset E_1(y)$, $\mathring{E}_2 \subset E_2(y)$, $\cdots$, $\mathring{E}_L \subset E_L(y)$, in the notations of 3.22. This will become precise when the index set with multiplicity is introduced later. The projections $p_1(y), p_2(y), \cdots, p_L(y)$, to be constructed, will be certain sub-projections of the spectral projections corresponding to $\mathring{E}_1, \mathring{E}_2, \cdots, \mathring{E}_L$, respectively. (See Definition 1.2.8 for the spectral projection.)

Following §3, a split of $\mathrm{SP}\phi_y$ into L sets $E_1(y), E_2(y), \cdots, E_L(y)$ will be called a GROUPING of $\mathrm{SP}\phi_y$. THE WORD "GROUPING" IS RESERVED ONLY FOR THIS PURPOSE.

Recall, from 4.6, $\mathrm{SP}\phi_{y'}$ can be written as a disjoint union

$$\mathrm{SP}\phi_{y'} = (\mathrm{SP}\phi_{y'} \cap U(y,1)) \bigcup (\mathrm{SP}\phi_{y'} \cap U(y,2)) \bigcup \cdots \bigcup (\mathrm{SP}\phi_{y'} \cap U(y,t_y)).$$

And the elements in each set $\mathrm{SP}\phi_{y'} \cap U(y,t)$ can be identified as a single element with multiplicity. This will serve as the index set for the groupings. To avoid confusion, the above decomposition is NOT called a "grouping" of $\mathrm{SP}\phi_{y'}$. It is called a DECOMPOSITION instead.

In the next few paragraphs, we apply §3 to construct a subdivision (Y,τ) of (Y,σ) and useful groupings for all vertices $y \in \mathrm{Vertex}(Y,\tau)$.

4.9. Let Δ be a simplex of (Y, σ) and

$$O(y_1), O(y_2), \cdots, O(y_i),$$

the list of all open sets in $\mathcal{O}$, each of which covers Δ. Suppose that

$$\theta(y_1) \leq \theta(y_2) \leq \cdots \leq \theta(y_i).$$

From 4.6, for any $y \in \Delta \subset \cap_{k=1}^{i} O(y_k)$, and $j \in \{1, 2, \cdots, i\}$,

$$\mathrm{SP}\phi_y \subset U(y_j, 1) \bigcup U(y_j, 2) \bigcup \cdots \bigcup U(y_j, t_{y_j}).$$

CLAIM: If $j < j' \in \{1, 2, \cdots, i\}$, then each open set $U(y_j, t)$ $(t = 1, 2 \cdots, t_{y_j})$ intersects with at most one of $\{U(y_{j'}, s)\}_{s=1}^{t_{y_{j'}}}$.
Proof of the Claim: Suppose that the claim is not true, that is, for some $t \in \{1, 2, \cdots, t_{y_j}\}$, there are two different $s_1, s_2 \in \{1, 2 \cdots, t_{y_{j'}}\}$ such that

$$U(y_j, t) \cap U(y_{j'}, s_1) \neq \emptyset \quad \text{and} \quad U(y_j, t) \cap U(y_{j'}, s_2) \neq \emptyset.$$

Together with the fact that diameter$(U(y_j, t)) \leq \theta(y_j)$ (see (b) in 4.6), it yields

$$dist(U(y_{j'}, s_1), U(y_{j'}, s_2)) \leq \theta(y_j).$$

This contradicts with (c) in 4.6 which gives

$$dist(U(y_{j'}, s_1), U(y_{j'}, s_2)) \geq 2\theta(y_{j'}) > \theta(y_j).$$

(Recall $\theta(y_j) \leq \theta(y_{j'})$.) This proves the claim.
Still suppose that $j < j'$. From the claim, we have the following. For each $y \in \Delta$, if two different elements of $\mathrm{SP}\phi_y$ are identified as a single element in the decomposition

$$U(y_j, 1) \bigcup U(y_j, 2) \bigcup \cdots \bigcup U(y_j, t_{y_j})$$

(i.e., if these two elements are in the same open set $U(y_j, t)$ for some $t \in \{1, 2, \cdots, t_{y_j}\}$), then these two elements are also identified as a single element in the decomposition

$$U(y_{j'}, 1) \bigcup U(y_{j'}, 2) \bigcup \cdots \bigcup U(y_{j'}, t_{y_{j'}})$$

(i.e., these two elements are also in the same open set $U(y_{j'}, s)$ for some $s \in \{1, 2, \cdots, t_{y_{j'}}\}$).
Therefore, the decompositions of $\mathrm{SP}\phi_y$ corresponding to y_1 and y_i are the finest and coarsest decompositions, respectively, among all the above decompositions (corresponding to $y_1, y_2, \cdots, y_i$). The coarsest decomposition will be used to decompose $\mathrm{SP}\phi_y$ into several sets. The elements in each of the sets will be

identified as a single element with multiplicity. Denote $\theta(y_i)$ by $\theta(\Delta)$. (Recall that $O(y_1), O(y_2), \cdots, O(y_i)$ is the list of all open sets in $\mathcal{O}$, each of which covers Δ. Therefore, $\theta(y_i)$ — the maximum of all $\{\theta(y_j)\}_{j=1}^{i}$ — depends only on Δ.)

Introduce the following notations.

$$\Lambda(\Delta,1)(y) = U(y_i,1) \cap \mathrm{SP}\phi_y,$$

$$\Lambda(\Delta,2)(y) = U(y_i,2) \cap \mathrm{SP}\phi_y,$$

$$\vdots$$

$$\Lambda(\Delta,t_\Delta)(y) = U(y_i,t_{y_i}) \cap \mathrm{SP}\phi_y,$$

where $t_\Delta = t_{y_i}$. Recall (see 4.6) that $\mathrm{SP}\phi_{y_i}$ is written as

$$\mathrm{SP}\phi_{y_i} = \{\lambda_1^{\sim w_1}, \lambda_2^{\sim w_2}, \cdots, \lambda_k^{\sim w_k}\},$$

where $k = t_\Delta = t_{y_i}$. Since $y \in \Delta \subset O(y_i)$,

$$\#(\Lambda(\Delta,t)(y)) = w_t, \qquad\qquad 1 \le t \le t_\Delta,$$

counting multiplicity. Define set

$$\Lambda(\Delta) = \{\Lambda(\Delta,1)^{\sim w_1}, \Lambda(\Delta,2)^{\sim w_2}, \cdots, \Lambda(\Delta,k)^{\sim w_k}\},$$

where $k = t_\Delta$. That is, identify all the elements of $\mathrm{SP}\phi_y$ in $\Lambda(\Delta,t)(y)$ as a single element (denoted by $\Lambda(\Delta,t)$) with the multiplicity.

As above, we will use $\Lambda(\Delta,t)(y)$ for two purposes. It is a subset of $\mathrm{SP}\phi_y$, or it is a single element in $\Lambda(\Delta)$ which repeats w_t times.

Strictly speaking, w_t $(t = 1, 2, \cdots, t_\Delta)$ should be written as $w_t(\Delta)$, and the set $\Lambda(\Delta)$ should be written as
$\{\Lambda(\Delta,1)^{\sim w_1(\Delta)}, \Lambda(\Delta,2)^{\sim w_2(\Delta)}, \cdots, \Lambda(\Delta,k)^{\sim w_k(\Delta)}\}$. When there is a danger of confusion, we will use $w_t(\Delta)$ instead of w_t.

4.10. Let $Y' \subset Y$ be a path connected subspace. Usually we will let Y' be either an open or a closed subset. Suppose that there are positive integers $u_1, u_2, \cdots, u_t$ and continuous maps

$$A(Y',i) : Y' \to P^{u_i}X, \quad i = 1, 2, \cdots, t,$$

such that $\{\mathrm{SP}\phi_y\}_{y \in Y'}$ can be decomposed as

$$\mathrm{SP}\phi_y = \{A(Y',1)(y), A(Y',2)(y), \cdots, A(Y',t)(y)\}$$

for all $y \in Y'$. We say that THE ABOVE DECOMPOSITION OF $\{\mathrm{SP}\phi_y\}_{y \in Y'}$ SATISFIES THE CONDITION (S) (S stands for separation) if

(S): there are mutually disjoint open sets $U_1, U_2, \cdots, U_t \subset X$ satisfying

$$A(Y',i)(y) \subset U_i, \qquad \forall y \in Y',\ i = 1, 2, \cdots, t.$$

Define

$$A(Y') = \{A(Y',1)^{\sim u_1}, A(Y',2)^{\sim u_2}, \cdots, A(Y',t)^{\sim u_t}\},$$

where $u_i = \#(A(Y',i))$, counting multiplicity. Again, $A(Y',s)(y)$ is used for two purposes. It is regarded as a subset of $\mathrm{SP}\phi_y$ or as a single element of $A(Y')$ with multiplicity u_s.

In fact, if $U_1, U_2, \cdots U_t$ are open sets, with mutually disjoint closure, such that $\mathrm{SP}\phi_y \subset U_1 \cup U_2 \cup \cdots \cup U_t, \forall y \in Y'$, then $\#(\mathrm{SP}\phi_y \cap U_i),\ y \in Y'$ are constants, denoted by u_i, (note that Y' is path connected). Furthermore, the maps

$$A(Y',i) : Y' \to P^{u_i}X, \quad i = 1, 2, \cdots t,$$

defined by $A(Y',i)(y) = \mathrm{SP}\phi_y \cap U_i$, are continuous, and they determine a decomposition of $\{\mathrm{SP}\phi_y\}_{y\in Y'}$ as

$$\mathrm{SP}\phi_y = \{A(Y',1)(y), A(Y',2)(y), \cdots, A(Y',t)(y)\}$$

satisfying the condition (S). (See Lemmas 1.2.9 and 1.2.10.)

For any $y \in Y'$, a grouping $E_1, E_2, \cdots, E_L$ of $\mathrm{SP}\phi_y$ induces a UNIQUE grouping $E_1^{A(Y')}, E_2^{A(Y')}, \cdots E_L^{A(Y')}$ of $A(Y')$, defined by

$$E_i^{A(Y')} = \{A(Y',1)^{\sim v_1}, A(Y',2)^{\sim v_2}, \cdots, A(Y',t)^{\sim v_t}\},\ i = 1, 2, \cdots, L,$$

where $v_j = \#(E_i \cap A(Y',j)(y))$, counting multiplicity. (Here the intersection of sets is defined as for the sets with multiplicity, as in 3.21.)

On the other hand, let $E_1, E_2, \cdots, E_L$ be a grouping of $A(Y')$. Define a grouping of the set $\mathrm{SP}\phi_y$, for any $y \in Y'$, in the following way. For any $j = 1, 2, \cdots, L$, if the part E_j (for the grouping of $A(Y')$) contains exactly w elements of $\{A(Y',s)^{\sim u_s}\}$ ($w \leq u_s$), then the part E_j (for the grouping of the set of $\mathrm{SP}\phi_y$) contains exactly w elements (counting multiplicity) which are contained in $A(Y',s)(y)$. Since these w elements are to be chosen, the induced grouping is not unique. But we will always fix one of them for use.

Let $E_1, E_2, \cdots, E_L$ be a grouping of

$$A(Y') = \{A(Y',1)^{\sim u_1}, A(Y',2)^{\sim u_2}, \cdots, A(Y',t)^{\sim u_t}\}.$$

Define $\mathring{E}_1, \mathring{E}_2, \cdots, \mathring{E}_L$ as in 3.22.

Although the subsets of $\mathrm{SP}\phi_y$ corresponding to E_i are not unique, the subsets of $\mathrm{SP}\phi_y$ corresponding to $\mathring{E}_i$ are unique. We denote them by $\mathring{E}_i|_y$. Also, $\#(\mathring{E}_i) = \#(\mathring{E}_i|_y)$ counting multiplicity. Note that we use $\mathring{E}_i|_y$ instead of $\mathring{E}_i(y)$ for the following reason (also see the next paragraph). We reserve the notation $\{E_i(y)\}_{i=1}^{L}$ for the grouping of $\mathrm{SP}\phi_y$ which is associated to a vertex y in a certain simplicial complex (Y, τ). (τ is a subdivision of σ.)

Suppose that $y \in Y'$. Let $E_1(y), E_2(y), \cdots E_L(y)$ be a grouping of $\mathrm{SP}\phi_y$. Then it induces a grouping $E_1^{A(Y')}(y), E_2^{A(Y')}(y), \cdots E_L^{A(Y')}(y)$ of $A(Y')$ as above. The sets $\mathring{E}_i^{A(Y')}(y)$ are well defined as subsets of $A(Y')$. (Warning: $\mathring{E}_i^{A(Y')}(y)$

are not subsets of $\mathrm{SP}\phi_y$.) Also, from the last paragraph, the sets $\overset{\circ}{E}{}_i^{A(Y)}(y)|_z$ are well defined as subsets of $\mathrm{SP}\phi_z$ for any $z \in Y'$ (may be different from y). Furthermore, $\#(\overset{\circ}{E}{}_i^{A(Y')}(y)) = \#(\overset{\circ}{E}{}_i^{A(Y')}(y)|_z)$.
In the next few paragraphs, each grouping of $\mathrm{SP}\phi_y$ can be referred as a grouping of $A(Y')$ for different space Y' and different decomposition $A(Y')$ provided that $y \in Y'$, or vise versa.

4.11. Note that the collection of sets $\{\Lambda(\Delta, i)\}_{i=1}^{t_\Delta}$ (in 4.9) could be regarded as a decomposition of $\mathrm{SP}\phi_y$, $y \in \Delta$ (see 4.6 also). And this decomposition satisfies condition (S) for Δ in place of Y'; therefore 4.10 can be applied to $\Lambda(\Delta)$ as $A(Y')$.
As mentioned in 4.8, we will introduce the groupings of $\mathrm{SP}\phi_y$ for all vertices of a certain subdivision of (Y, σ). As in section 3, for a simplex Δ of (Y, σ), once we have the subdivision $(\partial\Delta, \tau)$ of $(\partial\Delta, \sigma)$ and groupings for all vertices in $\mathrm{Vertex}(\partial\Delta, \tau)$, then we can define the subdivision (Δ, τ) of (Δ, σ) and introduce the groupings for all newly introduced vertices. One may notice that, in section 3, for different vertices, the index sets involved are the SAME. But in the setting here, the index sets $\mathrm{SP}\phi_y$ are DIFFERENT for different vertices y. So some special care should be taken.
Suppose that (Δ, σ) is a simplicial complex consisting of a single simplex Δ and all its faces. Suppose that there is a subdivision $(\partial\Delta, \tau)$ of $(\partial\Delta, \sigma)$ and the groupings $E_1(y), E_2(y), \cdots, E_L(y)$ of $\mathrm{SP}\phi_y$ for all vertices $y \in \mathrm{Vertex}(\partial\Delta, \tau)$ (see notation in 3.1–3.3). ATTENTION: When we introduce a grouping $E_1(z), E_2(z), \cdots, E_L(z)$ of $\mathrm{SP}\phi_z$ for any newly introduced vertex $z \in \mathrm{interior}(\Delta) = \Delta \backslash \partial\Delta$, THE FOLLOWING PROCEDURE WILL ALWAYS BE USED.
First, as in 4.10, we can regard the groupings $E_1(y), E_2(y), \cdots, E_L(y)$ of $\mathrm{SP}\phi_y$ as groupings $E_1^{\Lambda(\Delta)}(y), E_2^{\Lambda(\Delta)}(y), \cdots, E_L^{\Lambda(\Delta)}(y)$ of $\Lambda(\Delta)$ for all vertices $y \in \mathrm{Vertex}(\partial\Delta, \tau)$. (Then the set $E_i^{\Lambda(\Delta)}(y) \cap E_i^{\Lambda(\Delta)}(y') \cap \cdots$, as a subset of $\Lambda(\Delta)$, makes sense, for vertices $y, y', \cdots \in \mathrm{Vertex}(\partial\Delta, \tau)$. Also $\{\overset{\circ}{E}{}_i^{\Lambda(\Delta)}(y)\}_{i=1}^{L}$ are subsets of $\Lambda(\Delta)$.) Then we use these groupings of the SAME index set, $\Lambda(\Delta)$, applying the results from section 3 (see 3.32), to introduce subdivision (Δ, τ) of (Δ, σ) and groupings of $\Lambda(\Delta)$ for all newly introduced vertices $z \in \Delta \backslash \partial\Delta$. Finally, theses groupings of $\Lambda(\Delta)$ will induce the groupings $E_1(z), E_2(z), \cdots, E_L(z)$ of $\mathrm{SP}\phi_z$ as in 4.10 (not unique, but we fix one of them for our use). Furthermore, as in 4.10, $\overset{\circ}{E}{}_1^{\Lambda(\Delta)}(z), \overset{\circ}{E}{}_2^{\Lambda(\Delta)}(z), \cdots, \overset{\circ}{E}{}_L^{\Lambda(\Delta)}(z)$ are well defined subsets of $\Lambda(\Delta)$ and $\overset{\circ}{E}{}_1^{\Lambda(\Delta)}(z)|_{z'}, \overset{\circ}{E}{}_2^{\Lambda(\Delta)}(z)|_{z'}, \cdots, \overset{\circ}{E}{}_L^{\Lambda(\Delta)}(z)|_{z'}$ are well defined subsets of $\mathrm{SP}\phi_{z'}$ for any $z' \in \Delta$ (not necessarily a vertex).

4.12. Let Δ' be a face of Δ. Then for $y \in \Delta' \subset \Delta$, both $\Lambda(\Delta')$ and $\Lambda(\Delta)$ can be viewed as decompositions of $\mathrm{SP}\phi_y$. Recall, in 4.9, the decomposition corresponding to $\Lambda(\Delta)$ is the coarsest decomposition among those corresponding to $O(y_j)$ such that $O(y_j) \supset \Delta$ and that $O(y_j) \in \mathcal{O}$. Since $\Delta' \subset \Delta$, any open set in $\mathcal{O}$ which covers Δ will also cover Δ'. Therefore, the decomposition of $\mathrm{SP}\phi_y$

corresponding to $\Lambda(\Delta')$ is coarser than that corresponding to $\Lambda(\Delta)$. That is, each set $\Lambda(\Delta', s)(y)$ is a finite union of certain sets,

$$\Lambda(\Delta, t_1)(y) \bigcup \Lambda(\Delta, t_2)(y) \bigcup \cdots ,$$

as subsets of $\mathrm{SP}\phi_y$. (Notice that if $w'_s = w_s(\Delta')$ is the multiplicity appeared in $\Lambda(\Delta')$ for $\Lambda(\Delta', s)$ and $w_t = w_t(\Delta)$ is the multiplicity appeared in $\Lambda(\Delta)$ for $\Lambda(\Delta, t)$, then $w'_s = w_{t_1} + w_{t_2} + \cdots$, a finite sum.)

It follows that if $y \in \Delta' \subset \Delta$ and $E_1(y), E_2(y), \cdots, E_L(y)$ is a grouping of $\mathrm{SP}\phi_y$, then

$$\overset{\circ}{E}{}_j^{\Lambda(\Delta')}(y)|_{y'} \subset \overset{\circ}{E}{}_j^{\Lambda(\Delta)}(y)|_{y'},$$

regarded as subsets of $\mathrm{SP}\phi_{y'}$ for any $y' \in \Delta'$. Now we are ready to construct the subdivision (Y, τ) of (Y, σ), and the grouping for each vertex of (Y, σ) and each vertex of the complex (Y, τ) (after subdivision).

Since the notations $y_1, y_2, \cdots, y_\bullet$ have been already used for the open cover $\mathcal{O} = \{O(y_1), O(y_1), \cdots, O(y_\bullet)\}$, we use $z_1, z_2, \cdots$ to denote the points in Y, especially the vertices of certain simplicial structure.

4.13. Let $\Omega = \dim X + \dim Y$, $M = 2^L - 1$, where L is the number of the continuous maps $\{a_i\}$ appeared in the statement of 4.1. Note that all the multiplicities w_t appearing in any of $\Lambda(\Delta)$ do not exceed Ω, by 4.3 and the construction of $\Lambda(\Delta)$ (see 4.6 and 4.9).

For each vertex $z \in \mathrm{Vertex}(Y, \sigma)$, by the condition of Theorem 4.1, $\mathrm{SP}\phi_z$ and

$$\Theta(z) = \{a_1(z)^{\sim T_1}, a_2(z)^{\sim T_2}, \cdots, a_L(z)^{\sim T_L}\}$$

can be paired within η. Therefore, we can define a grouping $E_{pre,1}(z), E_{pre,2}(z), \cdots, E_{pre,L}(z)$ of $\mathrm{SP}\phi_z$, with $T_1, T_2, \cdots, T_L$ elements, respectively, counting multiplicity, such that

$$\mathrm{dist}(\lambda, a_i(z)) < \eta \tag{1}$$

if $\lambda \in E_{pre,i}(z)$, where η is as in 4.4. (We denote them by $E_{pre,i}$ because this grouping will be modified later.)

We can regard such a grouping of $\mathrm{SP}\phi_z$ as a grouping of $\Lambda(\Delta)$, where $\Delta \ni z$ is a simplex.

First we regard it as the grouping of $\Lambda(\{z\})$, where $\{z\}$ is the 0-dimensional simplex of (Y, σ) corresponding to vertex z. By Lemma 3.28, we can modify the grouping to satisfy the Assumption 3.27. Then this modified grouping of $\Lambda(\{z\})$ could induce a grouping on $\mathrm{SP}\phi_z$, for which the condition (1) above may not hold. But if we carefully choose the sets H_i in Lemma 3.28, we could still guarantee that any elements $\lambda \in E_i$ are close to $a_i(z)$ (see (2) below). In this subsection, we will also introduce the sets $H_i(\Delta)$ to serve as the sets H_i of Lemma 3.28 and Theorem 3.32, when we construct groupings on Δ from the groupings on $\partial\Delta$, by applying Theorem 3.32.

For each vertex $z_0 \in \mathrm{Vertex}(Y,\sigma)$, the notation $\{z_0\}$ is used to denote the corresponding zero dimensional simplex of (Y,σ). The above grouping induces a grouping $\{E_{pre,i}^{\Lambda(\{z_0\})}\}$ of

$$\Lambda(\{z_0\}) = \{\Lambda(\{z_0\},1)^{\sim w_1}, \Lambda(\{z_0\},2)^{\sim w_2}, \cdots, \Lambda(\{z_0\},t_{\{z_0\}})^{\sim w_{t_{\{z_0\}}}}\}.$$

Define subsets $H_1(\{z_0\}), H_2(\{z_0\}), \cdots, H_L(\{z_0\})$ of $\Lambda(\{z_0\})$ as follows. For any $i = 1,2,\cdots,L$, $H_i(\{z_0\})$ is the collection of all
$\Lambda(\{z_0\},t)^{\sim w_t}$ ($\subset \Lambda(\{z_0\})$) satisfying

$$\Lambda(\{z_0\},t)(z_0) \subset \left\{x:\ \mathrm{dist}(x,a_i(z_0)) < \eta + \frac{1}{(\dim Y+1)}\cdot\eta\right\}.$$

Note that each set $\Lambda(\{z_0\},t)(z_0)$ (as a subset of certain $U(y_j,t)$, from 4.9) has diameter at most $\frac{\eta}{2(\dim Y+1)} < \frac{\eta}{(\dim Y+1)}$ (see (b) in 4.6). Combining this fact with (1) above, we know that if $\lambda_1 \in E_{pre,i}(z_0)$ and $\lambda_1 \in \Lambda(\{z_0\},t)(z_0)$, then

$$\mathrm{dist}(\lambda,a_i(z_0)) < \eta + \frac{\eta}{(\dim Y+1)} \tag{2}$$

for any $\lambda \in \Lambda(\{z_0\},t)(z_0)$. That means $E_{pre,i}^{\Lambda(\{z_0\})} \subset H_i(\{z_0\})$.
By Lemma 3.28, the above grouping can be modified to another grouping $E_i^{\Lambda(\{z_0\})}$ of $\Lambda(\{z_0\})$ satisfying

$$\#(\overset{\circ}{E}_i{}^{\Lambda(\{z_0\})}) \geq T_i - M\Omega.$$

(This is Assumption 3.27.) And $E_i^{\Lambda(\{z_0\})} \subset H_i(\{z_0\})$ still holds, regarded as a grouping of $\Lambda(\{z_0\})$.
The above grouping of $\Lambda(\{z_0\})$ could induce a grouping $E_1(z_0), E_2(z_0), \cdots, E_L(z_0)$ of $\mathrm{SP}\phi_{z_0}$ (see 4.10 and 4.11). This grouping will be used as the grouping for vertex z_0. Even though (1) may not hold for λ in the new $E_i(z_0)$, (2) holds for any λ in the new $E_i(z_0)$, from the definition of $H_i(\{z_0\})$, and $E_i^{\Lambda(\{z_0\})} \subset H_i(\{z_0\})$.
For each simplex Δ of (Y,σ), let us also define the subsets $H_1(\Delta)$, $H_2(\Delta), \cdots$, $H_L(\Delta)$ of $\Lambda(\Delta)$ as follows. For each $j = 1,2,\cdots,L$, $H_j(\Delta)$ is the collection of all such $\Lambda(\Delta,t)^{\sim w_t}$ ($\subset \Lambda(\Delta)$) that $\Lambda(\Delta,t)(z)$, as a subset of $\mathrm{SP}\phi_z$, satisfies

$$\Lambda(\Delta,t)(z) \subset \{x:\ \mathrm{dist}(x,a_i(z)) < \eta + \frac{\dim(\Delta)+1}{(\dim Y+1)}\cdot\eta\}$$

for any $z \in \Delta$. These sets will serve as the sets $H_1, H_2, \cdots, H_L$ when we apply Theorem 3.32.
The following fact follows directly from the definition of $\Lambda(\Delta,t)$ and $H_i(\Delta)$, which will be used in 4.14:
Suppose that $z \in \Delta$. A grouping $E_1, E_2, \cdots E_L$ of $\mathrm{SP}\phi_z$, regarded as a grouping of $\Lambda(\Delta)$, satisfies $E_i \subset H_i(\Delta)$ if and only if for any $\lambda \in E_i$ (as a subset of

$\mathrm{SP}\phi_z$), for the index t satisfying $\lambda \in \Lambda(\Delta, t)(z)$ (such t exists; see 4.9), we have $\{\Lambda(\Delta, t)^{\sim w_t(\Delta)}\} \subset H_i(\Delta)$.

4.14. Beginning with the simplicial structure (Y, σ) and the above groupings for all 0-dimensional simplex (i.e., vertex) of (Y, σ), we will construct a subdivision (Y, τ) of (Y, σ) and the groupings for newly introduced vertices. We will refine (Y, σ), simplex by simplex, from the lowest dimension to the highest dimension by use of Theorem 3.32.

To avoid confusion, use $\Gamma, \Gamma_1, \Gamma_2, \Gamma', etc.$ to denote the simplices of (Y, τ), after subdivision, and reserve the notations $\Delta, \Delta', \Delta_1, etc.$ for the simplices of (Y, σ)—with original simplicial complex structure σ introduced in 4.7.

As the induction assumption, we suppose that there are a subdivision $(\partial\Delta, \tau)$ of $(\partial\Delta, \sigma)$ and the groupings of $\mathrm{SP}\phi_z$ for all vertices $z \in \mathrm{Vertex}(\partial\Delta, \tau)$ with the following properties.

(1) If Δ' is a proper face of (Δ, σ) (by a proper face of Δ, we mean a face Δ' with $\Delta' \subset \partial\Delta$) and $z \in \Delta'$, then the grouping of $\Lambda(\Delta')$, induced by the grouping of $\mathrm{SP}\phi_z$ satisfies

$$E_i \subset H_i(\Delta').$$

In other words, $E_i^{\Lambda(\Delta')}(z) \subset H_i(\Delta')$.

(2) Let Γ be a simplex of $(\partial\Delta, \tau)$ with vertices $z_0, z_1, \cdots, z_j$. If $\Gamma \subset \Delta'$, where Δ' is a proper face of Δ, then

$$\begin{aligned} \# \left(\mathring{E}_i^{\Lambda(\Delta')}(z_0) \cap \mathring{E}_i^{\Lambda(\Delta')}(z_1) \cap \cdots \cap \mathring{E}_i^{\Lambda(\Delta')}(z_j) \right) & \\ \geq T_i - [M\Omega + M\Omega \dim \Delta' (\dim \Delta' + 1)] & \\ (\geq T_i - [M\Omega + M\Omega \dim \partial\Delta (\dim \partial\Delta + 1)]) \, . & \end{aligned}$$

(3) For each vertex z of $(\partial\Delta, \tau)$, Assumption 3.27 holds. I.e.,

$$\#(\mathring{E}_i^{\Lambda(\Delta')}(z)) \geq T_i - M\Omega$$

for any proper face Δ' of Δ with $z \in \Delta'$.

(In the above conditions (1), (2) and (3), $\{E_i^{\Lambda(\Delta')}(z)\}_{i=1}^{L}$ are regarded as groupings of the set $\Lambda(\Delta')$ (with multiplicity); see 4.11.)

Now we define the subdivision (Δ, τ) of (Δ, σ) and the groupings for all newly introduced vertices. The restriction of the simplicial structure (Δ, τ) on $\partial\Delta$ will be the same as $(\partial\Delta, \tau)$, that is, we will only introduce new vertices inside interior$(\Delta) = \Delta \backslash \partial\Delta$. We need to define the groupings as groupings of $\Lambda(\Delta)$. Then they will induce groupings of $\mathrm{SP}\phi_z$.

Claim: For any vertex z of $(\partial\Delta, \tau)$, if the grouping of $\mathrm{SP}\phi_z$ is regarded as the grouping of $\Lambda(\Delta)$, then

(1') $E_i(z) \subset H_i(\Delta)$ for $i = 1, 2, \cdots, L$. In other words, $E_i^{\Lambda(\Delta)}(z) \subset H_i(\Delta)$.

Proof of the Claim: Let y be the point y_i in the definition of $\Lambda(\Delta)$ in 4.9. Then $\Delta \subset O(y) \in \mathcal{O}$. (We avoid the notation y_i, since i is used for E_i above. So we use y instead.)

Let $\lambda \in E_i(z)$. Since $z \in \partial\Delta$, there is a proper face Δ' of Δ such that $z \in \Delta'$. By (1) above, $E_i(z) \subset H_i(\Delta')$ (regarded as a grouping of $\Lambda(\Delta')$). From the end of 4.13, there is an index s such that $\lambda \in \Lambda(\Delta', s)(z)$ and that $\{\Lambda(\Delta', s)^{\sim w_s(\Delta')}\} \subset H_i(\Delta')$.
Recall, from 4.12, $\Lambda(\Delta', s)(z')$ is a finite union $\Lambda(\Delta, t_1)(z') \cup \Lambda(\Delta, t_2)(z') \cup \cdots$, for any $z' \in \Delta' \subset \Delta$. Hence, there is an index t such that

$$\lambda \in \Lambda(\Delta, t)(z) \subset \Lambda(\Delta', s)(z),$$

where both $\Lambda(\Delta, t)(z)$ and $\Lambda(\Delta', s)(z)$ are regarded as subsets of $\mathrm{SP}\phi_z$. To prove the claim, by the end of 4.13, we only need to prove $\{\Lambda(\Delta, t)^{\sim w_t(\Delta)}\} \subset H_i(\Delta)$. From the definition of $H_i(\Delta)$, this is equivalent to

$$(A) \qquad \Lambda(\Delta, t)(z_1) \subset \{x : \ \mathrm{dist}(x, a_i(z_1)) < \eta + \frac{\dim(\Delta) + 1}{(\dim Y + 1)} \cdot \eta\}$$

for any $z_1 \in \Delta$. From $\{\Lambda(\Delta', s)^{\sim w_s(\Delta')}\} \subset H_i(\Delta')$ and the definition of $H_i(\Delta')$, we have

$$\Lambda(\Delta', s)(z') \subset \{x : \ \mathrm{dist}(x, a_i(z')) < \eta + \frac{\dim(\Delta') + 1}{(\dim Y + 1)} \cdot \eta\}$$

for any $z' \in \Delta'$. In the above, if we choose $z' = z$—the vertex in the claim—(and note that $\Lambda(\Delta, t)(z) \subset \Lambda(\Delta', s)(z)$), then

$$(a) \qquad \Lambda(\Delta, t)(z) \subset \{x : \ \mathrm{dist}(x, a_i(z)) < \eta + \frac{\dim(\Delta') + 1}{(\dim Y + 1)} \cdot \eta\}.$$

On the other hand, from (d) in 4.6, we have

$$(b) \qquad \mathrm{diameter}(a_i(\Delta)) \le \mathrm{diameter}(a_i(O(y))) < \frac{\eta}{2(\dim Y + 1)}.$$

And from (b) in 4.6, we have

$$(c) \qquad \mathrm{diameter}(U(y, t)) < \frac{\eta}{2(\dim Y + 1)}.$$

From 4.6 and 4.9, $\Lambda(\Delta, t)(z_1) \subset U(y, t)$ for any $z_1 \in \Delta \subset O(y)$. Combining this with (c) above, for any $\mu \in \Lambda(\Delta, t)(z_1)$ ($z_1 \in \Delta$), we have

$$dist(\mu, \Lambda(\Delta, t)(z)) < \frac{\eta}{2(\dim Y + 1)}.$$

Then combining it with (a) above, we have

$$dist(\mu, a_i(z)) < \eta + \frac{(\dim(\Delta') + 1)\eta}{(\dim Y + 1)} + \frac{\eta}{2(\dim Y + 1)}.$$

Finally, combining it with (b),

$$dist(\mu, a_i(z_1)) < \eta + \frac{(\dim(\Delta')+1)\eta}{(\dim Y+1)} + \frac{\eta}{2(\dim Y+1)} + \frac{\eta}{2(\dim Y+1)}$$

$$\leq \eta + \frac{\dim(\Delta)+1}{(\dim Y+1)},$$

since $dim(\Delta') \leq dim(\Delta) - 1$. Note that $z_1 \in \Delta$ and $\mu \in \Lambda(\Delta, t)(z_1)$ are arbitrary, this proves (A) and the Claim.

Suppose that Γ is a simplex of $(\partial\Delta, \tau)$ with vertices $z_0, z_1, \cdots z_j$. Suppose that $\Gamma \subset \Delta'$, where Δ' is a face of Δ. As mentioned in 4.12,

$$\overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z)|_{z'} \subset \overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z)|_{z'}$$

as a subset of $\mathrm{SP}\phi_{z'}$ for all $z' \in \Delta'$ and all $z = z_0, z_1, \cdots, z_j$. Therefore, from (2) and (3) above, we have the following (2') and (3').

(2') $\#\left(\overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_0) \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_1) \cap \cdots \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_j)\right)$

$$= \#\left(\overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_0)|_{z'} \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_1)|_{z'} \cap \cdots \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z_j)|_{z'}\right)$$

$$\geq \#\left(\overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_0)|_{z'} \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_1)|_{z'} \cap \cdots \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_j)|_{z'}\right)$$

$$= \#\left(\overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_0) \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_1) \cap \cdots \cap \overset{\circ}{E}{}_i^{\Lambda(\Delta')}(z_j)\right)$$

$$\geq T_i - [M\Omega + M\Omega \dim \partial\Delta(\dim \partial\Delta + 1)],$$

for every simplex $\Gamma \subset (\partial\Delta, \tau)$ with vertices $z_0, z_1, \cdots, z_j$.

(3') The Assumption 3.27 holds for the grouping $\{E_i(z)\}_{i=1}^L$ regarded as a grouping of $\Lambda(\Delta)$, i.e.,

$$\#(\overset{\circ}{E}{}_i^{\Lambda(\Delta)}(z)) \geq T_i - M\Omega,$$

where z is a vertex of $(\partial\Delta, \tau)$.

Apply Theorem 3.32 to obtain a subdivision (Δ, τ) of (Δ, σ), and, for each newly introduced vertex $z \in \Delta$, a grouping $E_1(z), E_2(z), \cdots E_L(z)$ of $\mathrm{SP}\phi_z$ such that (1), (2) and (3) hold with the version obtained by replacing Δ' by Δ, and $\dim(\partial\Delta)$ by $\dim\Delta$. (As mentioned in 4.11, for each vertex z, we should first get the groupings of $\Lambda(\Delta)$, then this grouping induces a grouping of $\mathrm{SP}\phi_z$.)

Using Mathematical Induction, combined with 4.13, we obtain our subdivision (Y, τ) of (Y, σ) and the groupings.

We summarize what we obtained in 4.13 and 4.14 as in the following proposition.

Proposition: *There is a subdivision (Y, τ) of (Y, σ), and for all vertices $z \in Vertex(Y, \tau)$, there are groupings $E_1(z), E_2(z), \cdots, E_L(z)$ of $SP\phi_z$ of type*

$(T_1, T_2, \cdots, T_L)$ (i.e., $\#(E_i(z)) = T_i \ \ \forall i$) such that the following are true.
(1) If Δ is a simplex of (Y, σ) (before subdivision) and $z \in \Delta$, then the grouping $(E_1^{\Lambda(\Delta)}(z), E_2^{\Lambda(\Delta)}(z), \cdots E_L^{\Lambda(\Delta)}(z))$ of $\Lambda(\Delta)$, induced by the grouping $(E_1(z), E_2(z), \cdots, E_L(z))$ of $SP\phi_z$, satisfies

$$E_i^{\Lambda(\Delta)}(z) \subset H_i(\Delta).$$

(2) Let Γ be a simplex of (Y, τ) with vertices $z_0, z_1, \cdots, z_j$. If $\Gamma \subset \Delta$, where Δ is a simplex of (Y, σ) (before subdivision), then

$$\begin{aligned}\# \left(\mathring{E}_i^{\Lambda(\Delta)}(z_0) \cap \mathring{E}_i^{\Lambda(\Delta)}(z_1) \cap \cdots \cap \mathring{E}_i^{\Lambda(\Delta)}(z_j) \right) & \\ \geq T_i - [M\Omega + M\Omega \dim \Delta(\dim \Delta + 1)] & \\ (\geq T_i - [M\Omega + M\Omega \dim Y(\dim Y + 1)]) \, . &\end{aligned}$$

(We do not need the condition (3) any more.)

4.15. For the simplicial complex (Y, τ), there is a finite open cover

$$\{W(\Gamma) : \ \Gamma \text{ is a simplex of } \ (Y, \tau)\}$$

of Y, with the following properties.
(a) $W(\Gamma) \supset \text{interior}(\Gamma) = \Gamma \backslash \partial\Gamma$.
(b) If $W(\Gamma_1) \cap W(\Gamma_2) \neq \emptyset$, then either Γ_1 is a face of Γ_2 or Γ_2 is a face of Γ_1.
(Such open cover has been constructed in 1.4.2 (b).)
For any simplex Γ, we will construct an open set $O(\Gamma) \supset \Gamma$ and introduce a decomposition $\Xi(\Gamma)$ of $\{\mathrm{SP}\phi_y\}_{y \in O(\Gamma)}$, which is the finest possible decomposition satisfying the condition (S) for Γ in place Y' in 4.10.
Recall that $K = \text{rank}(P)$, and $y \mapsto \mathrm{SP}\phi_y$ defines a map $\mathrm{SP}\phi : Y \to P^K X$. We will prove the following easy fact.
CLAIM 1: $\mathrm{SP}\phi|_\Gamma := \overline{\cup_{z \in \Gamma} \mathrm{SP}\phi_z}$ $(\subset X)$ has at most K connected components.
(For $K = 1$, the claim says that the image of a connected space Γ under a continuous map $\mathrm{SP}\phi : \Gamma \to P^1 X = X$ is connected. This is a trivial fact.)
Proof of Claim 1: Suppose that by the contrary, $\mathrm{SP}\phi|_\Gamma$ has more than K connected components. Write $\mathrm{SP}\phi|_\Gamma = X_1 \cup X_2 \cup \cdots \cup X_{K+1}$, where $X_1, X_2, \cdots, X_{K+1}$ are mutually disjoint non empty closed subsets (which are not necessary connected).
There are open sets $U_1, U_2, \cdots, U_{K+1}$ with mutually disjoint closures such that $U_i \supset X_i$. Then for any $z \in \Gamma$, $\mathrm{SP}\phi_z \subset \cup_{i=1}^{K+1} U_i$. By Lemma 1.2.9, for each i, $\#(\mathrm{SP}\phi_z \cap U_i)$ is a nonzero constant. Hence $\#(\mathrm{SP}\phi_z) \geq K + 1$, contradicting with $\#(\mathrm{SP}\phi_z) = K = \text{rank}(P)$, counting multiplicity. This proves the claim.
We are back to our construction of open set $O(\Gamma)$ and decomposition $\Xi(\Gamma)$. Write $\mathrm{SP}\phi|_\Gamma = X_1 \cup X_2 \cup \cdots \cup X_t$, where $X_1, X_2, \cdots, X_t$ (with $t \leq K$) are mutually disjoint connected components of $\mathrm{SP}\phi|_\Gamma$. Choose open sets $U_1, U_2, \cdots, U_t$

with mutually disjoint closures such that $X_i \subset U_i$. By Lemma 1.2.9, there is an open set $O(\Gamma) \supset \Gamma$ such that $\mathrm{SP}\phi|_{O(\Gamma)} \subset \cup_{i=1}^t U_i$.
As in 4.10, define

$$\Xi(\Gamma, t)(z) = \mathrm{SP}\phi_z \cap U_i, \ \forall z \in O(\Gamma), \ i = 1, 2, \cdots, t.$$

This gives a decomposition

$$\mathrm{SP}\phi_z = \ \{\Xi(\Gamma, 1)(z), \Xi(\Gamma, 2)(z), \cdots, \Xi(\Gamma, t)(z)\}, \ \forall z \in O(\Gamma).$$

Let $c_i = \#(\Xi(\Gamma, i)$, counting multiplicity. And write

$$\Xi(\Gamma) := \ \{\Xi(\Gamma, 1)^{\sim c_1}, \Xi(\Gamma, 2)^{\sim c_2}, \cdots, \Xi(\Gamma, t)^{\sim c_t}\}.$$

Note that the above decomposition satisfies condition (S) in 4.10 as the decomposition of spectrum on $O(\Gamma)$ (not only on Γ). In 4.16 below, when we apply 4.10, we will use $U(\Gamma)$ (a subset of $O(\Gamma)$) in place of Y' of 4.10. Obviously, $\Xi(\Gamma)$ is the finest decomposition among all the decompositions of $(\mathrm{SP}\phi_z)_{z\in\Gamma}$ satisfying condition (S) on Γ, since each X_i is connected. In particular, if $z \in \Gamma \subset \Delta$, where Δ is a simplex of (Y, σ) (before subdivision), then the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma)$ is finer than the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Lambda(\Delta)$.
We will use the following fact later.
Claim 2: If $\Gamma' \subset \Gamma$ is a face, then for any $z \in O(\Gamma') \cap O(\Gamma)$, the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma')$ is finer than the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma)$.
Proof of Claim 2: The Claim follows from the definition of $\Xi(\Gamma)$ and the fact that any connected component of $\mathrm{SP}\phi|_{\Gamma'}$ is completely contained in a connected component of $\mathrm{SP}\phi|_{\Gamma}$.

4.16. For each simplex Γ, define $U(\Gamma) = W(\Gamma) \cap O(\Gamma)$.
$\{U(\Gamma);\ \Gamma$ is a simplex of $(Y, \tau)\}$ is an open covering of Y since $U(\Gamma) \supset$ interior(Γ).
For each $U = U(\Gamma)$, we will define mutually orthogonal projection valued functions

$$P_1^U, P_2^U, \cdots, P_L^U : \ U(\Gamma) \ni y \mapsto \{ \text{ sub-projections of } P(y)\}.$$

Then apply Proposition 3.2 of [DNNP] to construct the globally defined projections $p_1, p_2, \cdots, p_L$ for our Theorem 4.1.
(Attention: For each vertex z of (Y, τ), we have a grouping
$E_1(z), E_2(z), \cdots, E_L(z)$ of $\mathrm{SP}\phi_z$. It will induce a grouping of $\Xi(\Gamma)$, as in 4.10, if $\Gamma \ni z$. In the following construction of $P_i^U(y)$, this grouping will be used. That is, we will use the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma)$. The decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Lambda(\Delta)$ will NOT be used in the definition of $P_i^U(y)$ at all— it is only used in the estimation of rank(P_i^U).
In the definition of the grouping $E_1(z), E_2(z), \cdots, E_L(z)$ of $\mathrm{SP}\phi_z$, it involves the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Lambda(\Delta)$. But once it has been defined, it

makes sense by itself without the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Lambda(\Delta)$ as a reference (though $\mathring{E}_i^{\Lambda(\Delta)}(z)|_z$ only makes sense with the decomposition as the reference).)

Back to our construction. For each $y \in U$ $(= U(\Gamma))$ and each $i = 1, 2, \cdots, L$, define $P_i^U(y)$ to be the spectral projection of ϕ_y corresponding to

$$\left(\mathring{E}_i^{\Xi(\Gamma)}(z_0) \cap \mathring{E}_i^{\Xi(\Gamma)}(z_1) \cap \cdots \cap \mathring{E}_i^{\Xi(\Gamma)}(z_j)\right)|_y \,,$$

where $z_0, z_1, \cdots, z_j$ are all vertices of Γ, and the notations $\mathring{E}_i^{\Xi(\Gamma)}(z)$ and $\mathring{E}_i^{\Xi(\Gamma)}(z)|_y$ are as in 4.10. (That is, $\mathring{E}_i^{\Xi(\Gamma)}(z)$ is a subset of $\Xi(\Gamma)$ and $\mathring{E}_i^{\Xi(\Gamma)}(z)|_y$ is a subset of $\mathrm{SP}\phi_y$.)

By Lemma 1.2.9, the above functions $P_i^U(y)$ depend on y continuously. In fact, for each i and any $y \in U(\Gamma) \subset Y$, $P_i^U(y)$ is the spectral projection of ϕ_y corresponding to an open subset (of X)— in the notation of 4.15 (see the paragraph after the proof of Claim 1 in 4.15), the open subset is the union of all open subsets $U_j \subset X$ such that

$$\Xi(\Gamma, j) \in \mathring{E}_i^{\Xi(\Gamma)}(z_0) \cap \mathring{E}_i^{\Xi(\Gamma)}(z_1) \cap \cdots \cap \mathring{E}_i^{\Xi(\Gamma)}(z_j) (\subset \Xi(\Gamma)).$$

(Note that when we apply Lemma 1.2.9, we use the fact that $\{U_i\}$ have mutually disjoint closures and $\mathrm{SP}\phi_y \subset \bigcup U_i$ from 4.15.) Recall, the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma)$ is finer than any decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Lambda(\Delta)$, if $\Gamma \subset \Delta$. Therefore, $\mathring{E}_i^{\Xi(\Gamma)}(z_0)|_z \supset \mathring{E}_i^{\Lambda(\Delta)}(z_0)|_z$, regarded as a subset of $\mathrm{SP}\phi_z$, for any vertex $z_0 \in \Gamma$ and any point $z \in \Gamma$. By Condition (2) of the grouping (see 4.14),

$$\mathrm{rank}(P_j^U) \geq T_j - [M\Omega + M\Omega \dim Y(\dim Y + 1)]$$

for each U.

The projections $P_i^U, i = 1, 2, \cdots, L$ are mutually orthogonal, since they are spectral projections corresponding to mutually disjoint subsets of X.

Let Γ' be a face of Γ and $z \in U(\Gamma) \cap U(\Gamma')$. By Claim 2 in 4.15, opposite to the case of decompositions corresponding to $\Lambda(\Delta)$ and $\Lambda(\Delta')$, the decomposition of $\mathrm{SP}\phi_z$ corresponding to $\Xi(\Gamma')$ is finer than that corresponding to $\Xi(\Gamma)$. Therefore,

$$\mathring{E}_i^{\Xi(\Gamma)}(z_0)|_y \subset \mathring{E}_i^{\Xi(\Gamma')}(z_0)|_y$$

for all $z_0 \in \mathrm{Vertex}(\Gamma', \tau) \subset \mathrm{Vertex}(\Gamma, \tau)$. Combining it with the fact that $\mathrm{Vertex}(\Gamma', \tau) \subset \mathrm{Vertex}(\Gamma, \tau)$, we get

$$\left(\bigcap_{z_j \in \mathrm{Vertex}(\Gamma, \tau)} \mathring{E}_i^{\Xi(\Gamma)}(z_j)\right)|_y \subset \left(\bigcap_{z_j \in \mathrm{Vertex}(\Gamma', \tau)} \mathring{E}_i^{\Xi(\Gamma')}(z_j)\right)|_y.$$

Consequently,

$$P_i^{U(\Gamma)}(y) \leq P_i^{U(\Gamma')}(y) \qquad \text{if } y \in U(\Gamma) \cap U(\Gamma').$$

Finally, from the condition (1) of the groupings (see the proposition in the end of 4.14) and the definition of $H_i(\Delta)$, we have

$$\left(\overset{\circ}{E}{}_i^{\Xi(\Gamma)}(z)\right)|_y \subset \left\{\lambda;\ \mathrm{dist}(\lambda, a_i(y)) < \eta + \frac{(\dim Y+1)}{(\dim Y+1)} \cdot \eta = 2\eta\right\},$$

where Δ is any simplex of (Y,σ) satisfying $\Gamma \subset \Delta$. Therefore, $P_i^U(y)$ is the spectral projection of ϕ_y corresponding to a subset of

$$\{\lambda;\ \mathrm{dist}(\lambda, a_i(y)) \leq 2\eta\} \quad \subset \quad X.$$

We have proved the following lemma.

Lemma 4.17. *There is a collection $\mathcal{U}$ of finitely many open sets which covers Y. For each open set $U \in \mathcal{U}$, there are mutually orthogonal projection valued continuous functions*

$$P_1^U, P_2^U, \cdots, P_L^U :\ U \ni y \mapsto \{ \text{ sub-projections of } P(y)\}$$

with the following properties.
(1) If $U_1, U_2 \in \mathcal{U}$, and $U_1 \cap U_2 \neq \emptyset$, then either

$$P_i^{U_1}(z) \leq P_i^{U_2}(z)$$

is true for all $i = 1, 2, \cdots, L$ and all $z \in U_1 \cap U_2$, or

$$P_i^{U_2}(z) \leq P_i^{U_1}(z)$$

is true for all $i = 1, 2, \cdots, L$ and all $z \in U_1 \cap U_2$.
(2) rank$(P_i^U(z)) \geq T_i - [M\Omega + M\Omega \dim Y(\dim Y + 1)]$.
(3) Each $P_i^U(z)$ is a spectral projection of ϕ_z corresponding to a subset of

$$\{\lambda;\ \mathit{dist}(\lambda, a_i(z)) < 2\eta\}.$$

4.18. For $i = 1, 2, \cdots, L$, applying Proposition 3.2 of [DNNP] to $\{P_i^U\}_{U\in\mathcal{U}}$, there exist continuous projection valued functions

$$p_1^U, p_2^U, \cdots, p_L^U :\ Y \ni y \mapsto \{ \text{ sub-projections of } P(y)\}$$

such that

$$p_i(y) \leq \bigvee\{P_i^U(y);\ y \in U \in \mathcal{U}\}$$

and that

$$\mathrm{rank}(p_i) \quad \geq T_i - [M\Omega + M\Omega \dim Y(\dim Y + 1)] - \dim Y > T_i - T\ .$$

(Note that $T = 2^L(\dim X + \dim Y)^3$.)

By Condition (1) of 4.17, for each y,

$$\mathrm{span}\{P_i^U;\ y \in U \in \mathcal{U}\} = P_i^{U_0}$$

for a certain $U_0 \ni y$ which does not depend on i. Therefore, $\{p_i(y)\}_{i=1}^L$ are mutually orthogonal since $\{P_i^{U_0}\}_{i=1}^L$ are mutually orthogonal.

4.19. We will prove that the above projections $\{p_i\}_{i=1}^L$ and $p_0 = P - \sum_{i=1}^L p_i$ are as desired in Theorem 4.1. This is a routine calculation, as in the proof of Theorem 2.7 of [GL1] or the last part of the proof of Theorem 2.21 of [EG2]. (See 1.5.4 and 1.5.7 also.) Since we need an extra property of $p_0\phi p_0$ (described in 4.20 below), we write down the complete proof.

For each $y \in Y$, as mentioned in 4.18, there exists an open set $U_0 \in \mathcal{U}$ with $U_0 \ni y$ such that

$$\mathrm{span}\{P_i^U;\ y \in U \in \mathcal{U}\} = P_i^{U_0}, \qquad i = 1, 2, \cdots, L.$$

Let $P_i(y) = P_i^{U_0}(y)$. Then

$$p_i(y) \le P_i(y), \qquad i = 1, 2, \cdots, L,$$

and each $P_i(y)$ is the spectral projection corresponding to a certain subset of

$$\{\lambda;\ \lambda \in \mathrm{SP}\phi_y,\ \mathrm{dist}(\lambda, a_i(y)) < 2\eta\}\ .$$

Let mutually different elements $\mu_1, \mu_2, \cdots, \mu_s \in \mathrm{SP}\phi_y$ be the list of spectra which are not in the set of those spectra belonging to the projections $\{P_i(y)\}_{i=1}^L$. Let $q_1, q_2, \cdots, q_s$ be spectral projections corresponding to $\{\mu_1\}, \{\mu_2\}, \cdots, \{\mu_s\}$, respectively. (The rank of each q_i is the multiplicity of μ_i in $\mathrm{SP}\phi_y$.) Then

$$P(y) = \sum_{i=1}^L P_i(y) + \sum_{i=1}^s q_i.$$

Therefore,

$$p_0(y) = P(y) - \sum_{i=1}^L p_i(y) = \sum_{i=1}^L (P_i(y) - p_i(y)) + \sum_{i=1}^s q_i.$$

Since the spectra belonging to $P_i(y)$ are within distance 2η of $a_i(y)$, by the way η is chosen in 4.4, for each $f \in F$,

$$\|\phi(f)(y) - [\sum_{i=1}^L f(a_i(y))P_i + \sum_{i=1}^s f(\mu_i)q_i]\| < \frac{\varepsilon}{3}.$$

Therefore, for each $f \in F$, $\|p_0(y)\phi(f)(y) - \phi(f)(y)p_0(y)\| < \frac{2\varepsilon}{3}$, and

$$(*) \qquad \|p_0(y)\phi(f)(y)p_0(y) - [\sum_{i=1}^L f(a_i(y))(P_i(y) - p_i(y)) + \sum_{i=1}^s f(\mu_i)q_i]\| < \frac{\varepsilon}{3}.$$

Also, for all $f \in F$,

$$\|p_i(y)\phi(f)(y) - f(a_i(y))p_i(y)\| < \frac{\varepsilon}{3} \qquad \text{and}$$

$$\|\phi(f)(y)p_i(y) - f(a_i(y))p_i(y)\| < \frac{\varepsilon}{3}.$$

Let $P' = \sum_{i=1}^{L} p_i$. Then

$$\begin{aligned} &\|P'(y)\phi(f)(y)p_0(y)\| = \|\textstyle\sum_{i=1}^{L} p_i(y)\phi(f)(y)p_0(y)\| \\ &\le \|\textstyle\sum_{i=1}^{L}[p_i(y)\phi(f)(y) - p_i(y)f(a_i(y))]p_0(y)\| + \|\textstyle\sum_{i=1}^{L} p_i(y)f(a_i(y))p_0(y)\| \\ &\le \tfrac{\varepsilon}{3} + 0 = \tfrac{\varepsilon}{3}\ . \end{aligned}$$

Similarly, for all $f \in F$,

$$\|p_0(y)\phi(f)(y)P'(y)\| < \frac{\varepsilon}{3}.$$

Also,

$$\|P'(y)\phi(f)(y)P'(y) - \bigoplus_{i=1}^{L} f(a_i(y))p_i(y)\| < \frac{\varepsilon}{3}.$$

Combining all the above estimations, we have, for $f \in F$,

$$\|\phi(f)(y) - p_0(y)\phi(f)(y)p_0(y) \oplus \bigoplus_{i=1}^{L} f(a_i(y))p_i(y)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This ends the proof of Theorem 4.1.

$\square$

ATTENTION: In fact, we proved that the conclusion of Theorem 4.1 holds not only for f in the finite set F, but also for any f satisfying the condition that if $\mathrm{dist}(x, x') < 2\eta$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{3}$.

REMARK 4.20. The following is the $(*)$ from 4.19:

$$(*) \qquad \|p_0(y)\phi(f)(y)p_0(y) - [\sum_{i=1}^{L} f(a_i(y))(P_i(y) - p_i(y)) + \sum_{i=1}^{s} f(\mu_i)q_i]\| < \frac{\varepsilon}{3}.$$

Recall that for any $x, x' \in X$, if $\mathrm{dist}(x, x') < 2\eta$, then

$$\|f(x) - f(x')\| < \frac{\varepsilon}{3}$$

for all $f \in F$.

Note that $\xi_y : C(X) \to p_0(y)M_\bullet(\mathbb{C})p_0(y)$, defined by $\xi_y(f) = \sum_{i=1}^{L} f(a_i(y))(P_i(y) - p_i(y)) + \sum_{i=1}^{s} f(\mu_i)q_i$, is a homomorphism. By 1.2.18, we have the following claim.

Claim: Let $\{x_1, x_2, \cdots, x_r\}$ be an η-dense subset of X. Suppose that mutually orthogonal projections $p^1, p^2, \cdots, p^r \in (P - p_0)M_{K'}(C(Y))(P - p_0)$ satisfy

$$\mathrm{rank}(p^i) \geq \mathrm{rank}(p_0).$$

Let $\psi : C(X) \to (p_0 \oplus p^1 \oplus p^2 \oplus \cdots \oplus p^r)M_{K'}(C(Y))(p_0 \oplus p^1 \oplus p^2 \oplus \cdots \oplus p^r)$ be the positive linear map defined by

$$\psi(g) = p_0\phi(g)p_0 \oplus \sum_{i=1}^{r} g(x_i)p^i \ ,$$

for all $g \in C(X)$. Then $\psi(F)$ is weakly approximately constant to within ε. This fact will be used later.

Remark 4.21. The proof of Theorem 4.1 is very long and complicated. We point out that the following direct approaches will encounter difficulties. (These discussions have appeared in §1.5.)

1. One may let $P_i^U(y)$ be the spectral projections corresponding to the open sets

$$\{\lambda;\ \mathrm{dist}(\lambda, a_i(y)) < \eta\}$$

and make use of Proposition 3.2 of [DNNP] to construct the projection p_i. The trouble is that such $\{p_i\}_{i=1}^{L}$ are not mutually orthogonal since P_i^U are not mutually orthogonal.

2. For each sufficiently small neighborhood U, applying the theorem about spectral multiplicity from §2, one can construct mutually orthogonal projections $\{P_i^U(y)\}_{i=1}^{L}$ with relatively large rank such that each $P_i^U(y)$ is the spectral projection corresponding to a subset of

$$\{\lambda;\ \mathrm{dist}(\lambda, a_i(y)) < \eta\}.$$

But one cannot guarantee that the projection associated to $\bigvee\{P_i^U;\ U \ni y\}$ is orthogonal to the projection associated to $\bigvee\{P_j^U;\ U \ni y\}$, for $i \neq j$. So one still can not obtain orthogonal projections $\{p_i\}_{i=1}^{L}$.

3. One may try to define $p_1, p_2, \cdots, p_L$, one by one. For example, after $p_1(y)$ is defined, try to choose $P_2^U(y)$ to be orthogonal to $p_1(y)$ and to be the spectral projection of a certain subset of X. Then this subset can not be chosen to be a subset of $\{\lambda;\ \mathrm{dist}(\lambda, a_2(y)) < \eta\}$ since some spectra may have been taken out when $p_1(y)$ is defined. In fact, this subset can be chosen to be a subset of $\{\lambda;\ \mathrm{dist}(\lambda, a_2(y)) < 2\eta\}$. In this way, when we define $P_i^U(y)$, it will be a spectral projection corresponding to a subset of

$$\{\lambda;\ \mathrm{dist}(\lambda, a_i(y)) < i \cdot \eta\}.$$

In order for the theorem to hold, $L{\cdot}\eta$ needs to be small, which makes η depend on L. This is not useful at all for the application.

REMARK 4.22. Note that in 4.19, when we prove that the projections $\{p_i\}_{i=1}^L$ satisfy the desired conditions (i) and (ii) of Theorem 4.1, we only use the property that for any $y \in Y$, $p_i(y)$, $i = 1, 2, \cdots, L$, are subprojections of $P_i(y), i = 1, 2, \cdots, L$, respectively. This means, (i) and (ii) of Theorem 4.1 hold for any set of projections $\{p_i'\}_{i=1}^L$ with $p_i' \leq p_i$, $i = 1, 2, \cdots, L$. So we have the freedom to replace any p_i by its subprojection (with suitable rank). This fact is important for the discussion below and in 4.41 and 4.44.

In what follows, we will use the fact that, for the projections in $M_{K'}(C(Y))$ of rank at least $\dim(Y)$, cancellation always holds. That is , if three projections p, q and r in $M_\bullet(C(Y))$ satisfy that $\operatorname{rank}(p) > \dim(Y)$, $\operatorname{rank}(q) > \dim(Y)$ and $p \oplus r$ is Murry von Neumann equivalent to $q \oplus r$, then p is Murry von Neumann equivalent to q.

(a) In fact, in 4.19, $\operatorname{rank}(p_i)$ for our projections p_i satisfy the stronger condition (see 4.18):

$$\operatorname{rank}(p_i) \geq T_i - [M\Omega + M\Omega \dim Y(\dim Y + 1)] - \dim Y.$$

From Theorem 1.2 of [Hu, Chapter 8], there is a trivial projection $p_i' < p_i$ such that

$$\begin{aligned} \operatorname{rank}(p_i') \quad &\geq \operatorname{rank}(p_i) - \dim Y \\ &\geq T_i - [M\Omega + M\Omega \dim Y(\dim Y + 1)] - 2\dim Y. \end{aligned}$$

That is, $\operatorname{rank}(p_i')$ is still larger than $T_i - T$, where $T = 2^L(\dim X + \dim Y)^3$. (In fact it is larger than $T_i - T + 2\dim Y$.) In Theorem 4.1, replacing p_i by p_i', one makes all the projections $\{p_i\}_{i=1}^L$ trivial.

(b) Suppose that there is an $i_0 \in \{1, 2, \cdots, L\}$ such that $T_{i_0} > T + \dim Y$. Suppose that the projections $p_1, p_2, \cdots, p_L$ are trivial as in (a). In particular, suppose that $\operatorname{rank}(p_{i_0}) \geq T_{i_0} - T + 2\dim Y$ as mentioned in (a). By [Hu], $P \in M_{K'}(C(Y))$ (the total projection of the target algebra $PM_{K'}(C(Y))P$ in Theorem 4.1) can be written in the form

$$q \oplus (\text{trivial projection}),$$

where q is of rank $T_{i_0} - T + \dim Y$. It follows from [Hu], that there is a subprojection p_{i_0}' of p_{i_0} which is unitarily equivalent to q. Replacing p_{i_0} by p_{i_0}', and keeping all the other projections p_i, then P will be unitarily equivalent to a projection of the form

$$\bigoplus_{i=1}^L p_i \oplus \ (\text{trivial projection}).$$

Therefore, $p_0 = P - \bigoplus_{i=1}^L p_i$ is a trivial projection. (Note that $\operatorname{rank}(p_0) \geq \dim Y$.)

In other words, in Theorem 4.1, we can choose all the projections $p_0, p_1, \cdots, p_L$ to be trivial except one of them, p_{i_0}, where $i_0 \neq 0$. In particular, p_0 is a trivial projection, as comparing with (a) above.

The following theorem is proved in [EGL].

Theorem 4.23. ([EGL]) *Let $A = \lim_{n\to\infty}(A_n, \phi_{n,m})$ be an inductive limit C^*-algebra (not necessarily unital) with*

$$A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})),$$

where $X_{n,i}$ are simplicial complexs. Then one can write $A = \lim_{n\to\infty}(B_n, \psi_{n,m})$ with

$$B_n = \bigoplus_{i=1}^{t_n} M_{\{n,i\}}(C(Y_{n,i})),$$

where $Y_{n,i}$ are (not necessarily connected) simplicial complexs, with $\dim(Y_{n,i}) \leq \dim(X_{n,i})$, such that all the connecting maps $\psi_{n,m}$ are injective.
Furthermore, if $(A_n, \phi_{n,m})$ satisfies the very slow dimension growth condition, then so does $(B_n, \psi_{n,m})$.

4.24. Without loss of generality, in the rest of this article, we will assume that the connecting maps $\phi_{n,m}$ in the inductive limit system are injective. Without this assumption one can still prove all the theorems in this paper by modifying our arguments, and by passing to some good subsets of $X_{n,i}$. But this assumption makes the discussions much simpler. As mentioned in 1.1.5, we will suppose that the inductive limit algebra $A = \lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ satisfies the very slow dimension growth condition.

4.25. As a consequence of Theorem 4.1 and the lemma inside 1.5.11— a result due to Li—, one can obtain a decomposition for each (partial map of a) connecting map $\phi_{n,m}^{i,j}$ (m large enough), with the major part factoring though an interval algebra. But for our application, we need a certain part of the decomposition to be defined by point evaluations and (even if it is not large absolutely) to be relatively large compared to the "bad" part $p_0\phi p_0$, where p_0 is the projection in Theorem 4.1, and ϕ is the map corresponding to $\phi_{n,m}^{i,j}$ (see 1.2.18 and 1.2.19), i.e., $\phi = \phi_{n,m}^{i,j}|_{e_{11}A_n^i e_{11}}$.
Following Section 2 of [Li3] (see the proof of Theorem 2.28 in [Li3]), we can prove our main Decomposition Theorem (see Theorem 4.37 below). [Li3] only proves the special case that $X_{n,i}$ = graphs (one dimensional spaces). Although the idea behind Li's proof is reasonably simple and clear (see the explanation in 2.29 of [Li3]), the proof itself is complicated and long. It combines several difficulties together. For convenience in the higher dimensional case, we will give a slightly different approach. (See 1.5.25 for the explanation of the difference between our approach and Li's approach.) Our proof will be a little shorter, and perhaps easier to follow (hopefully). More importantly, using this approach, we

will be able to prove the Decomposition Theorem for any homomorphism provided that the homomorphism satisfies a certain quantitative condition (see Theorem 4.35 below). (Li's theorem is for the homomorphism $\phi_{n,m}$ with m sufficiently large.) This slightly stronger version of the theorem is needed in [EGL] to prove the Uniqueness Theorem. It should be emphasized that our proof is essentially the same as Li's proof in spirit.

The idea behind our proof is roughly as follows.

In [Li2, 2.18–2.19] (see 1.5.11), Li proves that for fixed $\eta > 0$, for m large enough, and for any (partial) connecting map $\phi^{i,j}_{n,m}$—denoted by ϕ—, there are L continuous maps $\beta_1, \beta_2, \cdots, \beta_L :\ Y\ (= X_{m,j}) \longrightarrow X\ (= X_{n,i})$, factoring through the interval $[0,1]$, such that for each $y \in Y$, the set $\mathrm{SP}\phi_y$ and the set

$$\Theta(y) = \left\{\beta_1(y)^{\sim L_2}, \beta_2(y)^{\sim L_2}, \cdots, \beta_{L-1}(y)^{\sim L_2}, \beta_L(y)^{\sim L_2+L_1}\right\}$$

can be paired within η, where L_2 could be very large compared with $L \cdot 2^L \cdot (\dim(X) + \dim(Y))^3$, if the inductive limit system satisfies the very slow dimension growth condition.

What we are going to prove is that, if $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within η, then they can still be paired within some small number (e.g., 2η), if one changes a number—a small number compared with L—of maps β_i to ARBITRARY maps (in particular, to constant maps), provided that X is path connected and ϕ has a certain spectral distribution property related to the number η and another number δ (see 4.26 below). (Note that, how many maps are allowed to be changed, also depends on η and δ.) (Those constant maps form the part of the homomorphism defined by point evaluations.) At first sight, it might seem impossible for this to be true. But, with the spectral distribution property of the homomorphism ϕ, Lemma 2.15 of [Li2] (see Lemma 4.29 below) says that if ϕ and another homomorphism ψ (in the application, ψ should be chosen to be a homomorphism with the family of spectral functions $\Theta(y)$, i.e., $\mathrm{SP}\psi_y = \Theta(y)$ for all $y \in Y$) are close on the level of $\mathrm{Aff}T$, then their spectra $\mathrm{SP}\phi_y$ and $\mathrm{SP}\psi_y$ can be paired within a small number. On the other hand, changing a very few spectral functions (no matter how large a change in each function), will NOT create a big change on the level of $\mathrm{Aff}T$ (see 4.28 and the claims in 4.31 below).

Since the results of [Li 2] are not of a quantitative nature—they are for connecting homomorphisms $\phi_{n,m}$ with m large—, we can not apply them (2.18 and 2.19 of [Li 2]) directly. So we repeat part of the arguments in [Li 2].

The above method will lead us to Lemma 4.33 (see 4.26–4.33 for details). Then our main decomposition theorems—Theorem 4.35 and Theorem 4.37— will be more or less consequences.

Finally we remark that, in our decomposition, we cannot require that both parts of the decomposition be homomorphisms as in 2.28 of [Li3], since in general, $C(X)$ is not stably generated (see [Lo]).

4.26. For the reader's convenience, we will quote some notations, terminologies and results from [Li1] and [Li2].

The following notation is inspired by a similar notation in [Li1].
For any $\eta>0,\ \delta>0$, a homomorphism $\phi\colon PM_k(C(X))P\to QM_{k'}(C(Y))Q$ is said to have the property sdp(η,δ) (SPECTRAL DISTRIBUTION PROPERTY WITH RESPECT TO η AND δ) if for any $\eta-$ball

$$B_\eta(x):=\{x'\in X;\ \mathrm{dist}(x',x)<\eta\}\subset X$$

and any point $y\in Y$,

$$\#(\mathrm{SP}\phi_y\cap B_\eta(x))\ \geq\ \delta\#(\mathrm{SP}\phi_y),$$

counting multiplicity.
(Attension: The property sdp(r,δ) in [Li1] corresponds to sdp$(\frac{1}{2r},\delta)$ above.)
Any homomorphism $\phi:\ \oplus M_k(C(X))\to\oplus M_l(C(Y))$ is said to have the property sdp(η,δ) if each partial map has sdp(η,δ).

4.27. The following notations can be found in Section 2 of [Li2]. Let X be a connected simplicial complex. For any closed set $X_1\subset X,\ M>0$, let

$$\chi_{X_1,M}(x)=\begin{cases}1 & \text{if } x\in X_1\\ 1-M\cdot\mathrm{dist}(x,X_1) & \text{if } \mathrm{dist}(x,X_1)\leq\frac{1}{M}\\ 0 & \text{if } \mathrm{dist}(x,X_1)\geq\frac{1}{M}\end{cases}.$$

For $\eta>0$ and $\delta>0$, let

$$H_1(\eta)=\{\chi_{X_1,\frac{32}{\eta}}:\ X_1\subset X\ \text{ closed }\}.$$

Then there is a finite set $H\subset H_1(\eta)$ such that for all $h\in H_1(\eta),\ \mathrm{dist}(h,H)<\frac{\delta}{8}$ (the distance is the distance defined by uniform norm). Denote such set by $H(\eta,\delta,X)$ $(\subset C(X))$. Although such a set is not unique, we fix one for each triple (η,δ,X) for our purpose. (As pointed out in [Li1], the existence of such finite set $H(\eta,\delta,X)$ follows from equi-continuity of the functions in $H_1(\eta)$.)

4.28. For a unital C^*-algebra A, let TA denote the space of all tracial states of A, i.e., $\tau\in TA$ if and only if τ is a positive linear map from A to $\mathbb{C}$, with $\tau(xy)=\tau(yx)$ and $\tau(\mathbf{1})=1$. AffTA is the collection of all the affine maps from TA to $\mathbb{R}$.
Any unital homomorphism $\phi:\ A\to B$ induces an affine map

$$\mathrm{Aff}T\phi:\ \mathrm{Aff}TA\longrightarrow\mathrm{Aff}TB.$$

It is well known, for any connected metrizable space X and any projection $P\in M_k(C(X))$,

$$\mathrm{Aff}T(PM_k(C(X))P)=\mathrm{Aff}T(C(X))=C_{\mathbb{R}}(X).$$

We would like to quote some easy facts about the Aff$T$ map from [Li1] and [Li2].

If $\phi : C(X) \to PM_l(C(Y))P$ is a unital homomorphism and rank$(P) = k$, then Aff$T\phi : C(X) \to C(Y)$ is given by

$$\mathrm{Aff}T\phi(f) = \frac{1}{k}\sum_{i=1}^{l} \phi(f)_{ii},$$

where each $\phi(f)_{ii}$ is the diagonal entry of $\phi(f) \in PM_l(C(Y))P \subset M_l(C(Y))$ at the place (i,i)

For a continuous map $\beta : Y \to X$, let $\beta^* : C(X) \to C(Y)$ be defined by

$$\beta^*(f) = f \circ \beta \quad (\in C(Y)) \quad \text{for any } f \in C(X).$$

Suppose that $\beta_1, \beta_2, \cdots \beta_l : Y \to X$ are continuous maps. If $\psi : C(X) \to M_l(C(Y))$ is a homomorphism with $\{\beta_i\}_{i=1}^{l}$ as the set of spectral functions, (e.g., ψ is defined by $\psi(f) = diag(\beta_1^*(f), \beta_2^*(f), \cdots, \beta_l^*(f)),$) then

$$\mathrm{Aff}T\psi(f) = \frac{1}{l}\sum_{i=1}^{l} \beta_i^*(f).$$

(Let $H \subset C_{\mathbb{R}}(X)$ be a finite subset satisfying $\|f\| \le 1$ for any $f \in H$. If one modifies the above homomorphism ψ to a new homomorphism ψ', by replacing k functions from the set of spectral functions $\{\beta_i\}_{i=1}^{l}$ by other functions (from Y to X), then

$$\|\mathrm{Aff}T\psi(f) - \mathrm{Aff}T\psi'(f)\| \le \frac{k}{l}, \ \forall f \in H.$$

In particular, this modification (from ψ to ψ') does not create a big change on the level of Aff$T$, provided that k is very small compared with l, as mentioned in 4.25.)

For a unital homomorphism $\phi : C(X) \to PM_l(C(Y))P$ with $rank(P) = k$, quoting from 1.9 of [Li1], we have

$$\mathrm{Aff}T\phi(f)(y) = \frac{1}{k}\sum_{x_i(y)\in \mathrm{SP}\phi_y} f(x_i(y)).$$

Consider $e_y : PM_l(C(Y))P \to P(y)M_l(\mathbb{C})P(y) \quad (\cong M_{rank(P)}(\mathbb{C}))$, which is the homomorphism defined by evaluation at the point y. Then from the above paragraph, we know that Aff$T(e_y \circ \phi)$ depends only on SPϕ_y. We can denote $e_y \circ \phi$ by $\phi|_y$ (this is the homomorphism $\phi|_{\{y\}}$ in 1.2.13 for the single point set $\{y\}$).

Lemma 4.29. ([Li2,2.15]) *Suppose that two unital homomorphisms $\phi : C(X) \to PM_k(C(Y))P$ and $\psi : C(X) \to QM_k(C(Y))Q$ with $rank(P) = rank(Q)$, satisfy the following two conditions:*
(1) ϕ has the property $sdp(\frac{\eta}{32}, \delta)$;
(2) $\|AffT\phi(h) - AffT\psi(h)\| < \frac{\delta}{4}$, for all $h \in H(\eta, \delta, X)$.

Then $SP\phi_y$ and $SP\psi_y$ can be paired within $\frac{\eta}{4}$ for any $y \in Y$.
(Notice that, no matter how small the δ is , the above conditions (1) and (2) do not imply the other assumption $\mathrm{rank}(P) = \mathrm{rank}(Q)$, which is necessary for our conclusion.)

Proof: If $P = Q$, this is exactly 2.15 of [Li2]. (Notice that, we use $\frac{\eta}{32}$ in place of $\frac{\eta}{8}$ of [Li2 2.15], so our conclusion is that, $\mathrm{SP}\phi_y$ and $\mathrm{SP}\psi_y$ can be paired within $\frac{\eta}{4}$ (instead of η). Also notice that the set H in [Li2 2.15] is chosen to be the same as the above set $H(\eta, \delta, X)$.)
To see the general case, fix $y \in Y$. We can consider two maps $\phi|_y$ and $\psi|_y$ which are unital homomorphisms from $C(X)$ to C^*-algebras which are isomorphic to the same C^*-algebra $M_{rank(P)}(\mathbb{C})$. (Note that $rank(P) = rank(Q)$.) The conditions (1) and (2) above imply the same conditions for $\phi|_y$ and $\psi|_y$, since $\mathrm{Aff}T\phi(h)(y) = \mathrm{Aff}T(\phi|_y)(h)$. Therefore, by 2.15 of [Li 2], $\mathrm{SP}\phi_y = SP(\phi|_y)$ and $\mathrm{SP}\psi_y = SP(\psi|_y)$ can be paired within $\frac{\eta}{4}$.
(If one checks the proof of 2.15 of [Li2] carefully, then he will easily recognize that the above Lemma is already proved there.)

$\square$

4.30. In the following paragraphs (4.30—4.32), we will apply the materials from 2.8 – 2.10 of [Li2].
For any $\eta > 0$ and $\delta > 0$, from 2.9 of [Li2], there exist a continuous map $\alpha: [0,1] \to X$, and a unital positive linear map $\xi: C[0,1] \to C(X)$ such that

$$\|\xi\circ\alpha^*(f) - f\| < \frac{\delta}{16},$$

for each $f \in H(\eta, \delta, X)$, where $\alpha^*: C(X) \to C[0,1]$ is induced by α. Furthermore, we can choose α such that image(α) is $\frac{\eta}{32}$-dense in X.
For $\alpha: [0,1] \to X$, there is a $\sigma > 0$ such that $|t - t'| < 2\sigma$ implies that

$$\mathrm{dist}(\alpha(t), \alpha(t')) < \frac{\eta}{32}.$$

For a fixed space X, the number σ depends only on η and δ, since so does the continuous map α. We denote the number σ by $\sigma(\eta, \delta)$.

4.31. Let $\tilde{H} = \alpha^*(H(\eta, \delta, X)) \subset C[0,1]$. For the finite set $\tilde{H}$ and $\frac{\delta}{16} > 0$, there is an integer N (as in Theorem 2.1 of [Li2]) such that for any positive linear map $\zeta: C[0,1] \to C(Y)$, and for any $r \geq N$, there are r continuous maps

$$\beta_1, \beta_2, \cdots, \beta_r: Y \longrightarrow [0,1]$$

such that

$$\left\| \zeta(f) - \frac{1}{r}\sum_{i=1}^{r} \beta_i^*(f) \right\| < \frac{\delta}{16}$$

for all $f \in \tilde{H}$, where $\beta_i^*: C[0,1] \to C(Y)$ is induced by β_i.

We will also assume $\frac{1}{N} < \frac{\delta}{64}$. Then we can prove the following claim.
Claim 1: For any $r \geq N$, if

$$\left\|\zeta(f) - \frac{1}{r}\sum_{i=1}^{r}\beta_i^*(f)\right\| < \frac{\delta}{16} \qquad \text{for all} \quad f \in \tilde{H},$$

then for any other two continuous maps $\tau_1, \tau_2 : Y \longrightarrow [0,1]$,

$$\left\|\zeta(f) - \frac{1}{r+2}\left(\sum_{i=1}^{r}\beta_i^*(f) + \tau_1^*(f) + \tau_2^*(f)\right)\right\| < \frac{\delta}{8}, \quad \forall f \in \tilde{H}$$

Proof of the claim: The claim follows from

$$\begin{aligned}
&\left\|\zeta(f) - \frac{1}{r+2}\left(\sum_{i=1}^{r}\beta_i^*(f) + \tau_1^*(f) + \tau_2^*(f)\right)\right\| \\
&\quad \leq \left\|\zeta(f) - \frac{1}{r}\sum_{i=1}^{r}\beta_i^*(f)\right\| + \left\|\frac{1}{r}\sum_{i=1}^{r}\beta_i^*(f) - \frac{1}{r+2}\sum_{i=1}^{r}\beta_i^*(f)\right\| \\
&\qquad\qquad + \left\|\frac{1}{r+2}(\tau_1^*(f) + \tau_2^*(f))\right\| \\
&\quad < \frac{\delta}{16} + 2\cdot\frac{\delta}{64} + 2\cdot\frac{\delta}{64} = \frac{\delta}{8}
\end{aligned}$$

for any $f \in \tilde{H}$. In the above estimation, we use the facts that $\|f\| \leq 1$, $\|\beta_i^*(f)\| \leq 1$ and $\|\tau_i^*(f)\| \leq 1$ for any $f \in \tilde{H}$.
In the above claim, if we replace the condition $r \geq N$ by the condition $r \geq mN$, then in the conclusion, we can allow $2m$ continuous maps $\tau_1, \tau_2, \cdots, \tau_{2m} : Y \longrightarrow [0,1]$, instead of two maps. Namely, the following claim can be proved in exactly the same way.
Claim 2: For any $r \geq mN$, if

$$\left\|\zeta(f) - \frac{1}{r}\sum_{i=1}^{r}\beta_i^*(f)\right\| < \frac{\delta}{16}, \qquad \forall\ f \in \tilde{H},$$

then for any $2m$ continuous maps $\tau_1, \tau_2, \cdots, \tau_{2m} : Y \longrightarrow [0,1]$,

$$\left\|\zeta(f) - \frac{1}{r+2m}\left(\sum_{i=1}^{r}\beta_i^*(f) + \sum_{i=1}^{2m}\tau_i^*(f)\right)\right\| < \frac{\delta}{8}, \quad \forall f \in \tilde{H}.$$

4.32. Let $n = \mathrm{int}\left(\frac{1}{\sigma(\eta,\delta)}\right) + 1$, where $\mathrm{int}(\cdot)$ denote the integer part of the number (see 1.1.7 (c)).
Divide $[0,1]$ into n intervals such that each of them has length at most $\sigma(\eta,\delta)$. Choose n points

$$t_1, t_2, \cdots, t_n,$$

one from each of the intervals. Let $x_i = \alpha(t_i) \in X,\ i = 1, 2, \cdots, m$. Then the set

$$\{x_1, x_2, \cdots, x_n\}$$

is $\frac{\eta}{16}$-dense in X by the way σ is chosen in 4.30.
From the above discussion, for fixed $\eta > 0, \delta > 0$, and the space X, we can find $\alpha, \xi, \sigma, N, n, H(\eta, \delta, X), \tilde{H}$, the set $\{t_1, t_2, \cdots, t_n\} \subset [0, 1]$, and the $\frac{\eta}{16}$-dense set $\{x_1, x_2, \cdots, x_n\} \subset X$. All of them depend only on η, δ, and the space X.

Lemma 4.33. *For any connected simplicial complex X, any numbers $\eta > 0$ and $\delta > 0$, there are integers n, N, a continuous map $\alpha :\ [0, 1] \to X$, and finitely many points $\{t_1, t_2, \cdots, t_n\} \subset [0, 1]$ with $\{\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_n)\}$ $\frac{\eta}{16}$-dense in X, such that the following is true. (Denote $L := n(N + 2)$.)*
If a unital homomorphism $\phi : C(X) \to PM_k(C(Y))P$ satisfies the following two conditions:
(i) ϕ has the property sdp$(\frac{\eta}{32}, \delta)$;
(ii) rank$\phi(\mathbf{1}) := K \geq L^2 = (n(N + 2))^2$,
and write $K = LL_2 + L_1$ with $L_2 = int(\frac{K}{L})$ and $0 \leq L_1 < L$, (note that $L \leq L_2$, since $K \geq L^2$,)
then there are L continuous functions

$$\beta_1, \beta_2, \cdots, \beta_n, \beta_{n+1}, \cdots, \beta_L :\ Y \longrightarrow [0, 1]$$

such that
(1) $\beta_i(y) = t_i$ for $1 \leq i \leq n$;
(2) For each $y \in Y$, $SP\phi_y$ and the set

$$\Theta(y) = \left\{\alpha\circ\beta_1(y)^{\sim L_2}, \alpha\circ\beta_2(y)^{\sim L_2}, \cdots, \alpha\circ\beta_{L-1}(y)^{\sim L_2}, \alpha\circ\beta_L(y)^{\sim L_2+L_1}\right\}$$

can be paired within $\frac{\eta}{2}$.
(3) If Y is a connected finite simplicial complex and $Y \neq \{pt\}$, then the map $\beta_{n+1} :\ Y \to [0, 1]$—the first nonconstant map above—, is a surjection.
(This lemma is similar to Lemma 2.18 of [Li2], but we require some of the functions β_i $(1 \leq i \leq n)$ to be constant functions.)
(Attention: To apply Theorem 4.1, one only needs SPϕ_y and $\Theta(y)$ to be paired within η. The advantage of using $\frac{\eta}{2}$ is the following. If ψ is another homomorphism such that SPψ_y and SPϕ_y can be paired within $\frac{\eta}{2}$ for any y, then we can apply Theorem 4.1 to both ϕ and ψ without requiring ψ to have the property sdp$(\frac{\eta}{32}, \delta)$. This observation will not be used in the proof of the main theorem of this paper. But it will be used in the proof of the Uniqueness Theorem in [EGL] (part II of the series), see 4.41–4.48 below.)

Proof: Follow the notations in 4.26 – 4.32. Let

$$\zeta :\ C[0, 1] \longrightarrow C(Y)$$

be defined by $\zeta = \text{Aff}T\phi{\circ}\xi$. Since $K - 2nL_2 \geq nL_2N$, there are $K - 2nL_2$ continuous maps

$$\gamma_1, \gamma_2, \cdots, \gamma_{K-2nL_2} : \; Y \longrightarrow [0,1]$$

such that

$$\left\| \zeta(f) - \frac{1}{K - 2nL_2} \sum_{i=1}^{K-2nL_2} \gamma_i^*(f) \right\| < \frac{\delta}{16}$$

for all $f \in \tilde{H}$. Let

$$\beta_1, \beta_2, \cdots, \beta_n : \; Y \longrightarrow [0,1]$$

be defined by $\beta_i(y) = t_i$. Then by Claim 2 of 4.31 (taking $m = nL_2$),

$$\left\| \zeta(f) - \frac{1}{K} \left(\sum_{i=1}^{K-2nL_2} \gamma_i^*(f) + 2L_2 \sum_{i=1}^{n} \beta_i^*(f) \right) \right\| < \frac{\delta}{8}$$

for all $f \in \tilde{H} \subset C[0,1]$. Therefore,

$$\left\| (\zeta{\circ}\alpha^*)(f) - \frac{1}{K} \left(\sum_{i=1}^{K-2nL_2} (\alpha{\circ}\gamma_i)^*(f) + 2L_2 \sum_{i=1}^{n} (\alpha{\circ}\beta_i)^*(f) \right) \right\| < \frac{\delta}{8}$$

for all $f \in H(\eta, \delta, X)$. On the other hand, by 4.30 and $\zeta = \text{Aff}T\phi{\circ}\xi$,

$$\|\text{Aff}T\phi(f) - (\zeta{\circ}\alpha^*)(f)\| < \frac{\delta}{16} \qquad \text{for} \;\; f \in H(\eta, \delta, X).$$

One can define a unital homomorphism $\psi : \; C(X) \to M_K(C(Y))$ with $\{\alpha{\circ}\gamma_i\}_{i=1}^{K-2nL_2} \cup \{(\alpha{\circ}\beta_i)^{\sim 2L_2}\}_{i=1}^{n}$ as the family of the spectral functions. Then from 4.28,

$$\text{Aff}T\psi(f) = \frac{1}{K} \left(\sum_{i=1}^{K-2nL_2} (\alpha{\circ}\gamma_i)^*(f) + 2L_2 \sum_{i=1}^{n} (\alpha{\circ}\beta_i)^*(f) \right).$$

Hence,

$$\|\text{Aff}T\phi(f) - \text{Aff}T\psi(f)\| \leq \frac{\delta}{8} + \frac{\delta}{16} < \frac{\delta}{4}$$

for all $f \in H(\eta, \delta, X)$. Note that $\text{rank}(P) = K$. By Lemma 4.29, $\text{SP}\phi_y$ and

$$\text{SP}\psi_y = \left\{ \alpha{\circ}\beta_1(y)^{\sim 2L_2}, \alpha{\circ}\beta_2(y)^{\sim 2L_2}, \cdots, \alpha{\circ}\beta_n(y)^{\sim 2L_2}, \alpha{\circ}\gamma_1(y), \cdots, \alpha{\circ}\gamma_{L-2nL_2}(y) \right\}$$

can be paired within $\frac{\eta}{4}$.

Note that in our lemma, we only need L_2 copies of each constant maps β_i $(i = 1, 2, \cdots, n)$. One may wonder why we put $2L_2$ copies of each of maps β_i in the above set. The reason is that, after taking out L_2 copies of β_i, we still want the set $\Theta(y)$ to have enough elements in each small interval of length σ, and the other L_2 copies of β_i can serve for this purpose.

Consider the following set (of $K - nL_2$ elements)

$$\{\beta_1^{\sim L_2}(y), \beta_2^{\sim L_2}(y), \cdots, \beta_n^{\sim L_2}(y),\ \gamma_1(y), \gamma_2(y), \cdots, \gamma_{K-2nL_2}(y)\}.$$

In each interval of $[0,1]$ of length σ, there are at least L_2 points (counting multiplicities) in the above set.
The following argument appeared in 2.18 of [Li2].
For each fixed y, we can rearrange all the elements in the above set in the increasing order. I.e., write them as $\gamma_1'(y), \gamma_2'(y), \cdots, \gamma_{K-nL_2}'(y)$ such that for each fixed y

$$\{\gamma_1'(y), \gamma_2'(y), \cdots, \gamma_{K-nL_2}'(y)\} = \\ = \{\beta_1(y)^{\sim L_2}, \beta_2(y)^{\sim L_2}, \cdots, \beta_n(y)^{\sim L_2},\ \gamma_1(y), \gamma_2(y), \cdots, \gamma_{K-2nL_2}(y)\}$$

(as a set with multiplicity), and such that

$$0 \le \gamma_1'(y) \le \gamma_2'(y) \le \cdots \le \gamma_{K-nL_2}'(y) \le 1.$$

It is easy to prove that $\gamma_i'(y),\ 1 \le i \le K - nL_2$ are continuous (real-valued) functions, using the following well known fact repeatedly: For any two real-valued continuous functions f and g, the functions $\max(f,g)$ and $\min(f,g)$ are also continuous.
We can put each group of L_2 consecutive functions of $\{\gamma_i'\}$ (beginning with smallest one) together except the last $L_2 + L_1$ functions which will be put into a single group—the last group. Then we replace all the functions in a same group by the smallest function in the group. Namely, let

$$\beta_{n+1} = \gamma_1', \beta_{n+2} = \gamma_{L_2+1}', \cdots, \beta_L = \gamma_{(L-n-1)L_2+1}' \ .$$

Then from the fact that in each interval of $[0,1]$ of length σ, there are at least L_2 points (counting multiplicity) in the set $\{\gamma_1'(y), \gamma_2'(y), \cdots, \gamma_{K-nL_2}'(y)\}$, we know that $\{\gamma_1'(y), \gamma_2'(y), \cdots, \gamma_{K-nL_2}'(y)\}$ and

$$\{\beta_{n+1}(y)^{\sim L_2}, \beta_{n+2}(y)^{\sim L_2}, \cdots, \beta_{L-1}(y)^{\sim L_2}, \beta_L(y)^{\sim L_2+L_1}\}$$

can be paired within 2σ. Recall that $|t-t'| < 2\sigma$ implies that $\mathrm{dist}(\alpha(t), \alpha(t')) < \frac{\eta}{16}$. Hence

$$\big\{\alpha{\circ}\beta_1(y)^{\sim 2L_2}, \alpha{\circ}\beta_2(y)^{\sim 2L_2}, \cdots, \alpha{\circ}\beta_n(y)^{\sim 2L_2}, \\ \alpha{\circ}\gamma_1(y), \alpha{\circ}\gamma_2(y), \cdots, \alpha{\circ}\gamma_{L-2nL_2}(y)\}$$

and

$$\Theta(y) = \big\{\alpha{\circ}\beta_1(y)^{\sim L_2}, \alpha{\circ}\beta_2(y)^{\sim L_2}, \cdots, \alpha{\circ}\beta_n(y)^{\sim L_2}, \\ \alpha{\circ}\beta_{n+1}(y)^{\sim L_2}, \cdots, \alpha{\circ}\beta_{L-1}(y)^{\sim L_2}, \alpha{\circ}\beta_L(y)^{\sim L_2+L_1}\big\}$$

can be paired within $\frac{\eta}{16}$. Therefore, $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\frac{\eta}{4}+\frac{\eta}{16}<\frac{5\eta}{16}$.
Note that $\{\alpha\circ\beta_1(y),\alpha\circ\beta_2(y),\cdots,\alpha\circ\beta_n(y)\}$ is $\frac{\eta}{16}$ dense in X. From the proof of Lemma 1.2.17, if we replace only one map (say β_{n+1}) by an arbitrary map from Y to $[0,1]$, then the new $\Theta(y)$ can be paired with the old $\Theta(y)$ to within $\frac{\eta}{8}$. As a consequence, we still have that $\mathrm{SP}\phi_y$ and the new $\Theta(y)$ can be paired within $\frac{5\eta}{16}+\frac{\eta}{8}<\frac{\eta}{2}$. In particular, if Y is a connected finite simplicial complex which is not a single point, then β_{n+1} could be chosen to be a surjection, again using the Peano curve.

$\square$

4.34. Fix a large positive integer J. We require that the decomposition in 4.1 to satisfy the condition

$$J\cdot(\mathrm{rank}(p_0)+2\dim(Y))\le rank(p_i),\ \forall i\ge 1.$$

To do so, we need $\mathrm{rank}(\phi(1))$ to be large enough. We describe it as follows.
For a connected simplicial complex X, and for numbers $\eta>0$ and $\delta>0$, let N,n and $\alpha:[0,1]\to X$ be as in Lemma 4.33. Let $L=n(N+2)$. Suppose that $\phi:C(X)\to M_k(C(Y))$ is a homomorphism. If ϕ has the property $\mathrm{sdp}(\frac{\eta}{32},\delta)$ and

$$\mathrm{rank}(\phi(1))\ge 2JL^2\cdot 2^L(\dim X+\dim Y+1)^3,$$

then there are continuous functions

$$\beta_1,\beta_2,\cdots,\beta_L:\ Y\longrightarrow[0,1]$$

(as in Lemma 4.33) such that $\mathrm{SP}\phi_y$ and the set

$$\left\{\alpha\circ\beta_1(y)^{\sim L_2},\alpha\circ\beta_2(y)^{\sim L_2},\cdots,\alpha\circ\beta_{L-1}(y)^{\sim L_2},\alpha\circ\beta_L(y)^{\sim L_2+L_1}\right\}$$

can be paired within $\frac{\eta}{2}$, where

$$L_2=\mathrm{int}\left(\frac{\mathrm{rank}\phi(\mathbf{1})}{L}\right)\ge 2JL\cdot 2^L(\dim X+\dim Y+1)^3,$$

and $0\le L_1<L$. For any given set $F\subset C(X)$, if η is chosen as in Theorem 4.1 (see 4.4), then by Theorem 4.1, there are mutually orthogonal projections $p_1,p_2,\cdots,p_L$ and $p_0=\phi(\mathbf{1})-\sum_{i=1}^L p_i$ such that
(1) For all $f\in F$ and $y\in Y$,

$$\|\phi(f)(y)-p_0(y)\phi(f)(y)p_0(y)\oplus\bigoplus_{i=1}^{L}f(a_i(y))p_i(y)\|<\varepsilon;$$

(2) For each $i=1,2,\cdots,L$, $\mathrm{rank}(p_i)\ge L_2-2^L(\dim X+\dim Y+1)^3$, and

$$J(\mathrm{rank}(p_0)+2\dim(Y))\le J(L\cdot 2^L(\dim X+\dim Y+1)^3+2\dim Y)\le\mathrm{rank}(p_i).$$

By [Hu], $\underbrace{p_0 \oplus p_0 \oplus \cdots \oplus p_0}_{J}$ is (unitarily) equivalent to a subprojection of p_i, since every complex vector bundle (over Y) of dimension $J \cdot \mathrm{rank}(p_0)$ is a subbundle of any vector bundle (over Y) of dimension at least $J \cdot \mathrm{rank}(p_0) + dim(Y)$. We denote this fact by $J[p_0] < [p_i]$.

Let $Q_0 = p_0$, $Q_1 = p_1 + p_2 + \cdots + p_n$ and $Q_2 = p_{n+1} + p_{n+2} + \cdots + p_L$. Then

$$\phi(\mathbf{1}) = Q_0 + Q_1 + Q_2.$$

Let $\phi_0 : C(X) \to Q_0 M_k(C(Y)) Q_0$, $\phi_1 : C(X) \to Q_1 M_k(C(Y)) Q_1$ and $\phi_2 : C(X) \to Q_2 M_k(C(Y)) Q_2$ be defined by

$$\phi_0(f)(y) = p_0 \phi(f)(y) p_0,$$

$$\phi_1(f) = \sum_{i=1}^{n} f(\alpha_\circ \beta_i(y)) p_i, \quad \text{and}$$

$$\phi_2(f)(y) = \sum_{i=n+1}^{L} f(\alpha_\circ \beta_i(y)) p_i.$$

Then we have the following facts.

(a) ϕ_2 is a homomorphism factoring through $C[0,1]$ as

$$\phi_2 : C(X) \xrightarrow{\xi_1} C[0,1] \xrightarrow{\xi_2} Q_2 M_k(C(Y)) Q_2.$$

Furthermore, if $Y \neq \{pt\}$, then ξ_2 is injective. (This follows from the surjection of β_{n+1}.)

(b) Note that

$$\alpha_\circ \beta_1(y) = x_1, \alpha_\circ \beta_2(y) = x_2, \cdots, \alpha_\circ \beta_n(y) = x_n$$

are n constant maps with $\{x_1, x_2, \cdots, x_n\}$ η-dense in X. By the claim in 4.20, $(\phi_0 \oplus \phi_1)(F)$ is approximately constant to within ε. (Note that ϕ_0 is not a homomorphism, it is a completely positive linear $*$-contraction.)

Furthermore, if $\eta < \varepsilon$, then the set $\{x_1, x_2, \cdots, x_n\}$ is ε-dense in X.

Therefore, we have proved the following theorem.

Theorem 4.35. *Let X be a connected finite simplicial complex, and $\varepsilon > \eta > 0$. For any $\delta > 0$, there is an integer $L > 0$ such that the following holds.*

Suppose that $F \subset C(X)$ is a finite set such that $dist(x, x') < 2\eta$ implies $|f(x) - f(x')| < \frac{\varepsilon}{3}$ for all $f \in F$.

If $\phi : C(X) \to M_k(C(Y))$ is a homomorphism with the property $sdp(\frac{\eta}{32}, \delta)$, and $rank(\phi(1)) \geq 2J \cdot L^2 \cdot 2^L (\dim X + \dim Y + 1)^3$, where Y is a connected finite simplicial complex and J is any fixed positive integer, then there are three mutually orthogonal projections $Q_0, Q_1, Q_2 \in M_k(C(Y))$, a map $\phi_0 \in Map(C(X), Q_0 M_k(C(Y)) Q_0)_1$ and two homomorphisms

$\phi_1 \in Hom(C(X), Q_1M_k(C(Y))Q_1)_1$ and $\phi_2 \in Hom(C(X), Q_2M_k(C(Y))Q_2)_1$ such that
(1) $\phi(\mathbf{1}) = Q_0 + Q_1 + Q_2$;
(2) $\|\phi(f) - \phi_0(f) \oplus \phi_1(f) \oplus \phi_2(f)\| < \varepsilon$ for all $f \in F$;
(3) The homomorphism ϕ_2 factors through $C[0,1]$ as

$$\phi_2 : C(X) \xrightarrow{\xi_1} C[0,1] \xrightarrow{\xi_2} Q_2M_k(C(Y))Q_2.$$

Furthermore, if $Y \neq \{pt\}$, then ξ_2 is injective;
(4) The set $(\phi_0 \oplus \phi_1)(F)$ is approximately constant to within ε;
(5) $Q_1 = p_1 + \cdots + p_n$, with $J[Q_0] \leq [p_i]$ $(i = 1, 2 \cdots n)$, ϕ_0 is defined by $\phi_0(f) = Q_0\phi(f)Q_0$, and ϕ_1 is defined by

$$\phi_1(f) = \sum_{i=1}^{n} f(x_i)p_i, \ \forall f \in C(X),$$

where $p_0, p_1, \cdots p_n$ are mutually orthogonal projections and $\{x_1, x_2, \cdots x_n\} \subset X$ is an ε-dense subset of X. (Again by $J[p] \leq [q]$, we mean that $\underbrace{p \oplus p \oplus \cdots \oplus p}_{J}$ is (unitarily) equivalent to a subprojection of q.)
Furthermore, we can choose any two of projections Q_0, Q_1, Q_2 to be trivial, if we wish. If $\phi(\mathbf{1})$ is trivial, then all of them can be chosen to be trivial projections. (This is remark 4.22.)

4.36. Let a simple C^*-algebra A be an inductive limit of matrix algebras over simplicial complexes $(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ with injective homomorphisms. Suppose that this inductive limit system possesses the very slow dimension growth condition.
In what follows, we will use the material from 1.2.19.
Fix A_n, finite set $F_n = \bigoplus_{i=1}^{t_n} F_n^i \subset A_n$, and $\varepsilon > 0$. Let $\varepsilon' = \frac{\varepsilon}{max_{1\leq i\leq t_n}\{[n,i]\}}$.
Let ${F'}^i \subset C(X_{n,i})$ be the finite set consisting of all the entries of elements in F_n^i $(\subset M_{[n,i]}(C(X_{n,i})))$. Let $\eta > 0$ $(\eta \leq \varepsilon)$ be such that if $x, x' \in X_{n,i}$ $(i = 1, 2 \cdots t_n)$ and $dist(x, x') < 2\eta$, then $|f(x) - f(x')| < \frac{\varepsilon'}{3}$ for any $f \in {F'}^i$.
For the above $\eta > 0$, there is a $\delta > 0$ such that for sufficiently large m, each partial map $\phi_{n,m}^{i,j} :\ A_n^i \to A_m^j$ has the property sdp$(\frac{\eta}{32}, \delta)$. (This is a consequence of simplicity of the algebra A and injectivity of $\phi_{n,m}$. See [DNNP], [Ell], [Li1-2] for details.)
For these numbers η and δ, and the simplicial complexes $X_{n,i}$, there are $L(i)$, $i = 1, 2, \cdots, t_n$, as in Theorem 4.35. (Note that the numbers L_i only depend on η, δ and the spaces.) Let $L = \max_i L(i)$. Fix a positive integer J. By the very slow dimension growth condition, there is an integer M such that for any $m \geq M$,

$$\frac{\mathrm{rank}\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})}{\mathrm{rank}(\mathbf{1}_{A_n^i})} > 2J \cdot L^2 \cdot 2^L(\dim X_{n,i} + \dim X_{m,j} + 1)^3.$$

As in 1.2.16, (also see 1.2.19) each partial map
$\phi^{i,j}_{n,m}: A^i_n \to \phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})A^j_m\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})$ can be written as $\phi'\otimes \mathbf{1}_{[n,k]}$ for some homomorphism $\phi': C(X_{n,i}) \to EA^j_mE$, where $E=\phi^{i,j}_{n,m}(e_{11})$, and e_{11} is the canonical matrix unit corresponding to the upper left corner. The map ϕ' also has the property $\mathrm{sdp}(\frac{\eta}{32},\delta)$.
Applying Theorem 4.35 to ${F'}^i\subset C(X_{n,i})$, η, δ, and ϕ' (as the above) and using 1.2.19, one can obtain the following Theorem.

Theorem 4.37. *For any A_n, finite set $F=\bigoplus_{i=1}^{t_n}F^i\subset A_n$, positive integer J, and number $\varepsilon>0$, there are an A_m, mutually orthogonal projections $Q_0,Q_1,Q_2\in A_m$ with $Q_0+Q_1+Q_2=\phi_{n,m}(\mathbf{1}_{A_n})$, a unital map $\psi_0\in Map(A_n,Q_0A_mQ_0)_1$, and unital homomorphisms $\psi_1\in Hom(A_n,Q_1A_mQ_1)_1$, $\psi_2\in Hom(A_n,Q_2A_mQ_2)_1$, such that*
(1) $\|\phi_{n,m}(f)-\psi_0(f)\oplus\psi_1(f)\oplus\psi_2(f)\|<\varepsilon$ for all $f\in F$;
(2) The set $(\psi_0\oplus\psi_1)(F)$ is weakly approximately constant to within ε;
(3) The homomorphism ψ_2 factors through $\bigoplus_{i=1}^{t_n}M_{[n,i]}(C[0,1])$ as

$$\psi_2: A_n \xrightarrow{\xi_1} \bigoplus_{i=1}^{t_n} M_{[n,i]}(C[0,1]) \xrightarrow{\xi_2} Q_2A_mQ_2,$$

and ξ_2 satisfies the following condition: if $X_{m,j}\neq\{pt\}$, then $\xi_2^{i,j}: M_{[n,i]}(C[0,1])\to A^j_m$ is injective;
(4) Each partial map $\psi_0^{i,j}: A^i_n\to Q_0^{i,j}A^j_mQ_0^{i,j}$ (where $Q_0^{i,j}=\psi_0^{i,j}(\mathbf{1}_{A^i_n})$) is of the form $\psi_0'\otimes id_{[n,k]}$ with $\psi_0': C(X_{n,i})\to q_0A^j_mq_0$ (where $q_0=\psi_0^{i,j}(e_{11})$ is a projection). Each partial map $\psi_1^{i,j}: A^i_n\to Q_1^{i,j}A^j_mQ_1^{i,j}$ (where $Q_1^{i,j}=\psi_1^{i,j}(\mathbf{1}_{A^i_n})$) is of the form $\psi_1'\otimes id_{[n,k]}$ and $\psi_1': A^i_n\to p^{i,j}A^j_mp^{i,j}$ (where $p^{i,j}=\psi_1^{i,j}(e_{11})$), satisfies the following

$$\psi_1'(f)=\sum_{i=1}^n f(x_i)p_i$$

for any $f\in C(X_{n,i})$, where $p_1,\cdots,p_n$ are mutually orthogonal projections with $p^{i,j}=p_1+\cdots+p_n$, and with $J\cdot[q_0]\le[p_s]$ $(s=1,2\cdots n)$ and $\{x_1,x_2,\cdots x_n\}\subset X_{n,i}$ is an ε-dense subset in $X_{n,i}$.
(When we apply this theorem in Section 6, Q_0+Q_1 will be chosen to be a trivial projection.)

Definition 4.38. Let $A=PM_l(C(X))P$, and L be a positive integer and $\eta>0$. A homomorphism $\lambda: A\to B=QM_{l_1}(C(Y))Q$ is said to be defined by point evaluations of size at least L at an η-dense subset if there are mutually orthogonal projections $Q_1,Q_2,\cdots,Q_n$ with $\mathrm{rank}(Q_i)\ge L$, an η-dense subset $\{x_1,x_2,\cdots,x_n\}\subset X$, and unital homomorphisms $\lambda_i: A\to Q_iBQ_i$, $i=1,2,\cdots,n$ such that
(1) $\lambda(\mathbf{1})=\sum_{i=1}^n Q_i$, and $\lambda=\bigoplus_{i=1}^n\lambda_i$;

(2) The homomorphisms λ_i factor through $P(x_i)M_l(\mathbb{C})P(x_i)$ ($\cong M_{\mathrm{rank}(P)}(\mathbb{C})$) as

$$\lambda_i = \lambda_i' \circ e_{x_i} : PM_l(C(X))P \xrightarrow{e_{x_i}} P(x_i)M_l(\mathbb{C})P(x_i) \xrightarrow{\lambda_i'} Q_iBQ_i,$$

where e_{x_i} are evaluation maps defined by $e_{x_i}(f) = f(x_i)$ and $\lambda_i' \in \mathrm{Hom}(M_{\mathrm{rank}(P)}(\mathbb{C}), Q_iBQ_i)_1$.

We will also call the above homomorphism λ to have the PROPERTY PE$(L,\ \eta)$. (PE stands for point evaluation.)

A homomorphism $\lambda : A \to B = QM_{l_1}(C(Y))Q$ is said TO CONTAIN A PART OF POINT EVALUATION AT POINT x OF SIZE AT LEAST L, if $\lambda = \lambda_1 \oplus \lambda'$, where λ_1 factor through $P(x)M_l(\mathbb{C})P(x)$ as

$$\lambda_1 = \lambda_1' \circ e_x : PM_l(C(X))P \xrightarrow{e_x} P(x)M_l(\mathbb{C})P(x) \xrightarrow{\lambda_1'} Q_1BQ_1,$$

and λ_1' is a unital homomorphism with $\mathrm{rank}(Q_1) \geq L$.

The following result is a corollary of Theorem 4.37, and will also be used in the proof of our main reduction theorem.

Corollary 4.39. *For any A_n, finite set $F = \bigoplus_{i=1}^{t_n} F^i \subset A_n$, positive integer J, any numbers $\varepsilon > 0$ and $\eta > 0$, and any projection $P = \oplus P^i \in \oplus A_n^i$, there are an A_m, mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$ with $Q_0+Q_1+Q_2 = \phi_{n,m}(\mathbf{1}_{A_n})$, a unital map $\psi_0 \in Map(A_n, Q_0A_mQ_0)_1$, and unital homomorphisms $\psi_1 \in Hom(A_n, Q_1A_mQ_1)_1$, $\psi_2 \in Hom(A_n, Q_2A_mQ_2)_1$, such that*

Part I:

(1) $\|\phi_{n,m}(f) - \psi_0(f) \oplus \psi_1(f) \oplus \psi_2(f)\| < \varepsilon$ for all $f \in F$;

(2) The homomorphism ψ_2 factors through a direct sum of matrix algebras over $C[0,1]$ as

$$\psi_2 : A_n \xrightarrow{\xi_1} \bigoplus_{i=1}^{t_n} M_{[n,i]}(C[0,1]) \xrightarrow{\xi_2} Q_2A_mQ_2,$$

and ξ_2 satisfies the condition that, if $X_{m,j} \neq \{pt\}$, then $\xi_2^{i,j} : M_{[n,i]}(C[0,1]) \to A_m^j$ is injective.

(3) For any blocks $A_n^i \subset A_n$, $A_m^j \subset A_m$, and for the partial maps $\psi_0^{i,j}$ and $\psi_1^{i,j}$, we have that $\psi_0^{i,j}(\mathbf{1}_{A_n^i}) := Q_0^{i,j}$ is a projection and $\psi_1^{i,j}$ has the property PE$(J \cdot rank(Q_0^{i,j}),\ \eta)$.

(4) The set $(\psi_0 \oplus \psi_1)(F)$ is weakly approximately constant to within ε.

Part II:

$\psi_0^{i,j}(P^i)$ and $\psi_0^{i,j}(\mathbf{1}_{A_n^i} - P^i)$ are mutually orthogonal projections, and the decomposition of $\phi_{n,m}' := \phi_{n,m}|_{PA_nP}$ as the direct sum of $\psi_0' := \psi_0|_{PA_nP}$, $\psi_1' := \psi_1|_{PA_nP}$, and $\psi_2' := \psi_2|_{PA_nP}$ satisfies the following conditions:

(1) $\|\phi_{n,m}'(f) - \psi_0'(f) \oplus \psi_1'(f) \oplus \psi_2'(f)\| < \varepsilon$ for all $f \in PFP = \oplus P^iF^iP^i$;

(2) The homomorphism ψ_2' factors through a C^*-algebra C which is a direct sum of matrix algebras over $C[0,1]$ as

$$\psi_2' : PA_nP \xrightarrow{\xi_1'} C \xrightarrow{\xi_2'} Q_2'A_mQ_2',$$

and ξ_2' satisfies the following condition, if $X_{m,j} \neq \{pt\}$, then $\xi_2^{i,j} : C^i \to A_m^j$ is injective, where $Q_2' = \psi_2(P)$.
(3) For any blocks $A_n^i \subset A_n$, $A_m^j \subset A_m$, and for the partial maps ${\psi_0'}^{i,j}$ and ${\psi_1'}^{i,j}$, we have that ${\psi_0'}^{i,j}(P^i) := {Q_0'}^{i,j}$ is a projection and ${\psi_1'}^{i,j}$ has property $PE(J \cdot rank({Q_0'}^{i,j}),\ \eta)$.

Proof: Obviously, the first part of the corollary follows from Theorem 4.37. To prove the second part, we only need to perturb $\psi_0 \in \mathrm{Map}(A, Q_0BQ_0)_1$ to $new\psi_0$ such that the restriction $new\psi_0|_D$ is a homomorphism, where

$$D := \bigoplus_i \mathbb{C} \cdot P^i \oplus \bigoplus_i \mathbb{C} \cdot (\mathbf{1}_{A_n^i} - P^i)$$

is a finite dimensional subalgebra of A_n.
By Lemma 1.6.8, such perturbation exists if ψ_0 is sufficiently multiplicative, which is automatically true if the set F is large enough and the number ε is small enough, using the next lemma.
(Note that $C := \xi_1(P)(\bigoplus_{i=1}^{t_n} M_{[n,i]}(C[0,1]))\xi_1(P)$ is still a direct sum of matrix algebras over $C[0,1]$, since all the projections in $M_\bullet(C[0,1])$ are trivial.)

$\square$

Lemma 4.40. *Let A be a unital C^*-algebra. Suppose that $G \subset A$ is a finite set containing $\mathbf{1}_A$, and $G_1 = G \times G := \{gh \mid g \in G, h \in G\}$. Suppose that $\delta > 0$, and $\delta' = \frac{1}{3}\frac{1}{\|G\|}\delta$, where $\|G\| = \max_{g\in G}\{\|g\|\}$.*
Suppose that B is a unital C^*-algebra and $p \in B$ is a projection. If a homomorphism $\phi \in Hom(A,B)$ and two maps $\phi_1 \in Map(A, pBp), \phi_2 \in Map(A, (1-p)B(1-p))$ satisfy

$$\|\phi(g) - \phi_1(g) \oplus \phi_2(g)\| < \delta', \quad \forall g \in G_1,$$

then both ϕ_1 and ϕ_2 are G-δ multiplicative.

Proof: The proof is straight forward, we omit it.
Theorem 4.37 and Corollary 4.39 will be used in the proof of our Main Reduction Theorem in this article. Theorem 4.35 will be used in the proof of the Uniqueness Theorem in [EGL]. The rest of this section will not be used in this paper. They are important to [EGL].

4.41. In the rest of this section, we will compare the decompositions of two different homomorphisms. Such comparison will be used in the proof of the Uniqueness Theorem in [EGL].
Let $X, \eta, \delta, \phi, \{\beta_i\}_{i=1}^L$, and $\Theta(y)$ be as in 4.34. (Take $J = 1$.) Suppose that $\phi : C(X) \to M_k(C(Y))$ is as in Theorem 4.35, and $\psi : C(X) \to M_k(C(Y))$ is another homomorphism with $\phi(\mathbf{1}) = \psi(\mathbf{1})$. If

$$\|\mathrm{Aff}T\phi(f) - \mathrm{Aff}T\psi(f)\| < \frac{\delta}{4}$$

for all $f \in H(\eta, \delta, X)$, then by Lemma 4.29, $\mathrm{SP}\phi_y$ and $\mathrm{SP}\psi_y$ can be paired within $\frac{\eta}{2}$. Since $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\frac{\eta}{2}$, $\mathrm{SP}\psi_y$ and $\Theta(y)$ can be paired within η. Similar to 4.34, by Theorem 4.1, there are mutually orthogonal projections $q_1, q_2, \cdots, q_n, q_{n+1}, \cdots, q_L$ and $q_0 = \psi(\mathbf{1}) - \sum_{i=1}^{L} q_i$ such that
(1) For all $y \in Y$ and $f \in F$,

$$\|\psi(f)(y) - q_0\psi(f)(y)q_0 \oplus \sum_{i=1}^{L} f(\alpha{\circ}\beta_i(y))q_i\| < \varepsilon.$$

(2) $\mathrm{rank}(q_0) + 2\dim(Y) \leq \mathrm{rank}(q_i)$.
As Remark 4.22, we can choose projections p_i for ϕ and q_i for ψ to be trivial projections with $\mathrm{rank}(p_i) = \mathrm{rank}(q_i)$. (Note that, in 4.34, the number L_2 and $L_2 + L_1$, which serve as T_i, $i = 1, 2, \cdots, L$ (i.e., $T_i = L_2$, for $1 \leq i \leq L - 1$, and $T_L = L_2 + L_1$) in Theorem 4.1, are very larger.) Therefore, there is a unitary $u \in M_k(C(Y))$ such that

$$uq_iu^* = p_i, \qquad i = 1, 2, \cdots, L.$$

Let $\tilde{\psi} = \mathrm{Ad}u{\circ}\psi$. Then

$$\|\tilde{\psi}(f)(y) - p_0\tilde{\psi}(f)(y)p_0 \oplus \sum_{i=1}^{L} f(\alpha{\circ}\beta_i(y))p_i\| < \varepsilon$$

for all $y \in Y$ and $f \in F$.
Note that the above decomposition has the same form as that of ϕ, even with the same projections p_i and the part $\sum_{i=1}^{L} f(\alpha{\circ}\beta_i(y))p_i$. Also, in the part $\sum_{i=1}^{L} f(\alpha{\circ}\beta_i(y))p_i$, there is a map defined by point evaluations:

$$\phi'(f) = \sum_{i=1}^{n} f(x_i)p_i,$$

with $\{x_1, x_2, \cdots, x_n\}$ η-dense in X, and $\mathrm{rank}(p_i) \geq \mathrm{rank}(p_0) + 2\dim(Y)$. This means that two different homomorphisms which are close at the level of AffT can be decomposed in the same way. This result will be useful in the proof of the Uniqueness Theorem for certain spaces X with $K_1(C(X))$ a torsion group. We summarize what we obtained as the following proposition which will be used in the proof of the Uniqueness Theorem for certain spaces X with $K_1(C(X))$ a torsion group.

Proposition 4.42. *Let X be a connected simplicial complex, $\varepsilon > 0$, and $F \subset C(X)$ be a finite set.*
Suppose that $\eta \in (0, \varepsilon)$ satisfies that if $dist(x, x') < 2\eta$, then $|f(x) - f(x')| < \frac{\varepsilon}{3}$ for all $f \in F$.
For any $\delta > 0$, there is an integer $L > 0$ and a finite set $H \subset AffT(C(X))(= C(X))$ such that the following holds.

If $\phi, \psi: C(X) \to M_k(C(Y))$ are homomorphisms with properties
(a) ϕ has $sdp(\frac{\eta}{32}, \delta)$;
(b) $rank(\phi(\mathbf{1})) \geq 2L^2 \cdot 2^L(\dim X + \dim Y + 1)^3$;
(c) $\phi(\mathbf{1}) = \psi(\mathbf{1})$ and

$$\|AffT\phi(h) - AffT\psi(h)\| < \frac{\delta}{4}, \ \forall h \in H,$$

then there are two orthogonal projections $Q_0, Q_1 \in M_k(C(Y))$, two maps $\phi_0, \psi_0 \in Map(C(X), Q_0M_k(C(Y))Q_0)_1$, a homomorphism $\phi_1 \in Hom(C(X), Q_1M_k(C(Y))Q_1)_1$, and a unitary $u \in M_k(C(Y))$ such that
(1) $\phi(\mathbf{1}) = \psi(\mathbf{1}) = Q_0 + Q_1$;
(2) $\|\phi(f) - \phi_0(f) \oplus \phi_1(f)\| < \varepsilon$, and $\|(Adu \circ \psi)(f) - \psi_0(f) \oplus \phi_1(f)\| < \varepsilon$ for all $f \in F$;
(3) ϕ_1 factors through $C[0,1]$.
(4) $Q_0 = p_0 + p_1 + \cdots + p_n$ with $rank(p_0) + 2\dim(Y) \leq rank(p_i)$ $(i = 1, 2 \cdots n)$, and ϕ_0 and ψ_0 are defined by

$$\phi_0(f) = p_0\phi(f)p_0 + \sum_{i=1}^{n} f(x_i)p_i, \ \forall f \in C(X),$$

$$\psi_0(f) = p_0(Adu \circ \psi)(f)p_0 + \sum_{i=1}^{n} f(x_i)p_i, \ \forall f \in C(X),$$

where $p_0, p_1, \cdots p_n$ are mutually orthogonal projections and $\{x_1, x_2, \cdots x_n\} \subset X$ is an ε-dense subset in X.
(Comparing with Theorem 4.35, the maps ϕ_0 and ϕ_1 in 4.35 have been put together to form the map ϕ_0 in the above proposition.)
(In [EGL], we will prove that the above ϕ_0 and ψ_0 are approximately unitarily equivalent to each other to within some small number (under the condition $KK(\phi) = KK(\psi)$), then so also are ϕ and ψ.)

4.43. The above proposition is not strong enough to prove the Uniqueness Theorem for homomorphisms from $C(S^1)$ to $M_k(C(Y))$, since $K_1(C(S^1))$ is infinite. Before we conclude this section, we introduce a result which can be used to deal with this case (i.e, the case S^1).
We will discuss briefly what the problem is, and how to solve the problem.
Suppose that ϕ and ψ are two homomorphisms from $C(S^1)$ to another C^*-algebra, For ϕ and ψ to be approximately unitarily equivalent to each other, they should agree not only on $\mathrm{Aff}T(C(S^1))$ and $K_*(C(S^1))$, but also on the determinant functions. That is, $\phi(z)\psi(z)^*$ should have only a small variation in the determinant, where $z \in C(S^1)$ is the standard generator. (All these things will be made precise in [EGL].) This idea has appeared in [Ell2] and [NT].
Roughly speaking, if ϕ and ψ agree (approximately) to within ε at the level of the determinant (this will also be made precise in [EGL]), then the maps $p_0\phi p_0$

and $p_0(\mathrm{Ad}u \circ \psi)p_0$ from Proposition 4.42 agree only to within $\frac{\mathrm{rank}(\phi(\mathbf{1}))}{\mathrm{rank}(p_0)}\varepsilon$ at the level of the determinant. So for the decomposition to be useful for the proof of the uniqueness theorem, $\mathrm{rank}(p_0)$ should not be too small compared with $\mathrm{rank}(\phi(\mathbf{1}))$. (This will be the property (2) of Theorem 4.45 below.) On the other hand, in the decompositions of ϕ and $\mathrm{Ad}u \circ \psi$, we also need the homomorphism defined by point evaluations, which by Proposition 4.42 is the same for both of these decompositions, to be large in order to absorb the parts $p_0\phi p_0$ and $p_0(\mathrm{Ad}u \circ \psi)p_0$. This will be the property (3) of Theorem 4.45. Therefore, $\mathrm{rank}(p_0)$ should not be too large either.

To do that, besides the property $\mathrm{sdp}(\frac{\eta}{32}, \delta)$, we also need sdp property for an extra pair $(\frac{\tilde{\eta}}{32}, \tilde{\delta})$, where $\tilde{\eta}$ depends on δ. For those readers who are familiar with [Ell2] and [NT], we encourage them to compare the sdp property for the two pairs $(\frac{\eta}{32}, \delta)$ and $(\frac{\tilde{\eta}}{32}, \tilde{\delta})$, with the conditions of Theorem 4 of [Ell2], and Lemma 2.3 and Theorem 2.4 of [NT], in the following way. In Theorem 4 of [Ell2] (see page 100 of [Ell2]), roughly speaking, the sentence on lines 16–20 corresponds to our property $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$, and the sentence on lines 21–22 corresponds to our property $\mathrm{sdp}(\frac{\eta}{32}, \delta)$. That is, $\frac{1}{m}$ corresponds to our $\frac{\eta}{32}$ (or $\frac{\eta}{16}$ in some sense), $\frac{3}{n}$ corresponds to our δ, $\frac{1}{n}$ corresponds to our $\frac{\tilde{\eta}}{32}$, and δ corresponds to our $\tilde{\delta}$. Similarly, in Lemma 2.3 of [NT], condition (1) corresponds to our $\mathrm{sdp}(\frac{\eta}{32}, \delta)$ and condition (2) corresponds to our $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$. Also in Theorem 2.4 of [NT], condition (2) corresponds to our $\mathrm{sdp}(\frac{\eta}{32}, \delta)$ and condition (3) corresponds to our $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$.

Such a construction will be given in 4.44 below.

In 4.44, we will first describe the condition that ϕ should satisfy. Then we will carry out the construction in three steps.

In Step 1, we will follow the procedure in 4.34, to decompose ϕ into $p_0\phi p_0 \oplus \phi_1$, corresponding to the property $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$ (not $\mathrm{sdp}(\frac{\eta}{32}, \delta)$). (Here, the map ϕ_1 is $\phi_1 \oplus \phi_2$ in the notation of 4.34 or 4.35.)

In Step 2, we will take a part $p'\phi_1 p'$ out of ϕ_1 and add it to $p_0\phi p_0$ to obtain $P_0\phi P_0$, where $P_0 = p_0 + p'$. The rest of ϕ_1 will be defined to be newϕ_1. In this way, we can get the projection P_0 with suitable size (neither too small nor too large). The size depends on δ, which explains why $\tilde{\eta}$ depends on δ.

In Step 3. we will prove that newϕ_1 can be decomposed again in such a way that the point evaluation part of its decomposition is sufficiently large that it can be used to control $P_0\phi P_0$, in the proof of the uniqueness theorem in [EGL]. (See the property (3) of Theorem 4.45.) The property $\mathrm{sdp}(\frac{\eta}{32}, \delta)$ is used in this step.

4.44. Let $F \subset C(X)$ be a finite set, $\varepsilon > 0$ and $\varepsilon_1 > 0$. Suppose that the positive number $\eta < \frac{\varepsilon_1}{4}$ satisfies the condition that, if $\mathrm{dist}(x, x') < 2\eta$, then

$$\|f(x) - f(x')\| < \frac{\varepsilon}{3}.$$

For any $\delta > 0$, consider the pair (η, δ) as in 4.33. Let N, n be as in 4.33. Instead of choosing $L = n(N+2)$, we choose

$$L \geq \max\{n(N+2), \tfrac{8}{\delta}, \tfrac{4}{\varepsilon}, \tfrac{4}{\varepsilon_1}\}.$$

Consider $\tilde{\varepsilon} = \frac{1}{8L} < \min(\varepsilon, \varepsilon_1)$. Let positive number $\tilde{\eta} < \frac{\eta}{4}$ satisfy that, if $\mathrm{dist}(x, x') < 2\tilde{\eta}$, then

$$\|f(x) - f(x')\| < \frac{\tilde{\varepsilon}}{3}.$$

Let $\tilde{\delta} > 0$ be any number. Then for the pair $(\tilde{\eta}, \tilde{\delta})$, there exists an integer $\tilde{L}$ playing the role of L as in Lemma 4.33. We can assume $\tilde{L} > L$. Let

$$\Lambda = 6\tilde{L}^2 \cdot 2^{\tilde{L}}(\dim X + M + 1)^3,$$

where M is a positive integer.

Now let Y be a simplicial complex with $\dim Y \leq M$, and $\phi : C(X) \to PM_k(C(Y))P$ be a unital homomorphism satisfying the following two conditions:

(a) ϕ has both $\mathrm{sdp}(\frac{\eta}{32}, \delta)$ and $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$;

(b) $\mathrm{rank}(P) \geq \Lambda$.

We will construct a decomposition for ϕ.

Step 1. By the discussion in 4.34 corresponding to $\mathrm{sdp}(\frac{\tilde{\eta}}{32}, \tilde{\delta})$, there is a set

$$\Theta(y) = \left\{\alpha\circ\beta_1(y)^{\sim L_2}, \alpha\circ\beta_2(y)^{\sim L_2}, \cdots, \alpha\circ\beta_{\tilde{L}-1}(y)^{\sim L_2}, \alpha\circ\beta_{\tilde{L}}(y)^{\sim L_2+L_1}\right\},$$

where

$$L_2 = \mathrm{int}\left(\frac{\mathrm{rank}(P)}{\tilde{L}}\right) \geq \mathrm{int}\left(\frac{\Lambda}{\tilde{L}}\right),$$

such that $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\frac{\tilde{\eta}}{2}$.

As in 4.34, there are mutually orthogonal projections p_0 and $P_1 = \sum_{i=1}^{\tilde{L}} p_i$ and a homomorphism $\phi_1 : C(X) \to P_1 M_k(C(Y))P_1$, such that

(1) $\|\phi(f) - p_0\phi(f)p_0 \oplus \phi_1(f)\| \leq \tilde{\varepsilon} \leq \frac{1}{8L}$,

(2) $\mathrm{rank}(p_0) \leq \tilde{L} \cdot 2^{\tilde{L}}(\dim X + M + 1)^3 \leq \mathrm{int}\left(\frac{\Lambda}{6\tilde{L}}\right)$,

where ϕ_1 is defined by

$$\phi_1(f)(y) = \sum_{i=1}^{\tilde{L}} f(\alpha\circ\beta_i(y))p_i$$

with $\mathrm{rank}(p_i) \geq L_2 - 2^{\tilde{L}}(\dim X + M + 1)^3$.

Step 2. We will take a part $p'\phi_1 p'$ out from ϕ_1 and add it into $p_0\phi p_0$, such that the projection $P_0 = p_0 + p'$ has rank about $\frac{\mathrm{rank}(P)}{L}$, which is neither too large nor too small. (Here we use L not $\tilde{L}$.)

There exists a projection p' satisfying the following two conditions.

(c) $p' = \sum_{i=1}^{\tilde{L}} p'_i$, with $p'_i < p_i$, $i = 1, 2, \cdots, \tilde{L}$.

(d) $\mathrm{rank}(p') = \mathrm{int}\left(\frac{\mathrm{rank}(P)}{L}\right)$ (here we use L, not $\tilde{L}$), where L was chosen in the beginning of this subsection.
We can make the above (d) hold for the following reason. First,

$$\mathrm{rank}(\sum_{i=1}^{\tilde{L}} p_i) \geq \mathrm{rank}(P) - \mathrm{int}\left(\frac{\Lambda}{6\tilde{L}}\right) > \mathrm{int}\left(\frac{\mathrm{rank}(P)}{L}\right) + \tilde{L}\dim(Y).$$

So one can choose non negative integers $k_1, k_2, \cdots, k_{\tilde{L}}$ such that $\sum_{i=1}^{\tilde{L}} k_i = \mathrm{int}\left(\frac{\mathrm{rank}(P)}{L}\right)$ and that $k_i \leq rank(p_i) - \dim(Y)$. Therefore, by [Hu], we can choose trivial projections $p_i' < p_i$ with $\mathrm{rank}(p_i') = k_i$.
Define

$$P_0 = p_0 \oplus p' \qquad \text{and} \qquad \mathrm{new}P_1 = P_1 \ominus p'.$$

Note that p' is a sub-projection of $P_1 = \sum_{i=1}^{\tilde{L}} p_i$. Define $\mathrm{new}\phi_1 : C(X) \to \mathrm{new}P_1 M_k(C(Y))\mathrm{new}P_1$ by

$$(\mathrm{new}\phi_1(f))(y) = \sum_{i=1}^{\tilde{L}} f(\alpha\circ\beta_i(y))(p_i \ominus p_i').$$

$\mathrm{new}P_1$ and $\mathrm{new}\phi_1$ are still denoted by P_1 and ϕ_1, respectively. Evidently, the following are true.

$$(1') \qquad \|\phi(f) - P_0\phi(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4L}.$$

$$(2') \qquad \frac{\mathrm{rank}(P)}{L} \leq \mathrm{rank}(P_0) \leq 2 \cdot \mathrm{int}\left(\frac{\mathrm{rank}(P)}{L}\right).$$

(Notice that, to get the above decomposition, one only needs the condition that $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\tilde{\eta}$ (see the way η is chosen in 4.4 for Theorem 4.1 and the way $\tilde{\eta}$ is chosen above). On the other hand, $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\frac{\tilde{\eta}}{2}$ in our case. So if ψ satisfies the condition that $\mathrm{SP}\psi_y$ and $\mathrm{SP}\phi_y$ can be paired within $\frac{\tilde{\eta}}{2}$, then the above decomposition also holds for ψ, as discussed in 4.41. In particular, for a certain unitary u, $\mathrm{Ad}u \circ \psi$ can have same form of decomposition as ϕ does— same projection P_0 and even exactly the same part of the above ϕ_1. This will be used in 4.46 and Proposition 4.47.)
STEP 3. Now, we can decompose ϕ_1 again to obtain a large part of the homomorphism defined by point evaluations, which will be used to absorb the part of $P_0\phi P_0$, in the proof of the uniqueness theorem in [EGL].
For the compact metric space X, and $\eta > 0$ (now we use η not $\tilde{\eta}$), there exists a finite η-dense subset $\{x_1, x_2, \cdots, x_m\}$ such that $\mathrm{dist}(x_i, x_j) \geq \eta$, if $i \neq j$.

(Such set could be chosen to be a maximum set of finite many points which have mutual distance at least η. Then the η-density of the set follows from the maximality.)

We will prove the following claim.

Claim: There are mutually orthogonal projections $q_1, q_2, \cdots, q_m < P_1$ with $\operatorname{rank}(q_i) > \operatorname{rank}(P_0) + \dim(Y)$, such that

$$\left\| \phi_1(f) - (P_1 - \sum_{i=1}^{m} q_i)\phi_1(f)(P_1 - \sum_{i=1}^{m} q_i) \oplus \sum_{i=1}^{m} f(x_i)q_i \right\| < \varepsilon$$

for all $f \in F$.

Proof of the claim:

First, we know that the set $(\mathrm{SP}\phi_1)_y$ is obtained by deleting $\operatorname{rank}(P_0)$ points (counting multiplicity) from the set $\Theta(y)$. Also $\mathrm{SP}\phi_y$ and $\Theta(y)$ can be paired within $\frac{\tilde{\eta}}{2}$. Recall that,

$$\Theta(y) = \left\{ \alpha_\circ\beta_1(y)^{\sim L_2}, \alpha_\circ\beta_2(y)^{\sim L_2}, \cdots, \alpha_\circ\beta_{\tilde{L}-1}(y)^{\sim L_2}, \alpha_\circ\beta_{\tilde{L}}(y)^{\sim L_2+L_1} \right\}$$

is the set corresponding to ϕ and the pair $(\tilde{\eta}, \tilde{\delta})$ in 4.33. And recall that $L_2 = \operatorname{int}\left(\frac{\operatorname{rank}(P)}{\tilde{L}}\right)$. From (a), ϕ has the property $\mathrm{sdp}(\frac{\eta}{32}, \delta)$. So $\Theta(y)$ has the property $\mathrm{sdp}(\frac{\eta}{32} + \frac{\tilde{\eta}}{2}, \delta)$. But $(\mathrm{SP}\phi_1)_y$ is obtained by deleting

$$\operatorname{rank}(P_0) \left(\leq 2 \cdot \operatorname{int}\left(\frac{\operatorname{rank}(P)}{L}\right) \leq \frac{\delta}{4}\operatorname{rank}(P) \right)$$

points from $\Theta(y)$. (Note that $\frac{1}{L} < \frac{\delta}{8}$.) Therefore, in the $(\frac{\eta}{32} + \frac{\tilde{\eta}}{2})$-ball of any point in X, $(\mathrm{SP}\phi_1)_y$ contains at least

$$\delta \cdot \operatorname{rank}(P) - \frac{\delta}{4}\operatorname{rank}(P) = \frac{3\delta}{4}\operatorname{rank}(P)$$

points (counting multiplicity). That is, ϕ_1 has the property $\mathrm{sdp}(\frac{\eta}{32} + \frac{\tilde{\eta}}{2}, \frac{3\delta}{4})$. Therefore ϕ_1 has the property $\mathrm{sdp}(\frac{\eta}{4}, \frac{3\delta}{4})$, since $\tilde{\eta} < \frac{\eta}{4}$.

Set $U_i = B_{\frac{\eta}{2}}(x_i)$, $i = 1, 2, \cdots, m$. Then U_i, $i = 1, 2, \cdots, m$ are mutually disjoint open sets, since $\operatorname{dist}(x_i, x_j) \geq \eta$, if $i \neq j$. By the property $\mathrm{sdp}(\frac{\eta}{4}, \frac{3\delta}{4})$ of ϕ_1, for any $y \in Y$,

$$\#(\mathrm{SP}(\phi_1)_y \cap U_i) \geq \frac{3\delta}{4}\operatorname{rank}(P) > \frac{2}{L}\operatorname{rank}(P) + 3\dim Y > \operatorname{rank}(P_0) + 3\dim(Y).$$

The claim follows from the following proposition:

Proposition. *Let X be a simplicial complex, and $F \subset C(X)$ a finite subset. Let $\varepsilon > 0$ and $\eta > 0$ be such that if $\operatorname{dist}(x, x') < 2\eta$, then $|f(x) - f(x')| < \frac{\varepsilon}{3}$ for any $f \in F$.*

Suppose that $U_1, U_2, \cdots, U_m$ are disjoint open neighborhoods of points $x_1, x_2, \cdots, x_m \in X$, respectively, such that $U_i \subset B_\eta(x_i)$ for all $1 \leq i \leq m$.

Suppose that $\phi : C(X) \to PM_\bullet(C(Y))P$ *is a unital homomorphism, where* Y *is a simplicial complex, such that*

$$\#(SP\phi_y \cap U_i) \geq k_i, \qquad \text{for } 1 \leq i \leq m \text{ and for all } y \in Y.$$

Then there are mutually orthogonal projections $q_1, q_2, \cdots, q_m \in PM_\bullet(C(Y))P$ *with* $rank(q_i) \geq k_i - dim(Y)$ *such that*

$$\|\phi(f) - p_0\phi(f)p_0 \oplus \sum_{i=1}^{m} f(x_i)q_i\| < \varepsilon, \qquad \text{for all } f \in F,$$

where $p_0 = P - \sum q_i$.
This is Proposition 1.5.7 of this paper (see 1.5.4—1.5.6 for the proof). Since the expert reader may skip §1.5, we point out that the above result was essentially proved in [EG2, Theorem 2.21].
So we obtain the projections q_i with

$$\mathrm{rank}(q_i) \geq \min_y(\#(\mathrm{SP}(\phi_1)_y \cap U_i)) - \dim(Y) \geq \mathrm{rank}(p_0) + 2\dim(Y).$$

Summarizing the above, we obtain the following theorem.

Theorem 4.45. *Let* $F \subset C(X)$ *be a finite set,* $\varepsilon > 0$, $\varepsilon_1 > 0$, *and let* M *be a positive integer (in the application in [EGL], we will let* $M = 3$*). Let the positive number* $\eta < \frac{\varepsilon_1}{4}$ *satisfy that, if* $dist(x, x') < 2\eta$, *then*

$$\|f(x) - f(x')\| < \frac{\varepsilon}{3} \qquad \text{for all } f \in F.$$

Let $\delta > 0$ *be any positive number. There is an integer* $L > \max\{\frac{8}{\delta}, \frac{4}{\varepsilon}, \frac{4}{\varepsilon_1}\}$ *satisfying the following condition. The rest of the theorem describes this condition. Suppose that* $\tilde{\eta} > 0$ *satisfies that, if* $dist(x, x') < 2\tilde{\eta}$, *then*

$$\|f(x) - f(x')\| < \frac{1}{24L} \qquad \text{for all } f \in F.$$

For any $\tilde{\delta} > 0$, *there is a positive integer* Λ *such that if a unital homomorphism* $\phi : C(X) \to PM_k(C(Y))P$ *(with* $\dim Y \leq M$*) satisfies the following conditions*
(a) ϕ *has the properties* $sdp(\frac{\eta}{32}, \delta)$ *and* $sdp(\frac{\tilde{\eta}}{32}, \tilde{\delta})$*;*
(b) $rank(P) \geq \Lambda$,
then there are projections $P_0, P_1 \in PM_k(C(Y))P$ *(with* $P_0 + P_1 = P$*) and a homomorphism* $\phi_1 : C(X) \to P_1M_k(C(Y))P_1$ *such that*
(1) $\|\phi(f) - P_0\phi(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4L}$ *for all* $f \in F$*;*
(2) $rank(P_0) \geq \frac{rank(P)}{L}$*;*
(3) There are mutually orthogonal projections $q_1, q_2, \cdots, q_m \in P_1M_k(C(Y))P_1$ *and an* η*-dense finite subset* $\{x_1, x_2, \cdots, x_m\} \subset X$ *with the following properties.*

(i) $rank(q_i) > rank(P_0) + 2\dim(Y),\ i = 1, 2, \cdots, m$;
(ii) $\|\phi_1(f) - (P_1 - \sum_{i=1}^m q_i)\phi_1(f)(P_1 - \sum_{i=1}^m q_i) \oplus \sum_{i=1}^m f(x_i)q_i\| < \varepsilon$ *for all* $f \in F$.

4.46. Let $\tilde\eta$ and $\tilde\delta$ be as in 4.44 (or 4.45), and $H(\tilde\eta, \tilde\delta, X) \subset C(X)$ the subset defined in 4.27. Suppose that $\phi : C(X) \to PM_k(C(Y))P$ satisfies the conditions (a) and (b) in Theorem 4.45. And suppose that $\psi : C(X) \to PM_k(C(Y))P$ is another homomorphism satisfying

$$\|\mathrm{Aff}T\phi(h) - \mathrm{Aff}T\psi(h)\| < \frac{\tilde\delta}{4},$$

for all $h \in H(\tilde\eta, \tilde\delta, X)$. Similar to 4.41, there is a unitary $u \in PM_k(C(Y))P$ such that

$$\|\mathrm{Ad}u{\circ}\psi(f) - P_0\mathrm{Ad}u{\circ}\psi(f)P_0 \oplus \phi_1\| < \frac{1}{4L}, \quad \forall f \in F$$

where P_0 and ϕ_1 are exactly the same as those for ϕ in Theorem 4.45. (See the end of step 2 of 4.44.)
So we have the following proposition.

Proposition 4.47. *Let $F \subset C(X)$ be a finite set, $\varepsilon > 0$, $\varepsilon_1 > 0$, and let M be a positive integer (in the application in [EGL], we will let $M = 3$). Let the positive number $\eta < \frac{\varepsilon_1}{4}$ satisfy that, if $dist(x, x') < 2\eta$, then*

$$\|f(x) - f(x')\| < \frac{\varepsilon}{3} \qquad \text{for all } f \in F.$$

Let $\delta > 0$ be any positive number. There is an integer $L > \max\{\frac{8}{\delta}, \frac{4}{\varepsilon}, \frac{4}{\varepsilon_1}\}$ satisfying the following condition. The rest of the proposition describes this condition.
Suppose that $\tilde\eta > 0$ satisfies that, if $dist(x, x') < 2\tilde\eta$, then

$$\|f(x) - f(x')\| < \frac{1}{24L} \qquad \text{for all } f \in F.$$

For any $\tilde\delta > 0$, there is a positive integer Λ and a finite set $H \subset AffT(C(X))(= C(X))$ such that if unital homomorphisms $\phi, \psi : C(X) \to PM_k(C(Y))P$ (with $\dim Y \le M$) satisfy the following conditions:
(a) ϕ has the properties $sdp(\frac{\eta}{32}, \delta)$ and $sdp(\frac{\tilde\eta}{32}, \tilde\delta)$;
(b) $rank(P) \ge \Lambda$;
(c) $\|AffT\phi(h) - AffT\psi(h)\| < \frac{\tilde\delta}{4},\ \forall h \in H$,
then there are projections $P_0, P_1 \in PM_k(C(Y))P$ (with $P_0 + P_1 = P$), a homomorphism $\phi_1 : C(X) \to P_1M_k(C(Y))P_1$ factoring through $C[0,1]$, and a unitary $u \in PM_k(C(Y))P$ such that
(1) $\|\phi(f) - P_0\phi(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4L}$ and
$\|(Adu \circ \psi)(f) - P_0(Adu \circ \psi)(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4L}$ for all $f \in F$;

(2) rank$(P_0) \geq \frac{rank(P)}{L}$;
(3) There are mutually orthogonal projections $q_1, q_2, \cdots, q_m \in P_1M_k(C(Y))P_1$ *and an* η*-dense finite subset* $\{x_1, x_2, \cdots, x_m\} \subset X$ *with the following properties.*
(i) rank$(q_i) >$ *rank*$(P_0) + 2\dim(Y)$;
(ii) $\|\phi_1(f) - (P_1 - \sum_{i=1}^m q_i)\phi_1(f)(P_1 - \sum_{i=1}^m q_i) \oplus \sum_{i=1}^m f(x_i)q_i\| < \varepsilon$ *for all* $f \in F$.
In order to be consistent in notation with the application in [EGL], let us rewrite the above proposition in the following form.

Proposition 4.47'. *For any finite set* $F \subset C(X)$, $\varepsilon > 0$, $\varepsilon_1 > 0$, *there is a number* $\eta > 0$ *with the property described below.*
For any $\delta > 0$, *there are an integer* $K > \frac{4}{\varepsilon}$ *and a a number* $\tilde{\eta} > 0$ *satisfying the following condition.*
For any $\tilde{\delta} > 0$, *there is a positive integer* L *and a finite set* $H \subset AffT(C(X))$*(*H *can be chosen to be* $H(\tilde{\eta}, \tilde{\delta}, X)$ *in 4.27) such that if two unital homomorphisms* $\phi, \psi: C(X) \to PM_k(C(Y))P$ *(with* $\dim Y \leq 3$*) satisfy the following conditions:*
(a) ϕ *has the properties sdp*$(\frac{\eta}{32}, \delta)$ *and sdp*$(\frac{\tilde{\eta}}{32}, \tilde{\delta})$;
(b) rank$(P) \geq L$;
(c) $\|AffT\phi(h) - AffT\psi(h)\| < \frac{\tilde{\delta}}{4}$, $\forall h \in H$,
then there are projections $P_0, P_1 \in PM_k(C(Y))P$ *(with* $P_0 + P_1 = P$*), a homomorphism* $\phi_1 : C(X) \to P_1M_k(C(Y))P_1$ *factoring through* $C[0,1]$, *and a unitary* $u \in PM_k(C(Y))P$ *such that*
(1) $\|\phi(f) - P_0\phi(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4K}$ *and*
$\|(Adu \circ \psi)(f) - P_0(Adu \circ \psi)(f)P_0 \oplus \phi_1(f)\| < \frac{1}{4K}$ *for all* $f \in F$;
(2) rank$(P_0) \geq \frac{rank(P)}{K}$;
(3) There are mutually orthogonal projections $q_1, q_2, \cdots, q_m \in P_1M_k(C(Y))P_1$ *and an* $\frac{\varepsilon_1}{4}$*-dense finite subset* $\{x_1, x_2, \cdots, x_m\} \subset X$ *with the following properties.*
(i) rank$(q_i) >$ *rank*$(P_0) + 2\dim(Y)$;
(ii) $\|\phi_1(f) - (P_1 - \sum_{i=1}^m q_i)\phi_1(f)(P_1 - \sum_{i=1}^m q_i) \oplus \sum_{i=1}^m f(x_i)q_i\| < \varepsilon$ *for all* $f \in F$.
(Notice that in the above statement, we change the notation L and Λ to K and L respectively. Also, in condition (3), we change η-density to $\frac{\varepsilon_1}{4}$-density.)

4.48. Proposition 4.47' will be used in the proof of the Uniqueness Theorem in [EGL]. Namely, we will prove that, under certain conditions about KK(ϕ) and KK(ψ) and the determinants of $\phi(z)$ and $\psi(z)$ (see (4) of Theorem 2.4 of [NT]), where $z \in C(S^1)$ is the standard generator,

$$P_0\phi(f)P_0 \oplus \sum_{i=1}^m f(x_i)q_i, \qquad f \in F$$

is approximately unitarily equivalent to

$$P_0 \mathrm{Ad} u_\circ \psi(f) P_0 \oplus \sum_{i=1}^{m} f(x_i) q_i, \qquad\qquad f \in F.$$

Therefore, $\{\phi(f),\ f \in F\}$ is approximately unitarily equivalent to $\{\psi(f),\ f \in F\}$. In [EGL], we need both of the following conditions:

$$\mathrm{rank}(P_0) \geq \frac{\mathrm{rank}(P)}{L} \qquad \text{and} \qquad [q_i] > [P_0] \ \text{ in } \ K_0(C(Y)).$$

In comparison with Theorem 2.4 of [NT], in the Uniqueness Theorem in [EGL], we also have a condition similar to (4) of Theorem 2.4 of [NT]. But this condition will be useful only when it is combined with the condition (2) above (see [EGL] for details).

5 Almost Multiplicative Maps

In this section, we study almost multiplicative maps

$$\phi \in \mathrm{Map}\left(M_l(C(X)), M_{l_1}(C(Y))\right),$$

where $X = T_{II,k}, T_{III,k}$, or S^2, and Y is a simplicial complex of dimension at most M with M a fixed number. In this section, all the simplicial complexes are assumed to have dimension at most M.

5.1. Suppose that $B_1, B_2, \cdots, B_n, \cdots$ are unital C^*-algebras. Let $B = \bigoplus_{n=1}^{+\infty} B_n$. Then the multiplier algebra $M(B)$ of B is $\prod_{n=1}^{+\infty} B_n$. The Six Term Exact Sequence associated to

$$0 \longrightarrow B \longrightarrow M(B) \longrightarrow M(B)/B \longrightarrow 0$$

breaks into two exact sequences

$$0 \longrightarrow K_0(B) \longrightarrow K_0(M(B)) \longrightarrow K_0(M(B)/B) \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow K_1(B) \longrightarrow K_1(M(B)) \longrightarrow K_1(M(B))/B \longrightarrow 0.$$

since each projection (or unitary) in $M_n(M(B)/B)$ can be lifted to a projection (or a unitary) in $M_n(M(B))$.
Furthermore,

$$K_0(B) = \bigoplus_{n=1}^{+\infty} K_0(B_n) \qquad \text{and} \qquad K_1(B) = \bigoplus_{n=1}^{+\infty} K_1(B_n).$$

But in general, it is NOT true that

$$K_0(M(B)) = \prod_{n=1}^{+\infty} K_0(B_n) \qquad \text{or} \qquad K_1(M(B)) = \prod_{n=1}^{+\infty} K_1(B_n).$$

In fact, $K_0(M(B))$ is a subgroup of $\prod_{n=1}^{+\infty} K_0(B_n)$. But $K_1(M(B))$ is more complicated. In the first part of this section, we will calculate the K-theory of $M(B)$ (and of $M(B)/B$) for the case

$$B_n = M_{k_n}(C(X_n)),$$

where X_n are simplicial complexes of dimension at most M. For convenience, we always suppose that the spaces X_n are connected.

5.2. Consider $S^1 = \{z;\ |z| = 1\} \subset \mathbb{C}$. Let

$$F:\ (S^1\backslash\{-1\}) \times [0,1] \longrightarrow S^1\backslash\{-1\}$$

be defined by

$$F(e^{i\theta}, t) = e^{it\theta}, \qquad -\pi < \theta < \pi.$$

Then $|t - t'| < \varepsilon$ implies

$$|F(x,t) - F(x,t')| < \pi\varepsilon.$$

This fact implies the following. If u and v are unitaries such that $\|u - v\| < 1$, then there is a path of unitaries u_t with $u_0 = u,\ u_1 = v$ such that $|t - t'| < \varepsilon$ implies $\|u_t - u_{t'}\| < \pi\varepsilon$.
Let $SU(n)(\subset U(n))$ denote the collection of $n \times n$ unitaries with determinant 1. Let $SU_n(X)$ denote the collection of continuous functions from X to $SU(n)$. Note $SU_n(X) \subset U_n(X) \subset M_n(C(X))$.
From the proof of Theorem 3.3 (and Lemma 3.1) of [Phi2] (in particular $(***)$ in Step 4 of 3.3 of [Phi2]), one can prove the following useful fact.

Lemma 5.3. ([Phi2]) *For each positive integer M, there is an $M' > 0$ satisfying the following condition. For any connected finite CW-complex X of dimension at most M, and $u, v \in SU_n(X)$, if u and v can be connected to each other in $U_n(X)$, then there is a path $u_t \in SU_n(X)$ such that*
1. $u_0 = u, u_1 = v$ and
2. $|t - t'| < \varepsilon$ implies $\|u_t - u_{t'}\| < M' \cdot \varepsilon$.
(Note that M' does not depend on n, the size of the unitaries.)

5.4. Let $B_n = M_{k_n}(C(X_n)),\ \dim(X_n) \le M$. Let $B = \bigoplus_{n=1}^{+\infty} B_n$. Then we can describe $K_0(M(B))$ as below. Let $(K_0(B_n), K_0(B_n)^+, \mathbf{1}_{B_n})$ be the scaled ordered K-group of B_n (see 1.2 of [EG2]). Let $\Pi_b K_0(B_n)$ be the subgroup of $\prod_{n=1}^{+\infty} K_0(B_n)$ consisting of elements

$$(x_1, x_2, \cdots, x_n, \cdots) \in \prod_{n=1}^{+\infty} K_0(B_n)$$

with the property that there is a positive integer L such that

$$-L[\mathbf{1}_{B_n}] < x_n < L[\mathbf{1}_{B_n}]\ \in\ K_0(B_n)$$

for all n.

Lemma 5.5. $K_0\left(\prod_{n=1}^{+\infty} B_n\right) = \Pi_b K_0(B_n)$.

Proof: Any element in $K_0\left(\prod_{n=1}^{+\infty} B_n\right)$ is of the form $[p]-[q]$, where $p, q \in M_L\left(\prod_{n=1}^{+\infty} B_n\right)$ are projections. Let

$$p = (p_1, p_2, \cdots, p_n, \cdots),\ q = (q_1, q_2, \cdots, q_n, \cdots) \in M_L\left(\prod_{n=1}^{+\infty} B_n\right).$$

Then $[p]-[q] \in K_0\left(\prod_{n=1}^{+\infty} B_n\right)$ corresponds to the element

$$([p_1]-[q_1], [p_2]-[q_2], \cdots, [p_n]-[q_n], \cdots) \in \Pi_b K_0(B_n).$$

We will prove that this correspondence is bijective.
Surjectivity: Let

$$([p_1]-[q_1], [p_2]-[q_2], \cdots, [p_n]-[q_n], \cdots) \in \Pi_b K_0(B_n).$$

Then there is an $L > M$ such that

$$-L[\mathbf{1}_{B_n}] < [p_n]-[q_n] < L[\mathbf{1}_{B_n}], \quad \forall n.$$

Therefore,

$$-L\cdot k_n \leq \mathrm{rank}(p_n) - \mathrm{rank}(q_n) < L\cdot k_n, \quad \forall n.$$

It is well known that (see [Hu]) any vector bundle of dimension $M+T$ over an M dimensional space has a T dimensional trivial sub-bundle. Thus one can replace p_n by p'_n, q_n by q'_n, with properties

$$[p'_n] < 2L[\mathbf{1}_{B_n}], \quad [q'_n] < 2L[\mathbf{1}_{B_n}] \qquad \text{and}$$

$$[p'_n]-[q'_n] = [p_n]-[q_n] \qquad \text{in } K_0(B_n).$$

$([p'_1]-[q'_1], [p'_2]-[q'_2], \cdots)$ is in the image of the correspondence, since every element $[p'_n] < 2L[\mathbf{1}_{B_n}]$ can be realized by a projection in $M_{4L}(B_n)$ (recall that $L > M$).
Injectivity. Let $p = (p_1, p_2, \cdots, p_n, \cdots)$ and $q = (q_1, q_2, \cdots, q_n, \cdots)$ be projections in $M_L\left(\prod_{n=1}^{+\infty} B_n\right)$. Suppose that for each n, $[p_n] = [q_n] \in K_0(B_n)$. We have to prove that $[(p_1, p_2, \cdots, p_n, \cdots)] = [(q_1, q_2, \cdots, q_n, \cdots)] \in K_0\left(\prod_{n=1}^{+\infty} B_n\right)$.
Without loss of generality, assume that $L > M$. Let $\mathbf{1}_n \in M_L(B_n)$ be the unit. By [Hu], for each n, the projection $p_n \oplus \mathbf{1}_n$ is unitary equivalent to $q_n \oplus \mathbf{1}_n$. That is, there is a unitary $u_n \in M_{2L}(B_n)$ such that $q_n \oplus \mathbf{1}_n = u_n(p_n \oplus \mathbf{1}_n)u_n^*$. Hence the unitary $u = (u_1, u_2, \cdots, u_n, \cdots) \in M_{2L}(\prod_{n=1}^{+\infty} B_n)$ satisfies $q \oplus \mathbf{1} = u(p \oplus \mathbf{1})u^*$. It follows that $[q] = [p]$.

□

5.6. Let X be a finite CW complex. Then $K_1(C(X)) = K^1(X)$ is defined to be the collection of homotopy equivalence classes of continuous maps from X to $U(\infty)$, denoted by $[X, U(\infty)]$, (or from X to $U(n)$, denoted by $[X, U(n)]$, for n large enough). Consider the fibration

$$SU(n) \longrightarrow U(n) \xrightarrow{b} S^1,$$

where $S^1 \subset \mathbb{C}$ is the unit circle, b is defined by sending a unitary to its determinant, and $SU(n)$ is the special unitary group consisting the unitaries of determinant 1. The fibration has a splitting $S^1 \xrightarrow{b^{-1}} U(n)$, defined by

$$S^1 \ni z \xmapsto{b^{-1}} \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in U(n).$$

One can identify $U(n)$ with $SU(n) \times S^1$ by $U(n) \ni u \mapsto ((b^{-1} \circ b(u))^* u, b(u)) \in SU(n) \times S^1$.

Therefore, $[X, U(n)] = [X, SU(n)] \oplus [X, S^1]$ as a group. We use notation $SK_1(C(X))$ or $SK^1(X)$ to denote $[X, SU(n)]$, n large enough, and $\pi^1(X)$ to denote $[X, S^1]$. Then

$$K_1(C(X)) = SK_1(C(X)) \oplus \pi^1(X).$$

(The splitting is not a natural splitting.)

5.7. Let $\{X_n\}$ be a sequence of connected finite CW complexes of dimension at most M. Let $B_n = M_{k_n}(C(X_n))$. Define a map

$$\tau: \ K_1\left(\prod_{n=1}^{+\infty} B_n\right) \longrightarrow \prod_{n=1}^{+\infty} K_1 B_n \qquad \text{by}$$

$$\tau[(u_1, u_2, \cdots, u_n, \cdots)] = ([u_1], [u_2], \cdots, [u_n], \cdots),$$

where $(u_1, u_2, \cdots, u_n, \cdots)$ is a unitary in $M_L\left(\prod_{n=1}^{+\infty} B_n\right)$. If $L \geq M$, then any element in $K_1(B_n)$ can be realized by a unitary in $M_L(B_n)$. Based on this fact, we know that τ is surjective. We will prove that

$$0 \longrightarrow Ker\tau \longrightarrow K_1\left(\prod_{n=1}^{+\infty} B_n\right) \longrightarrow \prod_{n=1}^{+\infty} K_1 B_n \longrightarrow 0$$

is a splitting exact sequence. A splitting

$$\tilde{\tau}: \ \prod_{n=1}^{+\infty} K_1 B_n \longrightarrow K_1\left(\prod_{n=1}^{+\infty} B_n\right)$$

will be defined such that $\tau \circ \tilde{\tau} = \mathrm{id}$ on $\prod_{n=1}^{+\infty} K_1 B_n$. By 5.6,

$$K_1 B_n = SK_1 B_n \oplus \pi^1(X_n).$$

Hence we define $\tilde{\tau}$ on $\prod_{n=1}^{+\infty} SK_1 B_n$ and $\prod_{n=1}^{+\infty} \pi^1(X_n)$ separately.
If $x \in \prod_{n=1}^{+\infty} SK_1 B_n$ is represented by a sequence of unitaries

$$u_1 \in M_L(B_1),\ u_2 \in M_L(B_2),\ \cdots,\ u_n \in M_L(B_n),\ \cdots,$$

each with determinant 1, then define

$$\tilde{\tau}(x) = [(u_1, u_2, \cdots, u_n, \cdots)] \in K_1\left(\prod_{n=1}^{+\infty} B_n\right).$$

To see that $\tilde{\tau}$ is well defined, let $v_1, v_2, \cdots, v_n, \cdots$ be another sequence with determinant 1 and

$$[u_n] = [v_n] \qquad\qquad \text{in } K_1(B_n).$$

Without loss of generality, we assume that $L > M$. By Lemma 5.3, for each n, there is a unitary path u_n such that $u_n(0) = u_n$, $u_n(1) = v_n$, and $\|u_n(t) - u_n(t')\| < M' \cdot |t - t'|$, $\forall t \in [0,1]$, where M' is a constant which does not depend on n. Obviously,

$$(u_1(t), u_2(t), \cdots, u_n(t), \cdots) \in (M_L(\prod_{n=1}^{+\infty} B_n)) \otimes C([0,1]).$$

Hence

$$[(u_1, u_2, \cdots, u_n, \cdots)] = [(v_1, v_2, \cdots, v_n, \cdots)].$$

That is, the above map is well defined.
(Warning: It is not enough to prove that each u_n can be connected to v_n, since a sequence of paths, each connecting u_n and v_n $(n = 1, 2, \cdots)$, only defines an element in $M_L(\prod_{n=1}^{+\infty}(B_n \otimes C[0,1]))$, but

$$\left(\prod_{n=1}^{+\infty} B_n\right) \otimes C[0,1] \ \subsetneq\ \prod_{n=1}^{+\infty}(B_n \otimes C[0,1]).\ )$$

The following claim is a well known folklore result in topology. Since we can not find a precise reference, we present a proof here.
Claim: For any connected simplicial complex X, the cohomotopy group $\pi^1(X)$ is a finitely generated free abelian group.
Proof of the claim. Let $X^{(1)}$ be the 1-skeleton of X. Then $X^{(1)}$ is homotopy equivalent to a finite wedge of S^1. Evidently, $\pi^1(X^{(1)})$ is a finitely generated free abelian group. (In comparison with the above cohomotopy group, we point out that the fundamental group $\pi_1(X^{(1)})$ of a finite wedge $X^{(1)}$ of S^1 is a free group (not a free abelian group).)

On the other hand, we can prove that

$$i^* : \pi^1(X) \to \pi^1(X^{(1)}),$$

induced by the inclusion $i : X^{(1)} \to X$, is an injective map as below. Once this is done, the claim follows from the result in group theory that any subgroup of a free abelian group is still a free abelian group.

Let us prove the injectivity of i^*. Suppose that $f,\ g :\ X \to S^1$ are two maps satisfying that

$$i^*([f]) = i^*([g]),$$

where $[f], [g] \in \pi^1(X)$ are elements represented by f and g, respectively. Then $f|_{X^{(1)}}$ is homotopic to $g|_{X^{(1)}}$. Let $F :\ X^{(1)} \times [0,1] \to S^1$ be a homotopy path connecting $f|_{X^{(1)}}$ and $g|_{X^{(1)}}$. That is

$$F|_{X^{(1)}\times\{0\}} = f|_{X^{(1)}} \quad \text{and} \quad F|_{X^{(1)}\times\{1\}} = g|_{X^{(1)}}.$$

We are going to extend the homotopy F to a homotopy on the entire space $X \times [0,1]$. The construction is done by induction. Suppose that F has been extended to a homotopy (let us still denote it by F) $F :\ X^{(n)} \times [0,1] \to S^1$ between $f|_{X^{(n)}}$ and $g|_{X^{(n)}}$ on the n-skeleton (where $n \geq 1$) of X. I.e., $F|_{X^{(n)}\times\{0\}} = f|_{X^{(n)}}$ and $F|_{X^{(n)}\times\{1\}} = g|_{X^{(n)}}$. We need to prove that it can be extended to a homotopy on the $(n+1)$-skeleton. Let Δ be any $(n+1)$-simplex. Then $\partial\Delta \subset X^{(n)}$. Let $G : \partial\Delta \times [0,1] \cup \Delta \times \{0\} \cup \Delta \times \{1\} \to S^1$ be defined by

$$G(x) = \begin{cases} F(x) & \text{if } x \in \partial\Delta \times [0,1] \\ f(x) & \text{if } x \in \Delta \times \{0\} \\ g(x) & \text{if } x \in \Delta \times \{1\}\ . \end{cases}$$

Then $G(x)$ is a continuous map from $\partial(\Delta \times [0,1])$ to S^1. Since $\pi_{n+1}(S^1) = 0$ and $\partial(\Delta \times [0,1]) = S^{n+1}$, G can be extended to a map $G : \Delta \times [0,1] \to S^1$. Define F on each simplex Δ to be this G. Then F is the desired extension. This ends the proof of the claim.

Let us go back to the construction of $\tilde{\tau}$ on $\prod_{n=1}^{+\infty} \pi^1(X_n)$. Let $x_1^0 \in X_1,\ x_2^0 \in X_2,\ \cdots, x_n^0 \in X_n,\ \cdots$, be chosen as the base points of the spaces. Let

$$\theta_{n,1}, \theta_{n,2}, \cdots,\ \theta_{n,t_n} :\ X_n \longrightarrow S^1$$

be the functions representing the generators

$$[\theta_{n,1}],\ [\theta_{n,2}],\ \cdots,\ [\theta_{n,t_n}]\ \in\ \pi^1(X_n).$$

Suppose that

$$\theta_{n,j}(x_n^0) = 1 \in S^1 \subset \mathbb{C}, \qquad j = 1, 2, \cdots, t_n.$$

For any element $(x_1, x_2, \cdots, x_n, \cdots) \in \prod_{n=1}^{+\infty} \pi^1(X_n)$, define $\tilde{\tau}(x)$ as below. Let $u_n \in B_n$ be defined by

$$u_n(y) = \begin{pmatrix} \theta_{n,1}(y)^{m_1} \cdot \theta_{n,2}(y)^{m_2} \cdots \theta_{n,t_n}(y)^{m_{t_n}} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{k_n \times k_n} \in M_{k_n}(\mathbb{C})$$

for each $y \in X_n$, where $m_1, m_2, \cdots, m_{t_n}$ are integers with

$$(*) \qquad x_n = m_1[\theta_{n,1}] + m_2[\theta_{n,2}] + \cdots + m_{t_n}[\theta_{n,t_n}] \in \pi^1(X_n).$$

Define

$$\tilde{\tau}(x) = [(u_1, u_2, \cdots, u_n, \cdots)] \in K_1\left(\prod_{n=1}^{+\infty} B_n\right).$$

Since each $\pi^1(X_n)$ is a free abelian group, the expression $(*)$ for x_n is unique. It is easy to check that $\tilde{\tau}$ on $\prod_{n=1}^{+\infty} \pi^1(X_n)$ is a well defined group homomorphism. It is straight forward to check that

$$\tau \circ \tilde{\tau} = \mathrm{id} : \prod_{n=1}^{+\infty} \pi^1(X_n) \longrightarrow \prod_{n=1}^{+\infty} \pi^1(X_n)$$

And that

$$\tau \circ \tilde{\tau} = \mathrm{id} : \prod_{n=1}^{+\infty} SK_1(B_n) \longrightarrow \prod_{n=1}^{+\infty} SK_1(B_n).$$

That is,

$$\tau \circ \tilde{\tau} = \mathrm{id} : \prod_{n=1}^{+\infty} K_1(B_n) \longrightarrow \prod_{n=1}^{+\infty} K_1(B_n).$$

The splitting $\tilde{\tau} : \prod_{n=1}^{+\infty} K_1(B_n) \longrightarrow K_1\left(\prod_{n=1}^{+\infty} B_n\right)$ of the exact sequence

$$0 \longrightarrow Ker(\tau) \longrightarrow K_1\left(\prod_{n=1}^{+\infty} B_n\right) \xrightarrow{\tau} \prod_{n=1}^{+\infty} K_1 B_n \longrightarrow 0$$

gives an isomorphism

$$K_1\left(\prod_{n=1}^{+\infty} B_n\right) = \prod_{n=1}^{+\infty} K_1 B_n \oplus Ker(\tau).$$

5.8. In order to identify $Ker(\tau)$, suppose that

$$u = [(u_1, u_2, \cdots, u_n, \cdots)] \in K_1\left(\prod_{n=1}^{+\infty} B_n\right)$$

satisfies that

$$\tau(u) = 0 \in \prod_{n=1}^{+\infty} K_1 B_n.$$

Note that any unitary matrix $v \in M_\bullet(\mathbb{C})$ can be connected to $\mathbf{1} \in M_\bullet(\mathbb{C})$, by a path $v(t)$ satisfying that, if $|t - t'| < \varepsilon$, then

$$\|v(t) - v(t')\| < 2\pi\varepsilon.$$

Based on this fact, we have

$$[(u_1, u_2, \cdots)] = [(u_1^*(x_1^0)u_1, u_2^*(x_2^0)u_2, \cdots)] \in K_1\left(\prod_{n=1}^{+\infty} B_n\right).$$

Therefore, without loss of generality, we assume that

$$u_n(x_n^0) = \mathbf{1} \in M_L(B_n),$$

where $x_n^0 \in X_n$ are the base points.
Since $\tau(u) = 0$, if we assume $L \geq M$, then each u_n can be connected to $\mathbf{1} \in M_L(B_n)$. This implies that the map

$$\text{determinant}(u_n):\ X_n \longrightarrow S^1$$

is homotopy trivial. Therefore, this map can be lifted to a unique map

$$\det(u_n):\ X_n \longrightarrow \mathbb{R}$$

such that $\det(u_n)(x_n^0) = 0 \in \mathbb{R}$ and

$$exp(2\pi i \det(u_n)) = \text{determinant}(u_n).$$

Let $\coprod X_n$ be the disjoint union of X_n and $\mathrm{Map}\left(\coprod X_n, \mathbb{R}\right)_0$ the set of all continuous maps $f : \coprod X_n \to \mathbb{R}$ with $f(x_n^0) = 0$ for all x_n^0. Let $\mathrm{Map}_b\left(\coprod X_n, \mathbb{R}\right)_0$ be the set of those maps with bounded images.
Define a map $d : Ker(\tau) \to \dfrac{\mathrm{Map}\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}{\mathrm{Map}_b\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}$ by

$$d(u) = [(\frac{\det(u_1)}{k_1}, \frac{\det(u_2)}{k_2}, \cdots, \frac{\det(u_n)}{k_n}, \cdots)].$$

We will prove that d is a well defined isomorphism.
Suppose that u can be represented by another unitary

$$(v_1, v_2, \cdots, v_n, \cdots) \in M_L\left(\prod_{n=1}^{+\infty} B_n\right)$$

with $v(x_n^0) = \mathbf{1} \in M_L(B_n)$. Then for the unit of a certain matrix algebra over $\prod_{n=1}^{+\infty} B_n$

$$\mathbf{1}_{L_1} \in M_{L_1}\left(\prod_{n=1}^{+\infty} B_n\right),$$

we have that, the element

$$(u_1 \oplus \mathbf{1}, u_2 \oplus \mathbf{1}, \cdots, u_n \oplus \mathbf{1}, \cdots) \in M_{L+L_1}\left(\prod_{n=1}^{+\infty} B_n\right)$$

can be connected to the element

$$(v_1 \oplus \mathbf{1}, v_2 \oplus \mathbf{1}, \cdots, v_n \oplus \mathbf{1}, \cdots) \in M_{L+L_1}\left(\prod_{n=1}^{+\infty} B_n\right)$$

by a unitary path

$$(u_1(t), u_2(t), \cdots, u_n(t), \cdots) \in \left(M_{L+L_1}\left(\prod_{n=1}^{+\infty} B_n\right)\right) \otimes C[0,1].$$

We need to prove that

$$\left(\frac{\det(u_1) - \det(v_1)}{k_1}, \frac{\det(u_2) - \det(v_2)}{k_2}, \cdots, \frac{\det(u_n) - \det(v_n)}{k_n}, \cdots\right)$$

has a uniformly bounded image in $\mathbb{R}$. This follows from the following fact. If two unitaries $w_1, w_2 \in M_{(L+L_1)k_n}(\mathbb{C})$ satisfying $|w_1 - w_2| < \varepsilon < \frac{1}{4}$, then

$$|\text{determinant}(w_1^* w_2) - 1| < \pi(L + L_1)k_n\varepsilon.$$

Now, we have to prove that d is an isomorphism.
Obviously, d is surjective. In fact, for any function

$$(f_1, f_2, \cdots, f_n, \cdots) \in \text{Map}\left(\prod_{n=1}^{+\infty} \mathbb{R}\right)_0,$$

let $u \in K_1\left(\prod_{n=1}^{+\infty} B_n\right)$ be the element represented by

$$(exp(2\pi i f_1), exp(2\pi i f_2), \cdots, exp(2\pi i f_n), \cdots) \in \prod_{n=1}^{+\infty} B_n = \prod_{n=1}^{+\infty} M_{k_n}(C(X_n)).$$

Then $d(u) = [(f_1, f_2, \cdots, f_n, \cdots)]$.
Finally, we have to prove that d is injective. Suppose that $u \in Ker(\tau)$ is represented by

$$(u_1, u_2, \cdots, u_n, \cdots) \in M_L\left(\prod_{n=1}^{+\infty} B_n\right)$$

satisfying

$$\left(\frac{\det(u_1)}{k_1}, \frac{\det(u_2)}{k_2}, \cdots, \frac{\det(u_n)}{k_n}, \cdots\right) \in \mathrm{Map}_b\left(\coprod X_n, \mathbb{R}\right)_0.$$

Let $f_n = \frac{\det(u_n)}{k_n} : X_n \to \mathbb{R}$ and

$$v_n = exp\frac{2\pi i f_n}{L} \in M_{Lk_n}(C(X_n)).$$

Then $v_n^* u_n \in SU_{Lk_n}(X_n)$, i.e., it has determinant 1 every where. Since $(f_1, f_2, \cdots, f_n, \cdots)$ is of uniformly bounded image, we know that

$$(u_1, u_2, \cdots, u_n, \cdots) \qquad \text{and} \qquad (v_1^* u_1, v_2^* u_2, \cdots v_n^* u_n, \cdots)$$

can be connected by a continuous path in

$$\left(M_L\left(\prod_{n=1}^{+\infty} B_n\right)\right) \otimes C[0,1].$$

Therefore, $u = [(v_1^* u_1, v_2^* u_2, \cdots v_n^* u_n, \cdots)]$. The latter is zero by Lemma 5.3 and the fact that $u \in Ker(\tau)$.
Summarizing the above, we obtain

Lemma 5.9. $K_1\left(\prod_{n=1}^{+\infty} B_n\right) = \prod_{n=1}^{+\infty} K_1 B_n \oplus \frac{Map(\coprod_{n=1}^{+\infty} X_n, I\!R)_0}{Map_b(\coprod_{n=1}^{+\infty} X_n, I\!R)_0}$

$$= \prod_{n=1}^{+\infty} SK_1 B_n \oplus \prod_{n=1}^{+\infty} \pi^1(X_n) \oplus \frac{Map\left(\coprod_{n=1}^{+\infty} X_n, I\!R\right)_0}{Map_b\left(\coprod_{n=1}^{+\infty} X_n, I\!R\right)_0}.$$

Corollary 5.10.

$$K_1\left(\prod_{n=1}^{+\infty} B_n \Big/ \bigoplus_{n=1}^{+\infty} B_n\right) = \left(\prod_{n=1}^{+\infty} K_1 B_n \Big/ \bigoplus_{n=1}^{+\infty} K_1 B_n\right) \oplus \frac{\mathrm{Map}\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}{\mathrm{Map}_b\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}.$$

5.11. From [Sch], for any C^*-algebra A in the bootstrap class and any C^*-algebra B (not necessarily separable), there is a splitting short exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow Tor(K_*(A), K_*(B)) \longrightarrow 0.$$

Let $A = C_0(W_k)$, where $W_k = T_{II,k}$ as in the introduction. W_k is used for $T_{II,k}$ only when involving mod k K-theory $K_*(B, \mathbb{Z}/k)$. From the definition

$$K_*(B, \mathbb{Z}/k) := K_*(A \otimes B),$$

one has

$$0 \longrightarrow K_0(B)\otimes \mathbb{Z}/k \longrightarrow K_0(B,\mathbb{Z}/k) \longrightarrow Tor(\mathbb{Z}/k, K_1(B)) \longrightarrow 0$$

and

$$0 \longrightarrow K_1(B)\otimes \mathbb{Z}/k \longrightarrow K_1(B,\mathbb{Z}/k) \longrightarrow Tor(\mathbb{Z}/k, K_0(B)) \longrightarrow 0.$$

Since $G\otimes \mathbb{Z}/k$ can be identified with the cokernel of

$$G \xrightarrow{\times k} G,$$

and $Tor(\mathbb{Z}/k, G)$ can be identified with the kernel of

$$G \xrightarrow{\times k} G,$$

one has the following well known exact sequences

$$K_0(B) \xrightarrow{\times k} K_0(B) \longrightarrow K_0(B,\mathbb{Z}/k) \longrightarrow K_1(B) \xrightarrow{\times k} K_1(B)$$

and

$$K_1(B) \xrightarrow{\times k} K_1(B) \longrightarrow K_1(B,\mathbb{Z}/k) \longrightarrow K_0(B) \xrightarrow{\times k} K_0(B).$$

5.12. Let $\{X_n\}_{n=1}^{+\infty}$, $B_n = M_{k_n}(C(X_n))$ be as above, and $B = \bigoplus_{n=1}^{+\infty} B_n$, $M(B) = \prod_{n=1}^{+\infty} B_n$, $Q(B) = M(B)/B$.
From Lemma 5.5 and Corollary 5.10, we have

$$K_0(Q(B)) = \Pi_b K_0(B_n)/\oplus K_0(B_n) \qquad \text{and}$$

$$K_1(Q(B)) = \left\{\frac{\prod_{n=1}^{+\infty} K_1(B_n)}{\bigoplus_{n=1}^{+\infty} K_1(B_n)}\right\} \bigoplus \left\{\frac{\mathrm{Map}\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}{\mathrm{Map}_b\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}\right\}.$$

It is easy to see that the map

$$\left\{\frac{\mathrm{Map}\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}{\mathrm{Map}_b\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}\right\} \xrightarrow{\times k} \left\{\frac{\mathrm{Map}\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}{\mathrm{Map}_b\left(\coprod_{n=1}^{+\infty} X_n, \mathbb{R}\right)_0}\right\}$$

is an isomorphism.
Any torsion element $x_n \in K_0(B_n)$ can be realized as a formal difference of two projections $p, q \in M_\infty(B_n)$ of the same rank. (The rank of a projection makes sense since X_n are connected. Also for any element $x \in K_0(B_n)$ represented by $[p]-[q]$, we define $\mathrm{rank}(x) = \mathrm{rank}(p) - \mathrm{rank}(q)$, which is always a (possibly negative) integer.) By [Hu], if a projection $p \in M_\bullet(C(X_n))$ has rank larger than $\dim(X_n)+r$, then p has a trivial sub projection of rank r. Therefore, any torsion element $x_n \in K_0(B_n)$ can be realized as a formal difference of two

projections $p, q \in M_L(B_n)$ if $L > \dim(X_n)$. Based on this fact, one can directly compute that

$$\mathrm{Kernel}\left(\Pi_b K_0(B_n) \xrightarrow{\times k} \Pi_b K_0(B_n)\right) = \mathrm{Kernel}\left(\prod_{n=1}^{+\infty} K_0(B_n) \xrightarrow{\times k} \prod_{n=1}^{+\infty} K_0(B_n)\right).$$

Fixed a positive integer k. let $x = (x_1, x_2, \cdots, x_n, \cdots) \in \prod_{n=1}^{+\infty} K_0(B_n)$. For each element $x_n \in K_0(B_n)$, one can write $\mathrm{rank}(x_n) = k \cdot M \cdot l_n + r_n$, where l_n is a (possibly negative) integer, M is the maximum of $\{\dim(X_n)\}_n$, and $0 < k \cdot M \le r_n < 2k \cdot M$. Let p_n be the trivial rank one projection in B_n. Then x_n can be written as $k \cdot M \cdot l_n[p_n] + [q_n]$, where q_n is a projection of rank r_n. Therefore, x can can be written as $x = x' + x''$, where $x' \in k(\prod_{n=1}^{+\infty} K_0(B_n))$ and $x'' \in \Pi_b K_0(B_n)$. As a consequence, one can compute that

$$\begin{aligned}&\mathrm{Cokernel}\left(\Pi_b K_0(B_n) \xrightarrow{\times k} \Pi_b K_0(B_n)\right) \\ &\qquad = \mathrm{Cokernel}\left(\prod_{n=1}^{+\infty} K_0(B_n) \xrightarrow{\times k} \prod_{n=1}^{+\infty} K_0(B_n)\right).\end{aligned}$$

Combined with 5.11, yields

$$K_0(Q(B), \mathbb{Z}/k) = \prod_{n=1}^{+\infty} K_0(B_n, \mathbb{Z}/k) \Big/ \bigoplus_{n=1}^{+\infty} K_0(B_n, \mathbb{Z}/k),$$

and

$$K_1(Q(B), \mathbb{Z}/k) = \prod_{n=1}^{+\infty} K_1(B_n, \mathbb{Z}/k) \Big/ \bigoplus_{n=1}^{+\infty} K_1(B_n, \mathbb{Z}/k).$$

5.13. Following [DG], denote

$$\underline{K}(A) = K_*(A) \ \oplus \ \bigoplus_{n=2}^{+\infty} K_*(A, \mathbb{Z}/n).$$

For any finite CW complex X and two KK-elements $\alpha, \beta \in KK(C(X), A)$, from [DL] (also see [DG]), we know that $\alpha = \beta$ if and only if

$$\alpha_* = \beta_* : \ \underline{K}(C(X)) \longrightarrow \underline{K}(A).$$

We will discuss the special cases of $X = \{pt\}, [0,1], T_{II,k}, T_{III,k}$ and S^2, where $T_{II,k}, T_{III,k}$ are defined in the Introduction. (See §4 of [EG2] for details.) (The case $X = \{pt\}$ or $[0,1]$ is similar to the case $X = S^2$, so we will not discuss the spaces $\{pt\}$ and $[0,1]$ separately.)
From [DL], there is an isomorphism

$$KK(C(X), B) \longrightarrow \mathrm{Hom}_\Lambda(\underline{K}(C(X)), \underline{K}(B)),$$

where $\mathrm{Hom}_\Lambda(\underline{K}(C(X)), \underline{K}(B))$ is the set of systems of group homomorphisms which is compatible with all the Bockstein Operations (see [DL] for details). For any fixed finite CW complex X, an element $\alpha \in KK(C(X), B)$ is determined by the system of maps

$$\alpha_n^* : \ K_*(C(X), \mathbb{Z}/n) \longrightarrow K_*(B, \mathbb{Z}/n), \ n = 0, 2, 3, \cdots$$

which are induced by α. In fact α would be determined by a few maps from the above list—all the other maps in the system $\{\alpha_n^*\}_{n=0}^{+\infty} : \underline{K}(C(X)) \to \underline{K}(B)$ would be completely determined by these few maps via the Bockstein Operations. We will choose those few maps for the cases $X = \{pt\}$, $[0, 1]$, S^2, $T_{II,k}$, or $T_{III,k}$.

1. $X = S^2$. Then

$$\underline{K}(C(S^2)) \longrightarrow \underline{K}(B)$$

is completely determined by

$$K_0(C(S^2)) \longrightarrow K_0(B)$$

via the Bockstein Operation

$$\begin{array}{ccc} K_0(C(S^2)) & \longrightarrow & K_0(C(S^2), \mathbb{Z}/k) \\ \downarrow & & \downarrow \\ K_0(B) & \longrightarrow & K_0(B, \mathbb{Z}/k) \end{array}$$

since the top horizontal map is surjective. (Note that $K_1(C(S^2))$ and $K_1(C(S^2), \mathbb{Z}/k)$ are trivial groups.) Therefore,

$$KK(C(S^2), B) \cong \mathrm{Hom}(K_0(C(S^2)), K_0(B)).$$

(This is also a well known consequence of the Universal Coefficient Theorem.) (The case $X = \{pt\}$ or $[0, 1]$ is similar to the above case.)

2. $X = T_{II,k}$. Let $rC(T_{II,k}) \cong \mathbb{C}$ and let $C_0(T_{II,k})$ be the ideal of $C(T_{II,k})$ consisting of the continuous functions vanishing at the base point. (See 1.6 of [EG2] and 1.1.7 for the notations.) Consider the splitting exact sequence

$$0 \longrightarrow K_0(C_0(T_{II,k})) \longrightarrow K_0(C(T_{II,k})) \longrightarrow K_0(rC(T_{II,k})) \longrightarrow 0.$$

Each KK-element $\alpha \in KK(C(T_{II,k}), B)$ induces two group homomorphisms

$$\alpha_0^0 : \ K_0(rC(T_{II,k})) \ (= \mathbb{Z}) \longrightarrow K_0(B) \qquad \text{and}$$

$$\alpha_k^1 : \ K_1(C(T_{II,k}), \mathbb{Z}/k) \ (= \mathbb{Z}/k) \longrightarrow K_1(B, \mathbb{Z}/k).$$

This induces a map

$$\begin{aligned} KK(C(T_{II,k}), B) &\longrightarrow \mathrm{Hom}(K_0(rC(T_{II,k})), K_0(B)) \bigoplus \mathrm{Hom}(K_1(C(T_{II,k}), \mathbb{Z}/k), K_1(B, \mathbb{Z}/k)) \\ &= \mathrm{Hom}(\mathbb{Z}, K_0(B)) \bigoplus \mathrm{Hom}(\mathbb{Z}/k, K_1(B, \mathbb{Z}/k)). \end{aligned}$$

It can be verified that any two homomorphisms

$$\alpha_0^0: \; K_0(rC(T_{II,k}))\ (=\mathbb{Z}) \longrightarrow K_0(B) \qquad \text{and}$$

$$\alpha_k^1: \; K_1(C(T_{II,k}),\mathbb{Z}/k)\ (=\mathbb{Z}/k) \longrightarrow K_1(B,\mathbb{Z}/k)$$

induces a unique system of homomorphisms in $\mathrm{Hom}_\Lambda(\underline{K}(C(T_{II,k})),\underline{K}(B))$. Therefore, the above map is an isomorphism.
Another way to see it, is as follows. Note that

$$K_1(C(T_{II,k}),\mathbb{Z}/k) = K_0(C_0(T_{II,k})) \subset K_0(C(T_{II,k})).$$

Considering

$$K_1(B) \xrightarrow{\times k} K_1(B) \longrightarrow K_1(B,\mathbb{Z}/k) \longrightarrow K_0(B) \xrightarrow{\times k} K_0(B),$$

we obtain

$$\begin{aligned}&\mathrm{Hom}(K_1(C(T_{II,k}),\mathbb{Z}/k),K_1(B,\mathbb{Z}/k))\\ &\qquad\cong \mathrm{Hom}(K_0(C_0(T_{II,k})),K_0(B)) \oplus \mathrm{Ext}(K_0(C_0(T_{II,k})),K_1(B)).\end{aligned}$$

Then from the Universal Coefficient Theorem,

$$\begin{aligned}&KK(C(T_{II,k}),B)\\ &\cong \mathrm{Hom}(K_0(C(T_{II,k})),K_0(B)) \oplus \mathrm{Ext}(K_0(C(T_{II,k})),K_1(B))\\ &\cong \mathrm{Hom}(K_0(rC(T_{II,k})),K_0(B)) \oplus \mathrm{Hom}(K_0(C_0(T_{II,k})),K_0(B))\\ &\qquad\qquad \oplus \mathrm{Ext}(K_0(C_0(T_{II,k})),K_1(B)).\end{aligned}$$

(Note that $K_1(C(T_{II,k}))=0$.) Hence one can see again, the map mentioned above is an isomorphism.
3. $X = T_{III,k}$. Also, let $rC(T_{III,k}) = \mathbb{C}$ and let $C_0(T_{III,k})$ be the ideal consisting of functions vanishing at the base point. Notice that

$$K_0(C(T_{III,k})) = \mathbb{Z} \qquad \text{and} \qquad K_0(C_0(T_{III,k}),\mathbb{Z}/k) = \mathbb{Z}/k.$$

By the splitting exact sequence

$$0 \to K_0(C_0(T_{III,k}),\mathbb{Z}/k) \to K_0(C(T_{III,k}),\mathbb{Z}/k) \to K_0(rC(T_{III,k}),\mathbb{Z}/k) \to 0$$

we know that each $\alpha \in KK(C(T_{III,k}),B)$ induces an element

$$\alpha_k^0: \; K_0(C_0(T_{III,k}),\mathbb{Z}/k) \longrightarrow K_0(B,\mathbb{Z}/k).$$

It can be proved that

$$\begin{aligned}&KK(C(T_{III,k}),B)\\ &\quad\cong \mathrm{Hom}(K_0(C(T_{III,k})),K_0(B)) \bigoplus \mathrm{Hom}(K_0(C_0(T_{III,k}),\mathbb{Z}/k),K_0(B,\mathbb{Z}/k))\\ &\quad= \mathrm{Hom}(\mathbb{Z},K_0(B)) \bigoplus \mathrm{Hom}(\mathbb{Z}/k,K_0(B,\mathbb{Z}/k)),\end{aligned}$$

as what we did for $T_{II,k}$.
(Notice that the map $K_1(C(T_{III,k})) \to K_1(B)$ is completely determined by the map $K_0(C_0(T_{III,k}), \mathbb{Z}/k) \to K_0(B, \mathbb{Z}/k)$.)
Summarizing the above, we have the following.
For any two elements $\alpha, \beta \in KK(C(X), B)$, $\alpha = \beta$ if and only if
(1) $\alpha_0^0 = \beta_0^0 : K_0(C(X)) \longrightarrow K_0(B)$, when $X = S^2$;
(2) $\alpha_0^0 = \beta_0^0 : K_0(rC(X)) \longrightarrow K_0(B)$ and $\alpha_k^1 = \beta_k^1 : K_1(C(X), \mathbb{Z}/k) \longrightarrow K_1(B, \mathbb{Z}/k)$, when $X = T_{II,k}$;
(3) $\alpha_0^0 = \beta_0^0 : K_0(C(X)) \longrightarrow K_0(B)$ and $\alpha_k^0 = \beta_k^0 : K_0(C_0(X), \mathbb{Z}/k) \longrightarrow K_0(B, \mathbb{Z}/k)$, when $X = T_{III,k}$.
Therefore, we have the following lemma.

Lemma 5.14. *Let $A = PM_l(C(X))P$, and X one of $\{pt\}, [0,1], T_{II,k}, T_{III,k}$, or S^2. Let $\alpha, \beta \in KK(A, B)$, where B is a C^*-algebra. Then $\alpha = \beta$ if and only if the following hold:*
1. When $X = \{pt\}$, $[0,1]$ or S^2,

$$\alpha_* = \beta_* : K_0(A) \longrightarrow K_0(B);$$

2. When $X = T_{II,k}$,

$$\alpha_* = \beta_* : K_0(A) \longrightarrow K_0(B) \qquad and$$

$$\alpha_* = \beta_* : K_1(A, \mathbb{Z}/k) \longrightarrow K_1(B, \mathbb{Z}/k);$$

3. When $X = T_{III,k}$,

$$\alpha_* = \beta_* : K_0(A) \longrightarrow K_0(B) \qquad and$$

$$\alpha_* = \beta_* : K_0(A, \mathbb{Z}/k) \longrightarrow K_0(B, \mathbb{Z}/k).$$

Combined with Theorem 6.1 of [DG], yields the following lemma.

Lemma 5.15. *Let $A = PM_l(C(X))P$, and X, one of $\{pt\}, [0,1], T_{II,k}, T_{III,k}$ or S^2, and let B be any C^*-algebra. Let $\phi, \psi \in Hom(A, B)$. Suppose that the following statements hold.*
1. When $X = \{pt\}, [0,1]$ or S^2,

$$[\phi]_* = [\psi]_* : K_0(A) \longrightarrow K_0(B);$$

2. When $X = T_{II,k}$,

$$[\phi]_* = [\psi]_* : K_0(A) \longrightarrow K_0(B) \qquad and$$

$$[\phi]_* = [\psi]_* : K_1(A, \mathbb{Z}/k) \longrightarrow K_1(B, \mathbb{Z}/k);$$

3. When $X = T_{III,k}$,

$$[\phi]_* = [\psi]_* : K_0(A) \longrightarrow K_0(B) \qquad and$$

$$[\phi]_* = [\psi]_* : \; K_0(A, \mathbb{Z}/k) \longrightarrow K_0(B, \mathbb{Z}/k).$$

It follows that, for any finite set $F \subset A$ and any number $\varepsilon > 0$, there exist $n \in \mathbb{N}$, $\mu \in Hom(A, M_n(B))$ with finite dimensional image and a unitary $u \in M_{n+1}(B)$ such that

$$\|u(\phi(a) \oplus \mu(a))u^* - \psi(a) \oplus \mu(a)\| < \varepsilon$$

for all $a \in F$.

5.16. Fix $A = PM_l(C(X))P$, $X = \{pt\}, [0,1], T_{II,k}, T_{III,k}$ or S^2. Then A is stably isomorphic to $C(X)$. By 5.14, an element $\alpha \in KK(A, B)$ is completely determined by

$$\alpha_0^0 : \; K_0(A) \to K_0(B),$$
$$\alpha_k^0 : \; K_0(A, \mathbb{Z}/k) \to K_0(B, \mathbb{Z}/k), \qquad \text{and}$$
$$\alpha_k^1 : \; K_1(A, \mathbb{Z}/k) \to K_1(B, \mathbb{Z}/k).$$

Note that, for any C^*-algebra A,

$$K_0(A \otimes C(W_k \times S^1)) \cong K_0(A) \oplus K_1(A) \oplus K_0(A, \mathbb{Z}/k) \oplus K_1(A, \mathbb{Z}/k).$$

Each projection $p \in M_\infty(A \otimes C(W_k \times S^1))$ defines an element

$$[p] \in K_0(A) \oplus K_1(A) \oplus K_0(A, \mathbb{Z}/k) \oplus K_1(A, \mathbb{Z}/k) \subset \underline{K}(A).$$

This defines a map from the set of projections in $\bigcup_{k=2}^\infty M_\infty(A \otimes C(W_k \times S^1))$ to $\underline{K}(A)$.
For any finite set $\mathcal{P} \subset \bigcup_{k=2}^\infty M_\infty(A \otimes C(W_k \times S^1))$ of projections, denoted by $\mathcal{P}\underline{K}(A)$ the finite subset of $\underline{K}(A)$ consisting of elements coming from the projections $p \in \mathcal{P}$, that is

$$\mathcal{P}\underline{K}(A) = \{[p] \in \underline{K}(A) |\; p \in \mathcal{P}\}.$$

In particular, if $A = PM_l(C(X))P, X = T_{II,k}$, or $T_{III,k}$, then we can choose a finite set of projections $\mathcal{P}_A \subset M_\bullet(A \otimes C(W_k \times S^1))$ such that the set $\{[p] \in K_0(A) \oplus K_1(A) \oplus K_0(A, \mathbb{Z}/k) \oplus K_1(A, \mathbb{Z}/k) \mid p \in \mathcal{P}_A\} = \mathcal{P}_A\underline{K}(A)$ generates $K_0(A) \oplus K_1(A) \oplus K_0(A, \mathbb{Z}/k) \oplus K_1(A, \mathbb{Z}/k) \subset \underline{K}(A)$. For $X = \{pt\}, [0,1]$ or S^2, choose $\mathcal{P}_A \subset M_\bullet(A)$ such that $\{[p] \in K_0(A) \mid p \in \mathcal{P}_A\}$ generates $K_0(A) \subset \underline{K}(A)$. We will use $\mathcal{P}$ to denote $\mathcal{P}_A$ if there is no danger of confusion.

5.17. Let $A = PM_l(C(X))P, X = \{pt\}, [0,1], T_{II,k}, T_{III,k}$ or S^2, and $\mathcal{P} \subset M_\bullet(A \otimes C(W_k \times S^1))$ or $\mathcal{P} \subset M_\bullet(A)$ be as in 5.16. There are a finite subset $G(\mathcal{P}) \subset A$ and a number $\delta(\mathcal{P}) > 0$ such that if B is any C^*-algebra and $\phi \in \mathrm{Map}(A, B)$ is $G(\mathcal{P}) - \delta(\mathcal{P})$ multiplicative, then

$$\|((\phi \otimes \mathrm{id})(p))^2 - (\phi \otimes \mathrm{id})(p)\| < \frac{1}{4}, \;\; \forall p \in \mathcal{P},$$

where id is the identity map on $M_\bullet(C(W_k\times S^1))$ or on $M_\bullet(\mathbb{C})$. Hence for any $p\in\mathcal{P}$, there is a projection $q\in M_\bullet(B\otimes C(W_k\times S^1))$ (or $q\in M_\bullet(B)$) such that

$$\|(\phi\otimes\mathrm{id})(p)-q\|<\frac{1}{2}.$$

So q defines an element in $\underline{K}(B)$. (If q' is another projection satisfying the same condition, then $\|q-q'\|<1$, hence q' is unitarily equivalent to q.)
Therefore, if ϕ is $G(\mathcal{P})-\delta(\mathcal{P})$ multiplicative, then it induces a map

$$\phi_*:\mathcal{P}\underline{K}(A)\to\underline{K}(B).$$

Note that such $G(\mathcal{P})$ and $\delta(\mathcal{P})$ could be defined for any finite set $\mathcal{P}\subset M_\infty(A)\cup M_\infty(A\otimes C(S^1))\cup\bigcup_{k=2}^\infty M_\infty(A\otimes C(W_k\times S^1))$ of projections.

Theorem 5.18. *Let X be one of the spaces $\{pt\},[0,1],T_{II,k},T_{III,k}$ or S^2. Let $A=PM_l(C(X))P$ and $\mathcal{P}$ be as in 5.16. For any finite set $F\subset A$, any positive number $\varepsilon>0$, and any positive integer M, there are a finite set $G\subset A$ ($G\supset G(\mathcal{P})$ large enough), a positive number $\delta>0$ ($\delta<\delta(\mathcal{P})$ small enough), and a positive integer L (large enough) such that the following statement is true.*
If $\phi,\psi\in Map(A,B)$ are G-δ multiplicative and

$$\phi_*=\psi_*:\ \mathcal{P}\underline{K}(A)\longrightarrow\underline{K}(B),$$

where $B=QM_\bullet(C(Y))Q$ with $\dim(Y)\le M$, then there is a homomorphism $\nu\in Hom(A,M_L(B))$, with finite dimensional image, and there is a unitary $u\in M_{L+1}(B)$ such that

$$\|u(\phi\oplus\nu)(a)u^*-(\psi\oplus\nu)(a)\|<\varepsilon,\quad\forall f\in F.$$

Proof: We first prove the theorem for the case $B=M_\bullet(C(Y))$. Then we apply Lemma 1.3.6 to reduce the general case to this special case.
We prove the theorem by contradiction.
Let $G(\mathcal{P})\subset G_1\subset G_2\subset\cdots\subset G_n\subset\cdots$ be a sequence of finite subsets with

$$\overline{\cup G_n}=\text{unit ball of }A.$$

Let $\delta(\mathcal{P})>\delta_1>\delta_2>\cdots>\delta_n>\cdots$ be a sequence of positive numbers with $\delta_n\to 0$. Let $L_1<L_2<\cdots<L_n<\cdots$ be a sequence of positive integers with $L_n\to+\infty$.
Suppose that the theorem does not hold for (G_n,δ_n,L_n). That is, there exist a C^*-algebra $B_n=M_{k_n}(C(Y_n))$ and two G_n-δ_n multiplicative maps

$$\phi_n,\psi_n:\ A\longrightarrow B_n$$

with $(\phi_n)_* = (\psi_n)_* : \mathcal{P}\underline{K}(A) \longrightarrow \underline{K}(B_n)$ and

$$\inf_{\nu,u} \sup_{a \in F} \|u(\phi_n \oplus \nu)(a)u^* - (\psi_n \oplus \nu)(a)\| \geq \varepsilon, \tag{$*$}$$

where ν runs over all subsets of $\mathrm{Hom}(A, M_{L_n}(B_n))$ consisting of those homomorphisms with finite dimensional images, and u runs over $U(M_{L_n+1}(B_n))$. The above $\{\phi_n\}_{n=1}^{+\infty}, \{\psi_n\}_{n=1}^{+\infty}$ induce two homomorphisms

$$\tilde{\phi}, \tilde{\psi} : \ A \longrightarrow \prod_{n=1}^{+\infty} B_n \Big/ \bigoplus_{n=1}^{+\infty} B_n = Q(B).$$

We will prove that $KK(\phi) = KK(\psi)$.
1. $X = \{pt\}, [0,1]$ or S^2. By Lemma 5.14, $KK(\tilde{\phi})$ is completely determined by

$$([\tilde{\phi}]_*)_0^0 : \ K_0(A) \longrightarrow K_0(Q(B)).$$

From 5.12,

$$K_0(Q(B)) = \Pi_b K_0(B_n) \Big/ \bigoplus_{n=1}^{+\infty} K_0(B_n).$$

That is, the above $([\tilde{\phi}]_*)_0^0$ is completely determined by the component $([\phi_n]_*)_0^0$. From the condition that

$$[\phi_n]_* = [\psi_n]_* : \ \mathcal{P}\underline{K}(A) \longrightarrow \underline{K}(B_n)$$

and the condition that the group generated by $\mathcal{P}\underline{K}(A)$ is $K_0(A)$, we know that $KK(\phi) = KK(\psi)$.
2. $X = T_{II,k}$. By Lemma 5.14, $KK(\tilde{\phi})$ is completely determined by

$$([\tilde{\phi}]_*)_0^0 : \ K_0(A) \longrightarrow K_0(Q(B)) \qquad \text{and}$$

$$([\tilde{\phi}]_*)_k^1 : \ K_1(A, \mathbb{Z}/k) \longrightarrow K_1(Q(B), \mathbb{Z}/k).$$

Furthermore, by 5.12,

$$K_1(Q(B), \mathbb{Z}/k) = \prod_{n=1}^{+\infty} K_1(B_n, \mathbb{Z}/k) \Big/ \bigoplus_{n=1}^{+\infty} K_1(B_n, \mathbb{Z}/k).$$

Again, $([\tilde{\phi}]_*)_0^0$ and $([\tilde{\phi}]_*)_k^1$ are completely determined by the components corresponding to $[\phi_n]_*$. And from $[\phi_n]_* = [\psi_n]_*$ on $\mathcal{P}\underline{K}(A)$, we obtain $KK(\tilde{\phi}) = KK(\tilde{\psi})$. (Note that, we also use the fact that $\mathcal{P}\underline{K}(A)$ generates a subgroup of $\underline{K}(A)$ containing $K_0(A)$ and $K_1(A, \mathbb{Z}/k)$.) (The subgroup of $\underline{K}(A)$ generated by $\mathcal{P}\underline{K}(A)$ also contains $K_1(A)$, though we do not use this fact.)
3. $X = T_{III,k}$. It can be proved that $KK(\tilde{\phi}) = KK(\tilde{\psi})$ as above. Note that $K_0(B_n, \mathbb{Z}/k) = \prod_{n=1}^{+\infty} K_0(B_n, \mathbb{Z}/k) / \bigoplus_{n=1}^{+\infty} K_0(B_n, \mathbb{Z}/k)$, by 5.12.

By Lemma 5.15, there are a positive integer L and a homomorphism $\tilde{\nu}$: $A \to M_L(\prod_{n=1}^{+\infty} B_n/\bigoplus_{n=1}^{+\infty} B_n)$ with finite dimensional image, and a unitary $\tilde{u} \in M_{L+1}(\prod_{n=1}^{+\infty} B_n/\bigoplus_{n=1}^{+\infty} B_n)$, such that

$$\|\tilde{u}(\tilde{\phi} \oplus \tilde{\nu})(a)\tilde{u}^* - (\tilde{\psi} \oplus \tilde{\nu})(a)\| < \frac{\varepsilon}{2}$$

for all $a \in F$. Since $\tilde{\nu}$ has finite dimensional image, one can find a sequence of homomorphisms

$$\nu_n : \ A \longrightarrow M_L(B_n)$$

of finite dimensional images such that $\{\nu_n\}_{n=1}^{+\infty}$ induces $\tilde{\nu}$. One can also lift $\tilde{u}$ to a sequence of unitaries $u_n \in M_{L+1}(B_n)$. Then if n is large enough, we have

$$\|u_n(\phi_n \oplus \nu_n)(a)u_n^* - (\psi_n \oplus \nu_n)(a)\| < \varepsilon$$

for all $a \in F$. This contradicts with $(*)$ if one choose n to satisfy $L_n \geq L$.
Now we apply Lemma 1.3.6 to prove the general case. Let G, δ and L_1 (in place of L) be as above for the case of full matrix algebras over $C(Y)$ with $\dim(Y) \leq M$. Choose $L = (2M+2)L_1 - 1$. We will verify that G, δ and L satisfies the condition of the theorem even for $B = QM_\bullet(C(Y))Q$—cutting down of full matrix algebras by projections, as follows.
Let $n = \text{rank}(Q) + \dim(Y)$ and $m = 2M + 1$. Then by Lemma 1.3.6, $QM_\bullet(C(Y))Q$ can be identified as a corner subalgebra of $M_n(C(Y))$, and $M_n(C(Y))$ can be identified as corner subalgebra of $M_m(QM_\bullet(C(Y))Q)$.
If $\phi, \psi \in \text{Map}_{G-\delta}(A, B)$ satisfy the condition in the theorem, then regarding B as a corner subalgebra of $M_n(C(Y))$, we can regard ϕ, ψ as elements in $\text{Map}_{G-\delta}(A, M_n(C(Y)))$ which still satisfy the condition. Hence from the above special case of the theorem, there are $\nu : A \to M_{L_1}(M_n(C(Y)))$ and a unitary $u_1 \in M_{L_1+1}(M_n(C(Y)))$ such that

$$\|u_1(\phi \oplus \nu)(a)u_1^* - (\psi \oplus \nu)(a)\| < \varepsilon, \quad \forall a \in F.$$

Also, $M_n(C(Y))$ can be regarded as a corner subalgebra of $M_m(QM_\bullet(C(Y))Q)$, so $\phi \oplus \nu$ and $\psi \oplus \nu$ can be regarded as maps from A to $M_{L_1+1}(M_m(QM_\bullet(C(Y))Q)) = M_{L+1}(QM_\bullet(C(Y))Q)$. Therefore, there is a unitary $u \in M_{L+1}(B)$

$$\|u(\phi \oplus \nu)(a)u^* - (\psi \oplus \nu)(a)\| < \varepsilon, \quad \forall a \in F.$$

$\square$

Remark 5.19. The theorem is not true for $X = S^1$, even if we assume that both ϕ and ψ are homomorphisms. A counterexample is given below .
Let $\phi_n, \ \psi_n : \ C(S^1) \to C[0,1]$ be defined by

$$\phi_n(f)(t) = f(e^{2\pi int}) \qquad \text{and} \qquad \psi_n(f)(t) = f(\mathbf{1}).$$

Then $KK(\phi_n) = KK(\psi_n)$. Let $F = \{z\}$ and $\varepsilon = \frac{1}{4}$, where $z \in C(S^1)$ is a canonical generator. One can prove that there is no integer L which is good for all (ϕ_n, ψ_n) as in Theorem 5.18, by using the variation of determinant.

5.20. If X is any finite CW complex such that $K_1(C(X))$ is a torsion group, then Theorem 5.18 holds for X — one needs to choose $\mathcal{P}_A$ accordingly, which is described below.
Suppose that $m_1, m_2, \cdots, m_i$ are the degrees of all the torsion elements in $K_0(A)$ and $K_1(A)$. Let m be the least common multiple of $m_1, m_2, \cdots, m_i$. Since $K_1(A)$ is a torsion group, similar to the discussion in 5.13, an element $\alpha \in KK(A,B)$ is completely determined by the K-theory maps

$$\alpha^0 : K_0(A) \longrightarrow K_0(B),$$

$$\alpha^0_p : K_0(A, \mathbb{Z}/p) \longrightarrow K_0(B, \mathbb{Z}/p),$$

$$\alpha^1_p : K_1(A, \mathbb{Z}/p) \longrightarrow K_1(B, \mathbb{Z}/p),$$

where p are all the numbers with $p|m$. (In particular, the map $\alpha^1 : K_1(A) \longrightarrow K_1(B)$ is determined by the above maps.)
One can choose $\mathcal{P}$ to be a finite set of projections in $M_\bullet(A) \bigcup \bigcup_{p|m} M_\bullet(A \otimes C(W_p \otimes S^1))$ such that the set $\mathcal{P}\underline{K}(A)$ (defined in 5.16) generates a sub group containing the group

$$K_0(A) \oplus \bigoplus_{k|m} K_*(A, \mathbb{Z}/k).$$

Similar to the proof of Theorem 5.18, we can prove the following theorem, since, to determine a KK-element $\alpha \in KK(A,B)$, one does not need the map from $K_1(A)$. ($G(\mathcal{P})$ and $\delta(\mathcal{P})$ can be chosen accordingly as in 5.17.)

Theorem 5.21. *Suppose that X is a finite CW complex with $K_1(X)$ a torsion group. Suppose that $A = PM_l(C(X))P$ and $\mathcal{P}$ are as in 5.20. For any finite set $F \subset A$, positive number $\varepsilon > 0$, and positive integer M, there are a finite set $G \subset A$ ($G \supset G(\mathcal{P})$ large enough), a positive number $\delta > 0$ ($\delta < \delta(\mathcal{P})$ small enough), and a positive integer L (large enough) such that the following statement is true.*
If $\phi, \psi \in Map(A,B)$ are $G(\mathcal{P})$-$\delta(\mathcal{P})$ multiplicative and

$$\phi_* = \psi_* : \mathcal{P}\underline{K}(A) \longrightarrow \underline{K}(B),$$

where $B = QM_\bullet(C(Y))Q$ with $\dim(Y) \le M$, then there is a homomorphism $\nu \in Hom(A, M_L(B))$ with finite dimensional image, and there is a unitary $u \in M_{L+1}(B)$ such that

$$\|u(\phi \oplus \nu)(a)u^* - (\psi \oplus \nu)(a)\| < \varepsilon$$

for all $a \in F$.

The following is a direct consequence of Theorem 5.18.

Corollary 5.22. *Let $A = C(X)$, where X is one of the spaces: $[0,1], S^2, T_{II,k}$ or $T_{III,k}$, and let $\mathcal{P}$ be as in 5.16. For any finite set $F \subset C(X)$, any positive number $\varepsilon > 0$ and any positive integer M, there are a finite set $G \subset C(X)$ ($G \supset G(\mathcal{P})$ large enough), positive numbers $\delta > 0$ ($\delta \leq \delta(\mathcal{P})$ small enough) and $\eta > 0$ (small enough) such that the following statement is true.*
Let $B = M_\bullet(C(Y))$ with $\dim(Y) \leq M$, and $p \in B$, a projection.
If $\phi, \psi \in Map(C(X), pBp)$ are G-δ multiplicative maps inducing the same maps $\phi_* = \psi_* : \mathcal{P}\underline{K}(C(X)) \longrightarrow \underline{K}(B)$, and $\{x_1, x_2, \cdots, x_n\}$ is an η-dense subset of X, and $q_1, q_2, \cdots, q_n$ are mutually orthogonal projections in $(\mathbf{1}-p)B(\mathbf{1}-p)$ with rank$(q_i) \geq$ rank(p), then there is a unitary

$$u \in (p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)B(p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)$$

such that

$$\left\| \phi(f) \oplus \sum_{i=1}^{n} f(x_i)q_i - u\left(\psi(f) \oplus \sum_{i=1}^{n} f(x_i)q_i\right)u^* \right\| < \varepsilon, \qquad \forall f \in F.$$

In particular, if ψ is a homomorphism, then there is a homomorphism $\tilde{\phi} \in Hom(C(X), (p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)B(p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n))$ (defined by $\tilde{\phi}(f) = u\left(\psi(f) \oplus \sum_{i=1}^{n} f(x_i)q_i\right)u^*$) such that

$$\left\| \tilde{\phi}(f) - \left(\phi(f) \oplus \sum_{i=1}^{n} f(x_i)q_i\right) \right\| < \varepsilon, \qquad \forall\ f \in F.$$

Proof: Since X is not the space of a single point, we can assume that X, as a metric space, satisfies that diameter$(X) = 1$. Apply Theorem 5.18 to the finite set $F \subset A$, the positive number $\frac{\varepsilon}{3}$ and the integer M to obtain G, δ, L as in Theorem 5.18. Choose a positive number $\eta < \frac{1}{8ML^2}$ such that if dist$(x, x') < 8ML^2 \cdot \eta$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{3}$ for all $f \in F$.
Let $\{x_1, x_2, \cdots, x_n\}$ be an η-dense subset of X and let $q_1, q_2, \cdots, q_n \in (1-p)B(1-p)$ be mutually orthogonal projections with rank$(q_i) \geq$ rank(p). Similar to the proof of Corollary 1.6.13, one can find a $8ML \cdot \eta$-dense subset $\{x_{k_1}, x_{k_2}, \cdots, x_{k_l}\} \subset \{x_1, x_2, \cdots, x_n\}$ and mutually orthogonal projections $Q_1, Q_2, \cdots, Q_l$ with rank$(Q_j) \geq ML \cdot$ rank(p), such that

$$\| \sum_{i=1}^{n} f(x_i)q_i - \sum_{j=1}^{l} f(x_{k_j})Q_j \| < \frac{\varepsilon}{3}, \quad \forall f \in F.$$

Since rank$(Q_i) \geq ML \cdot$ rank(p), it follows that $[Q_i] \geq L \cdot [p]$.
Again, similar to the proofs of Corollaries 1.6.12 and 1.6.13, it can be proved that a homomorphism $\nu \in$ Hom$(A, M_L(pBp))$ with finite dimensional image

(from 5.18) can be perturbed, at the expense of at most $\frac{\varepsilon}{3}$ on the finite set F, to a homomorphism ν' which is of the form

$$\nu'(f) = \sum_{j=1}^{l} f(x_{k_j}) q'_j$$

with $[q'_j] \leq [Q_j]$ (SOME OF THE PROJECTIONS q'_j COULD BE ZERO). Hence the corollary follows (see the proof of Corollary 1.6.12).

□

By the discussion in 1.2.19, we have the following corollary.

COROLLARY 5.23. *Let $A = M_l(C(X))$, where X is one of the spaces: $[0,1], S^2, T_{II,k}$ or $T_{III,k}$, and let $\mathcal{P}$ be as in 5.16. For any finite set $F \subset A$, any positive number $\varepsilon > 0$ and any positive integer M, there are a finite set $G \subset A$ ($G \supset G(\mathcal{P})$ large enough), numbers $\delta > 0$ ($\delta \leq \delta(\mathcal{P})$ small enough) and $\eta > 0$ (small enough) such that the following statement is true.*
Let $B = M_\bullet(C(Y))$ with $\dim(Y) \leq M$, and $p \in B$ a projection.
If $\phi, \psi \in Map(A, pBp)$ are G-δ multiplicative maps inducing the same map $\phi_* = \psi_* : \mathcal{P}\underline{K}(A) \longrightarrow \underline{K}(B)$, and $\{x_1, x_2, \cdots, x_n\}$ is an η-dense subset of X, and

$$q_1 = \underbrace{q'_1 \oplus q'_1 \oplus \cdots \oplus q'_1}_{l}, q_2 = \underbrace{q'_2 \oplus q'_2 \oplus \cdots \oplus q'_2}_{l}, \cdots, q_n = \underbrace{q'_n \oplus q'_n \oplus \cdots \oplus q'_n}_{l}$$

are mutually orthogonal projections in $(\mathbf{1}-p)B(\mathbf{1}-p)$ with rank$(q_i) \geq$ rank(p), then there is a unitary

$$u \in (p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)B(p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)$$

such that

$$\left\| \phi(f) \oplus \sum_{i=1}^{n} q'_i \otimes f(x_i) - u \left(\psi(f) \oplus \sum_{i=1}^{n} q'_i \otimes f(x_i) \right) u^* \right\| < \varepsilon, \qquad \forall f \in F.$$

In particular, if ψ is a homomorphism, then there is a homomorphism $\tilde{\phi} \in Hom(C(X), (p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n)B(p \oplus q_1 \oplus q_2 \oplus \cdots \oplus q_n))$ such that

$$\left\| \tilde{\phi}(f) - \left(\phi(f) \oplus \sum_{i=1}^{n} q'_i \otimes f(x_i) \right) \right\| < \varepsilon, \qquad \forall\ f \in F.$$

Proof: Thanks to Lemma 1.6.8, we can always assume that the two maps ϕ and ψ satisfy the condition that $\phi|_{M_l(\mathbb{C})}$ and $\psi|_{M_l(\mathbb{C})}$ are homomorphisms. Using the condition $\phi_* = \psi_* : \mathcal{P}\underline{K}(A) \longrightarrow \underline{K}(B)$, we can assume

$$\phi|_{M_l(\mathbb{C})} = \psi|_{M_l(\mathbb{C})}$$

after conjugating with a unit.
Now the corollary follows from the following claim.
Claim: For any finite set $F \subset A = M_l(C(X))$, any $\varepsilon > 0$, there are a finite set $G \subset A$ and a positive number $\delta > 0$ such that if a map $\phi : A \to B$ is G-δ multiplicative and $\phi|_{M_l(\mathbb{C})}$ is a homomorphism, then there are a map $\phi_1 : C(X) \to \phi(e_{11})B\phi(e_{11})$ and an identification of $\phi(\mathbf{1})B\phi(\mathbf{1}) \cong M_l(\phi(e_{11})B\phi(e_{11}))$ such that

$$\|\phi(f) - (\phi_1 \otimes \mathbf{1}_l)(f)\| < \varepsilon \qquad \forall f \in F.$$

Furthermore, if $G_1 \subset C(X)$ and $\delta_1 > 0$ are a pregiven finite set and a pregiven positive number, then one can modify the set G and the number δ so that the map ϕ_1 above can be chosen to be G_1-δ_1 multiplicative.
Proof of Claim: Suppose $\mathbf{1} \in F$. Let $F_1 = \{a_{ij} | (a_{ij})_{l\times l} \in F\} (\subset C(X))$ be the set of all entries of the elements in F. Let $G = \{(b_{ij})_{l\times l} = \sum b_{ij}e_{ij} \mid b_{ij} \in F_1 \cup G_1 \subset C(X)\} (\subset A)$ and $\delta = \min\left(\frac{\varepsilon}{2l^2}, \delta_1\right)$. Suppose that $\phi :\ M_l(C(X)) \to B$ is G-δ multiplicative. Let

$$\phi_1 = \phi|_{e_{11}M_l(C(X))e_{11}} :\ C(X) \to \phi(e_{11})B\phi(e_{11}).$$

Obviously the G-δ multiplicativity of ϕ implies the G_1-δ_1 multiplicativity of ϕ_1. Identify $\phi(\mathbf{1})B\phi(\mathbf{1}) \cong (\phi(e_{11})B\phi(e_{11})) \otimes M_l$ by sending $\phi(e_{ij})$ to $e_{ij} \in M_l \subset (\phi(e_{11})B\phi(e_{11})) \otimes M_l$. Under this identification, we have

$$(\phi_1 \otimes \mathbf{1}_l)(a) = \sum_{i,j} \phi(e_{i1})\phi_1(a_{ij})\phi(e_{1j}),$$

where $a = (a_{ij})_{l\times l} \in M_l(C(X))$. On the other hand, writing $a = \sum e_{1i}(a_{ij}e_{11})e_{1j}$ and using the G-δ multiplicativity of ϕ, we have

$$\begin{aligned}\|\phi(a) - (\phi_1 \otimes \mathbf{1}_l)(a)\| &\le \sum_{i,j} \|\phi(e_{1i}(a_{ij}e_{11})e_{1j}) - \phi(e_{i1})\phi_1(a_{ij})\phi(e_{1j})\| \\ &= \sum_{i,j} \|\phi(e_{1i}(a_{ij}e_{11})e_{1j}) - \phi(e_{i1})\phi(a_{ij}e_{11})\phi(e_{1j})\| \\ &\le \sum_{i,j} 2\delta = 2l^2\delta \le \varepsilon.\end{aligned}$$

This proves the Claim.
Applying the Claim, one can reduce the proof to the case $A = C(X)$ which is Corollary 5.22.

$\square$

Definition 5.24. Let A be a unital C^*-algebra, let

$$\mathcal{P} \subset M_\bullet(A) \cup M_\bullet(A \otimes C(S^1)) \cup \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1))$$

be a finite set of projections, and let $G(\mathcal{P})$, $\delta(\mathcal{P})$ be as in 5.17. A $G(\mathcal{P})-\delta(\mathcal{P})$ multiplicative map $\phi: A \to B$ is called QUASI-$\mathcal{P}\underline{K}$-HOMOMORPHISM if there is a homomorphism $\psi: A \to B$ with $\phi(\mathbf{1}_A) = \psi(\mathbf{1}_A)$ such that

$$[\phi]_* = [\psi]_* : \mathcal{P}\underline{K}(A) \to \underline{K}(B).$$

Using the above definition and Definition 4.38, we can restate the second part of Corollary 5.23 as below.

Lemma 5.25. *Let $A = M_l(C(X))$, where X is one of the spaces: $[0,1], S^2, T_{II,k}$ or $T_{III,k}$, and let $\mathcal{P}$ be as in 5.16. For any finite set $F \subset A$, any positive number $\varepsilon > 0$ and any positive integer M, there are a finite set $G \subset A$ ($G \supset G(\mathcal{P})$ large enough), positive numbers $\delta > 0$ ($\delta \le \delta(\mathcal{P})$ small enough) and $\eta > 0$ (small enough) such that the following statement is true.*
Let $B = M_\bullet(C(Y))$ with $\dim(Y) \le M$, and let $p \in B$ be a projection. If $\phi \in Map(A, pBp)$ is a G-δ multiplicative quasi-$\mathcal{P}\underline{K}$-homomorphism, and $\lambda \in Hom(A, (\mathbf{1}-p)B(\mathbf{1}-B))$ has the property PE(rank(p), η), then there is a homomorphism $\tilde{\phi} \in Hom(A,B)$ such that

$$\|\tilde{\phi}(f) - (\phi \oplus \lambda)(f)\| < \varepsilon, \quad \forall f \in F.$$

Furthermore if Y is a connected simplicial complex different from the single point space, then $\tilde{\phi}$ can be chosen to be injective.

Proof: The main body of the lemma is a restatement of Corollary 5.23. So we only need to prove the last sentence of the lemma. We need the following fact: Let $X = [0,1], S^2, T_{II,k}$ or $T_{III,k}$, and let Y be a connected finite simplicial complex different from $\{pt\}$. If $\lambda_1 : M_l(C(X)) \to p_1 M_\bullet(C(Y))p_1$ is a homomorphism defined by the point evaluation at a point $x_1 \in X$ as

$$M_l(C(X)) \xrightarrow{e_{x_1}} M_l(\mathbb{C}) \longrightarrow p_1 M_\bullet(C(Y))p_1,$$

then λ_1 is homotopic to an injective homomorphism $\lambda_1' : M_l(C(X)) \to p_1 M_\bullet(C(Y))p_1$. (Again, this fact can be proved by using the Peano Curve.)
Let η' be as the η desired in the main body of the lemma for $\frac{\varepsilon}{2}$ (in place of ε). We can also assume that η' satisfies the condition that if $\mathrm{dist}(x,x') < \eta'$, then $\|f(x) - f(x')\| < \frac{\varepsilon}{2}$ for all $f \in F$. Choose $\eta = \frac{\eta'}{4}$. Suppose that $\lambda \in \mathrm{Hom}(A, (1-p)B(1-p))$ has the property $\mathrm{PE}(\mathrm{rank}(p), \eta)$. Write $\lambda = \bigoplus_{i=1}^n \lambda_i$, where

$$\lambda_i : M_l(C(X)) \xrightarrow{e_{x_i}} M_l(\mathbb{C}) \xrightarrow{\phi_i} p_i B p_i,$$

are point evaluations at an η-dense set $\{x_1, x_2, \cdots, x_n\}$ and ϕ_i are unital homomorphisms.
Let p_1 be a projection with minimum rank among all the projections $p_1, p_2, \cdots, p_n$. Let $\phi \in \mathrm{Map}(A, pBp)$ be a G-δ multiplicative quasi-$\mathcal{P}\underline{K}$-homomorphism. Then $\phi \oplus \lambda_1 \in \mathrm{Map}(A, (p \oplus p_1)B(p \oplus p_1))$ is also a G-δ

multiplicative quasi-$\mathcal{P}\underline{K}$-homomorphism. Furthermore, from the above fact, it defines the same map on the level of $\mathcal{P}\underline{K}(A)$ as an injective homomorphism $\psi \in \mathrm{Hom}(A, (p \oplus p_1)B(p \oplus p_1))$. On the other hand, $\lambda' = \bigoplus_{i=2}^{n} \lambda_i$ has the properties PE(rank$(p), 2\eta$) and PE(rank$(p_1), 2\eta$), since $\lambda = \bigoplus_{i=1}^{n} \lambda_i$ has PE(rank$(p), \eta$), and rank$(p_1) \leq$ rank(p_i), $i = 2, \cdots, n$. Similar to the proof of Corollary 5.22, λ' can be perturbed to a homomorphism λ'' which has the property PE(rank(p) + rank$(p_1), 4\eta$) at the expense of at most $\frac{\varepsilon}{2}$ on the finite set F. Note that $4\eta = \eta'$, and ψ is injective. Hence the homomorphism $\mathrm{Ad}u \circ (\psi \oplus \lambda'')$ (for a certain unitary u), as desired in the main body of the lemma, is also injective.

$\square$

LEMMA 5.26. *Let X and Y be connected finite simplicial complexes. Suppose that $\phi_1 : PM_k(C(X))P \to Q_1M_l(C(Y))Q_1$ and $\phi_2 : PM_k(C(X))P \to Q_2M_l(C(Y))Q_2$ are unital homomorphisms, where P, Q_1, and Q_2 are projections with*

$$rank(Q_2) - rank(Q_1) \geq 2\dim(Y) \cdot rank(P).$$

Then there exists a homomorphism $\psi : PM_k(C(X))P \to M_\bullet(C(Y))$ such that

$$[\psi] = [\phi_2] - [\phi_1] \in KK(C(X), C(Y)).$$

Proof: First, we suppose that $A = C(X)$. As in Lemma 3.14 of [EG 2] (see Remark 1.6.21 above), we can assume that $\phi_1(C_0(X)) \subset M_l(C_0(Y))$ and $\phi_2(C_0(X)) \subset M_l(C_0(Y))$, where $C_0(X)$ and $C_0(Y)$ are sets of functions vanishing on fixed base points of X and Y, respectively. Hence ϕ_i defines an element $kk(\phi_i) \in kk(Y, X)$ (see [DN]). Furthermore, $[\phi_i] \in KK(C(X), C(Y))$ is completely determined by $kk(\phi_i)$ and $\phi_{i*}([\mathbf{1}_A]) \in K_0(B)$. Let $\alpha = kk(\phi_2) - kk(\phi_1) \in kk(Y, X)$ (note that $kk(Y, X)$ is an abelian group, see [DN]). Since rank$(Q_2)-$rank$(Q_1) \geq 2\dim(Y)$, by [Hu], there is a projection $Q_3 \in M_\bullet(C(Y))$ such that $[Q_3] = [Q_2] - [Q_1] \in K_0(C(Y))$. By Theorem 4.11 of [DN] or Lemma 3.16 of [EG2], there is a unital homomorphism $\psi : C(X) \to Q_3M_\bullet(C(Y))Q_3$ to realize $\alpha \in kk(Y, X)$. Obviously ψ is as desired.

For the general case, using the Dilation Lemma (Lemma 1.3.1), one can prove that $[\phi_i] \in KK(C(X), C(Y))$ can be realized by homomorphism $\phi_i' : C(X) \to M_\bullet(C(Y))$. This reduces the proof to the above case.

$\square$

REMARK 5.27. In the above lemma, if $Q_1 < Q_2$, then one can choose ψ to satisfy $\psi(\mathbf{1}_A) = Q_2 - Q_1$.

LEMMA 5.28. *Let X be a finite simplicial complex, and $A = PM_l(C(X))P$.*

For any finite set

$$\mathcal{P} \subset M_\bullet(A) \cup M_\bullet(A \otimes C(S^1)) \cup \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1)),$$

there are a finite set $G \subset A$ and a number $\delta > 0$, such that the following is true.
If Y is a simplicial complex, $Q > Q_1$ are two projections in $M_\bullet(C(Y))$ with $rank(Q) - rank(Q_1) \geq 2\dim(Y) rank(P)$, and two unital homomorphisms $\phi \in Hom(A, QM_\bullet(C(Y))Q)_1$, $\phi_1 \in Hom(A, Q_1 M_\bullet(C(Y))Q_1)_1$ and a unital map $\phi_2 \in Map(A, (Q-Q_1)M_\bullet(C(Y))(Q-Q_1))_1$ satisfy that

$$(*) \qquad \|\phi(f) - \phi_1(f) \oplus \phi_2(f)\| < \delta, \quad \forall g \in G,$$

then there is a homomorphism $\psi : A \to (Q-Q_1)M_\bullet(C(Y))(Q-Q_1)$ such that

$$[\psi]_* = [\phi_2]_* : \mathcal{P}\underline{K}(A) \to \underline{K}(C(Y)).$$

In other words, ϕ_2 is a quasi-$\mathcal{P}\underline{K}$-homomorphism.
(Notice that, from Lemma 4.40, if G is large enough and δ is small enough, then (*) above implies that
$\phi_2 \in \mathrm{Map}(A, (Q-Q_1)M_\bullet(C(Y))(Q-Q_1))_1$ is $G(\mathcal{P}) - \delta(\mathcal{P})$ multiplicative, and hence $[\phi_2]_* : \mathcal{P}\underline{K}(A) \to \underline{K}(C(Y))$ makes sense.)

Proof: If G is large enough and δ is small enough, then (*) implies

$$[\phi_2]_* = [\phi]_* - [\phi_1]_* : \mathcal{P}\underline{K}(A) \to \underline{K}(C(Y)).$$

Then the lemma follows from Lemma 5.26 and Remark 5.27.

$\square$

Remark 5.29. In Corollary 4.39, we can choose ψ_0 (or ψ_0') such that $\psi_0^{i,j}$ (or $\psi_0'^{i,j}$) is a quasi-$\mathcal{P}\underline{K}$-homomorphism for any pre-given set of projections

$$\mathcal{P} \subset M_\bullet(A) \cup M_\bullet(A \otimes C(S^1)) \cup \bigcup_{k=2}^{\infty} M_\bullet(A \otimes C(W_k \times S^1)).$$

To do so, by Lemma 5.28, one only needs to choose the projection $Q_0^{i,j}$ to have rank at least $2\dim(X_{m,j}) \cdot \mathrm{rank}(1_{A_n^i})$. But from the construction in 4.34, we have freedom to do so.

Lemma 5.30. *Fix a positive integer M. Suppose that $B = \bigoplus_{i=1}^{s} M_{l_i}(C(Y_i))$, where Y_i are the spaces: $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}, S^2$. For any finite set $G \subset B$ and positive number $\varepsilon > 0$, there exist a finite set $G_1 \subset B$, numbers $\delta_1 > 0$ and $\eta > 0$ such that the following is true.*
If a map $\alpha = \alpha_0 \oplus \alpha_1 : B \to A = \bigoplus_{j=1}^{t} M_{k_j}(C(X_j))$, with $\dim(X_j) \leq M$, satisfies the following conditions:

(1) α_0 is G_1-δ_1 multiplicative, $\{\alpha_0(\mathbf{1}_{B^i})\}_{i=1}^s$ are mutually orthogonal projections, and α_1 is a homomorphism with finite dimensional image (i.e., defined by point evaluations);
(2) For any block B^i with $Y_i = T_{II,k}, T_{III,k}$ or S^2 and any block A^j, the partial map $\alpha_0^{i,j}$ is quasi-$\mathcal{P}\underline{K}$-homomorphism, where $\mathcal{P}$ is the set of projections associated to B^i as in 5.16, and the homomorphism $\alpha_1^{i,j}$ has the property PE(rank$\alpha_0^{i,j}(\mathbf{1}_{B^i})$, η);

then there is a unital homomorphism $\alpha' : B \to \alpha(\mathbf{1}_B)A\alpha(\mathbf{1}_B)$ such that

$$\|\alpha'(g) - \alpha(g)\| < \varepsilon, \quad \forall g \in G.$$

Proof: We only need to perturb all the individual maps $\alpha^{i,j}$ to homomorphisms ${\alpha'}^{i,j}$ within $\alpha^{i,j}(\mathbf{1}_{B^i})A^j\alpha^{i,j}(\mathbf{1}_{B^i})$.
For a block of B^i with spectrum $\{pt\}$, $[0,1]$ or S^1, such perturbation exists by Lemma 1.6.1. For a block of B^i with spectrum $T_{II,k}, T_{III,k}$ or S^2, such perturbation exists by Lemma 5.25.

$\square$

Lemma 5.31. *Let M be a fixed positive integer. Let $B = M_l(C(Y)), Y = T_{II,k}, T_{III,k}$ or S^2. Let the set of projections $\mathcal{P} \subset M_\bullet(B) \cup M_\bullet(B \otimes C(W_k \times S^1))$ be as in 5.16.*
Let $A = RM_{l_1}(C(X))R$ with $\dim(X) \le M$, where $R \in M_{l_1}(C(X))$ is a projection. Let $\alpha : B \to A$ be an injective homomorphism. Let a finite set of projections $\mathcal{P}'$ be given by $\mathcal{P}' := (\alpha \otimes id)(\mathcal{P}) \subset M_\bullet(A) \cup M_\bullet(A \otimes C(W_k \times S^1))$.
Let $\eta > 0$. Choose $\eta_1 > 0$ such that if a finite set $\{x_1, x_2, \cdots, x_n\} \subset X$ is η_1-dense in X, then $\bigcup_{i=1}^n SP\alpha_{x_i}$ is η-dense in Y. (Such η_1 exists because of injectivity of α.)
For any finite subset $G_1 \subset B$ and any number $\delta_1 > 0$, there are a finite subset $G_2 \subset A$ and a number $\delta_2 > 0$ such that the following are true.
Let $C = M_\bullet(Z)$ with $\dim(Z) \le M$.
(1) If $\psi_0 : A \to Q_0CQ_0$ is a G_2-δ_2 multiplicative quasi-$\mathcal{P}'\underline{K}$-homomorphism and $\psi_0(\alpha(\mathbf{1}_B))$ is a projection, then $\psi_0 \circ \alpha$ is a G_1-δ_1 multiplicative quasi-$\mathcal{P}\underline{K}$-homomorphism.
(2) If $\psi_1 : A \to Q_1CQ_1$ has the property PE($J \cdot L$, η_1), where $J =$ rank(R), then $\psi_1 \circ \alpha :\ B \to \psi_1(\alpha(\mathbf{1}_B))C\psi_1(\alpha(\mathbf{1}_B))$ has property PE(L, η).
In particular, if ψ_1 has the property PE($J \cdot$ rank(Q_0), η_1), (this is the condition (3) of Corollary 4.39), then $\psi_1 \circ \alpha$ has the property PE(rank$((\psi_0 \circ \alpha)(\mathbf{1}_B))$, η). (Note that rank$((\psi_0 \circ \alpha)(\mathbf{1}_B)) \le$ rank(Q_0).) Consequently, if we further assume that G_1, δ_1 and η are as those chosen in 5.30 for a finite set $G \subset B$ and $\varepsilon > 0$, and Q_1 is orthogonal to Q_0, then there is a homomorphism $\psi : B \to (Q_0 \oplus Q_1)C(Q_0 \oplus Q_1)$ such that

$$\|\psi(g) - (\psi_0 \oplus \psi_1)(\alpha(g))\| < \varepsilon, \quad \forall g \in G.$$

Proof: (1) holds if we choose $G_2 \supset \alpha(G)$ and $\delta_2 < \delta_1$.
(2) follows from the following fact: if a homomorphism $\phi : RM_{l_1}(C(X))R \to M_t(C(Z))$ contains a part of point evaluation at a point $x \in X$ of size at least L (see Definition 4.38), then for any $y \in \mathrm{SP}\alpha_x \subset Y$, $\phi \circ \alpha$ contains a part of point evaluation at point y of size at least $\frac{L}{\mathrm{rank}(R)}$.

$\square$

The following two theorems are important for the proof of our main theorem.

THEOREM 5.32A. *Let M be a positive integer. Let $\lim\limits_{n\to\infty}(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ be a simple inductive limit with injective connecting homomorphisms $\phi_{n,m}$ and with* $\dim(X_{n,i}) \le M$, *for any n, i. Let $B = \bigoplus_{i=1}^{s} M_{l_i}(C(Y_i))$, where Y_i are the spaces: $\{pt\}, [0,1], S^1, T_{II,k}$, $T_{III,k}$, and S^2.*
Suppose that a homomorphism $\alpha : B \to A_n$ satisfies the following DICHOTOMY CONDITION*:*
For any block B^i of B and any block A_n^j of A_n, either the partial map $\alpha^{i,j}$: $B^i \to A_n^j$ is injective or it has a finite dimensional image.
Denote $\alpha(\mathbf{1}_B) := R(= \bigoplus R^i) \in A_n(= \bigoplus A_n^i)$. For any finite sets $G \subset B$ and $F \subset RA_nR$, any positive number $\varepsilon > 0$, and any positive integer L, there are A_m and mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_m$, with $\phi_{n,m}(R) = Q_0 + Q_1 + Q_2$, a unital map $\theta_0 \in Map(RA_nR, Q_0A_mQ_0)_1$, two unital homomorphisms $\theta_1 \in Hom(RA_nR, Q_1A_mQ_1)_1$ and
$\xi \in Hom(RA_nR, Q_2A_mQ_2)_1$ such that
(1) $\|\phi_{n,m}(f) - (\theta_0(f) + \theta_1(f) + \xi(f))\| < \varepsilon, \quad \forall f \in F$;
(2) there is a homomorphism $\alpha_1 : B \to (Q_0 \oplus Q_1)A_m(Q_0 \oplus Q_1)$ such that

$$\|\alpha_1(g) - (\theta_0 + \theta_1) \circ \alpha(g)\| < \varepsilon, \quad \forall g \in G;$$

(3) θ_0 is $F - \varepsilon$ multiplicative and θ_1 satisfies that for any nonzero projection (including any rank 1 projection) $e \in R^iA_n^iR^i$

$$\theta_1^{i,j}([e]) \ge L \cdot [\theta_0^{i,j}(R^i)],$$

(the condition (3) will be used when we apply Theorem 1.6.9 in the proof of the Main Theorem);
(4) ξ factors through a C^*-algebra C—a direct sum of matrix algebras over $C[0,1]$ or $\mathbb{C}$— as

$$\xi : \ RA_nR \xrightarrow{\xi_1} C \xrightarrow{\xi_2} Q_2A_mQ_2,$$

and the partial maps of ξ_2 satisfy the dichotomy condition;
(5) the partial maps of α_1 satisfies the dichotomy condition.

Proof: Let $E^{i,j} = \alpha^{i,j}(\mathbf{1}_{B^i}) \in A_n^j$. Let

$$I = \{(i,j) \mid \alpha^{i,j} : B^i \to A_n^j \text{ has finite dimensional image}\}.$$

Let the subalgebra $D \subset A_n = \bigoplus A_n^i$ be defined by

$$D = \bigoplus_j \left(\bigoplus_{(i,j)\in I} \alpha^{i,j}(B^i) \oplus \bigoplus_{(i,j)\notin I} \alpha^{i,j}(\mathbb{C}\cdot \mathbf{1}_{B^i}) \right) \subset \bigoplus_j A_n^j.$$

Notice that D is a finite dimensional subalgebra of A_n containing the mutually orthogonal projections $\{E^{i,j} = \alpha^{i,j}(\mathbf{1}_{B_i})\}_{i,j}$.
Apply part 2 of Corollary 4.39 for sufficiently large set $F' \subset RA_nR$, sufficiently small number $\varepsilon' > 0$ and $\eta' > 0$, and positive integer $J = L \cdot \max_i \operatorname{rank}(R^i)$, to obtain A_m and the decomposition $\theta_0\oplus\theta_1\oplus\xi$ of $\phi_{n,m}|_{RA_nR}$ as $\psi_0'\oplus\psi_1'\oplus\psi_2'$ in 4.39. By Lemma 1.6.8, we can assume that the restriction $\theta_0|_D$ is a homomorphism. The condition (1) follows if we choose $F' \supset F$, and $\varepsilon' < \varepsilon$.
The $F-\varepsilon$ multiplicativity of θ_0 in (3) follows from Lemma 4.40, if F' is large enough and ε' is small enough, and the desired property of θ_1 in (3) follows from the choice of J and Lemma 5.31.
To construct α_1 as desired in the condition (2), we need to construct

$$\alpha_1^{i,j,k}:\ B^i \to \theta^{j,k}(E^{i,j})A_m^k\theta^{j,k}(E^{i,j}),$$

where $\theta = \theta_0 \oplus \theta_1$, to satisfy

$$\|\alpha_1^{i,j,k}(g) - \theta^{j,k}\circ\alpha^{i,j}(g)\| < \varepsilon, \quad \forall g \in G.$$

The construction are divided into three cases.
1. If $(i,j) \in I$, then $\theta^{j,k}\circ\alpha^{i,j}$ is already a homomorphism and can be chosen to be $\alpha_1^{i,j,k}$.
2. If $(i,j) \notin I$, and $Y_i = [0,1]$ or S^1, then the existence of $\alpha_1^{i,j,k}$ follows from Lemma 1.6.1 and Lemma 4.40, if F' is large enough and ε' is small enough. (See Lemma 5.30 also.) In fact, in this case, the map $\theta_0^{j,k}\circ\alpha^{i,j}$ itself can be perturbed to a homomorphism. On the other hand, the homomorphism $\theta_1^{j,k}\circ\alpha^{i,j}$ is defined by the point evaluations on an η-dense set for a certain small number η. Evidently, such a homomorphism $\theta_1^{j,k}\circ\alpha^{i,j}$ from $M_{l_i}(C(S^1))$ or $M_{l_i}(C([0,1]))$ (to A_m^k) can be perturbed to an injective homomorphism, provided that η is sufficiently small and that the path connected simplicial complex $X_{m,k}$ is not the space of a single point. Therefore, in this case, the homomorphism $\alpha_1^{i,j,k}$ can be chosen to be injective.
3. If $(i,j) \notin I$, and $Y_i = T_{II,k}, T_{III,k}$ or S^2, then $\alpha^{i,j}$ is injective, and the existence of $\alpha_1^{i,j,k}$ follows from Lemma 5.30 and the choice of J, if F' is large enough and ε' is small enough, and if we choose η' to be the number η_1 in Lemma 5.31 corresponding to the η in Lemma 5.30. The homomorphism $\alpha_1^{i,j,k}$ can also be chosen to be injective, if $X_{m,k}$ is not the space of a single point, according to the last part of Lemma 5.25.
Finally, define the partial map $\alpha_1^{i,k}$ of α_1 to be $\bigoplus_j \alpha_1^{i,j,k}$ to complete the construction. Obviously, it follows, from the discussion of the injectivity in case 2 and case 3, that α_1 satisfies the dichotomy condition.

$\square$

Theorem 5.32b. *Let M be a positive integer. Let $\lim\limits_{n\to\infty}(A_n = \bigoplus_{i=1}^{k_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ be a simple inductive limit with injective connecting homomorphisms $\phi_{n,m}$ and with $\dim(X_{n,i}) \leq M$, for any n, i. Let $B = \bigoplus_{i=1}^{s} M_{l_i}(C(Y_i))$, where Y_i are the spaces:$\{pt\}, [0,1], S^1, T_{II,k}$, $T_{III,k}$, and S^2.*

Suppose that a homomorphism $\alpha : B \to A_n$ satisfies the following dichotomy condition*:*

For any block B^i of B and any block A_n^j of A_n, either the partial map $\alpha^{i,j}$: $B^i \to A_n^j$ is injective or it has a finite dimensional image.

For any finite sets $G \subset B$ and $F \subset A_n$, and any number $\varepsilon > 0$, there are A_m and mutually orthogonal projections $P, Q \in A_m$, with $\phi_{n,m}(\mathbf{1}_{A_n}) = P + Q$, a unital map $\theta \in Map(A_n, PA_mP)_1$, and a unital homomorphism $\xi \in Hom(A_n, QA_mQ)_1$ such that

(1) $\|\phi_{n,m}(f) - (\theta(f) \oplus \xi(f))\| < \varepsilon, \quad \forall f \in F$;

(2) there is a homomorphism $\alpha_1 : B \to PA_mP$ such that

$$\|\alpha_1(g) - (\theta \circ \alpha)(g)\| < \varepsilon, \quad \forall g \in G;$$

(3) $\theta(F)$ is weakly approximately constant to within ε;

(4) ξ factors through a C^*-algebra C—a direct sum of matrix algebras over $C[0,1]$ or $\mathbb{C}$— as

$$\xi : \ A_n \xrightarrow{\xi_1} C \xrightarrow{\xi_2} QA_mQ,$$

and the partial maps of ξ_2 satisfy the dichotomy condition;

(5) the partial maps of α_1 satisfy the dichotomy condition.

The proof is similar to the proof of Theorem 5.32a, we omit it.

6 The proof of the main theorem

In this section, we will combine §4, §5 and §1.6 to prove our Main Theorem — the Reduction Theorem.

The following is Proposition 3.1 of [D2].

Proposition 6.1. ([D2, 3.1]) *Consider the diagram*

$$\begin{array}{ccccccccccc}
A_1 & \xrightarrow{\phi_{1,2}} & A_2 & \xrightarrow{\phi_{2,3}} & \cdots & \longrightarrow A_n & \xrightarrow{\phi_{n,n+1}} & A_{n+1} & \longrightarrow \cdots \\
\uparrow{\scriptstyle\alpha_1} & \searrow^{\beta_1} & \uparrow{\scriptstyle\alpha_2} & \searrow^{\beta_2} & \cdots & \uparrow{\scriptstyle\alpha_n} & \searrow^{\beta_n} & \uparrow{\scriptstyle\alpha_{n+1}} & \searrow \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & \cdots & \longrightarrow B_n & \xrightarrow{\psi_{n,n+1}} & B_{n+1} & \longrightarrow \cdots
\end{array},$$

where A_n, B_n are C^*-algebras, $\phi_{n,n+1}, \psi_{n,n+1}$ are homomorphisms and α_n, β_n are linear $*$-contractions.

Suppose that $F_n \subset A_n$, $E_n \subset B_n$ are finite sets satisfying the following conditions.

$$\phi_{n,n+1}(F_n) \cup \alpha_{n+1}(E_{n+1}) \subset F_{n+1}, \quad \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \subset E_{n+1},$$

and $\overline{\bigcup_{n=1}^{\infty}(\phi_{n,\infty}(F_n))}$ *and* $\overline{\bigcup_{n=1}^{\infty}(\psi_{n,\infty}(E_n))}$ *are the unit balls of* $A = \lim(A_n, \phi_{n,m})$ *and* $B = \lim(B_n, \psi_{n,m})$*, respectively. Suppose that there is a sequence* $\varepsilon_1, \varepsilon_2, \cdots$ *of positive numbers with* $\sum \varepsilon_n < +\infty$ *such that* α_n *and* β_n *are* F_n-ε_n *multiplicative and* E_n-ε_n *multiplicative, respectively, and*

$$\|\phi_{n,n+1}(f) - \alpha_{n+1} \circ \beta_n(f)\| < \varepsilon_n, \quad \text{and} \quad \|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < \varepsilon_n$$

for all $f \in F_n$ *and* $g \in E_n$.
Then A *is isomorphic to* B.

Lemma 6.2. *Let* $\lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ *be a simple inductive limit* C^**-algebra with* $\phi_{n,m}$ *injective, where* $X_{n,i}$ *are path connected finite simplicial complexes with uniformly bounded dimensions. Let* C^**-algebra* C *be a direct sum of matrix algebras over the spaces:* $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$ *and* S^2*, and* $\phi: C \to A_n$ *be an injective homomorphism. Then for any finite set* $F \subset C$ *and* $\varepsilon > 0$*, there is a positive integer* $N > n$ *such that for any* $m > N$*, there is a homomorphism* $\psi: C \to A_m$ *satisfying the following conditions.*
(1) $\psi(\mathbf{1}_{C^i}) = (\phi_{n,m} \circ \phi)(\mathbf{1}_{C^i})$*, for any block* C^i *of* C.
(2) $\|\psi(f) - (\phi_{n,m} \circ \phi)(f)\| < \varepsilon, \quad \forall f \in F$.
(3) ψ *satisfies the following dichotomy condition:*
For any block C^i *of* C *and* A_m^j *of* A_m*, either* $\psi^{i,j}$ *is injective or* $\psi^{i,j}$ *has finite dimensional image.*
(For the proof of the main theorem of this article—Theorem 6.3 below, we only need this lemma for the case that C is a direct sum of matrix algebras over spaces $\{pt\}$ and $[0,1]$. The full generality of the lemma will be used in the proof of Corollary 6.11 below.)

Proof: We only need to prove for the case that C has only one block $C = M_k(C(X))$. And, by the discussion in 1.2.19, this case can further be reduced to the case $C = C(X)$.
For the finite set $F \subset C$, there is an $\eta > 0$ such that if $\mathrm{dist}(t, t') < 4\eta$, then

$$\|f(t) - f(t')\| < \varepsilon, \quad \forall f \in F.$$

Let (X, σ) be a simplicial decomposition of X such that for any simplex $\Delta \subset (X, \sigma)$, $\mathrm{diameter}(\Delta) < \eta$. We call a simplex Δ a top simplex if Δ is not a proper face of any simplex. Obviously, Δ is a top simplex if and only if the interior $\mathring{\Delta}$ is an open subset of X.

From [DNNP, Proposition 2.1], using the injectivity of ϕ and $\phi_{n,m}$, it follows that there is an integer $N > n$ such that for any open set $\mathring{\Delta}$ — the interior of a top simplex $\Delta \subset (X,\sigma)$, one has

$$SP(\phi_{n,m}\circ\phi)_y \cap \overset{\circ}{\Delta} \neq \emptyset.$$

We can define the homomorphism $\psi : C \to A_m$ for each block A_m^j of A_m separately. That is, we need to define $\psi^j : C \to A_m^j$, then let $\psi := \oplus\psi^j$.
If $\mathrm{SP}(A_m^j) = X_{m,j} = \{pt\}$, then the partial map $(\phi_{n,m}\circ\phi)^j$ has finite dimensional image, and we can define it to be ψ^j. Hence we assume that the connected finite simplicial complex $X_{m,j}$ is not the space of single point $\{pt\}$.
Let $\alpha = (\phi_{n,m}\circ\phi)^j : C \to A_m^j$.
Let Y be the union of all such top simplices Δ that $\overset{\circ}{\Delta} \cap \mathrm{SP}\alpha$ is uncountable.
Let Z be the union of all simplices Δ which are not top simplices. Both Y and Z are closed subset of X. Let $\Delta_1, \Delta_2, \cdots, \Delta_l$ be the list of all top simplices such that $\Delta_i \not\subset Y, i = 1, 2, \cdots, l$. Then

$$X = Y \cup \Delta_1 \cup \Delta_2 \cdots \cup \Delta_l.$$

(This fact will be used later.) (Here we use the fact that X is equal to the union of all top simplices, since each simplex is a face of a top simplex.)
For each Δ_i, $\overset{\circ}{\Delta}_i \cap \mathrm{SP}\alpha$ is a countable nonempty set. There is a point x_i and an open disk $U_i = B_{\varepsilon_i}(x_i) \ni x_i$ such that

$$\mathrm{SP}\alpha \cap U_i = \{x_i\}.$$

We can assume that $U_i \subset \overset{\circ}{\Delta}_i$. Obviously, $\partial\Delta_i$ is a deformation retract of $\Delta_i \backslash U_i$.
Set $\left(X \backslash \left(\cup_{i=1}^l U_i\right)\right) \cap \mathrm{SP}\alpha = T$. Then $\mathrm{SP}\alpha = T \cup \{x_1, x_2, \cdots, x_l\}$.
Define a function $g : T \to Y \cup Z (\subset X)$ as below.
Let $g' : Z \to Y \cup Z$ be the identity map, that is,

$$g'(z) = z, \qquad \forall z \in Z.$$

We will extend the map g' to a map (let us still denote it by g').

$$g' : X \backslash \left(\cup_{i=1}^l U_i\right) \longrightarrow Y \cup Z.$$

For each top simplex $\Delta \subset Y$, extend $g'|_{\partial\Delta}$ to a map $g' : \Delta \to \Delta$ satisfying

$$g'(T \cap \Delta) = \Delta.$$

(Such extension exists since $T \cap \overset{\circ}{\Delta}$ is uncountable, see Lemma 2.6 of [EGL].)
For any simplex Δ_i, $i = 1, 2, \cdots, l$, one can extend $g'|_{\partial\Delta}$ to a map $g' : \Delta_i \backslash U_i \to \partial\Delta_i$, since $\partial\Delta_i$ is a deformation retract of $\Delta_i \backslash U_i$.

Thus we obtain the extension $g' : X\setminus\left(\cup_{i=1}^{l}U_i\right) \longrightarrow Y\cup Z$. Let $g = g'|_T$. Then

$$g(T) \supset Y, \quad \text{ and } \quad \mathrm{dist}(g(x), x) < \eta, \;\; \forall x \in T.$$

Since $\mathrm{SP}\alpha = T \cup \{x_1, x_2, \cdots, x_l\}$, there are homomorphisms $\alpha_0 : C(T) \to A_m^j$ and $\alpha_i : \mathbb{C} = C(\{x_i\}) \to A_m^j$, $j = 1, 2, \cdots, l$, with mutually orthogonal images, such that

$$\alpha(f) = \alpha_0(f|_T) + \sum_{i=1}^{l}(f|_{\{x_i\}}), \qquad \forall f \in C(X).$$

Define $\beta_0 : C(Y\cup Z) \to A_m^j$ by

$$\beta_0(f) = \alpha_0(f \circ g), \qquad \forall f \in C(Y\cup Z),$$

where $g : T \to Y \cup Z$ is defined as above. For each Δ_i, there is a surjective map $g_i : X_{m,j} \to \Delta_i$, since $X_{m,j} \neq \{pt\}$. Define $\beta_i : C(\Delta_i) \to A_m^j$ by

$$\beta_i(f)(x) = f(g_i(x)) \cdot \alpha_i(\mathbf{1}_{\mathbb{C}}), \qquad \forall f \in C(\Delta_i), x \in X_{m,j}.$$

Then, obviously, we have
(1′) $\beta_0(\mathbf{1}_{C(Y\cup Z)}) = \alpha_0(\mathbf{1}_{C(T)})$, and $\beta_i(\mathbf{1}_{C(\Delta_i)}) = \alpha_i(\mathbf{1}_{\mathbb{C}})$, for $i = 1, 2, \cdots, l$.
From the way η is chosen and the properties that $\mathrm{dist}(g(x), x) < \eta$ for any $x \in T$, and that $\mathrm{diameter}(\Delta_i) < \eta$ for any $i = 1, 2, \cdots, l$, we have
(2′) $\|\beta_0(f|_{Y\cup Z}) - \alpha_0(f|_T)\| < \varepsilon$, and $\|\beta_i(f|_{\Delta_i}) - \alpha_i(f|_{\{x_i\}})\| < \varepsilon$ for $i = 1, 2, \cdots, l$, and $f \in F$.
Finally, let the partial homomorphism $\psi^j : C(X) \to A_m^j$ be defined by

$$\psi^j(f) = \beta_0(f|_{Y\cup Z}) + \sum_{i=1}^{l}\beta_i(f|_{\Delta_i}).$$

Since $T \subset \mathrm{SP}\alpha_0$ and the map $g : T \to Y\cup Z$ satisfies $g(T) \supset Y$, we have $\mathrm{SP}(\beta_0) \supset Y$. Hence $\mathrm{SP}\psi^j = \mathrm{SP}\beta_0 \cup \cup_{i=1}^{l}\mathrm{SP}\beta_i \supset Y \cup \cup_{i=1}^{l}\Delta_i = X$. That is, ψ^j is injective.
The property (1) follows from (1′) and (2) follows from (2′).

$\square$

We will use 5.32a, 5.32b, 1.6.9, 1.6.29, 1.6.30 to prove the following main theorem of this article.

Theorem 6.3. *Suppose that* $\lim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ *is a simple inductive limit* C^**-algebra with* $\dim(X_{n,i}) \leq M$ *for a fixed positive integer* M*. Then there is another inductive system* $(B_n = \bigoplus_{i=1}^{t_n} M_{\{n,i\}}(C(Y_{n,i})),\ \phi_{n,m})$ *with the same limit algebra as the above system, where all* $Y_{n,i}$ *are spaces of forms* $\{pt\}, [0,1], S^1, S^2, T_{II,k},$ *or* $T_{III,k}$.

Proof: Without loss of generality, assume that the spaces $X_{n,i}$ are connected finite simplicial complexes and the connecting maps $\phi_{n,m}$ are injective (see Theorem 4.23).

Let $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > 0$ be a sequence of positive numbers satisfying $\sum \varepsilon_n < +\infty$.

We need to construct the intertwining diagram

$$\begin{array}{ccccccccc}
F_1 & & F_2 & & & F_n & & F_{n+1} & \\
\bigcap & & \bigcap & & & \bigcap & & \bigcap & \\
A_{s(1)} & \xrightarrow{\phi_{s(1),s(2)}} & A_{s(2)} & \xrightarrow{\phi_{s(2),s(3)}} & \cdots\longrightarrow & A_{s(n)} & \xrightarrow{\phi_{s(n),s(n+1)}} & A_{s(n+1)} & \longrightarrow\cdots \\
\uparrow{\scriptstyle\alpha_1} & \searrow^{\beta_1} & \uparrow{\scriptstyle\alpha_2} & \searrow^{\beta_2} & & \uparrow{\scriptstyle\alpha_n} & \searrow^{\beta_n} & \uparrow{\scriptstyle\alpha_{n+1}} & \searrow \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & \cdots\longrightarrow & B_n & \xrightarrow{\psi_{n,n+1}} & B_{n+1} & \longrightarrow\cdots \\
\bigcup & & \bigcup & & & \bigcup & & \bigcup & \\
E_1 & & E_2 & & & E_n & & E_{n+1} &
\end{array}$$

satisfying the following conditions.

(0.1) $(A_{s(n)}, \phi_{s(n),s(m)})$ is a sub-inductive system of $(A_n, \phi_{n,m})$. $(B_n, \psi_{n,m})$ is an inductive system of matrix algebras over the spaces: $\{pt\}, [0,1], S^1$, $\{T_{II,k}\}_{i=2}^{\infty}, \{T_{III,k}\}_{i=2}^{\infty}, S^2$.

(0.2) Choose $\{a_{ij}\}_{j=1}^{\infty} \subset A_{s(i)}$ and $\{b_{ij}\}_{j=1}^{\infty} \subset B_i$ to be countable dense subsets of the unit balls of $A_{s(i)}$ and B_i, respectively. F_n are subsets of the unit balls of $A_{s(n)}$, and E_n are subsets of the unit balls of B_n satisfying

$$\phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{n+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2}, \cdots, a_{i\ n+1}\}) \subset F_{n+1}$$

and

$$\psi_{n,n+1}(E_n) \cup \beta_n(F_n) \cup \bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2}, \cdots, b_{i\ n+1}\}) \subset E_{n+1}.$$

(Here we use the convention that $\phi_{n,n} = id : A_n \to A_n$.)

(0.3) β_n are $F_n - 2\varepsilon_n$ multiplicative and α_n are homomorphisms.

(0.4) $\|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < 2\varepsilon_n$ for all $g \in E_n$, and $\|\phi_{s(n),s(n+1)}(f) - \alpha_{n+1} \circ \beta_n(f)\| < 12\varepsilon_n$ for all $f \in F_n$.

(0.5) For any block B_n^i of B_n and any block $A_{s(n)}^j$ of $A_{s(n)}$, the map $\alpha_n^{i,j}$ satisfies the following dichotomy condition:

either $\alpha_n^{i,j}$ is injective or $\alpha_n^{i,j}$ has a finite dimensional image.

The diagram will be constructed inductively.

First, let $B_1 = \{0\}$, $A_{s(1)} = A_1$, $\alpha_1 = 0$. Let $b_{1j} = 0 \in B_1$ for $j = 1, 2, \cdots$, and let $\{a_{1j}\}_{j=1}^{\infty}$ be a countable dense subset of the unit ball of $A_{s(1)}$. And let $E_1 = \{b_{11}\} = B_1$ and $F_1 = \{a_{11}\} \subset A_{s(1)}$.

As an inductive assumption, assume that we already have the diagram

$$
\begin{array}{ccccccc}
F_1 & & F_2 & & & & F_n \\
\bigcap & & \bigcap & & & & \bigcap \\
A_{s(1)} & \xrightarrow{\phi_{s(1),s(2)}} & A_{s(2)} & \xrightarrow{\phi_{s(2),s(3)}} & \cdots & \longrightarrow & A_{s(n)} \\
\uparrow{\scriptstyle \alpha_1} & \searrow^{\beta_1} & \uparrow{\scriptstyle \alpha_2} & \searrow^{\beta_2} & \cdots & \searrow^{\beta_{n-1}} & \uparrow{\scriptstyle \alpha_n} \\
B_1 & \xrightarrow{\psi_{1,2}} & B_2 & \xrightarrow{\psi_{2,3}} & \cdots & \longrightarrow & B_n \\
\bigcup & & \bigcup & & & & \bigcup \\
E_1 & & E_2 & & & & E_n
\end{array}
$$

and, for each $i = 1, 2 \cdots, n$, we have countable dense subsets $\{a_{ij}\}_{j=1}^{\infty} \subset$ unit ball of $A_{s(i)}$ and $\{b_{ij}\}_{j=1}^{\infty} \subset$ unit ball of B_i to satisfy the conditions (0.1)-(0.5) above. We have to construct the next piece of the diagram,

$$
\begin{array}{cccccc}
F_n \subset & A_{s(n)} & \xrightarrow{\phi_{s(n),s(n+1)}} & A_{s(n+1)} & \supset F_{n+1} & \\
 & \uparrow{\scriptstyle \alpha_n} & \searrow^{\beta_n} & \uparrow{\scriptstyle \alpha_{n+1}} & & \\
E_n \subset & B_n & \xrightarrow{\psi_{n,n+1}} & B_{n+1} & \supset E_{n+1} & ,
\end{array}
$$

to satisfy the conditions (0.1)-(0.5).

Our construction are divided into several steps. In order to provide the reader with a whole picture of the construction, we first give an outline of it. Then the detailed construction will follow.

OUTLINE OF THE CONSTRUCTION. We will construct the following diagram.

$$
\begin{tikzcd}
C \arrow[r, "\xi_2"] & P_1 A_{m_1} P_1 \arrow[rrr, "\phi_{m_1,s(n+1)}"] & & & \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1) \\
 & & & & \bigoplus \\
 & \approx\phi_{s(n),m_1} \quad \bigoplus & RA_{m_2}R \arrow[r, "\xi_3"] \arrow[ddrr, dashed, "\theta_0+\theta_1"] & D \arrow[r, "\xi_4"] & (Q_2) A_{s(n+1)} (Q_2) \\
 & & u & \approx\phi_{m_2,s(n+1)} & \bigoplus \\
A_{s(n)} \arrow[uuuu, "\xi_1"] \arrow[r, dashed, "\theta"] & P_0 A_{m_1} P_0 \arrow[uur, "\phi_{m_1,m_2}"] \arrow[dd, dashed, "\beta"] & & & \\
1 & 2 & RA_{m_2}R \arrow[rr, dashed, "\theta_0+\theta_1"] & & (Q_0+Q_1) A_{s(n+1)} (Q_0+Q_1) \\
B_n \arrow[uu, "\alpha_n"] \arrow[uur, "\alpha"] \arrow[r, "\psi"] & B \arrow[ur, "\lambda\circ\alpha'"] \arrow[urrr, "\alpha''"] & 3 & & Adu
\end{tikzcd}
$$

This large picture consists of several smaller diagrams, each of which is called a sub-diagram. There are two kinds of sub-diagrams. The sub-diagrams of the first kind are labeled by the numbers 1, 2, 3 and the letter u (in the centers of the sub-diagrams). These sub-diagrams are almost commutative in some sense. For example, the one in the center of the large picture, labeled by the letter u consists of two composite maps $(\theta_0 + \theta_1) \circ (\lambda \circ \alpha') \circ \beta$ and $(\theta_0 + \theta_1) \circ (\phi_{m_1,m_2}|_{P_0 A_{m_1} P_0})$. They are almost equal to each other on a given finite set up to unitary equivalence.

The sub-diagrams of the second kind are those two labeled by "$\approx \phi_{s(n),m_1}$" and "$\approx \phi_{m_2,s(n+1)}$". They describe the approximate decompositions of the given maps "$\phi_{s(n),m_1}$" and "$\phi_{m_2,s(n+1)}|_{RA_{m_2}R}$".

All the maps in the above picture are homomorphisms except β, θ, and $\theta_0 + \theta_1$ (which are represented by broken line arrows). These maps are linear $*$-contractions which are almost multiplicative on some given finite sets (i.e., on the sets $F_n \subset A_{s(n)}$, $F := \theta(F_n) \subset P_0 A_{m_1} P_0$, or a certain (large enough) finite subset $F' \subset RA_{m_2}R_2$) to within given small numbers (i.e., ε_n or some related small numbers).

The sub-diagrams labeled by the numbers $1, 2$, and 3 are approximately commutative on certain given finite sets (i.e., $E_n \subset B_n$, $G := \psi(E_n) \cup \beta(F) \subset B$ (F is from the above paragraph)) to within a small number (i.e., ε_n). The sub-diagram labeled by the letter u is approximately commutative on a finite set ($F := \theta(F_n)$) to within a given small number ($9\varepsilon_n$) up to unitary equivalence.

The sub-diagrams labeled by "$\approx \phi_{s(n),m_1}$" and "$\approx \phi_{m_2,s(n+1)}$" are approximate decompositions of $\phi_{s(n),m_1}$ and $\phi_{m_2,s(n+1)}|_{RA_{m_2}R}$, respectively. (E.g., the direct sum $\theta \oplus (\xi_2 \circ \xi_1)$ of the two maps θ and $\xi_2 \circ \xi_1$ is close to $\phi_{s(n),m_1}$ to within a small number ε_n on a given finite set F_n.)

The above decomposition of $\phi_{s(n),m_1}$ and the almost commutative sub-diagram labeled by the number 1 are obtained in Step 1 in the detailed proof, applying Theorem 5.32b to $A_{s(n)}$ and $\alpha_n : B_n \to A_{s(n)}$ (and to the finite sets E_n and F_n). The main purpose of this step is to make the set $\theta(F_n) := F$ weakly approximately constant to within ε_n (the other part $\xi_2 \circ \xi_1$ of the decomposition factors through an interval algebra C), which will be useful later when we apply Theorem 1.6.9. (If one assumes in the beginning that the set F_n is weakly approximately constant to within ε_n, then he does not need this step.)

The sub-diagrams labeled by 2 or u will be explained by another picture later. The almost commutative sub-diagram labeled by the number 3 and the decomposition of $\phi_{m_2,s(n+1)}|_{RA_{m_2}R}$ (i.e., "$\approx \phi_{m_2,s(n+1)}$" in the picture), are obtained in Step 4, applying Theorem 5.32a to $RA_{m_2}R$ and $\lambda \circ \alpha' : B \to RA_{m_2}R$ (and certain finite subsets of B and $RA_{m_2}R$). The purpose of applying Theorem 5.32a is to construct the map $\theta_0 + \theta_1$ to satisfy the condition in Theorem 1.6.9 for the two homotopic homomorphisms $\lambda \circ (\phi \oplus r)$ and $\phi_{m_1,m_2}|_{P_0 A_{m_1} P_0}$ in the next picture, and therefore to obtain the almost commutative sub-diagram up to unitary equivalence—the sub-diagram labeled by u—, (the other part $\xi_4 \circ \xi_3$ of the decomposition factors through an interval algebra D).

In order to get the parts of the sub-diagram labeled by 2 and u, we need to start with $\alpha : B_n \to P_0 A_{m_1} P_0$. We describe it in the next picture.

$$\begin{tikzcd}
 & RA_{m_2}R \arrow[r, dashed, "\theta_0+\theta_1"] & (Q_0+Q_1)A_{s(n+1)}(Q_0+Q_1) \arrow[loop below, "Adu"] \\
A(=P_0A_{m_1}P_0) \arrow[ur, "\phi_{m_1,m_2}"] \arrow[r, "\phi\oplus r"] \arrow[dr, dashed, "\beta"] \arrow[ur, phantom, "h", near end, swap] & M_L(A)\oplus r(A) \arrow[u, "\lambda"] \arrow[d, phantom, "4", swap] & \\
B_n \arrow[u, "\alpha"] \arrow[r, "\psi"] \arrow[ur, phantom, "2", near start] & B \arrow[u, "\alpha'"] &
\end{tikzcd}$$

By Corollary 1.6.29 (see 1.6.31 also), applied to the homomorphism α from the first picture, we obtain the almost commutative sub-diagrams labeled by the numbers 2 and 4. Then we apply Lemma 1.6.30 to obtain the sub-diagram labeled by the letter h which commutes up to homotopy equivalence. By Theorem 1.6.9 and the property of the map $\theta_0+\theta_1$ (from Theorem 5.32a), this sub-diagram leads to the sub-diagram labeled by u in the first picture.
With the first picture in mind, we define

$$B_{n+1} = C \oplus B \oplus D,$$

$$\psi_{n,n+1} = (\xi_1 \circ \alpha_n) \oplus \psi \oplus (\xi_3 \circ \phi_{m_1,m_2}|_{P_0A_{m_1}P_0} \circ \alpha),$$
$$\beta_n = \xi_1 \oplus (\beta \circ \theta) \oplus (\xi_3 \circ \phi_{m_1,m_2}|_{P_0A_{m_1}P_0} \circ \theta),$$

and

$$\alpha_{n+1} = (\phi_{m_1,s(n+1)}|_{P_1A_{m_1}P_1} \circ \xi_2) \oplus (\mathrm{Ad}u \circ \alpha'') \oplus \xi_4.$$

In the definitions of $\psi_{n,n+1}$ and α_{n+1}, we use solid line arrows only since these maps are supposed to be homomorphisms (but in the definition of β_n, we can use broken line arrows).
One can easily verify the conditions (0.1)–(0.5) except that the map $\phi_{m_1,s(n+1)}|_{P_1A_{m_1}P_1} \circ \xi_2$ may not automatically satisfy the dichotomy condition (0.5), for which we have to apply Lemma 6.2 to make some modification.
Details of the construction. The above outline can be used as a guide to understand the following construction. But the proof below is complete by itself. (We encourage readers to compare the following detailed proof with the two diagrams in the outline.)
Among the conditions in the induction assumption, only the dichotomy condition (0.5) of α_n is used in the following construction.
Step 1. By Theorem 5.32b, applied to $\alpha_n : B_n \to A_{s(n)}, E_n \subset B_n$, $F_n \subset A_{s(n)}$, and $\varepsilon > 0$, there are A_{m_1} $(m_1 > s(n))$, two orthogonal projections $P_0, P_1 \in A_{m_1}$ with $\phi_{s(n),m_1}(\mathbf{1}_{A_{s(n)}}) = P_0+P_1$ and P_0 trivial, a C^*-algebra C — a direct sum of matrix algebras over $C[0,1]$ or $\mathbb{C}$—, a unital map $\theta \in \mathrm{Map}(A_{s(n)}, P_0A_{m_1}P_0)_1$,

a unital homomorphism $\xi_1 \in \mathrm{Hom}(A_{s(n)}, C)_1$, an injective unital homomorphism $\xi_2 \in \mathrm{Hom}(C, P_1 A_{m_1} P_1)_1$ and a (not necessarily unital) homomorphism $\alpha \in \mathrm{Hom}(B_n, P_0 A_{m_1} P_0)$ such that
(1.1) $\|\phi_{s(n),m_1}(f) - \theta(f) \oplus (\xi_2 \circ \xi_1)(f)\| < \varepsilon_n$ for all $f \in F_n$.
(1.2) θ is F_n-ε_n multiplicative and $F := \theta(F_n)$ is weakly approximately constant to within ε_n.
(1.3) $\|\alpha(g) - \theta \circ \alpha_n(g)\| < \varepsilon_n$ for all $g \in E_n$.
(1.4) Both $\alpha : B_n \to P_0 A_{m_1} P_0$ and $\xi_2 : C \to P_1 A_{m_1} P_1$ satisfy the dichotomy condition in (0.5).
(Thus we finished the construction of the sub-diagrams labeled by the number "1" and "$\approx \phi_{s(n),m_1}$" of the large diagram in the outline.)
Let all the blocks of C be parts of C^*-algebra B_{n+1}. That is,

$$B_{n+1} = C \oplus (\text{some other blocks}).$$

The map $\beta_n : A_{s(n)} \to B_{n+1}$ and the homomorphism $\psi_{n,n+1} : B_n \to B_{n+1}$ are defined by

$$\beta_n = \xi_1 : A_{s(n)} \to C(\subset B_{n+1}) \quad \text{and} \quad \psi_{n,n+1} = \xi_1 \circ \alpha_n : B_n \to C\ (\ \subset B_{n+1})$$

for the blocks of C ($\subset B_{n+1}$). For this part, β_n is also a homomorphism.
STEP 2. Let $A = P_0 A_{m_1} P_0,\ F = \theta(F_n)$. Since P_0 is a trivial projection,

$$A \cong \bigoplus M_{l_i}(C(X_{m_1,i})).$$

Let $rA := \bigoplus M_{l_i}(\mathbb{C}) \subset A$, and $r : A \to rA$ be the homomorphism defined by evaluation at certain base points $x_i^0 \in X_{m_1,i}$ (see 1.1.7(h)).
Applying Corollary 1.6.29 (see Remark 1.6.31 also) to $\alpha : B_n \to A$ (notice that α satisfies the dichotomy condition), $E_n \subset B_n$ and $F \subset A$, we obtain the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\phi \oplus r} & M_L(A) \oplus r(A) \\ \uparrow{\scriptstyle \alpha} & \searrow^{\beta} & \uparrow{\scriptstyle \alpha'} \\ B_n & \xrightarrow{\psi} & B \end{array}$$

such that
(2.1) B is a direct sum of matrix algebras over $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$, or S^2.
(2.2) α' is an injective homomorphism, and β is an F-ε_n multiplicative map.
(2.3) $\phi : A \to M_L(A)$ is a unital simple embedding. $r : A \to r(A)$ is the homomorphism defined by evaluations as in 1.1.7(h).
(2.4) $\|\beta \circ \alpha(g) - \psi(g)\| < \varepsilon_n$ for all $g \in E_n$, $\|(\phi \oplus r)(f) - \alpha' \circ \beta(f)\| < \varepsilon_n$ for all $f \in F(:= \theta(F_n))$.
(Thus we finished the construction of the sub-diagrams labeled by the number "2" and "4" of the second diagram in the outline.)
Let all the blocks B be also parts of B_{n+1}, that is,

$$B_{n+1} = C \oplus B \oplus (\text{some other blocks}).$$

The maps $\beta_n : A_{s(n)} \to B_{n+1}, \psi_{n,n+1} : B_n \to B_{n+1}$ are defined by

$$\beta_n := \beta \circ \theta : \ A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\beta} B \ \ (\subset B_{n+1})$$

and

$$\psi_{n,n+1} := \psi : \ B_n \longrightarrow B \ \ (\subset B_{n+1})$$

for the blocks of $B(\subset B_{n+1})$. This part of β_n is F_n-$2\varepsilon_n$ multiplicative, since θ is F_n-ε_n multiplicative, β is F-ε_n multiplicative, and $F = \theta(F_n)$.

Step 3. By the simplicity of $\lim(A_n, \phi_{n,m})$, for m large enough, the homomorphism $\phi_{m_1,m}|_{P_0 A_{m_1} P_0}$ is $4M$-large in the sense of 1.6.16. By Lemma 1.6.30, applied to $\phi \oplus r : A \to M_L(A) \oplus r(A)$, there is an A_{m_2} and unital homomorphism $\lambda : M_L(A) \oplus r(A) \to RA_{m_2}R$, where $R = \phi_{m_1,m_2}(P_0)$ (write R as $\bigoplus_j R^j \in \bigoplus_j A^j_{m_i}$) such that the diagram

$$\begin{tikzcd}
 & RA_{m_2}R \\
A(= P_0 A_{m_1} P_0) \arrow[ur, "\phi_{m_1,m_2}"] \arrow[r, "\phi \oplus r"] & M_L(A) \oplus r(A) \arrow[u, "\lambda"']
\end{tikzcd}$$

satisfies the following conditions:

(3.1) For each block $A^j_{m_2}$, the partial map

$$\lambda^{\cdot,j} : \ M_L(A) \oplus r(A) \longrightarrow R^j A^j_{m_2} R^j$$

is non zero. Furthermore, either it is injective or it has finite dimensional image — depending on whether $\mathrm{SP}(A^j_{m_2})$ is a single point space.

(3.2) $\lambda \circ (\phi \oplus r)$ is homotopy equivalent to

$$\phi' := \phi_{m_1,m_2}|_A.$$

(Thus we finished the construction of the sub-diagram labeled by the letter "h" of the second diagram in the outline.)

Step 4. Applying Theorem 1.6.9 to the finite set $F \subset A$ (which is weakly approximately constant to within ε_n), and to two homotopic homomorphisms

$$\phi' \ \text{ and } \ \lambda \circ (\phi \oplus r) : \ A \longrightarrow RA_{m_2}R$$

(with $RA_{m_2}R$ in place of C), we obtain a finite set $F' \subset RA_{m_2}R$, $\delta > 0$ and $L > 0$ as in the Theorem 1.6.9.

Let $G := \psi(E_n) \cup \beta(F) \subset B$. By Theorem 5.32a, applied to $RA_{m_2}R$,

$$\lambda \circ \alpha' : B \longrightarrow RA_{m_2}R$$

(which satisfies the dichotomy condition by (2.2) and (3.1)), finite sets $G \subset B$, $F' \subset RA_{m_2}R$, $\min(\varepsilon_n, \delta) > 0$ (in place of ε), and $L > 0$, there are $A_{s(n+1)}$, mutually orthogonal projections $Q_0, Q_1, Q_2 \in A_{s(n+1)}$ with $\phi_{m_2,s(n+1)}(R) = Q_0 + Q_1 + Q_2$, a C^*-algebra D — a direct sum of matrix algebras over $C[0,1]$—, a unital map $\theta_0 \in \mathrm{Map}(RA_{m_2}R, Q_0A_{s(n+1)}Q_0)$ and four unital homomorphisms $\theta_1 \in \mathrm{Hom}(RA_{m_2}R, Q_1A_{s(n+1)}Q_1)_1$, $\xi_3 \in \mathrm{Hom}(RA_{m_2}R, D)_1, \xi_4 \in \mathrm{Hom}(D, Q_2A_{s(n+1)}Q_2)_1$, and $\alpha'' \in Hom(B, (Q_0+Q_1)A_{s(n+1)}(Q_0+Q_1))_1$ such that the following are true:

(4.1) $\|\phi_{m_2,s(n+1)}(f) - ((\theta_0 + \theta_1) \oplus (\xi_4 \circ \xi_3))(f)\| < \varepsilon_n$ for all $f \in F' \subset RA_{m_2}R$.

(4.2) $\|\alpha''(g) - (\theta_0 + \theta_1) \circ \lambda \circ \alpha'(g)\| < \varepsilon_n$ for all $g \in G$.

(4.3) θ_0 is F'-$\min(\varepsilon_n, \delta)$ multiplicative and θ_1 satisfies that

$$\theta_1^{i,j}([q]) > L \cdot [\theta_0^{i,j}(R^i)]$$

for any non zero projection $q \in R^iA_{m_2}R^i$.

(4.4) Both $\alpha'' : B \longrightarrow (Q_0 + Q_1)A_{s(n+1)}(Q_0 + Q_1)$ and $\xi_4 : D \to Q_2A_{s(n+1)}Q_2$ satisfy the dichotomy condition (0.5).

(Thus we finished the construction of the sub-diagrams labeled by the number "3" and "$\approx \phi_{m_2,s(n+1)}$" of the large diagram in the outline. Combined with Step 2 and Step 3, these two sub-diagrams will lead to the sub-diagram labeled by the letter "u" of the large diagram as below.)

By the end of 1.1.4, for any blocks A^i, $A^k_{m_2}$ and any non zero projection $e \in A^i$, $\phi^{i,k}_{m_1,m_2}(e) \in A^k_{m_2}$ is a non zero projection. As a consequence of (4.3), we have

$$[(\theta_1 \circ \phi')(e)] \geq L \cdot [\theta_0(R)] \ (= L \cdot [Q_0]),$$

(Recall that $\phi' = \phi_{m_1,m_2}|_A$). Therefore, θ_0 and θ_1 (in place of λ_0 and λ_1) satisfy the condition in Theorem 1.6.9. By Theorem 1.6.9, there is a unitary $u \in (Q_0 + Q_1)A_{s(n+1)}(Q_0 + Q_1)$ such that

$$\|(\theta_0 + \theta_1) \circ \phi'(f) - \mathrm{Ad}u \circ (\theta_0 + \theta_1) \circ \lambda \circ (\phi \oplus r)(f)\| < 8\varepsilon_n, \qquad \forall f \in F.$$

Combining it with the second inequality of (2.4), we have

(4.5) $\|(\theta_0 + \theta_1) \circ \phi'(f) - \mathrm{Ad}u \circ (\theta_0 + \theta_1) \circ \lambda \circ \alpha' \circ \beta(f)\| < 9\varepsilon_n, \qquad \forall f \in F.$

STEP 5. Finally, let all the blocks of D be the rest of B_{n+1}. Namely, let

$$B_{n+1} = C \oplus B \oplus D,$$

where C is from *Step 1*, B is from *Step 2*, and D is from *Step 4*.

We already have the definitions of $\beta_n : A_{s(n)} \to B_{n+1}$ and $\psi_{n,n+1} : B_n \to B_{n+1}$ for those blocks of $C \oplus B \subset B_{n+1}$ (from *Step 1* and *Step 2*). The definitions of β_n and $\psi_{n,n+1}$ for blocks of D, and the homomorphism $\alpha_{n+1} : C \oplus B \oplus D \to A_{s(n+1)}$ will be given below.

The part of $\beta_n : A_{s(n)} \to D \ (\subset B_{n+1})$ is defined by

$$\beta_n = \xi_3 \circ \phi' \circ \theta : A_{s(n)} \xrightarrow{\theta} A \xrightarrow{\phi'} RA_{m_2}R \xrightarrow{\xi_3} D.$$

(Recall that $A = P_0 A_{m_1} P_0$ and $\phi' = \phi_{m_1,m_2}|_A$.) Since θ is F_n-ε_n multiplicative, and ϕ' and ξ_3 are homomorphisms, we know that this part of β_n is F_n-ε_n multiplicative.
The part of $\psi_{n,n+1} : B_n \to D$ ($\subset B_{n+1}$) is defined by

$$\psi_{n,n+1} = \xi_3 \circ \phi' \circ \alpha : B_n \xrightarrow{\alpha} A \xrightarrow{\phi'} RA_{m_2}R \xrightarrow{\xi_3} D$$

which is a homomorphism. The homomorphism $\alpha_{n+1} : C \oplus B \oplus D \to A_{s(n+1)}$ is defined as follows.
Consider the composition

$$\phi'' \circ \xi_2 : C \xrightarrow{\xi_2} P_1 A_{m_1} P_1 \xrightarrow{\phi''} \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1),$$

where P_1 and ξ_2 are from *Step 1*, $\phi'' = \phi_{m_1,s(n+1)}|_{P_1 A_{m_1} P_1}$. Using the dichotomy condition of ξ_2, by Lemma 6.2, there is a homomorphism $\tau : C \to \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1)$ such that
(5.1) $\quad \|\tau(f) - (\phi'' \circ \xi_2)(f)\| < \varepsilon_n, \quad \forall f \in \xi_1(F_n) \subset C$, and
(5.2) τ satisfies the dichotomy condition (0.5).
Define

$$\alpha_{n+1}|_C = \tau : C \to \phi_{m_1,s(n+1)}(P_1) A_{s(n+1)} \phi_{m_1,s(n+1)}(P_1),$$
$$\alpha_{n+1}|_B = \mathrm{Ad}u \circ \alpha'' : B \xrightarrow{\alpha''} (Q_0 + Q_1) A_{s(n+1)} (Q_0 + Q_1) \qquad \circlearrowleft_{Adu} \quad ,$$

where α'' is from *Step 4*, and define

$$\alpha_{n+1}|_D = \xi_4 : D \longrightarrow Q_2 A_{s(n+1)} Q_2.$$

Finally, choose $\{a_{n+1\ j}\}_{j=1}^{\infty} \subset A_{s(n+1)}$ and $\{b_{n+1\ j}\}_{j=1}^{\infty} \subset B_{n+1}$ to be countable dense subsets of unit balls of $A_{s(n+1)}$ and B_{n+1}, respectively. And choose

$$F_{n+1} = \phi_{s(n),s(n+1)}(F_n) \cup \alpha_{n+1}(E_{n+1}) \cup \bigcup_{i=1}^{n+1} \phi_{s(i),s(n+1)}(\{a_{i1}, a_{i2}, \cdots, a_{i\ n+1}\})$$

and

$$E_{n+1} = \psi_{n,n+1}(E_n) \cup \beta_n(F_n) \cup \bigcup_{i=1}^{n+1} \psi_{i,n+1}(\{b_{i1}, b_{i2}, \cdots, b_{i\ n+1}\}).$$

Thus we obtain the following diagram:

$$\begin{tikzcd}
F_n \subset & A_{s(n)} \arrow[rr, "\phi_{s(n),s(n+1)}"] \arrow[drr, "\beta_n"] & & A_{s(n+1)} & \supset F_{n+1} \\
E_n \subset & B_n \arrow[u, "\alpha_n"] \arrow[rr, "\psi_{n,n+1}"] & & B_{n+1} \arrow[u, "\alpha_{n+1}"] & \supset E_{n+1}\ .
\end{tikzcd}$$

STEP 6. Now we need to verify all the conditions (0.1)–(0.5) for the above diagram.

(0.1)–(0.2) hold from the construction (see the constructions of B, C, D in *Step 1*, *Step 2* and *Step 4*, and E_{n+1}, F_{n+1} in the end of *Step 5*.)
(0.3) follows from the end of *Step 1*, the end of *Step 2*, and the part of the definition of β_n for D from *Step 5*.
(0.5) follows from (4.4) and (5.2).
So we only need to verify (0.4).
Combining (1.1) with (4.1), we have

$$\|\phi_{s(n),s(n+1)}(f) - [(\phi'' \circ \xi_2 \circ \xi_1) \oplus ((\theta_0 + \theta_1) \circ \phi' \circ \theta) \oplus (\xi_4 \circ \xi_3 \circ \phi' \circ \theta)](f)\| \\ < \varepsilon_n + \varepsilon_n = 2\varepsilon_n$$

for all $f \in F_n$ (recall that $\phi'' = \phi_{m_1,s(n+1)}|_{P_1 A_{m_1} P_1}, \phi' := \phi_{m_1,m_2}|_{P_0 A_{m_1} P_0}$).
Combined with (4.2), (4.5), (5.1) and the definitions of β_n and α_{n+1}, the proceeding inequality yields

$$\|\phi_{s(n),s(n+1)}(f) - (\alpha_{n+1} \circ \beta_n)(f)\| < 9\varepsilon_n + \varepsilon_n + 2\varepsilon_n = 12\varepsilon_n \qquad \forall f \in F_n.$$

Combining (1.3), the first inequality of (2.4), and the definitions of β_n and $\psi_{n,n+1}$, we have

$$\|\psi_{n,n+1}(g) - \beta_n \circ \alpha_n(g)\| < \varepsilon_n + \varepsilon_n = 2\varepsilon_n \qquad \forall g \in E_n.$$

So we obtain (0.4).
The theorem follows from Proposition 6.1.

$\square$

Remark 6.4. In the proof of the above theorem, if there is at least one block of B_{n+1} having spectrum of forms $S^1, T_{II,k}, T_{III,k}$, or S^2, then we can chose the map $\psi_{n,n+1}$ to be injective (e.g., the map ψ in *Step 2* can be chosen to be injective). Hence, in general, we can make the maps $\psi_{n,m}$, in the inductive system $(B_n, \psi_{n,m})$, injective. (Note that if no space of $S^1, T_{II,k}, T_{III,k}$ or S^2 appears, then it is easy to make the maps injective; see Theorem 2.2.1 of [Li2]).

Remark 6.5. By Lemma 1.3.3, our main result Theorem 6.3 also holds for general simple AH inductive limit C^*-algebras
$\lim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}, \phi_{n,m})$ with uniformly bounded dimensions of $X_{n,i}$, where $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections. That is, such an AH algebra can be written as an inductive limit of a system $(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i})) Q_{n,i}, \psi_{n,m})$, where $Y_{n,i}$ are the spaces:
$\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$ and S^2, and $Q_{n,i} \in M_{\{n,i\}}(C(Y_{n,i}))$ are projections.

6.6. Suppose that a simple C^*-algebra A is an inductive limit of matrix algebras over $X_{n,i}$, where $X_{n,i}$ are the spaces of forms $\{pt\}, [0,1], S^1, S^2, T_{II,k}$ or $T_{III,k}$. Suppose that $K_*(A)$ is torsion free. Then it can be proved that for each fixed algebra A_n, integer $N > 0$, there is an A_m such that

$$\frac{\mathrm{rank}\phi_{n,m}^{i,j}(\mathbf{1}_{A_n^i})}{\mathrm{rank}(\mathbf{1}_{A_n^i})} \geq N,$$

and that $(\phi_{n,m})_*(torK_*(A_n)) = 0$. Based on this, using the argument from §4 of [G2], we know that for any $F \subset A_n, \varepsilon > 0$, if N is large enough, then the above $\phi_{n,m}$ is homotopic to a homomorphism $\psi : A_n \to \phi_{n,m}(\mathbf{1}_{A_n})A_m\phi_{n,m}(\mathbf{1}_{A_n})$ satisfying $\psi(F) \subset_\varepsilon C$, where C is a direct sum of matrix algebras over spaces $\{pt\}, [0,1]$ and S^1. (See [G1] and the proof of Lemma 5.6 of [EGL] also.) Using the above fact, the following Corollary is a direct consequence of our Main Theorem and its proof. (In fact, since the algebras $M_k(\mathbb{C}), M_k(C[0,1])$ and $M_k(C(S^1))$ are stably generated, the proof is much simpler (see §3 of [Li3]).

Corollary 6.7. *Suppose that A is a simple C^*-algebra which is an inductive limit of an AH system with uniformly bounded dimensions of local spectra. If $K_*(A)$ is torsion free, then it is an inductive limit of matrix algebras over $C(S^1)$.*

Combining the above corollary with [El2] (see [NT] also), we have the following theorem.

Theorem 6.8. *Suppose that $A = \lim\limits_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ and $B = \lim\limits_{n\to\infty}(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i}M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m})$ are unital simple inductive limit algebras with uniformly bounded dimensions of local spectra $X_{n,i}$ and $Y_{n,i}$, respectively. Suppose that $K_*(A) = K_*(B)$ are torsion free. Suppose that there is an isomorphism of ordered groups*

$$\phi_0 : \ K_0A \longrightarrow K_0B$$

taking $[\mathbf{1}] \in K_0A$ into $[\mathbf{1}] \in K_0B$, that there is a group isomorphism

$$\phi_1 : \ K_1A \longrightarrow K_1B$$

and that there is an isomorphism between compact convex sets

$$\phi_T : TB \longrightarrow TA,$$

where TA and TB denote the simplices of tracial states of A and B, respectively. Suppose that ϕ_0 and ϕ_T are compatible, in the sense that

$$\tau(\phi_0 g) = \phi_T(\tau)(g), \qquad g \in K_0A, \ \tau \in TB.$$

It follows that there exists an isomorphism

$$\phi : \ A \longrightarrow B$$

giving rise to ϕ_0, ϕ_1, ϕ_T.

Remark 6.9. Since the C^*-algebras $C(T_{II,k}), C(T_{III,k})$ and $C(S^2)$ are not stably generated, our proof heavily depends on the results that, certain G-δ

multiplicative maps (with parts of point evaluations of sufficiently large sizes) are approximated by true homomorphisms in §5. We believe that such results should play important role in the future study of general simple C^*-algebras (with or without real rank zero property).

REMARK 6.10. From a result of J. Villadsen, [V1], one knows that the restriction on the dimensions of the spaces $X_{n,i}$ can not be removed.

In [G5]—an appendix to this article, we will show that the condition of uniformly bounded dimensions of local spectra can be replaced by the condition of very slow dimension growth. The main difficulty for this case is that we can not obtain the homomorphism from B_n to $A_{s(n)}$ as the homomorphism α_n in the above proof. (The α_n in this case will be only a sufficiently multiplicative map.) But we can still construct homomorphisms $\psi_n: B_n \to B_{n+1}$, if we carefully choose α_n and β_n. This case does not create essential difficulty, but makes the proof much longer. We refer it to [G5] , a separate appendix to this paper.

It could be an improvement if one can replace the very slow dimension growth condition by the slow dimension growth condition. The author believes that the theorem is also true for this case. In fact, if one can prove the corresponding decomposition results (see Section 4) for the AH-algebras with slow dimension growth, then the Main Theorem in this article would also hold, by the same proof as in [G5].

COROLLARY 6.11. *Suppose that* $A = \lim_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ *is a simple inductive limit C^*-algebras. Suppose that each of the spaces $X_{n,i}$ is of the forms: $\{pt\}, [0,1], S^1, S^2, T_{II,k}$ or $T_{III,k}$. And suppose that all the connecting maps $\phi_{n,m}$ are injective. For any $F \subset A_n, \varepsilon > 0$, if m is large enough, then there are two mutually orthogonal projections $P, Q \in A_m$ and two homomorphisms $\phi: A_n \to PA_mP$ and $\psi: A_n \to QA_mQ$ such that*

(1) $\|\phi_{n,m}(f) - (\phi\oplus\psi)(f)\| < \varepsilon$ for all $f \in F$;

(2) $\phi(F)$ is weakly approximately constant to within ε and $SPV(\phi) < \varepsilon$;

(3) ψ factors through matrix algebras over $C[0,1]$.

Furthermore, if for some i,j, the partial map $\phi_{n,m}^{i,j}: A_n^i \to A_m^j$ is homotopic to a homomorphism with finite dimensional image, then the part ϕ of the decomposition $\phi\oplus\psi$ corresponding to this partial map can be chosen to be zero (or, equivalently, $\phi_{n,m}^{i,j}$ itself is close to a homomorphism factoring through a matrix algebra over $C[0,1]$).

Proof: It follows from the corollary of 2.3 of [Su] that for any $M_l(C(X))$, $\varepsilon > 0$, there are $\varepsilon_1 > 0$ and a finite subset F of self adjoint elements of $M_l(C(X))$ (i.e., $F \subset (M_l(C(X)))_{s.a}$) such that for any homomorphism $\phi: M_l(C(X)) \to M_{l_1}(C(Y))$, if $\phi(F)$ is weakly approximately constant to within ε_1, then $\mathrm{SPV}(\phi) < \varepsilon$. Therefore, for the desired condition (2) above, we only need to make $\phi(F)$ weakly approximately constant to within $\min(\varepsilon, \varepsilon_1)$.

To simplify the notation, we still denote $\min(\varepsilon, \varepsilon_1)$ by ε.
Now, the main body of the corollary follows from Lemma 6.2 and Theorem 5.32b. Namely, first apply Lemma 6.2 to id : $A_n \to A_n$ (in place of ϕ) and A_n in place of both B and A_n to find A_{n_1} (in place of A_m) and homomorphism $\alpha : A_n \to A_{n_1}$ such that α satisfies the dichotomy condition and such that α is sufficiently close to ϕ_{n,n_1} on the finite set F. Then apply Lemma 5.32b to A_n and $F \subset A_n$ (in place of B and $G \subset B$), A_{n_1} and $\phi_{n,n_1}(F) \subset A_{n_1}$ (in place of A_n and $F \subset A$), and $\alpha : A_n \to A_{n_1}$ (in place of $\alpha : B \to A_n$) to construct the desired decomposition. (Note that we use the following trivial fact: If two maps $\phi_1, \phi_2 : A_n \to A_m$ are approximately equal to each other to within ε_1 on the finite set F and the set $\phi_1(F)$ is weakly approximately constant to within ε_2, then the set $\phi_2(F)$ is weakly approximately constant to within $2\varepsilon_1 + \varepsilon_2$.)
For the last part of the Corollary, one needs to notice the following facts.
(i) In the additional parts of Corollaries 5.22 and 5.23, if the homomorphisms ψ are homomorphisms with finite dimensional images, then the homomorphisms $\tilde{\phi}$ in the corollaries 5.22 and 5.23 are also homomorphisms with finite dimensional images.
(ii) In Lemma 5.28, if both ϕ and ϕ_1 are homomorphisms factoring through interval algebras (this condition implies that they are homotopic to homomorphisms with finite dimensional images), then the homomorphism ψ in Lemma 5.28 (with $[\psi]_* = [\phi_2]_*$) can be chosen to be a homomorphism with finite dimensional image.
With the above facts, if $\phi^{i,j}_{n,m}$ is homotopic to a homomorphism with finite dimensional image and if $X_{n,i} \neq S^1$, then the corresponding part of ϕ in our corollary could be chosen to be a homomorphism with finite dimensional image, and therefore it can also factor through matrix algebras over $C[0,1]$. So, we can put it together with the part ψ and hence the part ϕ disappears from the decomposition of this partial map. This proves the additional part for the case $X_{n,i} \neq S^1$.
For the case that $X_{n,i} = S^1$, the additional part of the corollary follows from the following claim.
Claim: For any unitary $u \in A^i_n$ and any $\varepsilon > 0$, there is an integer $N > n$ such that if $m > N$, and if $\phi^{i,j}_{n,m}(u)$ is in the path connected component of the unit in the unitary group of $\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})A^j_m\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})$, then there is a self adjoint element $a \in \phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})A^j_m\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})$ such that

$$\|\phi^{i,j}_{n,m}(u) - e^{2\pi i a}\| < \varepsilon.$$

(Obviously, if $\phi^{i,j}_{n,m}$ is homotopic to a homomorphism with finite dimensional image, then $\phi^{i,j}_{n,m}(u)$ is in the path connected component of the unit element in the unitary group of $\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})A^j_m\phi^{i,j}_{n,m}(\mathbf{1}_{A^i_n})$.)
The proof of the above claim is exactly the same as the proof of the main theorem of [Phi3]: the simple inductive limit C^*-algebra in our corollary has exponential rank at most $1+\varepsilon$. We omit the details.

We point out that, in [EGL], we will only need this result for the case $X_{m,j} = S^2$. Since $\dim(S^2) \leq 2$, $PM_\bullet(C(S^2))P$ has exponential rank at most $1+\varepsilon$. Therefore, the claim for the case $X_{m,j} = S^2$ ($X_{n,i} = S^1$) is trivial.

□

By Lemma 1.3.3, the above corollary also holds for the case of
$A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$, instead of $A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i}))$.

COROLLARY 6.12. *Suppose that $A = \lim\limits_{n\to\infty}(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$, $\phi_{n,m})$ is a simple inductive limit C^*-algebra. Suppose that each of the spaces $X_{n,i}$ is of the forms: $\{pt\}, [0,1], S^1, S^2, T_{II,k}$ or $T_{III,k}$. And suppose that all the connecting maps $\phi_{n,m}$ are injective. For any $F \subset A_n, \varepsilon > 0$, if m is large enough, then there are two mutually orthogonal projections $P, Q \in A_m$ and two homomorphisms $\phi : A_n \to PA_mP$ and $\psi : A_n \to QA_mQ$ such that*
(1) $\|\phi_{n,m}(f) - (\phi\oplus\psi)(f)\| < \varepsilon$ for all $f \in F$;
(2) $\phi(F)$ is weakly approximately constant to within ε and $SPV(\phi) < \varepsilon$;
(3) ψ factors through matrix algebras over $C[0,1]$.
Furthermore, if for some i, j, the partial map $\phi_{n,m}^{i,j} : A_n^i \to A_m^j$ is homotopic to a homomorphism with finite dimensional image, then the part ϕ of the decomposition $\phi\oplus\psi$ corresponding to this partial map can be chosen to be zero (or, equivalently, $\phi_{n,m}^{i,j}$ itself is close to a homomorphism factoring through a matrix algebra over $C[0,1]$).

Proof: By Lemma 1.3.3, there is an inductive system

$$\tilde{A} = \lim_{n\to\infty}(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{\{n,i\}}(C(X_{n,i})), \tilde{\phi}_{n,m})$$

such that each $P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ is a corner of $M_{\{n,i\}}(C(X_{n,i}))$ and $\phi_{n,m} = \tilde{\phi}_{n,m}|_{P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}}$. $\tilde{A}$ is simple since it is stably isomorphic to a simple C^*-algebra A. $\tilde{\phi}_{n,m}$ are injective since $\phi_{n,m}$ are injective. Apply Corollary 6.11 to $F \cup \{\mathbf{1}_{A_n^i}\}_{i=1}^{t_n} \subset A_n \subset \tilde{A}_n$ and $\frac{\varepsilon}{4} > 0$ to obtain $\tilde{\phi}$ and $\tilde{\psi}$ as the homomorphisms ϕ and ψ in Corollary 6.11. Since

$$\|(\tilde{\phi}+\tilde{\psi})(\mathbf{1}_{A_n^i}) - \tilde{\phi}_{n,m}(\mathbf{1}_{A_n^i})\| < \frac{\varepsilon}{4}, \quad \forall i,$$

there is a unitary $u \in \tilde{A}_m$ such that $\|u - \mathbf{1}\| < \frac{\varepsilon}{2}$ and

$$u((\tilde{\phi}+\tilde{\psi})(\mathbf{1}_{A_n^i}))u^* = \tilde{\phi}_{n,m}(\mathbf{1}_{A_n^i}) = \phi_{n,m}(\mathbf{1}_{A_n^i}), \quad \forall i$$

Finally, let

$$\phi = (\mathrm{Ad}u \circ \tilde{\phi})|_{A_n} \quad \text{and} \quad \psi = (\mathrm{Ad}u \circ \tilde{\psi})|_{A_n}$$

to obtain our corollary.

□

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