Derived Categories of Coherent Sheaves on Rational Homogeneous Manifolds

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Received: February 14, 2006
Revised: May 15, 2006

Communicated by Thomas Peternell

Abstract. One way to reformulate the celebrated theorem of Beilinson is that $(\mathcal{O}(-n), \dots, \mathcal{O})$ and $(\Omega^n(n), \dots, \Omega^1(1), \mathcal{O})$ are strong complete exceptional sequences in $D^b(Coh\,\mathbb{P}^n)$, the bounded derived category of coherent sheaves on $\mathbb{P}^n$. In a series of papers ([Ka1], [Ka2], [Ka3]) M. M. Kapranov generalized this result to flag manifolds of type A_n and quadrics. In another direction, Y. Kawamata has recently proven existence of complete exceptional sequences on toric varieties ([Kaw]).

Starting point of the present work is a conjecture of F. Catanese which says that on every rational homogeneous manifold $X = G/P$, where G is a connected complex semisimple Lie group and $P \subset G$ a parabolic subgroup, there should exist a complete strong exceptional poset (cf. def. 2.1.7 (B)) and a bijection of the elements of the poset with the Schubert varieties in X such that the partial order on the poset is the order induced by the Bruhat-Chevalley order (cf. conjecture 2.2.1 (A)). An answer to this question would also be of interest with regard to a conjecture of B. Dubrovin ([Du], conj. 4.2.2) which has its source in considerations concerning a hypothetical mirror partner of a projective variety Y: There is a complete exceptional sequence in $D^b(Coh\,Y)$ if and only if the quantum cohomology of Y is generically semisimple (the complete form of the conjecture also makes a prediction about the Gram matrix of such a collection). A proof of this conjecture would also support M. Kontsevich's homological mirror conjecture, one of the most important open problems in applications of complex geometry to physics today (cf. [Kon]).

The goal of this work will be to provide further evidence for F. Catanese's conjecture, to clarify some aspects of it and to supply new techniques. In section 2 it is shown among other things that

the length of every complete exceptional sequence on X must be the number of Schubert varieties in X and that one can find a complete exceptional sequence on the product of two varieties once one knows such sequences on the single factors, both of which follow from known methods developed by Rudakov, Gorodentsev, Bondal et al. Thus one reduces the problem to the case $X = G/P$ with G simple. Furthermore it is shown that the conjecture holds true for the sequences given by Kapranov for Grassmannians and quadrics. One computes the matrix of the bilinear form on the Grothendieck K-group $K_\circ(X)$ given by the Euler characteristic with respect to the basis formed by the classes of structure sheaves of Schubert varieties in X; this matrix is conjugate to the Gram matrix of a complete exceptional sequence. Section 3 contains a proof of theorem 3.2.7 which gives complete exceptional sequences on quadric bundles over base manifolds on which such sequences are known. This enlarges substantially the class of varieties (in particular rational homogeneous manifolds) on which those sequences are known to exist. In the remainder of section 3 we consider varieties of isotropic flags in a symplectic resp. orthogonal vector space. By a theorem due to Orlov (thm. 3.1.5) one reduces the problem of finding complete exceptional sequences on them to the case of isotropic Grassmannians. For these, theorem 3.3.3 gives generators of the derived category which are homogeneous vector bundles; in special cases those can be used to construct complete exceptional collections. In subsection 3.4 it is shown how one can extend the preceding method to the orthogonal case with the help of theorem 3.2.7. In particular we prove theorem 3.4.1 which gives a generating set for the derived category of coherent sheaves on the Grassmannian of isotropic 3-planes in a 7-dimensional orthogonal vector space. Section 4 is dedicated to providing the geometric motivation of Catanese's conjecture and it contains an alternative approach to the construction of complete exceptional sequences on rational homogeneous manifolds which is based on a theorem of M. Brion (thm. 4.1.1) and cellular resolutions of monomial ideals à la Bayer/Sturmfels. We give a new proof of the theorem of Beilinson on $\mathbb{P}^n$ in order to show that this approach might work in general. We also prove theorem 4.2.5 which gives a concrete description of certain functors that have to be investigated in this approach.

2000 Mathematics Subject Classification: 14M15, 14F05; 18E30

Keywords and Phrases: flag varieties, rational homogeneous manifolds, derived category

Contents

1 Introduction

The concept of derived category of an Abelian category $\mathcal{A}$, which gives a transparent and compact way to handle the totality of cohomological data attached to $\mathcal{A}$ and puts a given object of $\mathcal{A}$ and all of its resolutions on equal footing, was conceived by Grothendieck at the beginning of the 1960's and their internal structure was axiomatized by Verdier through the notion of triangulated category in his 1967 thesis (cf. [Ver1], [Ver2]). Verdier's axioms for distinguished triangles still allow for some pathologies (cf. [GeMa], IV.1, 7) and in [BK] it was suggested how to replace them by more satisfactory ones, but since the former are in current use, they will also be the basis of this text. One may consult [Nee] for foundational questions on triangulated categories.
However, it was only in 1978 that people laid hands on "concrete" derived categories of geometrical significance (cf. [Bei] and [BGG2]), and A. A. Beilinson constructed strong complete exceptional sequences of vector bundles for $D^b(Coh\,\mathbb{P}^n)$, the bounded derived category of coherent sheaves on $\mathbb{P}^n$. The terminology is explained in section 2, def. 2.1.7, below, but roughly the simplification brought about by Beilinson's theorem is analogous to the construction of a semi-orthonormal basis $(e_1, \dots, e_d)$ for a vector space equipped with a non-degenerate (non-symmetric) bilinear form χ (i.e., $\chi(e_i, e_i) = 1\ \forall i$, $\chi(e_j, e_i) = 0\ \forall j > i$)).
Beilinson's theorem represented a spectacular breakthrough and, among other things, his technique was applied to the study of moduli spaces of semi-stable sheaves of given rank and Chern classes on $\mathbb{P}^2$ and $\mathbb{P}^3$ by Horrocks,

Barth/Hulek, Drézet/Le Potier (cf. [OSS], [Po] and references therein).
Recently, A. Canonaco has obtained a generalization of Beilinson's theorem to weighted projective spaces and applied it to the study of canonical projections of surfaces of general type on a 3-dimensional weighted projective space (cf. [Can], cf. also [AKO]).
From 1984 onwards, in a series of papers [Ka1], [Ka2], [Ka3], M. M. Kapranov found strong complete exceptional sequences on Grassmannians and flag varieties of type A_n and on quadrics. Subsequently, exceptional sequences alongside with some new concepts introduced in the meantime such as helices, mutations, semi-orthogonal decompositions etc. were intensively studied, especially in Russia, an account of which can be found in the volume [Ru1] summarizing a series of seminars conducted by A. N. Rudakov in Moscow (cf. also [Bo], [BoKa], [Or]). Nevertheless, despite the wealth of new techniques introduced in the process, many basic questions concerning exceptional sequences are still very much open. These fall into two main classes: first questions of existence: E.g., do complete exceptional sequences always exist on rational homogeneous manifolds? (For toric varieties existence of complete exceptional sequences was proven very recently by Kawamata, cf. [Kaw].) Secondly, one often does not know if basic intuitions derived from semi-orthogonal linear algebra hold true in the framework of exceptional sequences, and thus one does not have enough flexibility to manipulate them, e.g.: Can every exceptional bundle on a variety X on which complete exceptional sequences are known to exist (projective spaces, quadrics...) be included in a complete exceptional sequence?
To round off this brief historical sketch, one should not forget to mention that derived categories have proven to be of geometrical significance in a lot of other contexts, e.g. through Fourier-Mukai transforms and the reconstruction theorem of Bondal-Orlov for smooth projective varieties with ample canonical or anti-canonical class (cf. [Or2]), in the theory of perverse sheaves and the generalized Riemann-Hilbert correspondence (cf. [BBD]), or in the recent proof of T. Bridgeland that birational Calabi-Yau threefolds have equivalent derived categories and in particular the same Hodge numbers (cf. [Brid]). Interest in derived categories was also extremely stimulated by M. Kontsevich's proposal for homological mirror symmetry ([Kon]) on the one side and by new applications to minimal model theory on the other side.
Let me now describe the aim and contents of this work. Roughly speaking, the problem is to give as concrete as possible a description of the (bounded) derived categories of coherent sheaves on rational homogeneous manifolds $X = G/P$, G a connected complex semisimple Lie group, $P \subset G$ a parabolic subgroup. More precisely, the following set of main questions and problems, ranging from the modest to the more ambitious, have served as programmatic guidelines:

P 1. Find generating sets of $D^b(Coh\, X)$ with as few elements as possible. (Here a set of elements of $D^b(Coh\, X)$ is called a *generating set* if the smallest full triangulated subcategory containing this set is equivalent to $D^b(Coh\, X)$).

We will see in subsection 2.3 below that the number of elements in a generating set is always bigger or equal to the number of Schubert varieties in X.
In the next two problems we mean by a complete exceptional sequence an ordered tuple $(E_1, \dots, E_n)$ of objects $E_1, \dots, E_n$ of $D^b(Coh\, X)$ which form a generating set and such that moreover $R^\bullet\mathrm{Hom}(E_i, E_j) = 0$ for all $i > j$, $R^\bullet\mathrm{Hom}(E_i, E_i) = \mathbb{C}$ (in degree 0) for all i. If in addition all extension groups in nonzero degrees between the elements E_i vanish we speak of a strong complete exceptional sequence. See section 2, def. 2.1.7, for further discussion.

P 2. Do there always exist complete exceptional sequences in $D^b(Coh\, X)$?

P 3. Do there always exist strong complete exceptional sequences in $D^b(Coh\, X)$?

Besides the examples found by Kapranov mentioned above, the only other substantially different examples I know of in answer to P 3. is the one given by A. V. Samokhin in [Sa] for the Lagrangian Grassmannian of totally isotropic 3-planes in a 6-dimensional symplectic vector space and, as an extension of this, some examples in [Kuz].
In the next problem we mean by a complete strong exceptional poset a set of objects $\{E_1, \dots, E_n\}$ of $D^b(Coh\, X)$ that generate $D^b(Coh\, X)$ and satisfy $R^\bullet\mathrm{Hom}(E_i, E_i) = \mathbb{C}$ (in degree 0) for all i and such that all extension groups in nonzero degrees between the E_i vanish, together with a partial order $\le$ on $\{E_1, \dots, E_n\}$ subject to the condition: $\mathrm{Hom}(E_j, E_i) = 0$ for $j \ge i$, $j \ne i$ (cf. def. 2.1.7 (B)).

P 4. Catanese's conjecture: On any $X = G/P$ there exists a complete strong exceptional poset $(\{E_1, \dots, E_n\}, \le)$ together with a bijection of the elements of the poset with the Schubert varieties in X such that $\le$ is the partial order induced by the Bruhat-Chevalley order (cf. conj. 2.2.1 (A)).

P 5. Dubrovin's conjecture (cf. [Du], conj. 4.2.2; slightly modified afterwards in [Bay]; cf. also [B-M]): The (small) quantum cohomology of a smooth projective variety Y is generically semi-simple if and only if there exists a complete exceptional sequence in $D^b(Coh\, Y)$ (Dubrovin also relates the Gram matrix of the exceptional sequence to quantum-cohomological data but we omit this part of the conjecture).

Roughly speaking, quantum cohomology endows the usual cohomology space with complex coefficients $H^*(Y)$ of Y with a new commutative associative multiplication $\circ_\omega : H^*(Y) \times H^*(Y) \to H^*(Y)$ depending on a complexified Kähler class $\omega \in H^2(Y, \mathbb{C})$, i.e. the imaginary part of ω is in the Kähler cone of Y (here we assume $H^{odd}(Y) = 0$ to avoid working with supercommutative rings). The condition that the quantum cohomology of Y is generically semi-simple means that for generic values of ω the resulting algebra is semi-simple. The validity of this conjecture would provide further evidence for the famous homological mirror conjecture by Kontsevich ([Kon]). However, we will not

deal with quantum cohomology in this work.
Before stating the results, a word of explanation is in order to clarify why we narrow down the focus to rational homogeneous manifolds:

- Exceptional vector bundles need not always exist on an arbitrary smooth projective variety; e.g., if the canonical class of Y is trivial, they never exist (see the explanation following definition 2.1.3).

- $D^b(Coh\,Y)$ need not be finitely generated, e.g., if Y is an Abelian variety (see the explanation following definition 2.1.3).

- If we assume that Y is Fano, then the Kodaira vanishing theorem tells us that all line bundles are exceptional, so we have at least some *a priori* supply of exceptional bundles.

- Within the class of Fano manifolds, the rational homogeneous spaces $X = G/P$ are distinguished by the fact that they are amenable to geometric, representation-theoretic and combinatorial methods alike.

Next we will state right away the main results obtained, keeping the numbering of the text and adding a word of explanation to each.
Let V be a $2n$-dimensional symplectic vector space and $\mathrm{IGrass}(k,V)$ the Grassmannian of k-dimensional isotropic subspaces of V with tautological subbundle $\mathcal{R}$. $\Sigma^\bullet$ denotes the Schur functor (see subsection 2.2 below for explanation).

Theorem 3.3.3. *The derived category $D^b(Coh(\mathrm{IGrass}(k,V)))$ is generated by the bundles $\Sigma^\nu\mathcal{R}$, where ν runs over Young diagrams Y which satisfy*

$$\begin{aligned}(\text{number of columns of } Y) &\le 2n-k\,,\\ k \ge (\text{number of rows of } Y) &\ge (\text{number of columns of } Y) - 2(n-k)\,.\end{aligned}$$

This result pertains to P 1. Moreover, we will see in subsection 3.3 that P 2. for isotropic flag manifolds of type C_n can be reduced to P 2. for isotropic Grassmannians. Through examples 3.3.6-3.3.8 we show that theorem 3.3.3 gives a set of bundles which is in special cases manageable enough to obtain from it a complete exceptional sequence. In general, however, this last step is a difficult combinatorial puzzle relying on Bott's theorem for the cohomology of homogeneous bundles and Schur complexes derived from tautological exact sequences on the respective Grassmannians.
For the notion of semi-orthogonal decomposition in the next theorem we refer to definition 2.1.17 and for the definition of spinor bundles Σ, $\Sigma^\pm$ for the orthogonal vector bundle $\mathcal{O}_{\mathcal{Q}}(-1)^\perp/\mathcal{O}_{\mathcal{Q}}(-1)$ we refer to subsection 3.2.

Theorem 3.2.7. *Let X be a smooth projective variety, $\mathcal{E}$ an orthogonal vector bundle of rank $r+1$ on X (i.e., $\mathcal{E}$ comes equipped with a quadratic form $q \in \Gamma(X, \mathrm{Sym}^2\mathcal{E}^\vee)$ which is non-degenerate on each fibre), $\mathcal{Q} \subset \mathbb{P}(\mathcal{E})$ the associated* *quadric bundle, and let $\mathcal{E}$ admit spinor bundles (see subsection 3.2). Then there is a semiorthogonal decomposition*

$$D^b(\mathcal{Q}) = \big\langle D^b(X) \otimes \Sigma(-r+1), D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-r+2), \\ \dots, D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-1), D^b(X) \big\rangle$$

for $r+1$ odd and

$$D^b(\mathcal{Q}) = \big\langle D^b(X) \otimes \Sigma^+(-r+1), D^b(X) \otimes \Sigma^-(-r+1), \\ D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-r+2), \dots, D^b(X) \otimes \mathcal{O}_{\mathcal{Q}}(-1), D^b(X) \big\rangle$$

for $r+1$ even.

This theorem is an extension to the relative case of a theorem of [Ka2]. It enlarges substantially the class of varieties (especially rational-homogeneous varieties) on which complete exceptional sequences are proven to exist (P 2). It will also be the substantial ingredient in subsection 3.4: Let V be a 7-dimensional orthogonal vector space, $\mathrm{IGrass}(3, V)$ the Grassmannian of isotropic 3-planes in V, $\mathcal{R}$ the tautological subbundle on it; L denotes the ample generator of $\mathrm{Pic}(\mathrm{IGrass}(3, V)) \simeq \mathbb{Z}$ (a square root of $\mathcal{O}(1)$ in the Plücker embedding). For more information cf. subsection 3.4.

THEOREM 3.4.1. *The derived category $D^b(Coh\,\mathrm{IGrass}(3, V))$ is generated as triangulated category by the following 22 vector bundles:*

$$\bigwedge^2 \mathcal{R}(-1),\ \mathcal{O}(-2),\ \mathcal{R}(-2) \otimes L,\ \mathrm{Sym}^2\mathcal{R}(-1) \otimes L,\ \mathcal{O}(-3) \otimes L,$$
$$\bigwedge^2 \mathcal{R}(-2) \otimes L,\ \Sigma^{2,1}\mathcal{R}(-1) \otimes L,\ \mathcal{R}(-1),\ \mathcal{O}(-2) \otimes L,\ \mathcal{O}(-1),$$
$$\mathcal{R}(-1) \otimes L,\ \bigwedge^2 \mathcal{R}(-1) \otimes L,\ \Sigma^{2,1}\mathcal{R} \otimes L,\ \mathrm{Sym}^2\mathcal{R}^\vee(-2) \otimes L,\ \bigwedge^2 \mathcal{R},\ \mathcal{O},$$
$$\Sigma^{2,1}\mathcal{R},\ \mathrm{Sym}^2\mathcal{R}^\vee(-2),\ \mathcal{O}(-1) \otimes L,\ \mathrm{Sym}^2\mathcal{R}^\vee(-1),\ \bigwedge^2 \mathcal{R} \otimes L,\ \mathcal{R} \otimes L.$$

This result pertains to P 1. again. It is worth mentioning that the expected number of elements in a complete exceptional sequence for $D^b(Coh\,\mathrm{IGrass}(3, V))$ is 8, the number of Schubert varieties in $\mathrm{IGrass}(3, V)$. In addition, one should remark that P 2. for isotropic flag manifold of type B_n or D_n can again be reduced to isotropic Grassmannians. Moreover, the method of subsection 3.4 applies to all orthogonal isotropic Grassmannians alike, but since the computations tend to become very large, we restrict our attention to a particular case.

Beilinson proved his theorem on $\mathbb{P}^n$ using a resolution of the structure sheaf of the diagonal and considering the functor $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_\Delta) \simeq \mathrm{id}_{D^b(Coh\,\mathbb{P}^n)}$ (here $p_1, p_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projections onto the two factors). The situation is complicated on general rational homogeneous manifolds X because resolutions of the structure sheaf of the diagonal $\Delta \subset X \times X$ analogous to those used in [Bei], [Ka1], [Ka2], [Ka3] to exhibit complete exceptional sequences, are not known. The preceding theorems are proved by "fibrational techniques". Section 4 outlines an alternative approach: In fact, M. Brion ([Bri]) constructed, for any rational homogeneous manifold X, a degeneration of the diagonal Δ_X into $\mathfrak{X}_0$, which is a union, over the Schubert varieties in X, of the products of a Schubert variety with its opposite Schubert variety (cf. thm. 4.1.1). It turns out that it is important to describe the functors $Rp_{2*}(p_1^*(-)\otimes^L \mathcal{O}_{\mathfrak{X}_0})$ which, contrary to what one might expect at first glance, are no longer isomorphic to the identity functor by Orlov's representability theorem [Or2], thm. 3.2.1 (but one might hope to reconstruct the identity out of $Rp_{2*}(p_1^*(-)\otimes^L \mathcal{O}_{\mathfrak{X}_0})$ and some infinitesimal data attached to the degeneration). For $\mathbb{P}^n$ this is accomplished by the following

Theorem 4.2.5. *Let $\{pt\} = L_0 \subset L_1 \subset \cdots \subset L_n = \mathbb{P}^n$ be a full flag of projective linear subspaces of $\mathbb{P}^n$ (the Schubert varieties in $\mathbb{P}^n$) and let $L^0 = \mathbb{P}^n \supset L^1 \supset \cdots \supset L^n = \{pt\}$ be a complete flag in general position with respect to the L_j.*
For $d \geq 0$ one has in $D^b(Coh\,\mathbb{P}^n)$

$$Rp_{2*}(p_1^*(\mathcal{O}(d))\otimes^L \mathcal{O}_{\mathfrak{X}_0}) \simeq \bigoplus_{j=0}^{n} \mathcal{O}_{L_j}\otimes H^0(L^j,\mathcal{O}(d))^\vee/H^0(L^{j+1},\mathcal{O}(d))^\vee\,.$$

Moreover, one can also describe completely the effect of $Rp_{2*}(p_1^*(-)\otimes^L \mathcal{O}_{\mathfrak{X}_0})$ on morphisms (cf. subsection 4.2 below).

The proof uses the technique of cellular resolutions of monomial ideals of Bayer and Sturmfels ([B-S]). We also show in subsection 4.2 that Beilinson's theorem on $\mathbb{P}^n$ can be recovered by our method with a proof that uses only $\mathfrak{X}_0$ (see remark 4.2.6).
It should be added that we will not completely ignore the second part of P 4. concerning Hom-spaces: In section 2 we show that the conjecture in P 4. is valid in full for the complete strong exceptional sequences found by Kapranov on Grassmannians and quadrics (cf. [Ka3]). In remark 2.3.8 we discuss a possibility for relating the Gram matrix of a strong complete exceptional sequence on a rational homogeneous manifold with the Bruhat-Chevalley order on Schubert cells.
Additional information about the content of each section can be found at the beginning of the respective section.
Acknowledgements. I would like to thank my thesis advisor Fabrizio Catanese for posing the problem and several discussions on it. Special thanks also to Michel Brion for filling in my insufficient knowledge of representation theory and algebraic groups on a number of occasions and for fruitful suggestions and discussions.

2 Tools and background: getting off the ground

This section supplies the concepts and dictionary that will be used throughout the text. We state a conjecture due to F. Catanese which was the motivational backbone of this work and discuss its relation to work of M. M. Kapranov. Moreover, we prove some results that are useful in the study of the derived categories of coherent sheaves on rational homogeneous varieties, but do not yet tackle the problem of constructing complete exceptional sequences on them: This will be the subject matter of sections 3 and 4.

2.1 Exceptional sequences

Throughout the text we will work over the ground field $\mathbb{C}$ of complex numbers. The classical theorem of Beilinson (cf. [Bei]) can be stated as follows.

Theorem 2.1.1. *Consider the following two ordered sequences of sheaves on* $\mathbb{P}^n = \mathbb{P}(V)$*,* V *an* $n+1$ *dimensional vector space:*

$$\mathfrak{B} = (\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O})$$
$$\mathfrak{B}' = \left(\Omega^n(n), \dots, \Omega^1(1), \mathcal{O}\right).$$

Then $D^b(Coh\mathbb{P}^n)$ *is equivalent as a triangulated category to the homotopy category of bounded complexes of sheaves on* $\mathbb{P}^n$ *whose terms are finite direct sums of sheaves in* $\mathfrak{B}$ *(and the same for* $\mathfrak{B}$ *replaced with* $\mathfrak{B}'$*).*
Moreover, one has the following stronger assertion: If $\Lambda = \bigoplus_{i=0}^{n+1} \wedge^i V$ *and* $S = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i V^*$ *are the* $\mathbb{Z}$*-graded exterior algebra of* V*, resp. symmetric algebra of* V^**, and* $K^b_{[0,n]}\Lambda$ *resp.* $K^b_{[0,n]}S$ *are the homotopy categories of bounded complexes whose terms are finite direct sums of free modules* $\Lambda[i]$*, resp.* $S[i]$*, for* $0 \le i \le n$*, and whose morphisms are homogeneous graded of degree* 0*, then*

$$K^b_{[0,n]}\Lambda \simeq D^b(Coh\mathbb{P}^n) \quad K^b_{[0,n]}S \simeq D^b(Coh\mathbb{P}^n)$$

as triangulated categories, the equivalences being given by sending $\Lambda[i]$ *to* $\Omega^i(i)$ *and* $S[i]$ *to* $\mathcal{O}(-i)$ *(*$\Lambda[i]$*,* $S[i]$ *have their generator in degree* i*).*

One would like to have an analogous result on any rational homogeneous variety X, i.e. a rational projective variety with a transitive Lie group action or equivalently (cf. [Akh], 3.2, thm. 2) a coset manifold G/P where G is a connected semisimple complex Lie group (which can be assumed to be simply connected) and $P \subset G$ is a parabolic subgroup. However, to give a precise meaning to this wish, one should first try to capture some formal features of Beilinson's theorem in the form of suitable definitions; thus we will recall next a couple of notions which have become standard by now, taking theorem 2.1.1 as a model.
Let $\mathcal{A}$ be an Abelian category.

Definition 2.1.2. A class of objects $\mathcal{C}$ *generates* $D^b(\mathcal{A})$ if the smallest full triangulated subcategory containing the objects of $\mathcal{C}$ is equivalent to $D^b(\mathcal{A})$. If $\mathcal{C}$ is a set, we will also call $\mathcal{C}$ a *generating set* in the sequel.

Unravelling this definition, one finds that this is equivalent to saying that, up to isomorphism, every object in $D^b(\mathcal{A})$ can be obtained by successively enlarging $\mathcal{C}$ through the following operations: Taking finite direct sums, shifting in $D^b(\mathcal{A})$ (i.e., applying the translation functor), and taking a cone Z of a morphism $u : X \to Y$ between objects already constructed: This means we complete u to a distinguished triangle $X \xrightarrow{u} Y \longrightarrow Z \longrightarrow X[1]$.
The sheaves $\Omega^i(i)$ and $\mathcal{O}(-i)$ in theorem 2.1.1 have the distinctive property of being "exceptional".

Definition 2.1.3. An object E in $D^b(\mathcal{A})$ is said to be *exceptional* if

$$\mathrm{Hom}(E, E) \simeq \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^i(E, E) = 0 \; \forall i \neq 0.$$

If Y is a smooth projective variety of dimension n, exceptional objects need not always exist (e.g., if Y has trivial canonical class this is simply precluded by Serre duality since then $\mathrm{Hom}(E, E) \simeq \mathrm{Ext}^n(E, E) \neq 0$).
What is worse, $D^b(Coh\, Y)$ need not even possess a finite generating set: In fact we will see in subsection 2.3 below that if $D^b(Coh\, Y)$ is finitely generated, then $A(Y) \otimes \mathbb{Q} = \bigoplus_{r=0}^{\dim Y} A^r(Y) \otimes \mathbb{Q}$, the rational Chow ring of Y, is finite dimensional (here $A^r(Y)$ denotes the group of cycles of codimension r on Y modulo rational equivalence). But, for instance, if Y is an Abelian variety, $A^1(Y) \otimes \mathbb{Q} \simeq \mathrm{Pic}\, Y \otimes \mathbb{Q}$ does not have finite dimension.
Recall that a vector bundle $\mathcal{V}$ on a rational homogeneous variety $X = G/P$ is called *G-homogeneous* if there is a G-action on $\mathcal{V}$ which lifts the G-action on X and is linear on the fibres. It is well known that this is equivalent to saying that $\mathcal{V} \simeq G \times_\varrho V$, where $\varrho : P \to \mathrm{GL}(V)$ is some representation of the algebraic group P and $G \times_\varrho V$ is the quotient of $G \times V$ by the action of P given by $p \cdot (g, v) := (gp^{-1}, \varrho(p)v)$, $p \in P$, $g \in G$, $v \in V$. The projection to G/P is induced by the projection of $G \times V$ to G; this construction gives a 1-1 correspondence between representations of the subgroup P and homogeneous vector bundles over G/P (cf. [Akh], section 4.2).
Then we have the following result (mentioned briefly in a number of places, e.g. [Ru1], 6., but without a precise statement or proof).

Proposition 2.1.4. *Let $X = G/P$ be a rational homogeneous manifold with G a simply connected semisimple group, and let $\mathcal{F}$ be an exceptional sheaf on X. Then $\mathcal{F}$ is a G-homogeneous bundle.*

Proof. Let us first agree that a deformation of a coherent sheaf $\mathcal{G}$ on a complex space Y is a triple $(\tilde{\mathcal{G}}, S, s_0)$ where S is another complex space (or germ), $s_0 \in S$, $\tilde{\mathcal{G}}$ is a coherent sheaf on $Y \times S$, flat over S, with $\tilde{\mathcal{G}}\mid_{Y \times \{s_0\}} \simeq \mathcal{G}$ and $\mathrm{Supp}\, \tilde{\mathcal{G}} \to S$ proper. Then one knows that, for the deformation with base a complex space germ, there is a versal deformation and its tangent space at the marked point

is $\mathrm{Ext}^1(\mathcal{G},\mathcal{G})$ (cf. [S-T]).
Let $\sigma : G\times X \to X$ be the group action; then $(\sigma^*\mathcal{F}, G, \mathrm{id}_G)$ is a deformation of $\mathcal{F}$ (flatness can be seen e.g. by embedding X equivariantly in a projective space (cf. [Akh], 3.2) and noting that the Hilbert polynomial of $\sigma^*\mathcal{F}\mid_{\{g\}\times X} = \tau_g^*\mathcal{F}$ is then constant for $g\in G$; here $\tau_g : X\to X$ is the automorphism induced by g). Since $\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) = 0$ one has by the above that $\sigma^*\mathcal{F}$ will be locally trivial over G, i.e. $\sigma^*\mathcal{F}\simeq \mathrm{pr}_2^*\mathcal{F}$ locally over G where $\mathrm{pr}_2 : G\times X\to X$ is the second projection ($\mathcal{F}$ is "rigid"). In particular $\tau_g^*\mathcal{F}\simeq\mathcal{F}\ \forall g\in G$.
Since the locus of points where $\mathcal{F}$ is not locally free is a proper algebraic subset of X and invariant under G by the preceding statement, it is empty because G acts transitively. Thus $\mathcal{F}$ is a vector bundle satisfying $\tau_g^*\mathcal{F}\simeq\mathcal{F}\ \forall g\in G$. Since G is semisimple and assumed to be simply connected, this is enough to imply that $\mathcal{F}$ is a G-homogeneous bundle (a proof of this last assertion due to A. Huckleberry is presented in [Ot2] thm. 9.9). □

Remark 2.1.5. In proposition 2.1.4 one must insist that G be simply connected as an example in [GIT], ch.1, §3 shows : The exceptional bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ is SL_{n+1}-homogeneous, but not homogeneous for the adjoint form PGL_{n+1} with its action $\sigma : PGL_{n+1}\times\mathbb{P}^n\to\mathbb{P}^n$ since the SL_{n+1}-action on $H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ does not factor through PGL_{n+1}.

Remark 2.1.6. It would be interesting to know which rational homogeneous manifolds X enjoy the property that exceptional objects in $D^b(Coh\,X)$ are actually just shifts of exceptional sheaves. It is straightforward to check that this is true on $\mathbb{P}^1$. This is because, if C is a curve, $D^b(Coh\,C)$ is not very interesting: In fancy language, the underlying abelian category is *hereditary* which means $\mathrm{Ext}^2(\mathcal{F},\mathcal{G}) = 0\ \forall\mathcal{F},\mathcal{G}\in\mathrm{obj}\,(Coh\,C)$. It is easy to see (cf. [Ke], 2.5) that then every object Z in $D^b(Coh\,C)$ is isomorphic to the direct sum of shifts of its cohomology sheaves $\bigoplus_{i\in\mathbb{Z}} H^i(Z)[-i]$ whence morphisms between objects Z_1 and Z_2 correspond to tuples $(\varphi_i, e_i)_{i\in\mathbb{Z}}$ with $\varphi_i : H^i(Z_1)\to H^i(Z_2)$ a sheaf morphism and $e_i\in\mathrm{Ext}^1(H^i(Z_1), H^{i-1}(Z_2))$ an extension class . Exceptional objects are indecomposable since they are simple.
The same property holds on $\mathbb{P}^2$ (and more generally on any Del Pezzo surface) by [Gor], thm. 4.3.3, and is conjectured to be true on $\mathbb{P}^n$ in general ([Gor], 3.2.7).

The sequences $\mathfrak{B}$ and $\mathfrak{B}'$ in theorem 2.1.1 are examples of complete strong exceptional sequences (cf. [Ru1] for the development of this notion).

Definition 2.1.7. (A) An n-tuple $(E_1,\dots,E_n)$ of exceptional objects in $D^b(\mathcal{A})$ is called an *exceptional sequence* if

$$\mathrm{Ext}^l(E_j,E_i) = 0 \quad \forall 1\le i<j\le n \text{ and } \forall l\in\mathbb{Z}\,.$$

If in addition

$$\mathrm{Ext}^l(E_j,E_i) = 0 \quad \forall 1\le i,j\le n \text{ and } \forall l\neq 0$$

we call $(E_1, \dots, E_n)$ a *strong exceptional sequence*. The sequence is *complete* if $E_1, \dots, E_n$ generate $D^b(\mathcal{A})$.

(B) In order to phrase conjecture 2.2.1 below precisely, it will be convenient to introduce also the following terminology: A set of exceptional objects $\{E_1, \dots, E_n\}$ in $D^b(\mathcal{A})$ that generates $D^b(\mathcal{A})$ and such that $\mathrm{Ext}^l(E_j, E_i) = 0$ for all $1 \le i, j \le n$ and all $l \ne 0$ will be called a *complete strong exceptional set*. A partial order $\le$ on a complete strong exceptional set is *admissible* if $\mathrm{Hom}(E_j, E_i) = 0$ for all $j \ge i, i \ne j$. A pair $(\{E_1, \dots, E_n\}, \le)$ consisting of a complete strong exceptional set and an admissible partial order on it will be called a *complete strong exceptional poset*.

(C) A *complete very strong exceptional poset* is a pair $(\{E_1, \dots, E_n\}, \le)$ where $\{E_1, \dots, E_n\}$ is a complete strong exceptional set and $\le$ is a partial order on this set such that $\mathrm{Hom}(E_j, E_i) = 0$ unless $i \ge j$.

Obviously every complete strong exceptional sequence is a complete strong exceptional poset (with the partial order being in fact a total order). I think it might be possible that for complete strong exceptional posets in $D^b(Coh\, X)$ which consist of vector bundles, X a rational homogeneous manifold, the converse holds, i.e. any admissible partial order can be refined to a total order which makes the poset into a complete strong exceptional sequence. But I cannot prove this.
Moreover, every complete very strong exceptional poset is in particular a complete strong exceptional poset. If we choose a total order refining the partial order on a complete very strong exceptional poset, we obtain a complete strong exceptional sequence.
Let me explain the usefulness of these concepts by first saying what kind of analogues of Beilinson's theorem 2.1.1 we can expect for $D^b(\mathcal{A})$ once we know the existence of a complete strong exceptional set.
Look at a complete strong exceptional set $\{E_1, \dots, E_n\}$ in $D^b(\mathcal{A})$ consisting of objects E_i, $1 \le i \le n$, of $\mathcal{A}$. If $K^b(\{E_1, \dots, E_n\})$ denotes the homotopy category of bounded complexes in $\mathcal{A}$ whose terms are finite direct sums of the E_i's, it is clear that the natural functor

$$\Phi_{(E_1, \dots, E_n)} : K^b(\{E_1, \dots, E_n\}) \to D^b(\mathcal{A})$$

(composition of the inclusion $K^b(\{E_1, \dots, E_n\}) \hookrightarrow K^b(\mathcal{A})$ with the localization $Q : K^b(\mathcal{A}) \to D^b(\mathcal{A})$) is an equivalence; indeed $\Phi_{(E_1, \dots, E_n)}$ is essentially surjective because $\{E_1, \dots, E_n\}$ is complete and $\Phi_{(E_1, \dots, E_n)}$ is fully faithful because $\mathrm{Ext}^p(E_i, E_j) = 0$ for all $p > 0$ and all i and j implies

$$\mathrm{Hom}_{K^b(\{E_1, \dots, E_n\})}(A, B) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(\Phi_{(E_1, \dots, E_n)}\, A, \Phi_{(E_1, \dots, E_n)}\, B)$$
$$\forall A,\, B \in \mathrm{obj}\, K^b(\{E_1, \dots, E_n\})$$

(cf. [AO], prop. 2.5).
Returning to derived categories of coherent sheaves and dropping the hypothesis that the E_i's be objects of the underlying Abelian category, we have the following stronger theorem of A. I. Bondal:

Theorem 2.1.8. *Let X be a smooth projective variety and $(E_1, \dots, E_n)$ a strong complete exceptional sequence in $D^b(Coh\, X)$. Set $E := \bigoplus_{i=1}^n E_i$, let $A := \mathrm{End}(E) = \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j)$ be the algebra of endomorphisms of E, and denote $\mathrm{mod} - A$ the category of right modules over A which are finite dimensional over $\mathbb{C}$.*
Then the functor

$$R\mathrm{Hom}^\bullet(E, -) : D^b(Coh\,(X)) \to D^b(\mathrm{mod} - A)$$

is an equivalence of categories (note that, for any object Y of $D^b(Coh\,(X))$, $R\mathrm{Hom}^\bullet(E, Y)$ has a natural action from the right by $A = \mathrm{Hom}(E, E)$).
Moreover, the indecomposable projective modules over A are (up to isomorphism) exactly the $P_i := \mathrm{id}_{E_i} \cdot A$, $i = 1, \dots, n$. We have $\mathrm{Hom}_{D^b(Coh\,(X))}(E_i, E_j) \simeq \mathrm{Hom}_A(P_i, P_j)$ and an equivalence

$$K^b(\{P_1, \dots, P_n\}) \xrightarrow{\sim} D^b(\mathrm{mod} - A)$$

where $K^b(\{P_1, \dots, P_n\})$ is the homotopy category of complexes of right A-modules whose terms are finite direct sums of the P_i's.

For a proof see [Bo], §§5 and 6. Thus whenever we have a strong complete exceptional sequence in $D^b(Coh\,(X))$ we get an equivalence of the latter with a homotopy category of projective modules over the algebra of endomorphisms of the sequence. For the sequences $\mathfrak{B}$, $\mathfrak{B}'$ in theorem 2.1.1 we recover Beilinson's theorem (although the objects of the module categories $K^b(\{P_1, \dots, P_n\})$ that theorem 2.1.8 produces in each of these cases will be different from the objects in the module categories $K^b_{[0,n]}S$, resp. $K^b_{[0,n]}\Lambda$, in theorem 2.1.1, the morphisms correspond and the respective module categories are equivalent).
Next suppose that $D^b(Coh\, X)$ on a smooth projective variety X is generated by an exceptional sequence $(E_1, \dots, E_n)$ that is not necessarily strong. Since extension groups in nonzero degrees between members of the sequence need not vanish in this case, one cannot expect a description of $D^b(Coh\, X)$ on a homotopy category level as in theorem 2.1.8. But still the existence of $(E_1, \dots, E_n)$ makes available some very useful computational tools, e.g. Beilinson type spectral sequences. To state the result, we must briefly review some basic material on an operation on exceptional sequences called *mutation*. Mutations are also needed in subsection 2.2 below. Moreover, the very concept of exceptional sequence as a weakening of the concept of strong exceptional sequence was first introduced because strong exceptionality is in general not preserved by mutations, cf. [Bo], introduction p.24 (exceptional sequences are also more flexible in other situations, cf. remark 3.1.3 below).

For A, $B \in \operatorname{obj} D^b(Coh\,X)$ set $\operatorname{Hom}^{\times}(A,B) := \bigoplus_{k\in\mathbb{Z}} \operatorname{Ext}^k(A,B)$, a graded $\mathbb{C}$-vector space. For a graded $\mathbb{C}$-vector space V, $(V^{\vee})^i := \operatorname{Hom}_{\mathbb{C}}(V^{-i},\mathbb{C})$ defines the grading of the dual, and if $X \in \operatorname{obj} D^b(Coh\,X)$, then $V\otimes X$ means $\bigoplus_{i\in\mathbb{Z}} V^i \otimes X[-i]$ where $V^i\otimes X[-i]$ is the direct sum of $\dim V^i$ copies of $X[-i]$.

DEFINITION 2.1.9. Let (E_1,E_2) be an exceptional sequence in $D^b(Coh\,X)$. The *left mutation* $L_{E_1}E_2$ (resp. the *right mutation* $R_{E_2}E_1$) is the object defined by the distinguished triangles

$$L_{E_1}E_2 \longrightarrow \operatorname{Hom}^{\times}(E_1,E_2)\otimes E_1 \xrightarrow{\text{can}} E_2 \longrightarrow L_{E_1}E_2[1]$$
$$(\text{resp.}\quad R_{E_2}E_1[-1] \longrightarrow E_1 \xrightarrow{\text{can}'} \operatorname{Hom}^{\times}(E_1,E_2)^{\vee}\otimes E_2 \longrightarrow R_{E_2}E_1 \quad).$$

Here can resp. can$'$ are the canonical morphisms ("evaluations").

THEOREM 2.1.10. *Let $\mathfrak{E} = (E_1,\dots,E_n)$ be an exceptional sequence in $D^b(Coh\,X)$. Set, for $i=1,\dots,n-1$,*

$$R_i\mathfrak{E} := \left(E_1,\dots,E_{i-1},E_{i+1},R_{E_{i+1}}E_i,E_{i+2},\dots,E_n\right),$$
$$L_i\mathfrak{E} := \left(E_1,\dots,E_{i-1},L_{E_i}E_{i+1},E_i,E_{i+2},\dots,E_n\right).$$

Then $R_i\mathfrak{E}$ and $L_i\mathfrak{E}$ are again exceptional sequences. R_i and L_i are inverse to each other; the R_i's (or L_i's) induce an action of Bd_n, the Artin braid group on n strings, on the class of exceptional sequences with n terms in $D^b(Coh\,X)$. If moreover $\mathfrak{E}$ is complete, so are all the $R_i\mathfrak{E}$'s and $L_i\mathfrak{E}$'s.

For a proof see [Bo], §2.
We shall see in example 2.1.13 that the two exceptional sequences $\mathfrak{B}$, $\mathfrak{B}'$ of theorem 2.1.1 are closely related through a notion that we will introduce next:

DEFINITION 2.1.11. Let $(E_1,\dots,E_n)$ be a complete exceptional sequence in $D^b(Coh\,X)$. For $i=1,\dots,n$ define

$$E_i^{\vee} := L_{E_1}L_{E_2}\dots L_{E_{n-i}}E_{n-i+1},$$
$${}^{\vee}E_i := R_{E_n}R_{E_{n-1}}\dots R_{E_{n-i+2}}E_{n-i+1}.$$

The complete exceptional sequences $(E_1^{\vee},\dots,E_n^{\vee})$ resp. $({}^{\vee}E_1,\dots,{}^{\vee}E_n)$ are called the *right* resp. *left dual* of $(E_1,\dots,E_n)$.

The name is justified by the following

PROPOSITION 2.1.12. *Under the hypotheses of definition 2.1.11 one has*

$$\operatorname{Ext}^k({}^{\vee}E_i,E_j) = \operatorname{Ext}^k(E_i,E_j^{\vee}) = \begin{cases} \mathbb{C} & \text{if } i+j=n+1,\ i=k+1 \\ 0 & \text{otherwise} \end{cases}$$

Moreover the right (resp. left) dual of $(E_1,\dots,E_n)$ is uniquely (up to unique isomorphism) defined by these equations.

The proof can be found in [Gor], subsection 2.6.

Example 2.1.13. Consider on $\mathbb{P}^n = \mathbb{P}(V)$, the projective space of lines in the vector space V, the complete exceptional sequence $\mathfrak{B}' = (\Omega^n(n), \dots, \Omega^1(1), \mathcal{O})$ and for $1 \le p \le n$ the truncation of the p-th exterior power of the Euler sequence

$$0 \longrightarrow \Omega^p_{\mathbb{P}^n} \longrightarrow \left(\bigwedge^p V^\vee\right) \otimes \mathcal{O}_{\mathbb{P}^n}(-p) \longrightarrow \Omega^{p-1}_{\mathbb{P}^n} \longrightarrow 0\,.$$

Let us replace $\mathfrak{B}'$ by $\big(\Omega^n(n), \dots, \Omega^2(2), \mathcal{O}, R_{\mathcal{O}}\Omega^1(1)\big)$, i.e., mutate $\Omega^1(1)$ to the right across $\mathcal{O}$. But in the exact sequence

$$0 \longrightarrow \Omega^1(1) \longrightarrow V^\vee \otimes \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

the arrow $\Omega^1(1) \to V^\vee \otimes \mathcal{O}$ is nothing but the canonical morphism $\Omega^1(1) \to \mathrm{Hom}(\Omega^1(1), \mathcal{O})^\vee \otimes \mathcal{O}$ from definition 2.1.9. Therefore $R_{\mathcal{O}}\Omega^1(1) \simeq \mathcal{O}(1)$.
Now in the mutated sequence $(\Omega^n(n), \dots, \Omega^2(2), \mathcal{O}, \mathcal{O}(1))$ we want to mutate in the next step $\Omega^2(2)$ across $\mathcal{O}$ and $\mathcal{O}(1)$ to the right. In the sequence

$$0 \longrightarrow \Omega^2(2) \longrightarrow \bigwedge^2 V^\vee \otimes \mathcal{O} \longrightarrow \Omega^1(2) \longrightarrow 0$$

the arrow $\Omega^2(2) \to \bigwedge^2 V^\vee \otimes \mathcal{O}$ is again the canonical morphism $\Omega^2(2) \to \mathrm{Hom}(\Omega^2(2), \mathcal{O})^\vee \otimes \mathcal{O}$ and $R_{\mathcal{O}}\Omega^2(2) \simeq \Omega^1(2)$ and then

$$0 \longrightarrow \Omega^1(2) \longrightarrow V^\vee \otimes \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0$$

gives $R_{\mathcal{O}(1)}R_{\mathcal{O}}\Omega^2(2) \simeq \mathcal{O}(2)$.
Continuing this pattern, one transforms our original sequence $\mathfrak{B}'$ by successive right mutations into $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(n))$ which, looking back at definition 2.1.11 and using the braid relations $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$, one identifies as the left dual of $\mathfrak{B}'$.

Here is Gorodentsev's theorem on generalized Beilinson spectral sequences.

Theorem 2.1.14. *Let X be a smooth projective variety and let $D^b(Coh\,X)$ be generated by an exceptional sequence $(E_1, \dots, E_n)$. Let $F : D^b(Coh\,X) \to \mathcal{A}$ be a covariant cohomological functor to some Abelian category $\mathcal{A}$.*
For any object A in $D^b(Coh\,X)$ there is a spectral sequence

$$\begin{aligned} E_1^{p,q} &= \bigoplus_{i+j=q} \mathrm{Ext}^{n+i-1}({}^\vee E_{n-p}, A) \otimes F^j(E_{p+1}) \\ &= \bigoplus_{i+j=q} \mathrm{Ext}^{-i}(A, E_{n-p}^\vee)^\vee \otimes F^j(E_{p+1}) \implies F^{p+q}(A) \end{aligned}$$

(with possibly nonzero entries for $0 \le p, q \le n-1$ only).

For the proof see [Gor], 2.6.4 (actually one can obtain A as a convolution of a complex over $D^b(Coh\,X)$ whose terms are computable once one knows the $\mathrm{Ext}^i({}^\vee E_j, A)$, but we don't need this).
In particular, taking in theorem 2.1.14 the dual exceptional sequences in example 2.1.13 and for F the functor that takes an object in $D^b(Coh\,\mathbb{P}^n)$ to its zeroth cohomology sheaf, we recover the classical Beilinson spectral sequence.
It is occasionally useful to split a derived category into more manageable building blocks before starting to look for complete exceptional sequences. This is the motivation for giving the following definitions.

Definition 2.1.15. Let $\mathcal{S}$ be a full triangulated subcategory of a triangulated category $\mathcal{T}$. The *right orthogonal* to $\mathcal{S}$ in $\mathcal{T}$ is the full triangulated subcategory $\mathcal{S}^\perp$ of $\mathcal{T}$ consisting of objects T such that $\mathrm{Hom}(S,T)=0$ for all objects S of $\mathcal{S}$. The *left orthogonal* ${}^\perp\mathcal{S}$ is defined similarly.

Definition 2.1.16. A full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is *right-* (resp. *left-*) *admissible* if for every $T \in \mathrm{obj}\,\mathcal{T}$ there is a distinguished triangle

$$S \longrightarrow T \longrightarrow S' \longrightarrow S[1] \quad \text{with} \quad S \in \mathrm{obj}\,\mathcal{S}\,,\ S' \in \mathrm{obj}\,\mathcal{S}^\perp$$
$$(\text{resp.} \quad S'' \longrightarrow T \longrightarrow S \longrightarrow S''[1] \quad \text{with} \quad S \in \mathrm{obj}\,\mathcal{S}\,,\ S'' \in \mathrm{obj}\,{}^\perp\mathcal{S}\,)$$

and *admissible* if it is both right- and left-admissible.

Other useful characterizations of admissibility can be found in [Bo], lemma 3.1 or [BoKa], prop. 1.5.

Definition 2.1.17. An n-tuple of admissible subcategories $(\mathcal{S}_1,\dots,\mathcal{S}_n)$ of a triangulated category $\mathcal{T}$ is *semi-orthogonal* if $\mathcal{S}_j$ belongs to $\mathcal{S}_i^\perp$ whenever $1 \le j < i \le n$. If $\mathcal{S}_1,\dots,\mathcal{S}_n$ generate $\mathcal{T}$ one calls this a *semi-orthogonal decomposition of* $\mathcal{T}$ and writes

$$\mathcal{T} = \langle \mathcal{S}_1,\dots,\mathcal{S}_n\rangle\ .$$

To conclude, we give a result that describes the derived category of coherent sheaves on a product of varieties.

Proposition 2.1.18. *Let X and Y be smooth, projective varieties and*

$$(\mathcal{V}_1,\dots,\mathcal{V}_m)$$

resp.

$$(\mathcal{W}_1,\dots,\mathcal{W}_n)$$

be (strong) complete exceptional sequences in $D^b(Coh(X))$ resp. $D^b(Coh(Y))$ where $\mathcal{V}_i$ and $\mathcal{W}_j$ are vector bundles on X resp. Y. Let π_1 resp. π_2 be the projections of $X\times Y$ on the first resp. second factor and put $\mathcal{V}_i \boxtimes \mathcal{W}_j := \pi_1^*\mathcal{V}_i \otimes \pi_2^*\mathcal{W}_j$. Let $\prec$ be the lexicographic order on $\{1,\dots,m\}\times\{1,\dots,n\}$. Then

$$(\mathcal{V}_i \boxtimes \mathcal{W}_j)_{(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}}$$

is a (strong) complete exceptional sequence in $D^b(Coh(X\times Y))$ where $\mathcal{V}_{i_1}\boxtimes\mathcal{W}_{j_1}$ precedes $\mathcal{V}_{i_2}\boxtimes\mathcal{W}_{j_2}$ iff $(i_1,j_1)\prec(i_2,j_2)$.

Proof. The proof is a little less straightforward than it might be expected at first glance since one does not know explicit resolutions of the structure sheaves of the diagonals on $X \times X$ and $Y \times Y$.
First, by the Künneth formula,

$$\begin{aligned}\mathrm{Ext}^k(\mathcal{V}_{i_2} \boxtimes \mathcal{W}_{j_2}, \mathcal{V}_{i_1} \boxtimes \mathcal{W}_{j_1}) &\simeq H^k(X \times Y, (\mathcal{V}_{i_1} \otimes \mathcal{V}_{i_2}^\vee) \boxtimes (\mathcal{W}_{j_1} \otimes \mathcal{W}_{j_2}^\vee)) \\ &\simeq \bigoplus_{k_1+k_2=k} H^{k_1}(X, \mathcal{V}_{i_1} \otimes \mathcal{V}_{i_2}^\vee) \otimes H^{k_2}(Y, \mathcal{W}_{j_1} \otimes \mathcal{W}_{j_2}^\vee) \\ &\simeq \bigoplus_{k_1+k_2=k} \mathrm{Ext}^{k_1}(\mathcal{V}_{i_2}, \mathcal{V}_{i_1}) \otimes \mathrm{Ext}^{k_2}(\mathcal{W}_{j_2}, \mathcal{W}_{j_1})\end{aligned}$$

whence it is clear that $(\mathcal{V}_i \boxtimes \mathcal{W}_j)$ will be a (strong) exceptional sequence for the ordering $\prec$ if $(\mathcal{V}_i)$ and $(\mathcal{W}_j)$ are so.
Therefore we have to show that $(\mathcal{V}_i \boxtimes \mathcal{W}_j)$ generates $D^b(Coh(X \times Y))$ (see [BoBe], lemma 3.4.1). By [Bo], thm. 3.2, the triangulated subcategory $\mathcal{T}$ of $D^b(Coh(X \times Y))$ generated by the $\mathcal{V}_i \boxtimes \mathcal{W}_j$'s is admissible, and thus by [Bo], lemma 3.1, it suffices to show that the right orthogonal $\mathcal{T}^\perp$ is zero. Let $Z \in \mathrm{obj}\, \mathcal{T}^\perp$ so that we have

$$\begin{aligned}\mathrm{Hom}_{D^b(Coh(X \times Y))}(\mathcal{V}_i \boxtimes \mathcal{W}_j, Z[l_1 + l_2]) = 0 \quad \forall i \in \{1, \dots, m\}\,, \\ \forall j \in \{1, \dots, n\}\, \forall l_1, l_2 \in \mathbb{Z}.\end{aligned}$$

But

$$\begin{aligned}&\mathrm{Hom}_{D^b(Coh(X \times Y))}(\mathcal{V}_i \boxtimes \mathcal{W}_j, Z[l_1 + l_2]) \\ &\quad\simeq \mathrm{Hom}_{D^b(Coh(X \times Y))}\left(\pi_1^*\mathcal{V}_i, R\mathcal{H}om^\bullet_{D^b(Coh(X \times Y))}(\pi_2^*\mathcal{W}_j, Z[l_1])[l_2]\right) \\ &\quad\simeq \mathrm{Hom}_{D^b(Coh(X))}\left(\mathcal{V}_i, R\pi_{1*}R\mathcal{H}om^\bullet_{D^b(Coh(X \times Y))}(\pi_2^*\mathcal{W}_j, Z[l_1])[l_2]\right)\end{aligned}$$

using the adjointness of $\pi_1^* = L\pi_1^*$ and $R\pi_{1*}$. But then

$$R\pi_{1*}R\mathcal{H}om^\bullet_{D^b(Coh(X \times Y))}(\pi_2^*\mathcal{W}_j, Z[l_1]) = 0 \quad \forall j \in \{1, \dots, m\}\, \forall l_1 \in \mathbb{Z}$$

because the $\mathcal{V}_i$ generate $D^b(Coh(X))$ and hence there is no non-zero object in the right orthogonal to $\langle \mathcal{V}_1, \dots, \mathcal{V}_n \rangle$. Let $U \subset X$ and $V \subset Y$ be affine open sets. Then

$$\begin{aligned}0 = R\Gamma\left(U, R\pi_{1*}R\mathcal{H}om^\bullet_{D^b(Coh(X \times Y))}(\pi_2^*\mathcal{W}_j, Z[l + l_1])\right) \\ \simeq R\mathrm{Hom}^\bullet\,(\mathcal{W}_j, R\pi_{2*}(Z[l]\mid_{U \times Y})[l_1])\ \forall l, l_1 \in \mathbb{Z}\end{aligned}$$

whence $R\pi_{2*}(Z[l]\mid_{U \times Y}) = 0$ since the $\mathcal{W}_j$ generate $D^b(Coh(Y))$ (using thm. 2.1.2 in [BoBe]). Therefore we get

$$R\Gamma(U \times V, Z) = 0\,.$$

But $R^i\Gamma(U \times V, Z) = \Gamma(U \times V, H^i(Z))$ and thus all cohomology sheaves of Z are zero, i.e. $Z = 0$ in $D^b(Coh(X \times Y))$. □

Remark 2.1.19. This proposition is very useful for a treatment of the derived categories of coherent sheaves on rational homogeneous spaces from a systematic point of view. For if $X = G/P$ with G a connected semisimple complex Lie group, $P \subset G$ a parabolic subgroup, it is well known that one has a decomposition

$$X \simeq S_1/P_1 \times \ldots \times S_N/P_N$$

where $S_1, \ldots, S_N$ are connected simply connected simple complex Lie groups and $P_1, \ldots, P_N$ corresponding parabolic subgroups (cf. [Akh], 3.3, p. 74). Thus for the construction of complete exceptional sequences on any G/P one can restrict oneself to the case where G is simple.

2.2 Catanese's conjecture and the work of Kapranov

First we fix some notation concerning rational homogenous varieties and their Schubert varieties that will remain in force throughout the text unless otherwise stated. References for this are [Se2], [Sp].

G is a complex semi-simple Lie group which is assumed to be connected and simply connected with Lie algebra $\mathfrak{g}$.

$H \subset G$ is a fixed maximal torus in G with Lie algebra the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

$R \subset \mathfrak{h}^*$ is the root system associated to $(\mathfrak{g}, \mathfrak{h})$ so that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^{\alpha}$$

with $\mathfrak{g}^{\alpha}$ the eigen-subspace of $\mathfrak{g}$ corresponding to $\alpha \in \mathfrak{h}^*$. Choose a base $S = \{\alpha_1, \ldots, \alpha_r\}$ for R; R^+ denotes the set of positive roots w.r.t. S, $R^- := -R^+$, so that $R = R^+ \cup R^-$, and ϱ is the half-sum of the positive roots.

$\mathrm{Aut}(\mathfrak{h}^*) \supset W := \langle s_\alpha \mid s_\alpha$ the reflection with vector α leaving R invariant$\rangle \simeq N(H)/H$ is the Weyl group of R.

Let $\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}^{\alpha}$, $\mathfrak{b}^- := \mathfrak{h} \oplus \bigoplus_{\alpha < 0} \mathfrak{g}^{\alpha}$ be opposite Borel subalgebras of $\mathfrak{g}$ corresponding to $\mathfrak{h}$ and S, and $\mathfrak{p} \supset \mathfrak{b}$ a parabolic subalgebra corresponding uniquely to a subset $I \subset S$ (then

$$\mathfrak{p} = \mathfrak{p}(I) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}^{\alpha} \oplus \bigoplus_{\alpha \in R^-(I)} \mathfrak{g}^{\alpha}$$

where $R^-(I) := \{\alpha \in R^- \mid \alpha = \sum_{i=1}^r k_i \alpha_i$ with $k_i \leq 0$ for all i and $k_j = 0$ for all $\alpha_j \in I\}$). Let B, B^-, $P = P(I) \supset B$ be the corresponding connected subgroups of G with Lie algebras $\mathfrak{b}$, $\mathfrak{b}^-$, $\mathfrak{p}$.

$X := G/P$ is the rational homogeneous variety corresponding to G and P.

$l(w)$ is the length of an element $w \in W$ relative to the set of generators $\{s_\alpha \mid \alpha \in S\}$, i.e. the least number of factors in a decomposition

$$w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \dots s_{\alpha_{i_l}}\,,\ \alpha_{i_j} \in S\,;$$

A decomposition with $l = l(w)$ is called reduced. One has the Bruhat order $\le$ on W, i.e. $x \le w$ for $x, w \in W$ iff x can be obtained by erasing some factors of a reduced decomposition of w. W_P is the Weyl group of P, the subgroup of W generated by the simple reflections s_α with $\alpha \notin I$. In each coset $wW_P \in W/W_P$ there exists a unique element of minimal length and W^P denotes the set of minimal representatives of W/W^P. One has $W^P = \{w \in W \mid l(ww') = l(w) + l(w')\ \forall w' \in W_P\}$.
For $w \in W^P$, C_w denotes the double coset BwP/P in X, called a Bruhat cell, $C_w \simeq \mathbb{A}^{l(w)}$. Its closure in X is the Schubert variety X_w. $C_w^- = B^- wP/P$ is the opposite Bruhat cell of codimension $l(w)$ in X, $X^w = \overline{C_w^-}$ is the Schubert variety opposite to X_w.
There is the extended version of the Bruhat decomposition

$$G/P = \bigsqcup_{w \in W^P} C_w$$

(a paving of X by affine spaces) and for $v,\ w \in W^P$: $v \le w \Leftrightarrow X_v \subseteq X_w$; we denote the boundaries $\partial X_w := X_w \backslash C_w$, $\partial X^w := X^w \backslash C_w^-$, which have pure codimension 1 in X_w resp. X^w.

Moreover, we need to recall some facts and introduce further notation concerning representations of the subgroup $P = P(I) \subset G$, which will be needed in subsection 3 below. References are [A], [Se2], [Sp], [Ot2], [Stei].

The spaces $\mathfrak{h}_\alpha := [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{h}$, $\alpha \in R$, are 1-dimensional, one has $\mathfrak{g} = \bigoplus_{\alpha \in S} \mathfrak{h}_\alpha \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}^\alpha \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}^\alpha$ and there is a unique $H_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(H_\alpha) = 2$.
Then we have the weight lattice $\Lambda := \{\omega \in \mathfrak{h}^* \mid \omega(H_\alpha) \in \mathbb{Z}\ \forall \alpha \in R\}$ (which one identifies with the character group of H) and the set of dominant weights $\Lambda^+ := \{\omega \in \mathfrak{h}^* \mid \omega(H_\alpha) \in \mathbb{N}\ \forall \alpha \in R\}$. $\{\omega_1, \dots, \omega_r\}$ denotes the basis of $\mathfrak{h}^*$ dual to the basis $\{H_{\alpha_1}, \dots, H_{\alpha_r}\}$ of $\mathfrak{h}$. The ω_i are the fundamental weights. If $(\cdot, \cdot)$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form, they can also be characterized by the equations $2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ (Kronecker delta). It is well known that the irreducible finite dimensional representations of $\mathfrak{g}$ are in one-to-one correspondence with the $\omega \in \Lambda^+$, these ω occurring as highest weights.
I recall the Levi-Malčev decomposition of $P(I)$ (resp. $\mathfrak{p}(I)$): The algebras

$$\mathfrak{s}_P := \bigoplus_{\alpha \in S \backslash I} \mathfrak{h}_\alpha \oplus \bigoplus_{\alpha \in R^-(I)} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$$

resp.

$$\mathfrak{l}_P := \bigoplus_{\alpha\in S}\mathfrak{h}_\alpha \oplus \bigoplus_{\alpha\in R^-(I)} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha})$$

are the semisimple resp. reductive parts of $\mathfrak{p}(I)$ containing $\mathfrak{h}$, the corresponding connected subgroups of G will be denoted S_P resp. L_P. The algebra

$$\mathfrak{u}_P := \bigoplus_{\alpha\in R^-\backslash R^-(I)} \mathfrak{g}^{-\alpha}$$

is an ideal of $\mathfrak{p}(I)$, $\mathfrak{p}(I) = \mathfrak{l}_P \oplus \mathfrak{u}_P$, and the corresponding normal subgroup $R_u(P)$ is the unipotent radical of P. One has

$$P = L_P \ltimes R_u(P)\,,$$

the Levi-Malčev decomposition of P. The center Z of the Levi subgroup L_P is $Z = \{g \in H \mid \alpha(g) = 1\,\forall\alpha \in S\backslash I\}$. The connected center corresponds to the Lie algebra $\bigoplus_{\alpha\in I}\mathfrak{h}_\alpha$ and is isomorphic to the torus $(\mathbb{C}^*)^{|I|}$. One has

$$P = Z \cdot S_P \ltimes R_u(P)\,.$$

Under the hypothesis that G is simply connected, also S_P is simply connected.
If $r : P \to \mathrm{GL}(V)$ is an irreducible finite-dimensional representation, $R_u(P)$ acts trivially, and thus those r are in one-to-one correspondence with irreducible representations of the reductive Levi-subgroup L_P and as such possess a well-defined highest weight $\omega \in \Lambda$. Then the irreducible finite dimensional representations of $P(I)$ correspond bijectively to weights $\omega \in \mathfrak{h}^*$ such that ω can be written as $\omega = \sum_{i=1}^r k_i\omega_i$, $k_i \in \mathbb{Z}$, such that $k_j \in \mathbb{N}$ for all j such that $\alpha_j \notin I$. We will say that such an ω is the highest weight of the representation $r : P \to \mathrm{GL}(V)$.
The homogeneous vector bundle on G/P associated to r will be $G \times_r V := G \times V/\{(g,v) \sim (gp^{-1}, r(p)v)\,,\ p \in P,\ g \in G,\ v \in V\}$ as above. However, for a character $\chi : H \to \mathbb{C}$ (which will often be identified with $d\chi \in \mathfrak{h}^*$), $\mathcal{L}(\chi)$ will denote the homogeneous line bundle on G/B whose fibre at the point $e\cdot B$ is the one-dimensional representation of B corresponding to the character $-\chi$. This has the advantage that $\mathcal{L}(\chi)$ will be ample iff $d\chi = \sum_{j=1}^r k_j\omega_j$ with $k_j > 0$, $k_j \in \mathbb{Z}$ for all j, and it will also prove a reasonable convention in later applications of Bott's theorem.

The initial stimulus for this work was a conjecture due to F. Catanese. This is variant (A) of conjecture 2.2.1. Variant (B) is a modification of (A) due to the author, but closely related.

Conjecture 2.2.1. (A) On any rational homogeneous variety $X = G/P$ there exists a complete strong exceptional poset (cf. def. 2.1.7 (B)) and a bijection of the elements of the poset with the Schubert varieties in X such that the partial order of the poset is the one induced by the Bruhat-Chevalley order.

(B) For any $X = G/P$ there exists a strong complete exceptional sequence $\mathfrak{E} = (E_1, \dots, E_n)$ in $D^b(Coh\, X)$ with $n = |W^P|$, the number of Schubert varieties in X (which is the topological Euler characteristic of X).
Moreover, since there is a natural partial order $\le_{\mathfrak{E}}$ on the set of objects in $\mathfrak{E}$ by defining that $E' \le_{\mathfrak{E}} E$ for objects E and E' of $\mathfrak{E}$ iff there are objects $F_1, \dots, F_r$ of $\mathfrak{E}$ such that $\mathrm{Hom}(E', F_1) \neq 0$, $\mathrm{Hom}(F_1, F_2) \neq 0$, ..., $\mathrm{Hom}(F_r, E) \neq 0$ (the order of the exceptional sequence $\mathfrak{E}$ itself is a total order refining $\le_{\mathfrak{E}}$), there should be a relation between the Bruhat order on W^P and $\le_{\mathfrak{E}}$ (for special choice of $\mathfrak{E}$).
If $P = P(\alpha_i)$, some $i \in \{1, \dots, r\}$, is a maximal parabolic subgroup in G and G is simple, then one may conjecture more precisely: There exists a strong complete exceptional sequence $\mathfrak{E} = (E_1, \dots, E_n)$ in $D^b(Coh\, X)$ and a bijection

$$b : \{E_1, \dots, E_n\} \to \{X_w \mid w \in W^P\}$$

such that

$$\mathrm{Hom}(E_i, E_j) \neq 0 \iff b(E_j) \subseteq b(E_i)\,.$$

We would like to add the following two questions:

(C) Does there always exist on X a complete very strong exceptional poset (cf. def. 2.1.7 (C)) and a bijection of the elements of the poset with the Schubert varieties in X such that the partial order of the poset is the one induced by the Bruhat-Chevalley order?

(D) Can we achieve that the E_i's in (A), (B) and/or (C) are homogeneous vector bundles?

It is clear that, if the answer to (C) is positive, this implies (A). Moreover, the existence of a complete very strong exceptional poset entails the existence of a complete strong exceptional sequence.
For P maximal parabolic, part (B) of conjecture 2.2.1 is stronger than part (A). We will concentrate on that case in the following.
In the next subsection we will see that, at least upon adopting the right point of view, it is clear that the number of terms in any complete exceptional sequence in $D^b(Coh\, X)$ must equal the number of Schubert varieties in X.
To begin with, let me show how conjecture 2.2.1 can be brought in line with results of Kapranov obtained in [Ka3] (and [Ka1], [Ka2]) which are summarized in theorems 2.2.2, 2.2.3, 2.2.4 below.
One more piece of notation: If L is an m-dimensional vector space and $\lambda =$

$(\lambda_1, \dots, \lambda_m)$ is a non-increasing sequence of integers, then $\Sigma^\lambda L$ will denote the space of the irreducible representation $\varrho_\lambda : \mathrm{GL}(L) \to \mathrm{GL}(\Sigma^\lambda L)$ of $\mathrm{GL}(L) \simeq \mathrm{GL}_m\mathbb{C}$ with highest weight λ. Σ^λ is called the Schur functor associated to λ; if $\mathcal{E}$ is a rank m vector bundle on a variety Y, $\Sigma^\lambda\mathcal{E}$ will denote the vector bundle $P_{\mathrm{GL}}(\mathcal{E}) \times_{\varrho_\lambda} \Sigma^\lambda(L) := P_{\mathrm{GL}}(\mathcal{E}) \times \Sigma^\lambda(L)/\{(f, w) \sim (fg^{-1}, \varrho_\lambda(g)w),\ f \in P_{\mathrm{GL}}(\mathcal{E}),\ w \in \Sigma^\lambda L,\ g \in \mathrm{GL}_m\mathbb{C}\}$ where $P_{\mathrm{GL}}(\mathcal{E})$ is the principal $\mathrm{GL}_m\mathbb{C}$- bundle of local frames in $\mathcal{E}$.

Theorem 2.2.2. *Let* $\mathrm{Grass}(k, V)$ *be the Grassmanian of k-dimensional subspaces of an n-dimensional vector space V, and let $\mathcal{R}$ be the tautological rank k subbundle on* $\mathrm{Grass}(k, V)$. *Then the bundles $\Sigma^\lambda\mathcal{R}$ where λ runs over $Y(k, n-k)$, the set of Young diagrams with no more than k rows and no more than $n-k$ columns, are all exceptional, have no higher extension groups between each other and generate $D^b(Coh\,\mathrm{Grass}(k, V))$.*
Moreover, $\mathrm{Hom}(\Sigma^\lambda\mathcal{R}, \Sigma^\mu\mathcal{R}) \neq 0$ *iff $\lambda_i \geq \mu_i\ \forall i = 1, \dots, k$. (Thus these $\Sigma^\lambda\mathcal{R}$ form a strong complete exceptional sequence in $D^b(Coh\,\mathrm{Grass}(k, V))$ when appropriately ordered).*

Theorem 2.2.3. *If V is an n-dimensional vector space, $1 \leq k_1 < \dots < k_l \leq n$ a strictly increasing sequence of integers, and* $\mathrm{Flag}(k_1, \dots, k_l; V)$ *the variety of flags of subspaces of type $(k_1, \dots, k_l)$ in V, and if $\mathcal{R}_{k_1} \subset \dots \subset \mathcal{R}_{k_l}$ denotes the tautological flag of subbundles, then the bundles*

$$\Sigma^{\lambda_1}\mathcal{R}_{k_1} \otimes \dots \otimes \Sigma^{\lambda_l}\mathcal{R}_{k_l}$$

where λ_j, $j = 1, \dots, l-1$, runs over $Y(k_j, k_{j+1} - k_j)$, the set of Young diagrams with no more than k_j rows and no more than $k_{j+1} - k_j$ columns, and λ_l runs over $Y(k_l, n - k_l)$, form a strong complete exceptional sequence in $D^b(Coh\,\mathrm{Flag}(k_1, \dots, k_l; V)$ if we order them as follows:
Choose a total order $\prec_j$ on each of the sets $Y(k_j, k_{j+1} - k_j)$ and $\prec_l$ on $Y(k_l, n - k_l)$ such that if $\lambda \prec_j \mu$ (or $\lambda \prec_t \mu$) then the Young diagram of λ is not contained in the Young diagram of μ; endow the set $Y = Y(k_l, n - k_l) \times Y(k_{l-1}, k_l - k_{l-1}) \times \dots \times Y(k_1, k_2 - k_1)$ with the resulting lexicographic order $\prec$. Then $\Sigma^{\lambda_1}\mathcal{R}_{k_1} \otimes \dots \otimes \Sigma^{\lambda_l}\mathcal{R}_{k_l}$ precedes $\Sigma^{\mu_1}\mathcal{R}_{k_1} \otimes \dots \otimes \Sigma^{\mu_l}\mathcal{R}_{k_l}$ iff $(\lambda_l, \dots, \lambda_1) \prec (\mu_l, \dots, \mu_1)$.

Theorem 2.2.4. *Let V be again an n-dimensional vector space and $Q \subset \mathbb{P}(V)$ a nonsingular quadric hypersurface.*
If n is odd and Σ denotes the spinor bundle on Q, then the following constitutes a strong complete exceptional sequence in $D^b(Coh\,Q)$:

$$(\Sigma(-n+2),\ \mathcal{O}_Q(-n+3), \dots, \mathcal{O}_Q(-1),\ \mathcal{O}_Q)$$

and $\mathrm{Hom}(\mathcal{E}, \mathcal{E}') \neq 0$ *for two bundles $\mathcal{E}$, $\mathcal{E}'$ in this sequence iff $\mathcal{E}$ precedes $\mathcal{E}'$ in the ordering of the sequence.*
If n is even and Σ^+, Σ^- denote the spinor bundles on Q, then

$$(\Sigma^+(-n+2),\ \Sigma^-(-n+2),\ \mathcal{O}_Q(-n+3), \dots, \mathcal{O}_Q(-1),\ \mathcal{O}_Q)$$

is a strong complete exceptional sequence in $D^b(Coh\,Q)$ *and* $\mathrm{Hom}(\mathcal{E},\mathcal{E}') \neq 0$ *for two bundles* $\mathcal{E}$, $\mathcal{E}'$ *in this sequence iff* $\mathcal{E}$ *precedes* $\mathcal{E}'$ *in the ordering of the sequence with the one exception that* $\mathrm{Hom}(\Sigma^+(-n+2), \Sigma^-(-n+2)) = 0$.

Here by Σ (resp. Σ^+, Σ^-), we mean the homogeneous vector bundles on $Q = \mathrm{Spin}_n\mathbb{C}/P(\alpha_1)$, α_1 the simple root corresponding to the first node in the Dynkin diagram of type B_m, $n = 2m+1$, (resp. the Dynkin diagram of type D_m, $n = 2m$), that are the duals of the vector bundles associated to the irreducible representation of $P(\alpha_1)$ with highest weight ω_m (resp. highest weights ω_m, ω_{m-1}). We will deal more extensively with spinor bundles in subsection 3.2 below.
First look at theorem 2.2.2. It is well known (cf. [BiLa], section 3.1) that if one sets

$$I_{k,n} := \{\underline{i} = (i_1, \dots, i_k) \in \mathbb{N}^k \mid 1 \leq i_1 < \dots < i_k \leq n\}$$

and if $V_i := \langle v_1, \dots, v_i \rangle$ where $(v_1, \dots, v_n)$ is a basis for V, then the Schubert varieties in $\mathrm{Grass}(k, V)$ can be identified with

$$X_{\underline{i}} := \left\{ L \in \mathrm{Grass}(k,V) \mid \dim(L \cap V_{i_j}) \geq j \quad \forall 1 \leq j \leq k \right\}, \quad \underline{i} \in I_{k,n}$$

and the Bruhat order is reflected by

$$X_{\underline{i}} \subseteq X_{\underline{i'}} \iff i_j \leq i'_j \quad \forall 1 \leq j \leq k;$$

and the $\underline{i} \in I_{k,n}$ bijectively correspond to Young diagrams in $Y(k, n-k)$ by associating to $\underline{i}$ the Young diagram $\lambda(\underline{i})$ defined by

$$\lambda(\underline{i})_t := i_{k-t+1} - (k-t+1) \quad \forall 1 \leq t \leq k\,.$$

Then containment of Schubert varieties corresponds to containment of associated Young diagrams. Thus conjecture 2.2.1 (B) is verified by the strong complete exceptional sequence of theorem 2.2.2.
In the case of $\mathrm{Flag}(k_1, \dots, k_l; V)$ (theorem 2.2.3) one can describe the Schubert subvarieties and the Bruhat order as follows (cf. [BiLa], section 3.2): Define

$$I_{k_1,\dots,k_l} = \left\{ \left(\underline{i}^{(1)}, \dots, \underline{i}^{(l)}\right) \in I_{k_1,n} \times \dots \times I_{k_l,n} \mid \underline{i}^{(j)} \subset \underline{i}^{(j+1)} \quad \forall 1 \leq j \leq l-1 \right\}$$

Then the Schubert varieties in $\mathrm{Flag}(k_1, \dots, k_l; V)$ can be identified with the

$$X_{\left(\underline{i}^{(1)},\dots,\underline{i}^{(l)}\right)} := \{(L_1, \dots, L_l) \in \mathrm{Flag}(k_1, \dots, k_l; V) \subset \mathrm{Grass}(k_1, V) \times \dots$$
$$\dots \times \mathrm{Grass}(k_l, V) \mid L_j \in X_{\underline{i}^{(j)}} \quad \forall 1 \leq j \leq l \Big\}$$

for $\left(\underline{i}^{(1)}, \dots, \underline{i}^{(l)}\right)$ running over $I_{k_1,\dots,k_l}$ (keeping the preceding notation for the Grassmannian). The Bruhat order on the Schubert varieties may be identified with the following partial order on $I_{k_1,\dots,k_l}$:

$$\left(\underline{i}^{(1)}, \dots, \underline{i}^{(l)}\right) \leq \left(\underline{j}^{(1)}, \dots, \underline{j}^{(l)}\right) \iff i^{(t)} \leq j^{(t)} \quad \forall 1 \leq t \leq l.$$

To set up a natural bijection between the set Y in theorem 2.2.3 and $I_{k_1,\dots,k_l}$ associate to $\mathbf{i} := \left(\underline{i}^{(1)}, \dots, \underline{i}^{(l)}\right)$ the following Young diagrams: $\lambda_l(\mathbf{i}) \in Y(k_l, n-k_l)$ is defined by

$$(\lambda_l(\mathbf{i}))_t := i^{(l)}_{k_l-t+1} - (k_l - t + 1) \quad \forall 1 \le t \le k_l.$$

Now since $\underline{i}^{(j)} \subset \underline{i}^{(j+1)} \quad \forall 1 \le j \le l-1$ one can write

$$i^{(j)}_s = i^{(j+1)}_{r(s)}, \; s = 1, \dots, k_j$$

where $1 \le r(1) < \ldots < r(k_j) \le k_{j+1}$. One then defines $\lambda_j(\mathbf{i}) \in Y(k_j, k_{j+1} - k_j)$ by

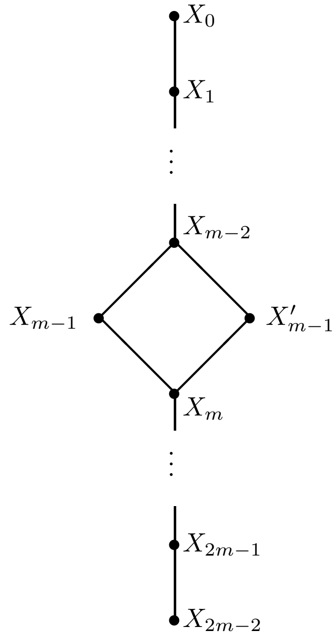
$$(\lambda_j(\mathbf{i}))_t := r(k_j - t + 1) - (k_j - t + 1) \quad \forall 1 \le t \le k_j.$$

However it is not clear to me in this case how to relate the Bruhat order on $I_{k_1,\dots,k_l}$ with the vanishing or non-vanishing of Hom-spaces between members of the strong complete exceptional sequence in theorem 2.2.3 (there is an explicit combinatorial criterion for the non-vanishing of

$$\mathrm{Hom}\left(\Sigma^{\lambda_1}\mathcal{R}_{k_1} \otimes \dots \otimes \Sigma^{\lambda_l}\mathcal{R}_{k_l}, \Sigma^{\mu_1}\mathcal{R}_{k_1} \otimes \dots \otimes \Sigma^{\mu_l}\mathcal{R}_{k_l}\right)$$

formulated in [Ka3], 3.12, but if this relates in any perspicuous way to the Bruhat order is not clear). In this respect, for the time being, conjecture 2.2.1 (parts (A) and (B)) must remain within the confines of wishful thinking.

If in the set-up of theorem 2.2.4 $Q \subset \mathbb{P}(V)$, $\dim V = n = 2m+1$ odd, is a smooth quadric hypersurface, then there are $2m$ Schubert varieties in Q and the Bruhat order on them is linear (cf. [BiLa], pp. 139/140), so the strong complete exceptional sequence of theorem 2.2.4 satisfies conjecture 2.2.1. (B). The case of a smooth quadric hypersurface $Q \subset \mathbb{P}(V)$ with $\dim V = n = 2m$ even, is more interesting. The Bruhat order on the set of Schubert varieties can be depicted in the following way (cf. [BiLa], p. 142/143):



Here $X_0, \dots, X_{m-2}$, X_{m-1}, X'_{m-1}, $X_m, \dots, X_{2m-2}$ are labels for the Schubert varieties in Q and the subscript denotes the codimension in Q. The strong complete exceptional sequence

$$(\Sigma^+(-2m+2),\ \Sigma^-(-2m+2),\ \mathcal{O}_Q(-2m+3), \dots, \mathcal{O}_Q(-1),\ \mathcal{O}_Q)$$

does not verify conjecture 2.2.1 (B), but we claim that there is a strong complete exceptional sequence in the same braid group orbit (see thm. 2.1.10) that does. In fact, by [Ott], theorem 2.8, there are two natural exact sequences on Q

$$0 \longrightarrow \Sigma^+(-1) \longrightarrow \mathrm{Hom}(\Sigma^+(-1), \mathcal{O}_Q)^\vee \otimes \mathcal{O}_Q \longrightarrow \Sigma^- \longrightarrow 0$$
$$0 \longrightarrow \Sigma^-(-1) \longrightarrow \mathrm{Hom}(\Sigma^-(-1), \mathcal{O}_Q)^\vee \otimes \mathcal{O}_Q \longrightarrow \Sigma^+ \longrightarrow 0$$

where the (injective) arrows are the canonical morphisms of definition 2.1.9; one also has $\dim \mathrm{Hom}(\Sigma^+(-1), \mathcal{O}_Q)^\vee = \dim \mathrm{Hom}(\Sigma^-(-1), \mathcal{O}_Q)^\vee = 2^{m-1}$. (Caution: the spinor bundles in [Ott] are the duals of the bundles that are called spinor bundles in this text which is clear from the discussion in [Ott], p.305!). It follows that if in the above strong complete exceptional sequence we mutate $\Sigma^-(-2m+2)$ across $\mathcal{O}_Q(-2m+3), \dots, \mathcal{O}_Q(-m+1)$ to the right and afterwards mutate $\Sigma^+(-2m+2)$ across $\mathcal{O}_Q(-2m+3), \dots, \mathcal{O}_Q(-m+1)$ to the right, we will obtain the following complete exceptional sequences in $D^b(Coh\, Q)$:

If m is odd:

$$\big(\mathcal{O}_Q(-2m+3), \dots, \mathcal{O}_Q(-m+1),\ \Sigma^+(-m+1),\ \Sigma^-(-m+1),$$
$$\mathcal{O}_Q(-m+2), \dots, \mathcal{O}_Q(-1),\ \mathcal{O}_Q\big)\,,$$

if m is even:

$$\big(\mathcal{O}_Q(-2m+3), \dots, \mathcal{O}_Q(-m+1),\ \Sigma^-(-m+1),\ \Sigma^+(-m+1),\\ \mathcal{O}_Q(-m+2), \dots, \mathcal{O}_Q(-1),\ \mathcal{O}_Q\big).$$

One finds (e.g. using theorem 2.2.4 and [Ott], thm.2.3 and thm. 2.8) that these exceptional sequences are again strong and if we let the bundles occurring in them (in the order given by the sequences) correspond to $X_0, \dots, X_{m-2},\ X_{m-1},\ X'_{m-1},\ X_m, \dots, X_{2m-2}$ (in this order), then the above two strong complete exceptional sequences verify conjecture 2.2.1. (B).

2.3 Information detected on the level of K-theory

The cellular decomposition of X has the following impact on $D^b(Coh\,X)$.

Proposition 2.3.1. *The structure sheaves $\mathcal{O}_{X_w}$, $w \in W^P$, of Schubert varieties in X generate $D^b(Coh\,X)$ as a triangulated category.*

Since we have the Bruhat decomposition and each Bruhat cell is isomorphic to an affine space, the proof of the proposition will follow from the next lemma.

Lemma 2.3.2. *Let Y be a reduced algebraic scheme, $U \subset Y$ an open subscheme with $U \simeq \mathbb{A}^d$, for some $d \in \mathbb{N}$, $Z := Y \backslash U$, $i : U \hookrightarrow Y$, $j : Z \hookrightarrow Y$ the natural embeddings. Look at the sequence of triangulated categories and functors*

$$D^b(Coh\,Z) \xrightarrow{\ j_*\ } D^b(Coh\,Y) \xrightarrow{\ i^*\ } D^b(Coh\,U)$$

(thus j_* is extension by 0 outside Z which is exact, and i^* is the restriction to U, likewise exact). Suppose $Z_1, \dots, Z_n \in \operatorname{obj} D^b(Coh\,Z)$ generate $D^b(Coh\,Z)$. Then $D^b(Coh\,Y)$ is generated by $j_*Z_1, \dots, j_*Z_n$, $\mathcal{O}_Y$.

Proof. $D^b(Coh\,Y)$ is generated by $Coh\,Y$ so it suffices to prove that each coherent sheaf $\mathcal{F}$ on Y is isomorphic to an object in the triangulated subcategory generated by $j_*Z_1, \dots, j_*Z_n$, $\mathcal{O}_Y$. By the Hilbert syzygy theorem $i^*\mathcal{F}$ has a resolution

$$(*) \quad 0 \to \mathcal{L}_t \to \dots \to \mathcal{L}_0 \to i^*\mathcal{F} \to 0$$

where the $\mathcal{L}_i$ are finite direct sums of $\mathcal{O}_U$. We recall the following facts (cf. [FuLa] , VI, lemmas 3.5, 3.6, 3.7):

(1) For any coherent sheaf $\mathcal{G}$ on U there is a coherent extension $\overline{\mathcal{G}}$ to Y.

(2) Any short exact sequence of coherent sheaves on U is the restriction of an exact sequence of coherent sheaves on Y.

(3) If $\mathcal{G}$ is coherent on U and $\overline{\mathcal{G}}_1$, $\overline{\mathcal{G}}_2$ are two coherent extensions of $\mathcal{G}$ to Y, then there are a coherent sheaf $\overline{\mathcal{G}}$ on Y and homomorphisms $\overline{\mathcal{G}} \xrightarrow{f} \overline{\mathcal{G}}_1$, $\overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{G}}_2$ which restrict to isomorphisms over U.

Note that in the set-up of the last item we can write

$$0 \to \ker(f) \to \overline{\mathcal{G}} \xrightarrow{f} \overline{\mathcal{G}}_1 \to \operatorname{coker}(f) \to 0\,,$$
$$0 \to \ker(g) \to \overline{\mathcal{G}} \xrightarrow{g} \overline{\mathcal{G}}_2 \to \operatorname{coker}(g) \to 0$$

and $\ker(f)$, $\operatorname{coker}(f)$, $\ker(g)$, $\operatorname{coker}(g)$ are sheaves with support in Z, i.e. in the image of j_*. Thus they will be isomorphic to an object in the subcategory generated by $j_*Z_1, \dots, j_*Z_n$. In conclusion we see that if one coherent extension $\overline{\mathcal{G}}_1$ of $\mathcal{G}$ is isomorphic to an object in the subcategory generated by $j_*Z_1, \dots, j_*Z_n$, $\mathcal{O}_Y$, the same will be true for any other coherent extension $\overline{\mathcal{G}}_2$. The rest of the proof is now clear: We split $(*)$ into short exact sequences and write down extensions of these to Y by item (2) above. Since the $\mathcal{L}_i$ are finite direct sums of $\mathcal{O}_U$ one deduces from the preceding observation that $\mathcal{F}$ is indeed isomorphic to an object in the triangulated subcategory generated by $j_*Z_1, \dots, j_*Z_n$, $\mathcal{O}_Y$. ☐

Remark 2.3.3. On $\mathbb{P}^n$ it is possible to prove Beilinson's theorem with the help of proposition 2.3.1. Indeed the structure sheaves of a flag of linear subspaces $\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^{n-1}}, \dots, \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^0}\}$ admit the Koszul resolutions

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$
$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^{n-2}} \to 0$$
$$\vdots$$
$$0 \to \mathcal{O}(-n) \to \mathcal{O}(-(n-1))^{\oplus n} \to \dots \to \mathcal{O}(-1)^{\oplus n} \to \mathcal{O} \to \mathcal{O}_{\mathbb{P}^0} \to 0$$

from which one concludes inductively that $(\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O})$ generates $D^b(Coh\,\mathbb{P}^n)$.

Next we want to explain a point of view on exceptional sequences that in particular makes obvious the fact that the number of terms in any complete exceptional sequence on $X = G/P$ equals the number $|W^P|$ of Schubert varieties in X.

Definition 2.3.4. Let $\mathcal{T}$ be a triangulated category. The *Grothendieck group* $K_\circ(\mathcal{T})$ *of* $\mathcal{T}$ is the quotient of the free abelian group on the isomorphism classes $[A]$ of objects of $\mathcal{T}$ by the subgroup generated by expressions

$$[A] - [B] + [C]$$

for every distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ in $\mathcal{T}$.

If $\mathcal{T} = D^b(\mathcal{A})$, $\mathcal{A}$ an Abelian category, then we also have $K_\circ(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$, i.e. the free abelian group on the isomorphism classes of objects of $\mathcal{A}$ modulo relations $[D'] - [D] + [D'']$ for every short exact sequence $0 \to D' \to D \to D'' \to 0$ in $\mathcal{A}$, and it is clear that in this case $K_\circ(D^b(\mathcal{A})) \simeq K_\circ(\mathcal{A})$ (to a complex $A \in \operatorname{obj} D^b(\mathcal{A})$ one associates

$\sum_{i\in\mathbb{Z}}(-1)^i[H^i(A)] \in K_\circ(\mathcal{A})$ which is a map that is additive on distinguished triangles by the long exact cohomology sequence and hence descends to a map $K_\circ(D^b(\mathcal{A})) \to K_\circ(\mathcal{A})$; the inverse map is induced by the embedding $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$).

Let now Y be some smooth projective variety. Then to $Z_1, Z_2 \in \mathrm{obj}\, D^b(Coh\, Y)$ one can assign the integer $\sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(Z_1, Z_2)$, a map which is biadditive on distinguished triangles. Set $K_\circ(Y) := K_\circ(Coh\, Y)$.

Definition 2.3.5. The (in general nonsymmetric) bilinear pairing

$$\begin{aligned} \chi : K_\circ(Y) \times K_\circ(Y) &\to \mathbb{Z} \\ ([Z_1],[Z_2]) &\mapsto \sum_{i\in\mathbb{Z}}(-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(Z_1, Z_2) \end{aligned}$$

is called the *Euler bilinear form* (cf.[Gor]).

Proposition 2.3.6. *Suppose that the derived category $D^b(Coh\, Y)$ of a smooth projective variety Y is generated by an exceptional sequence $(E_1, \dots, E_n)$. Then $K_\circ(Y) \simeq \mathbb{Z}^n$ is a free $\mathbb{Z}$-module of rank n with basis given by $([E_1], \dots, [E_n])$. The Euler bilinear form χ is unimodular with Gram matrix with respect to the basis $([E_1], \dots, [E_n])$:*

$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & * & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix};$$

in other words, $([E_1], \dots, [E_n])$ is a semi-orthonormal basis w.r.t. χ.
Moreover, $n = \mathrm{rk}\, K_\circ(Y) = \bigoplus_{r=0}^{\dim Y} \mathrm{rk}\, A^r(Y)$, where $A^r(Y)$ is the group of codimension r algebraic cycles on Y modulo rational equivalence (so that $A(Y) = \bigoplus_r A^r(Y)$ is the Chow ring of Y).

Proof. Since the E_i, $i = 1, \dots, n$, generate $D^b(Coh\, Y)$ in the sense of definition 2.1..2 it is clear that the $[E_i]$ generate $K_\circ(Y)$ (note that for X, X', $X'' \in \mathrm{obj}\, D^b(Coh\, Y)$ we have $[X[n]] = (-1)^n[X]$, $n \in \mathbb{Z}$, $[X' \oplus X''] = [X'] + [X'']$ and for every distinguished triangle $X' \to X \to X'' \to X'[1]$ one has $[X''] = [X] - [X']$).

$[E_1] \neq 0$ because $\chi([E_1], [E_1]) = 1$ since E_1 is exceptional. Assume inductively that $[E_1], \dots, [E_i]$ are linearly independent in $K_\circ(Y) \otimes \mathbb{Q}$. We claim $[E_{i+1}] \notin \langle [E_1], \dots, [E_i] \rangle_{\mathbb{Q}}$. Indeed otherwise $[E_{i+1}] = \sum_{j=1}^{i} \lambda_j [E_j]$; since $[E_{i+1}] \neq 0$ there is $l := \min\{j \mid \lambda_j \neq 0\}$. Then

$$\chi([E_{i+1}], [E_l]) = \chi(\sum_{j=l}^{i} \lambda_j [E_j], [E_l]) = \lambda_l \neq 0$$

(using $\mathrm{Ext}^k(E_j, E_i) = 0\ \forall k \in \mathbb{Z}\ \forall i < j$) contradicting the fact that $\chi([E_{i+1}], [E_l]) = 0$ since $l < i+1$. Thus the $([E_1], \dots, [E_n])$ form a free $\mathbb{Z}$-basis

of $K_\circ(Y)$. The remaining assertions concerning χ are obvious from the above arguments.
The last equality follows from the fact that the Grothendieck Chern character ch gives an isomorphism

$$\mathrm{ch} : K_\circ(Y) \otimes \mathbb{Q} \to A(Y) \otimes \mathbb{Q}$$

(cf. [Ful], 15.2.16 (b)). □

Corollary 2.3.7. *If $(E_1, \dots, E_n)$ is an exceptional sequence that generates $D^b(Coh\, X)$, X a rational homogeneous variety, then $n = |W^P|$, the number of Schubert varieties X_w in X.*

Proof. It suffices to show that the $[\mathcal{O}_{X_w}]$'s likewise form a free $\mathbb{Z}$-basis of $K_\circ(X)$. One way to see this is as follows: By proposition 2.3.1 it is clear that the $[\mathcal{O}_{X_w}]$ generate $K_\circ(X)$. $K_\circ(X)$ is a ring for the product $[\mathcal{F}] \cdot [\mathcal{G}] := \sum_{i\in\mathbb{Z}}(-1)^i[\mathcal{T}or_i^X(\mathcal{F}, \mathcal{G})]$ and

$$\begin{aligned} \beta : K_\circ(X) \times K_\circ(X) &\to \mathbb{Z} \\ ([\mathcal{F}], [\mathcal{G}]) &\mapsto \sum_{i\in\mathbb{Z}}(-1)^i h^i(X, [\mathcal{F}] \cdot [\mathcal{G}]) \end{aligned}$$

is a symmetric bilinear form. One can compute that $\beta([\mathcal{O}_{X_x}], [\mathcal{O}_{X^y}(-\partial X^y)]) = \delta_x^y$ (Kronecker delta) for $x, y \in W^P$, cf. [BL], proof of lemma 6, for details. □

It should be noted at this point that the constructions in subsection 2.1 relating to semi-orthogonal decompositions, mutations etc. all have their counterparts on the K-theory level and in fact appear more natural in that context (cf. [Gor], §1).

Remark 2.3.8. Suppose that on $X = G/P$ we have a strong complete exceptional sequence $(E_1, \dots, E_n)$. Then the Gram matrix Γ of χ w.r.t. the basis $([E_1], \dots, [E_n])$ on $K_\circ(X) \simeq \mathbb{Z}^n$ is upper triangular with ones on the diagonal and (i, j)-entry equal to $\dim_{\mathbb{C}} \mathrm{Hom}(E_i, E_j)$. Thus with regard to conjecture 2.2.1 it would be interesting to know the Gram matrix Γ' of χ in the basis given by the $[\mathcal{O}_{X_w}]$'s, $w \in W^P$, since Γ and Γ' will be conjugate.
The following computation was suggested to me by M. Brion. Without loss of generality one may reduce to the case $X = G/B$ using the fibration $\pi : G/B \to G/P$: Indeed, the pull-back under π of the Schubert variety X_{wP}, $w \in W^P$, is the Schubert variety $X_{w\, w_{0,P}}$ in G/B where $w_{0,P}$ is the element of maximal length of W_P, and $\pi^*\mathcal{O}_{X_{wP}} = \mathcal{O}_{X_{w\, w_{0,P}}}$. Moreover, by the projection formula and because $R\pi_*\mathcal{O}_{G/B} = \mathcal{O}_{G/P}$, we have $R\pi_* \circ \pi^* \simeq \mathrm{id}_{D^b(Coh\, G/P)}$ and

$$\chi(\pi^*\mathcal{E}, \pi^*\mathcal{F}) = \chi(\mathcal{E}, \mathcal{F})$$

for any $\mathcal{E}$, $\mathcal{F} \in \mathrm{obj}\, D^b(Coh\, G/P)$.
Therefore, let $X = G/B$ and let x, $y \in W$. The first observation is that

$X_y = w_0 X^{w_0 y}$ and $\chi(\mathcal{O}_{X_x}, \mathcal{O}_{X_y}) = \chi(\mathcal{O}_{X_x}, \mathcal{O}_{X^{w_0 y}})$. This follows from the facts that there is a connected chain of rational curves in G joining g to id_G (since G is generated by images of homomorphisms $\mathbb{C} \to G$ and $\mathbb{C}^* \to G$) and that flat families of sheaves indexed by open subsets of $\mathbb{A}^1$ yield the same class in $K_\circ(X)$, thus $[\mathcal{O}_{X^{w_0 y}}] = [\mathcal{O}_{w_0 X^{w_0 y}}]$. We have

$$\begin{aligned} R\mathrm{Hom}^\bullet(\mathcal{O}_{X_x}, \mathcal{O}_{X^{w_0 y}}) &\simeq R\Gamma(X, R\mathcal{H}om^\bullet(\mathcal{O}_{X_x}, \mathcal{O}_{X^{w_0 y}})) \\ &\simeq R\Gamma(X, R\mathcal{H}om^\bullet(\mathcal{O}_{X_x}, \mathcal{O}_X) \otimes^L \mathcal{O}_{X^{w_0 y}}) \end{aligned}$$

(cf. [Ha1], prop. 5.3/5.14). Now Schubert varieties are Cohen-Macaulay, in fact they have rational singularities (cf. [Ra1]), whence

$$\begin{aligned} R\mathcal{H}om^\bullet(\mathcal{O}_{X_x}, \mathcal{O}_X) &\simeq \mathcal{E}xt^{\mathrm{codim}(X_x)}(\mathcal{O}_{X_x}, \mathcal{O}_X)[-\mathrm{codim}(X_x)] \\ &\simeq \omega_{X_x} \otimes \omega_X^{-1}[-\mathrm{codim}(X_x)]\,. \end{aligned}$$

But $\omega_{X_x} \otimes \omega_X^{-1} \simeq \mathcal{L}(\varrho)|_{X_x}(-\partial X_x)$ ($\mathcal{L}(\varrho)$ is the line bundle associated to the character ϱ), cf. [Ra1], prop. 2 and thm. 4. Now X_x and $X^{w_0 y}$ are Cohen-Macaulay and their scheme theoretic intersection is proper in X and reduced ([Ra1], thm. 3) whence $\mathcal{T}or_i^X(\mathcal{O}_{X_x}, \mathcal{O}_{X^{w_0 y}}) = 0$ for all $i \geq 1$ (cf. [Bri], lemma 1). Since ∂X_x is likewise Cohen-Macaulay by [Bri], lemma 4, we get by the same reasoning $\mathcal{T}or_i^X(\mathcal{O}_{\partial X_x}, \mathcal{O}_{X^{w_0 y}}) = 0$ for all $i \geq 1$. Thus by the exact sequence

$$0 \to \mathcal{O}_{X_x}(-\partial X_x) \to \mathcal{O}_{X_x} \to \mathcal{O}_{\partial X_x} \to 0$$

and the long exact sequence of Tor's we see that $\mathcal{T}or_i^X(\mathcal{O}_{X_x}(-\partial X_x), \mathcal{O}_{X^{w_0 y}}) = 0$ for all $i \geq 1$.
Therefore

$$R\mathrm{Hom}^\bullet(\mathcal{O}_{X_x}, \mathcal{O}_{X^{w_0 y}}) \simeq R\Gamma(X, \mathcal{L}(\varrho)|_{X_x}(-\partial X_x)[-\mathrm{codim}(X_x)] \otimes \mathcal{O}_{X^{w_0 y}})$$

so that setting $X_x^{w_0 y} := X_x \cap X^{w_0 y}$ and $(\partial X_x)^{w_0 y} := \partial X_x \cap X^{w_0 y}$

$$\chi(\mathcal{O}_{X_x}, \mathcal{O}_{X_y}) = (-1)^{\mathrm{codim}(X_x)} \chi\left(\mathcal{L}(\varrho)|_{X_x^{w_0 y}}(-(\partial X_x)^{w_0 y})\right)\,.$$

This is 0 unless $w_0 y \leq x$ (because $X_x^{w_0 y}$ is non-empty iff $w_0 y \leq x$, see [BL], lemma 1); moreover if $w_0 y \leq x$ there are no higher h^i in the latter Euler characteristic by [BL], prop. 2. In conclusion

$$\chi(\mathcal{O}_{X_x}, \mathcal{O}_{X_y}) = \begin{cases} (-1)^{\mathrm{codim}(X_x)} h^0\left(\mathcal{L}(\varrho)|_{X_x^{w_0 y}}(-(\partial X_x)^{w_0 y})\right) & \text{if} \quad w_0 y \leq x \\ 0 \quad \text{otherwise} & \end{cases}$$

though the impact of this on conjecture 2.2.1 ((A) or (B)) is not clear to me. Cf. also [Bri2] for this circle of ideas.

3 Fibrational techniques

The main idea pervading this section is that the theorem of Beilinson on the structure of the derived category of coherent sheaves on projective space ([Bei]) and the related results of Kapranov ([Ka1], [Ka2], [Ka3]) for Grassmannians, flag varieties and quadrics, generalize without substantial difficulty from the absolute to the relative setting, i.e. to projective bundles etc. For projective bundles, Grassmann and flag bundles this has been done in [Or]. We review these results in subsection 3.1; the case of quadric bundles is dealt with in subsection 3.2. Aside from being technically a little more involved, the result follows rather mechanically combining the techniques from [Ka3] and [Or]. Thus armed, we deduce information on the derived category of coherent sheaves on isotropic Grassmannians and flag varieties in the symplectic and orthogonal cases; we follow an idea first exploited in [Sa] using successions of projective and quadric bundles.

3.1 The theorem of Orlov on projective bundles

Let X be a smooth projective variety, $\mathcal{E}$ a vector bundle of rank $r+1$ on X. Denote by $\mathbb{P}(\mathcal{E})$ the associated projective bundle † and $\pi : \mathbb{P}(\mathcal{E}) \to X$ the projection. Set $D^b(\mathcal{E}) := D^b(Coh(\mathbb{P}(\mathcal{E})))$, $D^b(X) := D^b(Coh(X))$. There are the functors $R\pi_* : D^b(\mathcal{E}) \to D^b(X)$ (note that $R\pi_* : D^+(Coh(\mathbb{P}(\mathcal{E}))) \to D^+(Coh(X))$, where $D^+(-)$ denotes the derived category of complexes bounded to the left in an abelian category, maps $D^b(\mathcal{E})$ to $D^b(X)$ using $R^i\pi_*(\mathcal{F}) = 0\,\forall i > \dim \mathbb{P}(\mathcal{E})\,\forall \mathcal{F} \in \mathrm{Ob}\, Coh(\mathbb{P}(\mathcal{E}))$ and the spectral sequence in hypercohomology) and $\pi^* : D^b(X) \to D^b(\mathcal{E})$ (π is flat, hence π^* is exact and passes to the derived category without taking the left derived functor).
We identify $D^b(X)$ with a full subcategory in $D^b(\mathcal{E})$ via π^* (cf. [Or], lemma 2.1). More generally we denote by $D^b(X)\otimes\mathcal{O}_{\mathcal{E}}(m)$ for $m \in \mathbb{Z}$ the subcategory of $D^b(\mathcal{E})$ which is the image of $D^b(X)$ in $D^b(\mathcal{E})$ under the functor $\pi^*(-)\otimes\mathcal{O}_{\mathcal{E}}(m)$, where $\mathcal{O}_{\mathcal{E}}(1)$ is the relative hyperplane bundle on $\mathbb{P}(\mathcal{E})$. Then one has the following result (cf. [Or], thm. 2.6):

Theorem 3.1.1. *The categories $D^b(X)\otimes\mathcal{O}_{\mathcal{E}}(m)$ are all admissible subcategories of $D^b(\mathcal{E})$ and we have a semiorthogonal decomposition*

$$D^b(\mathcal{E}) = \left\langle D^b(X) \otimes \mathcal{O}_{\mathcal{E}}(-r), \dots, D^b(X) \otimes \mathcal{O}_{\mathcal{E}}(-1), D^b(X) \right\rangle .$$

We record the useful

Corollary 3.1.2. *If $D^b(X)$ is generated by a complete exceptional sequence*

$$(E_1, \dots, E_n)\,,$$

then $D^b(\mathcal{E})$ is generated by the complete exceptional sequence

$$(\pi^*E_1 \otimes \mathcal{O}_{\mathcal{E}}(-r), \dots, \pi^*E_n \otimes \mathcal{O}_{\mathcal{E}}(-r), \pi^*E_1 \otimes \mathcal{O}_{\mathcal{E}}(-r+1), \\ \dots, \pi^*E_1, \dots, \pi^*E_n)\,.$$

†Here and in the following $\mathbb{P}(\mathcal{E})$ denotes $\mathrm{Proj}(\mathrm{Sym}^\bullet(\mathcal{E}^\vee))$, i.e. the bundle of 1-dimensional subspaces in the fibres of $\mathcal{E}$, and contrary to Grothendieck's notation NOT the bundle $\mathrm{Proj}(\mathrm{Sym}^\bullet \mathcal{E})$ of hyperplanes in the fibres of $\mathcal{E}$ which might be less intuitive in the sequel.

Proof. This is stated in [Or], cor. 2.7; for the sake of completeness and because the method will be used repeatedly in the sequel, we give a proof. One just checks that

$$\mathrm{Ext}^k(\pi^* E_i \otimes \mathcal{O}_{\mathcal{E}}(-r_1), \pi^* E_j \otimes \mathcal{O}_{\mathcal{E}}(-r_2)) = 0$$

$\forall k, \forall 1 \le i,\, j \le n\, \forall 0 \le r_1 < r_2 \le r$ and $\forall k$, $\forall 1 \le j < i \le n$, $r_1 = r_2$. Indeed,

$$\begin{aligned}\mathrm{Ext}^k(\pi^* E_i \otimes \mathcal{O}_{\mathcal{E}}(-r_1), \pi^* E_j \otimes \mathcal{O}_{\mathcal{E}}(-r_2)) &\simeq \mathrm{Ext}^k(\pi^* E_i, \pi^* E_j \otimes \mathcal{O}_{\mathcal{E}}(r_1 - r_2)) \\ &\simeq \mathrm{Ext}^k(E_i, E_j \otimes R\pi_*(\mathcal{O}_{\mathcal{E}}(r_1 - r_2)))\end{aligned}$$

where for the second isomorphism we use that $R\pi_*$ is right adjoint to π^*, and the projection formula (cf. [Ha2], II, prop. 5.6). When $r_1 = r_2$ and $i > j$ then $R\pi_* \mathcal{O}_{\mathcal{E}} \simeq \mathcal{O}_X$ and $\mathrm{Ext}^k(E_i, E_j) = 0$ for all k because $(E_1, \dots, E_n)$ is exceptional. If on the other hand $0 \le r_1 < r_2 \le r$ then $-r \le r_1 - r_2 < 0$ and $R\pi_*(\mathcal{O}_{\mathcal{E}}(r_1 - r_2)) = 0$.
It remains to see that each $\pi^* E_i \otimes \mathcal{O}_{\mathcal{E}}(-r_1)$ is exceptional. From the above calculation it is clear that this follows exactly from the exceptionality of E_i. $\square$

Remark 3.1.3. From the above proof it is clear that even if we start in corollary 3.1.2 with a strong complete exceptional sequence $(E_1, \dots, E_n)$ (i.e. $\mathrm{Ext}^k(E_i, E_j) = 0 \forall i,\, j \forall k \neq 0$), the resulting exceptional sequence on $\mathbb{P}(\mathcal{E})$ need not again be strong: For example take $X = \mathbb{P}^1$ with strong complete exceptional sequence $(\mathcal{O}(-1), \mathcal{O})$ and $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(h)$, $h \ge 2$, so that $\mathbb{P}(\mathcal{E}) = \mathbb{F}_h \xrightarrow{\pi} \mathbb{P}^1$ is a Hirzebruch surface. Then $(\pi^*\mathcal{O}(-1) \otimes \mathcal{O}_{\mathcal{E}}(-1), \mathcal{O}_{\mathcal{E}}(-1), \pi^*\mathcal{O}(-1) \otimes \mathcal{O}_{\mathcal{E}}, \mathcal{O}_{\mathcal{E}})$ is an exceptional sequence on $\mathbb{F}_h$ that generates $D^b(Coh(\mathbb{F}_h))$, but it is not a strong one since $\mathrm{Ext}^1(\mathcal{O}_{\mathcal{E}}(-1), \mathcal{O}_{\mathcal{E}}) \simeq H^1(\mathbb{P}^1, \pi_* \mathcal{O}_{\mathcal{E}}(1)) \simeq H^1(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(-h)) \simeq \mathrm{Sym}^{h-2}\, \mathbb{C}^2 \neq 0$.

Analogous results hold for relative Grassmannians and flag varieties. Specifically, if $\mathcal{E}$ is again a rank $r+1$ vector bundle on a smooth projective variety X, denote by $\mathrm{Grass}_X(k, \mathcal{E})$ the relative Grassmannian of k-planes in the fibres of $\mathcal{E}$ with projection $\pi : \mathrm{Grass}_X(k, \mathcal{E}) \to X$ and tautological subbundle $\mathcal{R}$ of rank k in $\pi^*\mathcal{E}$. Denote by $Y(k, r+1-k)$ the set of partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ with $0 \le \lambda_k \le \lambda_{k-1} \le \dots \le \lambda_1 \le r+1-k$ or equivalently the set of Young diagrams with at most k rows and no more than $r+1-k$ columns. For $\lambda \in V(k, r+1-k)$ we have the Schur functor Σ^λ and bundles $\Sigma^\lambda \mathcal{R}$ on $\mathrm{Grass}_X(k, \mathcal{E})$. Moreover, as before we can talk about full subcategories $D^b(X) \otimes \Sigma^\lambda \mathcal{R}$ of $D^b(Coh(\mathrm{Grass}_X(k, \mathcal{E})))$. Choose a total order $\prec$ on $Y(k, r+1-k)$ such that if $\lambda \prec \mu$ then the Young diagram of λ is not contained in the Young diagram of μ, i.e. $\exists i : \mu_i < \lambda_i$. Then one has (cf. [Or], p. 137):

Theorem 3.1.4. *There is a semiorthogonal decomposition*

$$D^b(Coh(\mathrm{Grass}_X(k, \mathcal{E}))) = \left\langle \dots, D^b(X) \otimes \Sigma^\lambda \mathcal{R}, \dots, D^b(X) \otimes \Sigma^\mu \mathcal{R}, \dots \right\rangle$$

$(\lambda \prec \mu)$.
If $(E_1, \ldots, E_n)$ is a complete exceptional sequence in $D^b(X)$, then

$$\big(\ldots, \pi^* E_1 \otimes \Sigma^\lambda \mathcal{R}, \ldots, \pi^* E_n \otimes \Sigma^\lambda \mathcal{R},\\ \ldots, \pi^* E_1 \otimes \Sigma^\mu \mathcal{R}, \ldots, \pi^* E_n \otimes \Sigma^\mu \mathcal{R}, \ldots\big)$$

is a complete exceptional sequence in $D^b(Coh(\mathrm{Grass}_X(k, \mathcal{E})))$. Here all $\pi^* E_i \otimes \Sigma^\lambda \mathcal{R}$, $i \in \{1, \ldots, n\}$, $\lambda \in Y(k, r+1-k)$ occur in the list, and $\pi^* E_i \otimes \Sigma^\lambda \mathcal{R}$ precedes $\pi^* E_j \otimes \Sigma^\mu \mathcal{R}$ iff $\lambda \prec \mu$ or $\lambda = \mu$ and $i < j$.

More generally, we can consider for $1 \le k_1 < \ldots < k_t \le r+1$ the variety $\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E})$ of relative flags of type $(k_1, \ldots, k_t)$ in the fibres of $\mathcal{E}$, with projection π and tautological subbundles $\mathcal{R}_{k_1} \subset \ldots \subset \mathcal{R}_{k_t} \subset \pi^* \mathcal{E}$. If we denote again by $Y(a, b)$ the set of Young diagrams with at most a rows and b columns, we consider the sheaves $\Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}$ on $\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E})$ with $\lambda_k \in Y(k_t, r+1-k_t)$ and $\lambda_j \in Y(k_j, k_{j+1} - k_j)$ for $j = 1, \ldots, t-1$ and subcategories $D^b(X) \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}$ of $D^b(Coh(\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E})))$. Choose a total order $\prec_j$ on each of the sets $Y(k_j, k_{j+1} - k_j)$ and $\prec_t$ on $Y(k_t, r+1-k_t)$ with the same property as above for the relative Grassmannian, and endow the set $Y = Y(k_t, r+1-k_t) \times \ldots \times Y(k_1, k_2 - k_1)$ with the resulting lexicographic order $\prec$.

Theorem 3.1.5. *There is a semiorthogonal decomposition*

$$D^b(Coh(\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E}))) = \big\langle \ldots, D^b(X) \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t},\\ \ldots, D^b(X) \otimes \Sigma^{\mu_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\mu_t} \mathcal{R}_{k_t}, \ldots \big\rangle$$

$((\lambda_t, \ldots, \lambda_1) \prec (\mu_t, \ldots, \mu_1))$.
If $(E_1, \ldots, E_n)$ is a complete exceptional sequence in $D^b(X)$, then

$$\big(\ldots, \pi^* E_1 \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}, \ldots, \pi^* E_n \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}, \ldots,\\ \pi^* E_1 \otimes \Sigma^{\mu_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\mu_t} \mathcal{R}_{k_t}, \ldots, \pi^* E_n \otimes \Sigma^{\mu_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\mu_t} \mathcal{R}_{k_t}, \ldots\big)$$

is a complete exceptional sequence in $D^b(Coh(\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E})))$. Here all $\pi^* E_i \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}$, $i \in \{1, \ldots, n\}$, $(\lambda_t, \ldots, \lambda_1) \in Y$ occur in the list, and $\pi^* E_i \otimes \Sigma^{\lambda_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\lambda_t} \mathcal{R}_{k_t}$ precedes $\pi^* E_j \otimes \Sigma^{\mu_1} \mathcal{R}_{k_1} \otimes \ldots \otimes \Sigma^{\mu_t} \mathcal{R}_{k_t}$ iff $(\lambda_t, \ldots, \lambda_1) \prec (\mu_t, \ldots, \mu_1)$ or $(\lambda_t, \ldots, \lambda_1) = (\mu_t, \ldots, \mu_1)$ and $i < j$.

Proof. Apply theorem 3.1.4 iteratively to the succession of Grassmann bundles

$$\mathrm{Flag}_X(k_1, \ldots, k_t; \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(k_2, \ldots, k_t; \mathcal{E})}(k_1, \mathcal{R}_{k_2})\\ \to \mathrm{Flag}_X(k_2, \ldots, k_t; \mathcal{E}) = \mathrm{Grass}_{\mathrm{Flag}_X(k_3, \ldots, k_t; \mathcal{E})}(k_2, \mathcal{R}_{k_3}) \to \ldots \to X$$

∎

3.2 The theorem on quadric bundles

Let us now work out in detail how the methods of Orlov ([Or]) and Kapranov ([Ka2], [Ka3]) yield a result for quadric bundles that is analogous to theorems 3.1.1, 3.1.4, 3.1.5.
As in subsection 3.1, X is a smooth projective variety with a vector bundle $\mathcal{E}$ of rank $r+1$ endowed with a symmetric quadratic form $q \in \Gamma(X, \mathrm{Sym}^2 \mathcal{E}^\vee)$ which is nondegenerate on each fibre; $\mathcal{Q} := \{q = 0\} \subset \mathbb{P}(\mathcal{E})$ is the associated quadric bundle:

$$\begin{tikzcd} \mathcal{Q} \arrow[r, hook] \arrow[dr, "\pi = \Pi|_{\mathcal{Q}}"'] & \mathbb{P}(\mathcal{E}) \arrow[d, "\Pi"] \\ & X \end{tikzcd}$$

Write $D^b(X) := D^b(Coh\, X)$, $D^b(\mathcal{Q}) := D^b(Coh\, \mathcal{Q})$, $D^b(\mathcal{E}) := D^b(Coh\, \mathbb{P}(\mathcal{E}))$.

Lemma 3.2.1. *The functor*

$$\pi^* = L\pi^* : D^b(X) \to D^b(\mathcal{Q})$$

is fully faithful.

Proof. Since $\mathcal{Q}$ is a locally trivial fibre bundle over X with rational homogeneous fibre, we have $\pi_* \mathcal{O}_{\mathcal{Q}} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{\mathcal{Q}} = 0$ for $i > 0$. The right adjoint to $L\pi^*$ is $R\pi_*$, and $R\pi_* \circ L\pi^*$ is isomorphic to the identity on $D^b(X)$ because of the projection formula and $R\pi_* \mathcal{O}_{\mathcal{Q}} = \mathcal{O}_X$. Hence $L\pi^*$ is fully faithful (and equal to π^* since π is flat). $\square$

Henceforth $D^b(X)$ is identified with a full subcategory of $D^b(\mathcal{Q})$.
We will now define two bundles of graded algebras, $\mathcal{A} = \bigoplus_{n \ge 0} \mathcal{A}_n$ and $\mathcal{B} = \bigoplus_{n \ge 0} \mathcal{B}_n$, on X. Form the tensor algebra $\mathrm{T}^\bullet(\mathcal{E}[h])$ where h is an indeterminate with $\deg h = 2$ and germs of sections in $\mathcal{E}$ have degree 1 and take the quotient modulo the two-sided ideal $\mathcal{I}$ of relations with $\mathcal{I}(x) := \langle e \otimes e - q(e)h,\ e \otimes h - h \otimes e \rangle_{e \in \mathcal{E}(x)}$, $(x \in X)$. This quotient is $\mathcal{A}$, the bundle of graded Clifford algebras of the orthogonal vector bundle $\mathcal{E}$. On the other hand, $\mathcal{B}$ is simply defined as $\bigoplus_{n \ge 0} \pi_* \mathcal{O}_{\mathcal{Q}}(n)$, the relative coordinate algebra of the quadric bundle $\mathcal{Q}$.
For each graded left $\mathcal{A}$-module $\mathcal{M} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}_i$ with $\mathcal{M}_i$ vector bundles on X we get a complex $L^\bullet(\mathcal{M})$ of bundles on $\mathcal{Q}$

$$L^\bullet(\mathcal{M}) : \ldots \to \pi^* \mathcal{M}_j \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{Q}}(j) \xrightarrow{d^j} \pi^* \mathcal{M}_{j+1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{Q}}(j+1) \to \ldots$$

with differentials given as follows: For $x \in X$ and $e \in \mathcal{E}(x)$ we get a family of mappings

$$d^j(x,e) : \mathcal{M}_j(x) \to \mathcal{M}_{j+1}(x)$$

given by left multiplication by e on $\mathcal{M}_j(x)$ and linear in e which globalize to mappings $\Pi^*\mathcal{M}_j \otimes \mathcal{O}_{\mathcal{E}}(j) \to \Pi^*\mathcal{M}_{j+1} \otimes \mathcal{O}_{\mathcal{E}}(j+1)$. When restricted to $\mathcal{Q}$ two successive maps compose to 0 and we get the required complex.

We recall at this point the relative version of Serre's correspondence (cf. e.g. [EGA], II, §3):

Theorem 3.2.2. *Let $Mod_{\mathcal{E}}^X$ be the category whose objects are coherent sheaves over X of graded $\mathrm{Sym}^\bullet \mathcal{E}^\vee$-modules of finite type with morphisms*

$$\mathrm{Hom}_{Mod_{\mathcal{E}}^X}(\mathcal{M},\mathcal{N}) := \varinjlim_n \mathrm{Hom}_{\mathrm{Sym}^\bullet \mathcal{E}^\vee}(\bigoplus_{i\geq n} \mathcal{M}_i, \bigoplus_{i \geq n} \mathcal{N}_i)$$

(the direct limit running over the groups of homomorphisms of sheaves of graded modules over $\mathrm{Sym}^\bullet \mathcal{E}^\vee$ which are homogeneous of degree 0). If $\mathcal{F} \in \mathrm{obj}(Coh(\mathbb{P}(\mathcal{E})))$ set

$$\alpha(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \Pi_*(\mathcal{F}(n)) .$$

Then the functor $\alpha : Coh(\mathbb{P}(\mathcal{E}))) \to Mod_{\mathcal{E}}^X$ is an equivalence of categories with quasi-inverse $(-)^\sim$ which is an additive and exact functor.

The key remark is now that $L^\bullet(\mathcal{A}^\vee)$ is exact since it arises by applying the Serre functor $(-)^\sim$ to the complex $P^\bullet$ given by

$$\dots \xrightarrow{\ d\ } \mathcal{A}_2^\vee \otimes \mathcal{B}[-2] \xrightarrow{\ d\ } \mathcal{A}_1^\vee \otimes \mathcal{B}[-1] \xrightarrow{\ d\ } \mathcal{A}_0^\vee \otimes \mathcal{B} \to \mathcal{O}_X \to 0.$$

Here, if $(e_1, \dots, e_{r+1})$ is a local frame of $\mathcal{E} = \mathcal{A}_1$ and $(e_1^\vee, \dots, e_{r+1}^\vee)$ is the corresponding dual frame for $\mathcal{E}^\vee = \mathcal{B}_1$, the differential d is $\sum_{i=1}^{r+1} l_{e_i}^\vee \otimes l_{e_i^\vee}$, where $l_{e_i} : \mathcal{A}[-1] \to \mathcal{A}$ is left multiplication by e_i and analogously $l_{e_i^\vee} : \mathcal{B}[-1] \to \mathcal{B}$. This complex is exact since it is so fibrewise as a complex of vector bundles; the fibre over a point $x \in X$ is just Priddy's generalized Koszul complex associated to the dual quadratic algebras $\mathcal{B}(x) = \oplus_i H^0(\mathcal{Q}(x), \mathcal{O}_{\mathcal{Q}(x)}(i))$ and $\mathcal{A}(x)$, the graded Clifford algebra of the vector space $\mathcal{E}(x)$. See [Ka3], 4.1 and [Pri].

Define bundles Ψ_i, $i \geq 0$, on $\mathcal{Q}$ by a twisted truncation, i.e., by the requirement that

$$0 \to \Psi_i \to \pi^*\mathcal{A}_i^\vee \to \pi^*\mathcal{A}_{i-1}^\vee \otimes \mathcal{O}_{\mathcal{Q}}(1) \to \dots \to \pi^*\mathcal{A}_0^\vee \otimes \mathcal{O}_{\mathcal{Q}}(i) \to 0$$

be exact. Look at the fibre product

$$\begin{array}{ccc} \Delta \subset \mathcal{Q} \times_X \mathcal{Q} & \xrightarrow{\ p_2\ } & \mathcal{Q} \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathcal{Q} & \xrightarrow{\ \pi\ } & X \end{array}$$

together with the relative diagonal Δ. The goal is to cook up an infinite to the left but eventually periodic resolution of the sheaf $\mathcal{O}_\Delta$ on $\mathcal{Q}\times_X\mathcal{Q}$, then truncate it in a certain degree and identify the remaining kernel explicitly.
Write $\Psi_i\boxtimes\mathcal{O}(-i)$ for $p_1^*\Psi_i\otimes p_2^*\mathcal{O}_{\mathcal{Q}}(-i)$ and consider the maps $\Psi_i\boxtimes\mathcal{O}(-i)\to\Psi_{i-1}\boxtimes\mathcal{O}(-i+1)$ induced by the maps of complexes

$$\begin{tikzcd}
(\pi^*\mathcal{A}_i^\vee\otimes\mathcal{O})\boxtimes\mathcal{O}(-i) \arrow[r]\arrow[d] & (\pi^*\mathcal{A}_{i-1}^\vee\otimes\mathcal{O}(1))\boxtimes\mathcal{O}(-i) \arrow[r]\arrow[d] & \dots \\
(\pi^*\mathcal{A}_{i-1}^\vee\otimes\mathcal{O})\boxtimes\mathcal{O}(-i+1) \arrow[r] & (\pi^*\mathcal{A}_{i-2}^\vee\otimes\mathcal{O}(1))\boxtimes\mathcal{O}(-i+1) \arrow[r] & \dots
\end{tikzcd}$$

where the vertical arrows are given by $\sum_{i=1}^{r+1}(\pi^*r_{e_i}^\vee\otimes\mathrm{id})\boxtimes\widetilde{l_{e_i^\vee}}$; here again we're using the local frames $(e_1,\dots,e_{r+1})$, resp. $(e_1^\vee,\dots,e_{r+1}^\vee)$, $r_{e_i}:\mathcal{A}[-1]\to\mathcal{A}$ is right multiplication by e_i and $\widetilde{l_{e_i^\vee}}$ is the map induced by $l_{e_i^\vee}:\mathcal{B}[-1]\to\mathcal{B}$ between the associated sheaves (via the Serre correspondence).
This is truly a map of complexes since right and left Clifford multiplication commute with each other. Moreover, we obtain a complex, infinite on the left side

$$R^\bullet:\ \dots\to\Psi_i\boxtimes\mathcal{O}(-i)\to\dots\to\Psi_2\boxtimes\mathcal{O}(-2)\to\Psi_1\boxtimes\mathcal{O}(-1)\to\mathcal{O}_{\mathcal{Q}\times_X\mathcal{Q}}\,.$$

LEMMA 3.2.3. *The complex $R^\bullet$ is a left resolution of $\mathcal{O}_\Delta$, $\Delta\subset\mathcal{Q}\times_X\mathcal{Q}$ being the diagonal.*

Proof. Consider $\mathcal{B}^2:=\bigoplus_i\mathcal{B}_i\otimes_{\mathcal{O}_X}\mathcal{B}_i$, the "Segre product of $\mathcal{B}$ with itself" (i.e. the homogeneous coordinate ring of $\mathcal{Q}\times_X\mathcal{Q}$ under the (relative) Segre morphism). Look at the following double complex $D^{\bullet\bullet}$ of $\mathcal{B}^2$-modules:

$$\begin{tikzcd}
\dots \arrow[r] & \bigoplus_i\mathcal{B}_{i+2}\otimes\mathcal{B}_{i-2} & & \\
\dots \arrow[r] & \bigoplus_i\mathcal{A}_1^\vee\otimes\mathcal{B}_{i+1}\otimes\mathcal{B}_{i-2} \arrow[r]\arrow[u] & \bigoplus_i\mathcal{B}_{i+1}\otimes\mathcal{B}_{i-1} & \\
\dots \arrow[r] & \bigoplus_i\mathcal{A}_2^\vee\otimes\mathcal{B}_i\otimes\mathcal{B}_{i-2} \arrow[r]\arrow[u] & \bigoplus_i\mathcal{A}_1^\vee\otimes\mathcal{B}_i\otimes\mathcal{B}_{i-1} \arrow[r]\arrow[u] & \bigoplus_i\mathcal{B}_i\otimes\mathcal{B}_i
\end{tikzcd}$$

Here the columns correspond to the right resolutions of $\Psi_0\boxtimes\mathcal{O}$, $\Psi_1\boxtimes\mathcal{O}(-1)$, $\Psi_2\boxtimes\mathcal{O}(-2)$ etc. (starting from the right) if we pass from complexes of coherent sheaves on $\mathcal{Q}\times_X\mathcal{Q}$ to complexes of graded $\mathcal{B}^2$-modules via Serre's theorem. For example, the left-most column in the above diagram arises from

$$\begin{aligned}(\pi^*\mathcal{A}_2^\vee\otimes\mathcal{O}_{\mathcal{Q}})\boxtimes\mathcal{O}(-2)&\to(\pi^*\mathcal{A}_1^\vee\otimes\mathcal{O}_{\mathcal{Q}}(1))\boxtimes\mathcal{O}(-2)\\&\to(\pi^*\mathcal{A}_0^\vee\otimes\mathcal{O}_{\mathcal{Q}}(2))\boxtimes\mathcal{O}(-2)\to 0\end{aligned}$$

The horizontal arrows in the above diagram then come from the morphisms of complexes defining the differentials in $R^\bullet$.
The associated total complex $Tot^\bullet(D^{\bullet\bullet})$ has a natural augmentation $a : Tot^\bullet(D^{\bullet\bullet}) \to \bigoplus_i \mathcal{B}_{2i}$ arising from the multiplication maps $\mathcal{B}_{i+j} \otimes \mathcal{B}_{i-j} \to \mathcal{B}_{2i}$ and corresponding to the augmentation $R^\bullet \to \mathcal{O}_\Delta$.
Claim: a is a quasi-isomorphism. For this note that $D^{\bullet\bullet}$ is the direct sum over i of double complexes

$$\begin{tikzcd}
\dots \arrow[r] & \mathcal{B}_{i+2} \otimes \mathcal{B}_{i-2} & & \\
\dots \arrow[r] & \mathcal{A}_1^\vee \otimes \mathcal{B}_{i+1} \otimes \mathcal{B}_{i-2} \arrow[u] \arrow[r] & \mathcal{B}_{i+1} \otimes \mathcal{B}_{i-1} & \\
\dots \arrow[r] & \mathcal{A}_2^\vee \otimes \mathcal{B}_{i} \otimes \mathcal{B}_{i-2} \arrow[u] \arrow[r] & \mathcal{A}_1^\vee \otimes \mathcal{B}_{i} \otimes \mathcal{B}_{i-1} \arrow[u] \arrow[r] & \mathcal{B}_{i} \otimes \mathcal{B}_{i}
\end{tikzcd}$$

which are bounded ($\mathcal{B}$ is positively graded) and whose rows are just Priddy's resolution $P^\bullet$ in various degrees and thus the total complex of the above direct summand of $D^{\bullet\bullet}$ is quasi-isomorphic to $\mathcal{A}_0^\vee \otimes \mathcal{B}_{2i} \otimes \mathcal{B}_0 = \mathcal{B}_{2i}$. Thus $Tot^\bullet(D^{\bullet\bullet})$ is quasi-isomorphic to $\bigoplus_i \mathcal{B}_{2i}$. $\square$

The next step is to identify the kernel of the map $\Psi_{r-2} \boxtimes \mathcal{O}(-r+2) \to \Psi_{r-3} \boxtimes \mathcal{O}(-r+3)$. For this we have to talk in more detail about spinor bundles.
Let $\mathrm{Cliff}(\mathcal{E}) = \mathcal{A}/(h-1)\mathcal{A}$ be the Clifford bundle of the orthogonal vector bundle $\mathcal{E}$. This is just $\mathrm{Cliff}(\mathcal{E}) := \mathrm{T}^\bullet\, \mathcal{E}/I(\mathcal{E})$ where $I(\mathcal{E})$ is the bundle of ideals whose fibre at $x \in X$ is the two-sided ideal $I(\mathcal{E}(x))$ in $T^\bullet(\mathcal{E}(x))$ generated by the elements $e \otimes e - q(e)1$ for $e \in \mathcal{E}(x)$. $\mathrm{Cliff}(\mathcal{E})$ inherits a $\mathbb{Z}/2$-grading, $\mathrm{Cliff}(\mathcal{E}) = \mathrm{Cliff}^{\mathrm{even}}(\mathcal{E}) \oplus \mathrm{Cliff}^{\mathrm{odd}}(\mathcal{E})$.
For $\boxed{r+1 \text{ odd}}$ we will now make the following assumption (A):

> There exists a bundle of irreducible $\mathrm{Cliff}^{\mathrm{even}}(\mathcal{E})$-modules $S(\mathcal{E})$, self-dual up to twist by a line bundle L on X, i.e.
> $S(\mathcal{E})^\vee \simeq S(\mathcal{E}) \otimes L$,
> together with an isomorphism of sheaves of algebras on X
> $\mathrm{Cliff}(\mathcal{E}) \simeq \mathcal{E}nd(S(\mathcal{E})) \oplus \mathcal{E}nd(S(\mathcal{E}))$
> such that
> $\mathrm{Cliff}^{\mathrm{even}}(\mathcal{E}) \simeq \mathcal{E}nd(S(\mathcal{E}))$.

For $\boxed{r+1 \text{ even}}$ we assume (A'):

There exist bundles of irreducible $\mathrm{Cliff}^{\mathrm{even}}(\mathcal{E})$-modules $S^+(\mathcal{E})$, $S^-(\mathcal{E})$, which for $\boxed{r+1\equiv 0(4)}$ satisfy
$S^+(\mathcal{E})^\vee \simeq S^+(\mathcal{E})\otimes L$
$S^-(\mathcal{E})^\vee \simeq S^-(\mathcal{E})\otimes L$
and for $\boxed{r+1\equiv 2(4)}$ satisfy
$S^+(\mathcal{E})^\vee \simeq S^-(\mathcal{E})\otimes L$
$S^-(\mathcal{E})^\vee \simeq S^+(\mathcal{E})\otimes L$, L a line bundle on X,
together with an isomorphism of sheaves on algebras on X
$\mathrm{Cliff}(\mathcal{E})\simeq \mathcal{E}nd(S^+(\mathcal{E})\oplus S^-(\mathcal{E}))$
such that
$\mathrm{Cliff}^{\mathrm{even}}(\mathcal{E})\simeq \mathrm{E}nd(S^+(\mathcal{E}))\oplus \mathcal{E}nd(S^-(\mathcal{E}))$.

We will summarize this situation by saying that the orthogonal vector bundle $\mathcal{E}$ *admits spinor bundles* $S(\mathcal{E})$ resp. $S^+(\mathcal{E})$, $S^-(\mathcal{E})$.
Conditions (A) resp. (A') will be automatically satisfied in the applications in subsection 3.4.
Then for $\boxed{r+1 \text{ even}}$

$$\mathcal{M}^- := S^-(\mathcal{E})\oplus S^+(\mathcal{E})\oplus S^-(\mathcal{E})\oplus\dots$$

and

$$\mathcal{M}^+ := S^+(\mathcal{E})\oplus S^-(\mathcal{E})\oplus S^+(\mathcal{E})\oplus\dots$$

are graded left $\mathcal{A}$-modules (the grading starting from 0); one defines bundles Σ^+, Σ^- on $\mathcal{Q}$ by the requirement that

$$0\to(\Sigma^\pm)^\vee\to L^\bullet(\mathcal{M}^\pm)\ \text{ for }\ \boxed{r+1\equiv 0(\mathrm{mod}\,4)},$$
$$0\to(\Sigma^\mp)^\vee\to L^\bullet(\mathcal{M}^\pm)\ \text{ for }\ \boxed{r+1\equiv 2(\mathrm{mod}\,4)}$$

be exact.
For $\boxed{r+1 \text{ odd}}$ let $\mathcal{M}$ be the graded left $\mathcal{A}$- module (grading starting from 0)

$$\mathcal{M} := S(\mathcal{E})\oplus S(\mathcal{E})\oplus S(\mathcal{E})\oplus\dots$$

and define the bundle Σ on $\mathcal{Q}$ by the requirement that

$$0\to(\Sigma)^\vee\to L^\bullet(\mathcal{M})$$

be exact.
From the definition

$$\Sigma^\pm =: \Sigma^\pm(\mathcal{O}_{\mathcal{Q}}(-1)^\perp/\mathcal{O}_{\mathcal{Q}}(-1)) \text{ resp. } \Sigma =: \Sigma(\mathcal{O}_{\mathcal{Q}}(-1)^\perp/\mathcal{O}_{\mathcal{Q}}(-1))$$

are spinor bundles for the orthogonal vector bundle $\mathcal{O}_{\mathcal{Q}}(-1)^\perp/\mathcal{O}_{\mathcal{Q}}(-1)$ on $\mathcal{Q}$.

LEMMA 3.2.4. *The kernel*

$$\ker(\Psi_{r-2}\boxtimes\mathcal{O}(-r+2)\to\Psi_{r-3}\boxtimes\mathcal{O}(-r+3))$$

is equal to

(i) for $\boxed{r+1}$ *odd:*

$$(\Sigma(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma(-r+1)$$

(ii) for $\boxed{r+1 \equiv 2(\mathrm{mod}\, 4)}$*:*

$$((\Sigma^+(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma^+(-r+1)) \oplus ((\Sigma^-(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma^-(-r+1))$$

(iii) and for $\boxed{r+1 \equiv 0(\mathrm{mod}\, 4)}$*:*

$$((\Sigma^+(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma^-(-r+1)) \oplus ((\Sigma^-(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma^+(-r+1))$$

Proof. For $i \geq r$ $\mathcal{A}_i \overset{\mathrm{mult}(h)}{\longrightarrow} \mathcal{A}_{i+2}$ is an isomorphism because $(e_{i_1} \cdot \ldots \cdot e_{i_k} h^m)$, $1 \leq i_1 < \ldots < i_k \leq r+1$, $m \in \mathbb{N}$ is a local frame for $\mathcal{A}$ if $(e_1, \ldots, e_{r+1})$ is one for $\mathcal{E}$, and the map $\mathcal{A}_i \to \mathrm{Cliff}^{\mathrm{par}(i)}(\mathcal{E})$ induced by $\mathcal{A} \to \mathcal{A}/(h-1)\mathcal{A}$ is then an isomorphism where

$$\mathrm{par}(i) := \begin{cases} \text{even}, & i \equiv 0(\mathrm{mod}\, 2) \\ \text{odd}, & i \equiv 1(\mathrm{mod}\, 2) \end{cases} .$$

Because $L^\bullet(\mathcal{A}^\vee)$ is exact, Ψ_i is also the cokernel of

$$(*) \quad \ldots \to \pi^* \mathcal{A}^\vee_{i+3} \otimes \mathcal{O}_{\mathcal{Q}}(-3) \to \pi^* \mathcal{A}^\vee_{i+2} \otimes \mathcal{O}_{\mathcal{Q}}(-2) \to \pi^* \mathcal{A}^\vee_{i+1} \otimes \mathcal{O}_{\mathcal{Q}}(-1)$$

Since $\ker(\Psi_{r-2} \boxtimes \mathcal{O}(-r+2) \to \Psi_{r-3} \boxtimes \mathcal{O}(-r+3)) = \mathrm{coker}(\Psi_r \boxtimes \mathcal{O}(-r) \to \Psi_{r-1} \boxtimes \mathcal{O}(-r+1))$ we conclude that a left resolution of the kernel in lemma 3.2.4 is given by $Tot^\bullet(E^{\bullet\bullet})$ where $E^{\bullet\bullet}$ is the following double complex:

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
\ldots & \to & (\pi^* \mathrm{Cliff}^{\mathrm{par}(r+2)} \mathcal{E}^\vee(-2)) \boxtimes \mathcal{O}(-r) & \longrightarrow & (\pi^* \mathrm{Cliff}^{\mathrm{par}(r+1)} \mathcal{E}^\vee(-2)) \boxtimes \mathcal{O}(-r+1) \\
 & & \downarrow & & \downarrow \\
\ldots & \to & (\pi^* \mathrm{Cliff}^{\mathrm{par}(r+1)} \mathcal{E}^\vee(-1)) \boxtimes \mathcal{O}(-r) & \longrightarrow & (\pi^* \mathrm{Cliff}^{\mathrm{par}(r)} \mathcal{E}^\vee(-1)) \boxtimes \mathcal{O}(-r+1)
\end{array}$$

Here the columns (starting from the right) are the left resolutions $(*)$ of $\Psi_{r-1} \boxtimes \mathcal{O}(-r+1)$, $\Psi_r \boxtimes \mathcal{O}(-r)$, etc. and the rows are defined through the morphisms of complexes defining the differentials $\Psi_{r-1} \boxtimes \mathcal{O}(-r+1) \to \Psi_r \boxtimes \mathcal{O}(-r)$ etc. in the resolution $R^\bullet$. For odd $r+1$ we have $\mathrm{Cliff}^{odd}(\mathcal{E}) \simeq \mathrm{Cliff}^{even}(\mathcal{E}) \simeq \mathcal{E}nd(S(\mathcal{E})) \simeq S(\mathcal{E})^\vee \otimes S(\mathcal{E})$ whence our double complex becomes

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
\dots \arrow[r] & \pi^* S(\mathcal{E})^\vee(-2) \boxtimes \pi^* S(\mathcal{E})(-r) \arrow[r] \arrow[d] & \pi^* S(\mathcal{E})^\vee(-2) \boxtimes \pi^* S(\mathcal{E})(-r+1) \arrow[d] \\
\dots \arrow[r] & \pi^* S(\mathcal{E})^\vee(-1) \boxtimes \pi^* S(\mathcal{E})(-r) \arrow[r] & \pi^* S(\mathcal{E})^\vee(-1) \boxtimes \pi^* S(\mathcal{E})(-r+1)
\end{tikzcd}
$$

and is thus isomorphic as a double complex to $L^\bullet(\mathcal{M})^\vee(-1) \boxtimes (L^\bullet(\mathcal{M})^\vee(-r+1)\otimes\pi^* L^{-1}) \simeq (L^\bullet(\mathcal{M})^\vee(-1)\otimes\pi^* L^{-1})\boxtimes L^\bullet(\mathcal{M})^\vee(-r+1)$, i.e. quasi-isomorphic to $(\Sigma(-1) \otimes \pi^* L^{-1}) \boxtimes \Sigma(-r+1)$. The cases for even $r+1$ are considered similarly. $\square$

Lemma 3.2.5. *Consider the following two ordered sets of sheaves on $\mathcal{Q}$:*

$$
\begin{aligned}
\mathfrak{S} &= \{\Sigma(-r+1) \prec \mathcal{O}_\mathcal{Q}(-r+2) \prec \dots \prec \mathcal{O}_\mathcal{Q}(-1) \prec \mathcal{O}_\mathcal{Q}\} \quad (r+1 \text{ odd}),\\
\mathfrak{S}' &= \left\{\Sigma^+(-r+1) \prec \Sigma^-(-r+1) \prec \dots \prec \mathcal{O}_\mathcal{Q}(-1) \prec \mathcal{O}_\mathcal{Q}\right\} \quad (r+1 \text{ even}).
\end{aligned}
$$

If $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{V}_2 \in \mathfrak{S}$ (resp.: $\in \mathfrak{S}'$) with $\mathcal{V}_1 \prec \mathcal{V}_2$, $\mathcal{V}_1 \neq \mathcal{V}_2$, we have the following identities

$$
\begin{aligned}
R^i\pi_*(\mathcal{V}\otimes\mathcal{V}^\vee) &= 0\,, \quad \forall i \neq 0,\\
R^i\pi_*(\mathcal{V}_1\otimes\mathcal{V}_2^\vee) = 0\ \forall i \in \mathbb{Z}, &\quad R^i\pi_*(\mathcal{V}_2\otimes\mathcal{V}_1^\vee) = 0\ \forall i \neq 0\,.
\end{aligned}
$$

and the canonical morphism $R^0\pi_*(\mathcal{V}\otimes\mathcal{V}^\vee) \to \mathcal{O}_X$ is an isomorphism.

Proof. In the absolute case (where the base X is a point) this is a calculation in [Ka3] , prop. 4.9., based on Bott's theorem. The general assertion follows from this by the base change formula because the question is local on X and we can check this on open sets $U \subset X$ which cover X and over which $\mathcal{Q}$ is trivial. $\square$

As in subsection 3.1, for $\mathcal{V} \in \mathfrak{S}$ (resp. $\in \mathfrak{S}'$), we can talk about subcategories $D^b(X) \otimes \mathcal{V}$ of $D^b(\mathcal{Q})$ as the images of $D^b(X)$ in $D^b(\mathcal{Q})$ under the functor $\pi^*(-) \otimes \mathcal{V}$.

Proposition 3.2.6. *Let $\mathcal{V}$, $\mathcal{V}_1, \mathcal{V}_2$ be as in lemma 3.2.5. The subcategories $D^b(X) \otimes \mathcal{V}$ of $D^b(\mathcal{Q})$ are all admissible subcategories. Moreover, for $A \in$ $\mathrm{obj}(D^b(X) \otimes \mathcal{V}_2)$, $B \in \mathrm{obj}(D^b(X) \otimes \mathcal{V}_1)$ we have $R\mathrm{Hom}(A, B) = 0$.*

Proof. Let $A = \pi^*A'\otimes\mathcal{V}_2$, $B = \pi^*B'\otimes\mathcal{V}_1$. Using lemma 3.2.5 and the projection formula we compute

$$
\begin{aligned}
R^i\mathrm{Hom}(\pi^*A'\otimes\mathcal{V}_2, \pi^*B'\otimes\mathcal{V}_1) &\simeq R^i\mathrm{Hom}(\pi^*A', \pi^*B'\otimes\mathcal{V}_1\otimes\mathcal{V}_2^\vee)\\
&\simeq R^i\mathrm{Hom}(A', B'\otimes R\pi_*(\mathcal{V}_1\otimes\mathcal{V}_2^\vee)) \simeq 0\,.
\end{aligned}
$$

If we repeat the same calculation with $\mathcal{V}$ instead of $\mathcal{V}_1$ and $\mathcal{V}_2$ we find that $R^i\mathrm{Hom}(\pi^*A'\otimes\mathcal{V},\pi^*B'\otimes\mathcal{V})\simeq R^i\mathrm{Hom}(A',B')$. This shows that the categories $D^b(X)\otimes\mathcal{V}$ are all equivalent to $D^b(X)$ as triangulated subcategories of $D^b(\mathcal{Q})$. It follows from [BoKa] , prop. 2.6 and thm. 2.14, together with lemma 3.2.1 that the $D^b(X)\otimes\mathcal{V}$ are admissible subcategories of $D^b(\mathcal{Q})$. □

Theorem 3.2.7. *Let X be a smooth projective variety, $\mathcal{E}$ an orthogonal vector bundle on X, $\mathcal{Q}\subset\mathbb{P}(\mathcal{E})$ the associated quadric bundle, and let assumptions (A) resp. (A') above be satisfied.*
Then there is a semiorthogonal decomposition

$$\begin{aligned} D^b(\mathcal{Q}) = \big\langle & D^b(X)\otimes\Sigma(-r+1), D^b(X)\otimes\mathcal{O}_{\mathcal{Q}}(-r+2),\\ & \dots, D^b(X)\otimes\mathcal{O}_{\mathcal{Q}}(-1), D^b(X)\big\rangle \end{aligned}$$

for $r+1$ odd and

$$\begin{aligned} D^b(\mathcal{Q}) = \big\langle & D^b(X)\otimes\Sigma^+(-r+1), D^b(X)\otimes\Sigma^-(-r+1),\\ & D^b(X)\otimes\mathcal{O}_{\mathcal{Q}}(-r+2),\dots, D^b(X)\otimes\mathcal{O}_{\mathcal{Q}}(-1), D^b(X)\big\rangle \end{aligned}$$

for $r+1$ even.

Proof. By proposition 3.2.6 the categories in question are semiorthogonal and it remains to see that they generate $D^b(\mathcal{Q})$. For ease of notation we will consider the case of odd $r+1$, the case of even $r+1$ being entirely similar.
From lemmas 3.2.3 and 3.2.4 we know that in the situation of the fibre product

$$\begin{array}{ccc} \Delta\subset\mathcal{Q}\times_X\mathcal{Q} & \xrightarrow{p_2} & \mathcal{Q}\\ {\scriptstyle p_1}\downarrow & & {\scriptstyle\pi}\downarrow\\ \mathcal{Q} & \xrightarrow{\pi} & X \end{array}$$

we have a resolution

$$\begin{aligned} 0\to(\Sigma(-1)\otimes\pi^*L^{-1})\boxtimes\Sigma(-r+1)\to\Psi_{r-2}\boxtimes\mathcal{O}_{\mathcal{Q}}(-r+2)\to\dots\\ \dots\to\Psi_1\boxtimes\mathcal{O}(-1)\to\mathcal{O}_{\mathcal{Q}\times_X\mathcal{Q}}\to\mathcal{O}_\Delta\to 0 \end{aligned}$$

and tensoring this with $p_1^*\mathcal{F}$ ($\mathcal{F}$ a coherent sheaf on $\mathcal{Q}$)

$$\begin{aligned} 0\to(\Sigma(-1)\otimes\pi^*L^{-1}\otimes\mathcal{F})\boxtimes\Sigma(-r+1)\to(\Psi_{r-2}\otimes\mathcal{F})\boxtimes\mathcal{O}_{\mathcal{Q}}(-r+2)\to\dots\\ \dots\to(\Psi_1\otimes\mathcal{F})\boxtimes\mathcal{O}(-1)\to\mathcal{F}\boxtimes\mathcal{O}_{\mathcal{Q}}\to p_1^*\mathcal{F}|_\Delta\to 0 \end{aligned}$$

and applying $Rp_{2\,*}$ we obtain a spectral sequence

$$\begin{aligned} E_1^{ij} &= R^ip_{2\,*}((\Psi_{-j}\otimes\mathcal{F})\boxtimes\mathcal{O}_{\mathcal{Q}}(j)) \quad -r+2<j\le 0\\ &= R^ip_{2\,*}((\Sigma(-1)\otimes\pi^*L^{-1}\otimes\mathcal{F})\boxtimes\Sigma(-r+1)) \quad j=-r+1\\ &= 0 \quad \text{otherwise} \end{aligned}$$

and $E_1^{ij} \Rightarrow R^{i+j}p_{2\,*}(p_1^*\mathcal{F}|_\Delta)$ which is $= \mathcal{F}$ for $i+j=0$ and $=0$ otherwise. But since cohomology commutes with flat base extension (cf. [EGA], III, §1, prop. 1.4.15), we have $R^ip_{2\,*}p_1^*\mathcal{G} \simeq \pi^*R^i\pi_*\mathcal{G}$ for any coherent $\mathcal{G}$ on $\mathcal{Q}$. This together with the projection formula shows that all E_1^{ij} belong to one of the admissible subcategories in the statement of theorem 3.2.7. This finishes the proof because $D^b(\mathcal{Q})$ is generated by the subcategory $Coh(\mathcal{Q})$. □

Corollary 3.2.8. *If $D^b(X)$ is generated by a complete exceptional sequence*

$$(E_1, \dots, E_n)\,,$$

then $D^b(\mathcal{Q})$ is generated by the complete exceptional sequence

$$(\pi^*E_1 \otimes \Sigma(-r+1), \dots, \pi^*E_n \otimes \Sigma(-r+1), \pi^*E_1 \otimes \mathcal{O}_{\mathcal{Q}}(-r+2), \dots, \pi^*E_n)$$

for $r+1$ odd and

$$\big(\pi^*E_1 \otimes \Sigma^+(-r+1), \dots, \pi^*E_n \otimes \Sigma^+(-r+1), \dots, \pi^*E_1 \otimes \Sigma^-(-r+1), \\ \dots, \pi^*E_n \otimes \Sigma^-(-r+1), \pi^*E_1 \otimes \mathcal{O}_{\mathcal{Q}}(-r+2), \dots, \pi^*E_1, \dots, \pi^*E_n\big)$$

for $r+1$ even.

Proof. Using lemma 3.2.5, one proves this analogously to corollary 3.1.2; we omit the details. □

3.3 Application to varieties of isotropic flags in a symplectic vector space

We first fix some notation: Let V be a $\mathbb{C}$-vector space of even dimension $2n$ with a nondegenerate skew symmetric bilinear form $\langle\cdot,\cdot\rangle$. For $1 \le k_1 < \ldots < k_t \le n$ we denote $\mathrm{IFlag}(k_1, \dots, k_t; V) := \{(L_{k_1}, \dots, L_{k_t}) \mid L_{k_1} \subset \ldots L_{k_t} \subset V$ isotropic subspaces of V with $\dim L_{k_j} = k_j,\ 1 \le j \le t\}$ the (partial) flag variety of isotropic flags of type $(k_1, \dots, k_t)$ in V; moreover, for $1 \le k \le n$, put $\mathrm{IGrass}(k, V) := \mathrm{IFlag}(k; V)$, the Grassmann manifold of isotropic k-planes in V. As usual, we have the tautological flag of subbundles $\mathcal{R}_{k_1} \subset \ldots \subset \mathcal{R}_{k_t} \subset V \otimes \mathcal{O}_{\mathrm{IFlag}(k_1,\dots,k_t;V)}$ on $\mathrm{IFlag}(k_1, \dots, k_t; V)$ and the tautological subbundle $\mathcal{R}$ on $\mathrm{IGrass}(k, V)$.

Remark 3.3.1. Via the projection $\mathrm{IFlag}(k_1, \dots, k_t; V) \to \mathrm{IGrass}(k_t, V)$, the variety $\mathrm{IFlag}(k_1, \dots, k_t; V)$ identifies with $\mathrm{Flag}_{\mathrm{IGrass}(k_t,V)}(k_1, \dots, k_{t-1}; \mathcal{R})$, the relative variety of flags of type $(k_1, \dots, k_{t-1})$ in the fibres of the tautological subbundle $\mathcal{R}$ on $\mathrm{IGrass}(k_t, V)$. Therefore, by theorem 3.1.5, if we want to exhibit complete exceptional sequences in the derived categories of coherent sheaves on all possible varieties of (partial) isotropic flags in V, we can reduce to finding them on isotropic Grassmannians. Thus we will focus on the latter in the sequel.

Now look at the following diagram (the notation will be explained below)

$$\begin{tikzcd}
\mathrm{IFlag}(1,2,\dots,k-1,k;V) \simeq \mathbb{P}(\mathcal{E}_{k-1}) \simeq \mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,2,\dots,k;\mathcal{R}) \arrow[d, "\pi_{k-1}"] \arrow[dddddr, "\pi"] & \\
\mathrm{IFlag}(1,2,\dots,k-1;V) \simeq \mathbb{P}(\mathcal{E}_{k-2}) \arrow[d, "\pi_{k-2}"] & \\
\vdots \arrow[d, "\pi_2"] & \\
\mathrm{IFlag}(1,2;V) \simeq \mathbb{P}(\mathcal{E}_1) \arrow[d, "\pi_1"] & \\
\mathrm{IFlag}(1;V) \simeq \mathbb{P}(V) & \mathrm{IGrass}(k,V)
\end{tikzcd}$$

Since for $1 \le i \le j \le k-1$ the i-dimensional tautological subbundle on $\mathrm{IFlag}(1,2,\dots,j;V)$ pulls back to the i-dimensional tautological subbundle on $\mathrm{IFlag}(1,2,\dots,j+1;V)$ under π_j, we denote all of them by the same symbol $\mathcal{R}_i$ regardless of which space they live on, if no confusion can arise.
Since any line in V is isotropic, the choice of a 1-dimensional isotropic $L_1 \subset V$ comes down to picking a point in $\mathbb{P}(V)$ whence the identification $\mathrm{IFlag}(1;V) \simeq \mathbb{P}(V)$ above; the space $L_1^{\perp}/L_1$ is again a symplectic vector space with skew form induced from $\langle\cdot,\cdot\rangle$ on V, and finding an isotropic plane containing L_1 amounts to choosing a line L_2/L_1 in $L_1^{\perp}/L_1$. Thus $\mathrm{IFlag}(1,2;V)$ is a projective bundle $\mathbb{P}(\mathcal{E}_1)$ over $\mathrm{IFlag}(1;V)$ with $\mathcal{E}_1 = \mathcal{R}_1^{\perp}/\mathcal{R}_1$, and on $\mathbb{P}(\mathcal{E}_1) \simeq \mathrm{IFlag}(1,2;V)$ we have $\mathcal{O}_{\mathcal{E}_1}(-1) \simeq \mathcal{R}_2/\mathcal{R}_1$. Of course, $\mathrm{rk}\,\mathcal{E}_1 = 2n-2$.
Continuing this way, we successively build the whole tower of projective bundles over $\mathbb{P}(V)$ in the above diagram where

$$\mathcal{E}_j \simeq \mathcal{R}_j^{\perp}/\mathcal{R}_j\,, \quad j=1,\dots,k-1\,, \quad \mathrm{rk}\,\mathcal{E}_j = 2n-2j$$
$$\text{and} \quad \mathcal{O}_{\mathcal{E}_j}(-1) \simeq \mathcal{R}_{j+1}/\mathcal{R}_j\,.$$

Moreover, $\mathrm{IFlag}(1,2,\dots,k-1,k;V)$ is just $\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,\dots,k;\mathcal{R})$, the relative variety of complete flags in the fibres of the tautological subbundle $\mathcal{R}$ on $\mathrm{IGrass}(k,V)$; the flag of tautological subbundles in $V \otimes \mathcal{O}_{\mathrm{IFlag}(1,\dots,k;V)}$ on $\mathrm{IFlag}(1,\dots,k;V)$ and the flag of relative tautological subbundles in $\pi^*\mathcal{R}$ on $\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,2,\dots,k;\mathcal{R})$ correspond to each other under this isomorphism, and we do not distinguish them notationally.
For $\lambda = (\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k$ define the line bundle $\mathcal{L}(\lambda)$ on the variety $\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,\dots,k;\mathcal{R})$ by

$$\mathcal{L}(\lambda) := (\mathcal{R}_1)^{\otimes(-\lambda_1)} \otimes (\mathcal{R}_2/\mathcal{R}_1)^{\otimes(-\lambda_2)} \otimes \dots \otimes (\pi^*\mathcal{R}/\mathcal{R}_{k-1})^{\otimes(-\lambda_k)}\,.$$

(This notation is consistent with that of $\mathcal{L}(\chi)$ in subsection 2.2 which will be further explained in the comment after the proof of lemma 3.3.2 below).

Repeatedly applying corollary 3.1.2 to the above tower of projective bundles, we find that the following sheaves constitute a complete exceptional sequence in $D^b(Coh\,(\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,\dots,k;\mathcal{R})))$:

$$(\sharp) \qquad (\mathcal{L}(\lambda)) \quad \text{with} \quad \begin{aligned} -2n+1 &\le \lambda_1 \le 0, \\ -2n+3 &\le \lambda_2 \le 0, \\ &\vdots \\ -2n+2k-1 &\le \lambda_k \le 0. \end{aligned}$$

Here $\mathcal{L}(\lambda)$ precedes $\mathcal{L}(\mu)$ according to the ordering of the exceptional sequence iff $(\lambda_k, \lambda_{k-1}, \dots, \lambda_1) \prec (\mu_k, \mu_{k-1}, \dots, \mu_1)$ where $\prec$ is the lexicographic order on $\mathbb{Z}^k$.

Let us record here the simple

LEMMA 3.3.2. *The set of full direct images $R\pi_*\mathcal{L}(\lambda)$, as $\mathcal{L}(\lambda)$ varies among the bundles $(\sharp)$, generates the derived category $D^b(Coh(\mathrm{IGrass}(k,V)))$.*

Proof. As in lemma 3.2.1, $R\pi_*\mathcal{O}_{\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,\dots,k;\mathcal{R})} \simeq \mathcal{O}_{\mathrm{IGrass}(k,V)}$, and $R\pi_* \circ \pi^*$ is isomorphic to the identity functor on $D^b(Coh(\mathrm{IGrass}(k,V)))$ by the projection formula. Thus, since the bundles in $(\sharp)$ generate the derived category upstairs, if E is an object in $D^b(Coh(\mathrm{IGrass}(k,V)))$, π^*E will be isomorphic to an object in the smallest full triangulated subcategory containing the objects $(\sharp)$, i.e. starting from the set $(\sharp)$ and repeatedly enlarging it by taking finite direct sums, shifting in cohomological degree and completing distinguished triangles by taking a mapping cone, we can reach an object isomorphic to π^*E. Hence it is clear that the objects $R\pi_*\mathcal{L}(\lambda)$ will generate the derived category downstairs because $R\pi_*\pi^*E \simeq E$. $\square$

Now the fibre of $\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,\dots,k;\mathcal{R})$ over a point $x \in \mathrm{IGrass}(k,V)$ is just the full flag variety $\mathrm{Flag}(1,\dots,k;\mathcal{R}(x))$ which is a quotient of $\mathrm{GL}_k\mathbb{C}$ by a Borel subgroup B; the $\lambda \in \mathbb{Z}^k$ can be identified with weights or characters of a maximal torus $H \subset B$ and the restriction of $\mathcal{L}(\lambda)$ to the fibre over x is just the line bundle associated to the character λ, i.e. $\mathrm{GL}_k\mathbb{C} \times_B \mathbb{C}_{-\lambda}$, where $\mathbb{C}_{-\lambda}$ is the one-dimensional B-module in which the torus H acts via the character $-\lambda$ and the unipotent radical $R_u(B)$ of B acts trivially, and $\mathrm{GL}_k\mathbb{C} \times_B \mathbb{C}_{-\lambda} := \mathrm{GL}_k\mathbb{C} \times \mathbb{C}_{-\lambda}/\{(g,v) \sim (gb^{-1}, bv)\,,\ b \in B\}$. Thus we can calculate the $R\pi_*\mathcal{L}(\lambda)$ by the following (relative) version of Bott's theorem (cf. [Wey], thm. 4.1.4 or [Akh], §4.3 for a full statement):

Let $\varrho := (k-1, k-2, \dots, 0)$ (the half sum of the positive roots) and let $W = \mathfrak{S}_k$, the symmetric group on k letters (the Weyl group), act on $\mathbb{Z}^k$ by permutation of entries:

$$\sigma((\lambda_1,\dots,\lambda_k)) := \left(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(k)}\right)\,.$$

The dotted action of $\mathfrak{S}_k$ on $\mathbb{Z}^k$ is defined by

$$\sigma^\bullet(\lambda) := \sigma(\lambda+\varrho) - \varrho\,.$$

Then the theorem of Bott asserts in our case:

- Either there exists $\sigma \in \mathfrak{S}_k$, $\sigma \neq \mathrm{id}$, such that $\sigma^\bullet(\lambda) = \lambda$. Then $R^i\pi_*\mathcal{L}(\lambda) = 0 \; \forall i \in \mathbb{Z}$;

- or there exists a unique $\sigma \in \mathfrak{S}_k$ such that $\sigma^\bullet(\lambda) =: \mu$ is non-increasing (i.e., μ is a dominant weight). Then

$$R^i\pi_*\mathcal{L}(\lambda) = 0 \quad \text{for} \quad i \neq l(\sigma) ,$$
$$R^{l(\sigma)}\pi_*\mathcal{L}(\lambda) = \Sigma^\mu \mathcal{R}^\vee ,$$

where $l(\sigma)$ is the length of the permutation σ (the smallest number of transpositions the composition of which gives σ) and Σ^μ is the Schur functor.

As a first consequence, note that the objects $R\pi_*\mathcal{L}(\lambda)$ all belong -up to shift in cohomological degree- to the abelian subcategory of $D^b(Coh(\mathrm{IGrass}(k,V)))$ consisting of coherent sheaves. We would like to determine the homogeneous bundles that arise as direct images of the bundles ($\sharp$) in this way. The following theorem gives us some information (though it is not optimal).

Theorem 3.3.3. *The derived category $D^b(Coh(\mathrm{IGrass}(k,V)))$ is generated by the bundles $\Sigma^\nu\mathcal{R}$, where ν runs over Young diagrams Y which satisfy*

$$(\text{number of columns of } Y) \leq 2n-k ,$$
$$k \geq (\text{number of rows of } Y) \geq (\text{number of columns of } Y) - 2(n-k) .$$

Proof. Note that if λ satisfies the inequalities in ($\sharp$), then for $\delta := \lambda + \varrho$ we have

$$(\sharp\sharp) \qquad \begin{aligned} -(2n-k) &\leq \delta_1 \leq k-1 , \\ -(2n-k-1) &\leq \delta_2 \leq k-2 , \\ &\vdots \\ -(2n-2k+1) &\leq \delta_k \leq 0 . \end{aligned}$$

First of all one remarks that for $\sigma^\bullet(\lambda) = \sigma(\delta) - \varrho$ to be non-increasing, it is necessary and sufficient that $\sigma(\delta)$ be strictly decreasing. We assume this to be the case in the following. Since the maximum possible value for $\sigma(\delta)_1$ is $k-1$, and the minimum possible value for $\sigma(\delta)_k$ is $-(2n-k)$, we find that for $\sigma^\bullet(\lambda) =: \mu$

$$0 \geq \mu_1 \geq \ldots \geq \mu_k \geq -(2n-k) ;$$

putting $\nu = (\nu_1, \ldots, \nu_k) := (-\mu_k, -\mu_{k-1}, \ldots, -\mu_1)$ and noticing that $\Sigma^\mu\mathcal{R}^\vee \simeq \Sigma^\nu\mathcal{R}$, we find that the direct images $R^i\pi_*\mathcal{L}(\lambda)$, $i \in \mathbb{Z}$, $\mathcal{L}(\lambda)$ as in ($\sharp$), will form a subset of the set of bundles $\Sigma^\nu\mathcal{R}$ on $\mathrm{IGrass}(k,V)$ where ν runs over the set of Young diagrams with no more than $2n-k$ columns and no more than k rows.

But in fact we are only dealing with a proper subset of the latter: Suppose that

$$\sigma(\delta)_k = -(2n-k-a+1)\,, \quad 1 \le a \le k-1\,.$$

Then the maximum possible value for $\sigma(\delta)_a$ is $k-a-1$. For in any case an upper bound for $\sigma(\delta)_a$ is $k-a$ because $\sigma(\delta)_1$ can be at most $k-1$ and the sequence $\sigma(\delta)$ is strictly decreasing. But in case this upper bound for $\sigma(\delta)_a$ is attained, the sequence $\sigma(\delta)$ must start with

$$\sigma(\delta)_1 = k-1\,,\ \sigma(\delta)_2 = k-2\,, \dots,\ \sigma(\delta)_a = k-a\,,$$

in other words, we can only have

$$\sigma(\delta)_1 = \delta_1\,, \dots,\ \sigma(\delta)_a = \delta_a\,.$$

But this is impossible since $\delta_{a+1}, \dots, \delta_k$ are all $\ge -(2n-k-a) > -(2n-k-a+1)$ and thus we could not have $\sigma(\delta)_k = -(2n-k-a+1)$. Hence $\sigma(\delta)_a$ is at most $k-a-1$, that is to say in $\sigma^\bullet(\lambda) = \sigma(\delta) - \varrho = \mu$ we have $\mu_a = \sigma(\delta)_a - (k-a) < 0$; or in terms of $\nu = (-\mu_k, \dots, -\mu_1)$ we can say that if the Young diagram $Y(\nu)$ of ν has $2n-k-a+1$ columns, $1 \le a \le k-1$, it must have at least $k-a+1$ rows; or that the Young diagram $Y(\nu)$ satisfies

$$(\text{number of rows of } Y(\nu)) \ge (\text{number of columns of } Y(\nu)) - 2(n-k)$$

where the inequality is meaningless if the number on the right is ≤ 0. Thus by lemma 3.3.2 this concludes the proof of theorem 3.3.3. $\square$

Remark 3.3.4. By thm. 2.2.2, in $D^b(Coh(\mathrm{Grass}(k,V)))$ there is a complete exceptional sequence consisting of the $\Sigma^{\tilde\nu}\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}}$ is the tautological subbundle on $\mathrm{Grass}(k,V)$ and $\tilde\nu$ runs over Young diagrams with at most $2n-k$ columns and at most k rows. Looking at $\mathrm{IGrass}(k,V)$ as a subvariety $\mathrm{IGrass}(k,V) \subset \mathrm{Grass}(k,V)$ we see that the bundles in theorem 3.3.3 form a proper subset of the restrictions of the $\Sigma^{\tilde\nu}\tilde{\mathcal{R}}$ to $\mathrm{IGrass}(k,V)$.

Before making the next remark we have to recall two ingredients in order to render the following computations transparent:

The first is the Littlewood-Richardson rule to decompose $\Sigma^\lambda \otimes \Sigma^\mu$ into irreducible factors where λ, μ are Young diagrams (cf. [FuHa], §A.1). It says the following: Label each box of μ with the number of the row it belongs to. Then expand the Young diagram λ by adding the boxes of μ to the rows of λ subject to the following rules:

(a) The boxes with labels $\le i$ of μ together with the boxes of λ form again a Young diagram;

(b) No column contains boxes of μ with equal labels.

(c) When the integers in the boxes added are listed from right to left and from top down, then, for any $0 \le s \le$ (number of boxes of μ), the first s entries of the list satisfy: Each label l ($1 \le l \le$ (number of rows of μ)-1) occurs at least as many times as the label $l+1$.

Then the multiplicity of Σ^ν in $\Sigma^\lambda \otimes \Sigma^\mu$ is the number of times the Young diagram ν can be obtained by expanding λ by μ according to the above rules, forgetting the labels.
The second point is the calculation of the cohomology of the bundles $\Sigma^\lambda\mathcal{R}$ on the variety $\mathrm{IGrass}(k,V)$, V a $2n$-dimensional symplectic vector space (cf. [Wey], cor. 4.3.4). Bott's theorem gives the following prescription:
Look at the sequence

$$\mu = (-\lambda_k, -\lambda_{k-1}, \dots, -\lambda_1, 0, \dots, 0) \in \mathbb{Z}^n$$

considered as a weight of the root system of type C_n. Let W be the Weyl group of this root system which is a semi-direct product of $(\mathbb{Z}/2\mathbb{Z})^n$ with the symmetric group $\mathfrak{S}_n$ and acts on weights by permutation and sign changes of entries. Let $\varrho := (n, n-1, \dots, 1)$ be the half sum of the positive roots for type C_n. The dotted action of W on weights is defined as above by $\sigma^\bullet(\mu) := \sigma(\mu+\varrho) - \varrho$. Then

- either there is $\sigma \in W$, $\sigma \neq \mathrm{id}$, such that $\sigma^\bullet(\mu) = \mu$. then all cohomology groups
$$H^\bullet(\mathrm{IGrass}(k,V), \Sigma^\lambda\mathcal{R}) = 0\,.$$
- or there is a unique $\sigma \in W$ such that $\sigma^\bullet(\mu) =: \nu$ is dominant (a non-increasing sequence of non-negative integers). Then the only non-zero cohomology group is
$$H^{l(\sigma)}(\mathrm{IGrass}(k,V), \Sigma^\lambda\mathcal{R}) = V_\nu\,,$$
where $l(\sigma)$ is the length of the Weyl group element σ and V_ν is the space of the irreducible representation of $\mathrm{Sp}_{2n}\mathbb{C}$ with highest weight ν.

Remark 3.3.5. The $R^i\pi_*\mathcal{L}(\lambda)$, $i \in \mathbb{Z}$, $\mathcal{L}(\lambda)$ as in ($\sharp$), are not in general exceptional: For example, take $k = n = 3$, so that we are dealing with $\mathrm{IGrass}(3,V)$, the Lagrangian Grassmannian of maximal isotropic subspaces in a 6-dimensional symplectic space V. Then $\mathcal{L}((0,-3,0))$ is in ($\sharp$). Adding $\varrho = (2,1,0)$ to $(0,-3,0)$ we get $(2,-2,0)$ and interchanging the last two entries and subtracting ϱ again, we arrive at $(0,-1,-2)$ which is non-increasing whence

$$R^1\pi_*\mathcal{L}((0,-3,0)) = \Sigma^{2,1,0}\mathcal{R}\,,$$

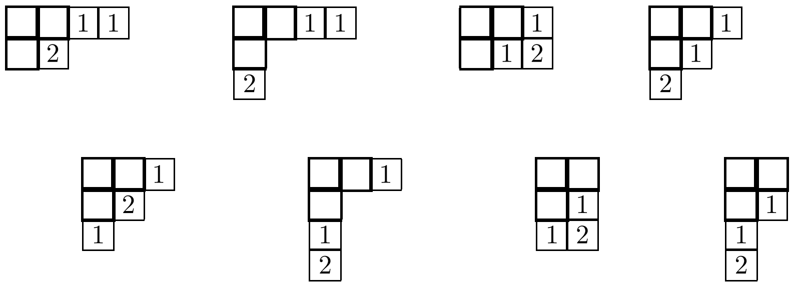
all other direct images being 0. To calculate

$$\begin{aligned}\mathrm{Ext}^\bullet\left(\Sigma^{2,1,0}\mathcal{R}, \Sigma^{2,1,0}\mathcal{R}\right) &= H^\bullet\left(\mathrm{IGrass}(3,V), \Sigma^{2,1,0}\mathcal{R} \otimes \Sigma^{0,-1,-2}\mathcal{R}\right)\\ &= H^\bullet\left(\mathrm{IGrass}(3,V), \Sigma^{2,1,0}\mathcal{R} \otimes \Sigma^{2,1,0}\mathcal{R} \otimes \left(\bigwedge^3 \mathcal{R}^\vee\right)^{\otimes 2}\right)\end{aligned}$$

we use the Littlewood-Richardson rule and Bott's theorem as recalled above: One gets that

$$\begin{aligned}\Sigma^{2,1,0}\mathcal{R} \otimes \Sigma^{0,-1,-2}\mathcal{R} = \Sigma^{2,0,-2}\mathcal{R} &\oplus \Sigma^{2,-1,-1}\mathcal{R} \oplus \Sigma^{1,1,-2}\mathcal{R}\\ &\oplus (\Sigma^{1,0,-1}\mathcal{R})^{\oplus 2} \oplus \Sigma^{0,0,0}\mathcal{R}\end{aligned}$$

in view of the fact that if we expand $\lambda = (2,1,0)$ by $\mu = (2,1,0)$ we get the following Young diagrams according to the Littlewood-Richardson rule:



Thus calculating the cohomology of $\Sigma^{2,1,0}\mathcal{R} \otimes \Sigma^{0,-1,-2}\mathcal{R}$ by the version of Bott's theorem recalled above one finds that

$$\mathrm{Hom}\left(\Sigma^{2,1,0}\mathcal{R}, \Sigma^{2,1,0}\mathcal{R}\right) = \mathbb{C} \quad \mathrm{Ext}^1\left(\Sigma^{2,1,0}\mathcal{R}, \Sigma^{2,1,0}\mathcal{R}\right) = V_{1,1,0} \oplus V_{2,0,0} \neq 0$$

the other Ext groups being 0.

Next we want to show by some examples that, despite the fact that theorem 3.3.3 does not give a complete exceptional sequence on $\mathrm{IGrass}(k, V)$, it is sometimes -for small values of k and n- not so hard to find one with its help.

Example 3.3.6. Choose $k = n = 2$, i.e. look at $\mathrm{IGrass}(2, V)$, $\dim V = 4$. Remarking that $\mathcal{O}(1)$ on $\mathrm{IGrass}(2, V)$ in the Plücker embedding equals $\bigwedge^{\mathrm{top}}\mathcal{R}^\vee$ and applying theorem 3.3.3 one finds that the following five sheaves generate $D^b(Coh(\mathrm{IGrass}(2, V)))$:

$$\mathcal{O}\,,\ \mathcal{R}\,,\ \bigwedge^2\mathcal{R} = \mathcal{O}(-1)\,,\ \Sigma^{2,1}\mathcal{R} = \mathcal{R}(-1)\,,\ \mathcal{O}(-2)\,;$$

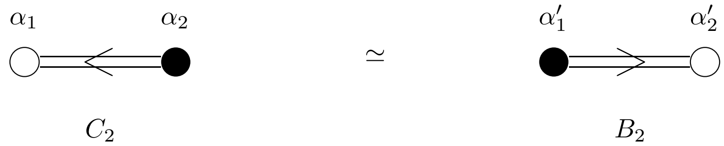
The real extra credit that one receives from working on the Lagrangian Grassmannian $\mathrm{IGrass}(2, V)$ is that $\mathcal{R} = \mathcal{R}^\perp$ and the tautological factor bundle can be identified with $\mathcal{R}^\vee \simeq \mathcal{R}(1)$, i.e. one has an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{R}(1) \longrightarrow 0\,.$$

Twisting by $\mathcal{O}(-1)$ in this sequence shows that of the above five sheaves, $\mathcal{R}(-1)$ is in the full triangulated subcategory generated by the remaining four; moreover, it is a straightforward computation with Bott's theorem that

$$(\mathcal{O}(-2),\ \mathcal{O}(-1),\ \mathcal{R},\ \mathcal{O})$$

is a strong exceptional sequence in $D^b(Coh(\mathrm{IGrass}(2, V)))$; but this is also complete, i.e., it generates this derived category by the preceding considerations. In fact, this does not come as a surprise. IGrass$(2, V)$ is isomorphic to a quadric hypersurface in $\mathbb{P}^4$, more precisely it is a hyperplane section of the Plücker quadric Grass$(2, V) \subset \mathbb{P}^5$. By [Ott], thm. 1.4 and ex. 1.5, the spinor bundles on the Plücker quadric are the dual of the tautological subbundle and the tautological factor bundle on Grass$(2, V)$ and these both restrict to the spinor bundle $\mathcal{R}^\vee$ on IGrass$(2, V) \subset \mathbb{P}^4$ (let us renew here the warning from subsection 2.2 that the spinor bundles in [Ott] are the duals of the bundles that we choose to call spinor bundles in this work). We thus recover the result of [Ka3], §4, in a special case. Note that the identification of IGrass$(2, V)$ with a quadric hypersurface in $\mathbb{P}^4$ also follows more conceptually from the isomorphism of marked Dynkin diagrams



corresponding to the isomorphism $\mathrm{Sp}_4\mathbb{C}/P(\alpha_2) \simeq \mathrm{Spin}_5\mathbb{C}/P(\alpha_1')$ (cf. [Stei], prop. p. 16 and [FuHa], §23.3). Recalling the one-to-one correspondence between conjugacy classes of parabolic subgroups of a simple complex Lie group G and subsets of the set of simple roots, the notations $P(\alpha_2)$ resp. $P(\alpha_1')$ are self-explanatory.

Example 3.3.7. Along the same lines which are here exposed in general, A. V. Samokhin treated in [Sa] the particular case of IGrass$(3, V)$, $\dim V = 6$, and using the identification of the tautological factor bundle with $\mathcal{R}^\vee$ on this Lagrangian Grassmannian and the exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{R}^\vee \longrightarrow 0\,.$$

together with its symmetric and exterior powers found the following strong complete exceptional sequence for $D^b(Coh(\mathrm{IGrass}(3, V)))$:

$$(\mathcal{R}(-3),\, \mathcal{O}(-3),\, \mathcal{R}(-2),\, \mathcal{O}(-2),\, \mathcal{R}(-1),\, \mathcal{O}(-1),\, \mathcal{R},\, \mathcal{O})$$

and we refer to [Sa] for details of the computation.

In general I conjecture that on any Lagrangian Grassmannian IGrass(n, V), $\dim V = 2n$, every "relation" between the bundles in theorem 3.3.3 in the derived category $D^b(Coh\,\mathrm{IGrass}(n, V))$ (that is to say that one of these bundles is in the full triangulated subcategory generated by the remaining ones) should follow using the Schur complexes (cf. [Wey], section 2.4) derived from the exact sequence $0 \to \mathcal{R} \to V \otimes \mathcal{O} \to \mathcal{R}^\vee \to 0$ (and the Littlewood-Richardson rule).
Let us conclude this subsection by giving an example which, though we do not find a complete exceptional sequence in the end, may help to convey the sort of combinatorial difficulties that one encounters in general.

EXAMPLE 3.3.8. For a case of a non-Lagrangian isotropic Grassmannian, look at $\mathrm{IGrass}(2, V)$, $\dim V = 6$. Theorem 3.3.3 says that $D^b(Coh\,\mathrm{IGrass}(2, V))$ is generated by the following 14 bundles:

$$(*) \qquad \mathrm{Sym}^a\,\mathcal{R}(-b)\,, \quad 0 \le a \le 3\,, \quad 0 \le b \le 4 - a\,.$$

By corollary 2.3.7, the number of terms in a complete exceptional sequence must be 12 in this case (in general for $\mathrm{IGrass}(k, V) = \mathrm{Sp}_{2n}\,\mathbb{C}/P(\alpha_k)$, $\dim V = 2n$, one has that $W^{P(\alpha_k)}$, the set of minimal representatives of the quotient $W/W_{P(\alpha_k)}$ can be identified with k-tuples of integers $(a_1, \dots, a_k)$ such that

$$1 \le a_1 < a_2 < \dots < a_k \le 2n \text{ and}$$
$$\text{for } 1 \le i \le 2n\,, \text{ if } i \in \{a_1, \dots, a_k\} \quad \text{then } 2n + 1 - i \notin \{a_1, \dots, a_k\}$$

(see [BiLa], §3.3) and these are

$$\frac{2n(2n-2)\dots(2n-2(k-1))}{1 \cdot 2 \cdot \dots \cdot k} = 2^k \binom{n}{k}$$

in number). Without computation, we know by a theorem of Ramanan (cf. [Ot2], thm 12.3) that the bundles in $(*)$ are all simple since they are associated to irreducible representations of the subgroup $P(\alpha_2) \subset \mathrm{Sp}_6\,\mathbb{C}$.
Moreover the bundles

$$\Sigma^{c_1,c_2}\mathcal{R} \ \text{ and } \ \Sigma^{d_1,d_2}\mathcal{R} \ \text{ with } 0 \le c_2 \le c_1 \le 3\,,\ 0 \le d_2 \le d_1 \le 3$$

have no higher extension groups between each other: By the Littlewood-Richardson rule, every irreducible summand $\Sigma^{e_1,e_2}\mathcal{R}$ occurring in the decomposition of $\Sigma^{d_1,d_2}\mathcal{R} \otimes \Sigma^{c_1,c_2}\mathcal{R}^\vee$ satisfies $-3 \le e_2 \le e_1 \le 3$ and hence for $\mu := (-e_2, -e_1, 0) \in \mathbb{Z}^3$ and $\varrho = (3, 2, 1)$ we find that $\mu + \varrho$ is either a strictly decreasing sequence of positive integers or two entries in $\mu + \varrho$ are equal up to sign or one entry in $\mu + \varrho$ is 0. In each of these cases, Bott's theorem as recalled before remark 3.3.5 tells us that $H^i(\mathrm{IGrass}(2, V), \Sigma^{e_1,e_2}\mathcal{R}) = 0\ \forall i > 0$. Combining this remark with the trivial observation that for $\mathrm{Sym}^a\mathcal{R}(-b)$, $\mathrm{Sym}^c\mathcal{R}(-d)$ in the set $(*)$ with $b, d \ge 1$ we have

$$\mathrm{Ext}^i(\mathrm{Sym}^a\mathcal{R}(-b), \mathrm{Sym}^c\mathcal{R}(-d)) = \mathrm{Ext}^i(\mathrm{Sym}^a\mathcal{R}(-b+1), \mathrm{Sym}^c\mathcal{R}(-d+1)) \ \ \forall i$$

we infer that for $\mathcal{A}$, $\mathcal{B}$ bundles in the set $(*)$ we can only have

$$\mathrm{Ext}^j(\mathcal{A}, \mathcal{B}) \neq 0, \ \text{ some } \ j > 0$$

if $\mathcal{A}$ occurs in the set

$$S_1 := \{\mathcal{O}(-4),\ \mathcal{R}(-3),\ \mathrm{Sym}^2\mathcal{R}(-2),\ \mathrm{Sym}^3\mathcal{R}(-1)\}$$

and $\mathcal{B}$ is in the set

$$S_2 := \{\mathcal{O},\ \mathcal{R},\ \mathrm{Sym}^2\mathcal{R},\ \mathrm{Sym}^3\mathcal{R}\}$$

or vice versa. By explicit calculation (which amounts to applying Bott's theorem another 32 more times) we find that the only non-vanishing higher extension groups between two bundles in $(*)$ are the following:

$$\mathrm{Ext}^1(\mathrm{Sym}^3\mathcal{R}, \mathcal{R}(-3)) = \mathbb{C}, \ \ \mathrm{Ext}^1(\mathrm{Sym}^2\mathcal{R}, \mathrm{Sym}^2\mathcal{R}(-2)) = \mathbb{C}$$
$$\mathrm{Ext}^1(\mathrm{Sym}^3\mathcal{R}, \mathrm{Sym}^2\mathcal{R}(-2)) = V, \ \ \mathrm{Ext}^1(\mathcal{R}, \mathrm{Sym}^3\mathcal{R}(-1)) = \mathbb{C}$$
$$\mathrm{Ext}^1(\mathrm{Sym}^2\mathcal{R}, \mathrm{Sym}^3\mathcal{R}(-1)) = V, \ \ \mathrm{Ext}^1(\mathrm{Sym}^3\mathcal{R}, \mathrm{Sym}^3\mathcal{R}(-1)) = V_{2,0,0}.$$

Thus in this case the set of bundles $(*)$ does not contain a strong complete exceptional sequence. It does not contain a complete exceptional sequence, either, since

$$\mathrm{Hom}(\mathcal{R}(-3), \mathrm{Sym}^3\mathcal{R}) \neq 0, \ \ \mathrm{Hom}(\mathrm{Sym}^2\mathcal{R}(-2), \mathrm{Sym}^2\mathcal{R}) \neq 0$$
$$\mathrm{Hom}(\mathrm{Sym}^2\mathcal{R}(-2), \mathrm{Sym}^3\mathcal{R}) \neq 0, \ \ \mathrm{Hom}(\mathrm{Sym}^3\mathcal{R}(-1), \mathcal{R}) \neq 0$$
$$\mathrm{Hom}(\mathrm{Sym}^3\mathcal{R}(-1), \mathrm{Sym}^2\mathcal{R}) \neq 0, \ \ \mathrm{Hom}(\mathrm{Sym}^3\mathcal{R}(-1), \mathrm{Sym}^3\mathcal{R}) \neq 0.$$

On the other hand one has on $\mathrm{IGrass}(2, V)$ the following exact sequences of vector bundles:

$$0 \to \mathcal{R}^\perp \to V \otimes \mathcal{O} \to \mathcal{R}^\vee \to 0 \tag{1}$$
$$0 \to \mathcal{R} \to \mathcal{R}^\perp \to \mathcal{R}^\perp/\mathcal{R} \to 0. \tag{2}$$

The second exterior power of the two term complex $0 \to \mathcal{R} \to \mathcal{R}^\perp$ gives an acyclic complex resolving $\bigwedge^2(\mathcal{R}^\perp/\mathcal{R})$ which is isomorphic to $\mathcal{O}_{\mathrm{IGrass}(2,V)}$ via the mapping induced by the symplectic form $\langle\cdot,\cdot\rangle$. Thus we get the exact sequence

$$0 \to \mathrm{Sym}^2\mathcal{R} \to \mathcal{R} \otimes \mathcal{R}^\perp \to \bigwedge^2 \mathcal{R}^\perp \to \mathcal{O} \to 0. \tag{3}$$

The second symmetric power of the two term complex $0 \to \mathcal{R}^\perp \to V \otimes \mathcal{O}$ yields the exact sequence

$$0 \to \bigwedge^2 \mathcal{R}^\perp \to \mathcal{R}^\perp \otimes V \to \mathrm{Sym}^2 V \otimes \mathcal{O} \to \mathrm{Sym}^2 \mathcal{R}^\vee \to 0. \tag{4}$$

Note also that $\mathcal{R}^\vee \simeq \mathcal{R}(1)$ and $\mathrm{Sym}^2 \mathcal{R}^\vee \simeq \mathrm{Sym}^2 \mathcal{R}(2)$. Since $\mathcal{R} \otimes \mathcal{R}(-1) \simeq \mathrm{Sym}^2 \mathcal{R}(-1) \oplus \mathcal{O}(-2)$ sequence (1) gives

$$0 \to \mathcal{R}^\perp \otimes \mathcal{R}(-2) \to V \otimes \mathcal{R}(-2) \to \mathrm{Sym}^2 \mathcal{R}(-1) \oplus \mathcal{O}(-2) \to 0 \tag{5}$$

and

$$0 \to \mathcal{R}^\perp(-2) \to V \otimes \mathcal{O}(-2) \to \mathcal{R}(-1) \to 0. \tag{6}$$

Moreover twisting by $\mathcal{O}(-2)$ in (3) and (4) yields

$$0 \to \mathrm{Sym}^2\mathcal{R}(-2) \to \mathcal{R} \otimes \mathcal{R}^\perp(-2) \to \bigwedge^2 \mathcal{R}^\perp(-2) \to \mathcal{O}(-2) \to 0 \tag{7}$$
$$0 \to \bigwedge^2 \mathcal{R}^\perp(-2) \to \mathcal{R}^\perp \otimes V(-2) \to \mathrm{Sym}^2 V \otimes \mathcal{O}(-2) \to \mathrm{Sym}^2 \mathcal{R} \to 0. \tag{8}$$

What (5), (6), (7), (8) tell us is that $\mathrm{Sym}^2\,\mathcal{R}(-2)$ is in the full triangulated subcategory generated by $\mathcal{O}(-2),\ \mathrm{Sym}^2\,\mathcal{R}(-1),\ \mathcal{R}(-2),\ \mathrm{Sym}^2\,\mathcal{R},\ \mathcal{R}(-1)$. Thus the derived category $D^b(Coh\,\mathrm{IGrass}(2,V))$ is generated by the bundles in $(*)$ without $\mathrm{Sym}^2\,\mathcal{R}(-2)$, which makes a total of 13 bundles.
But even in this simple case I do not know how to pass on to a complete exceptional sequence because there is no method at this point to decide which bundles in $(*)$ should be thrown away and what extra bundles should be let in to obtain a complete exceptional sequence.

3.4 Calculation for the Grassmannian of isotropic 3-planes in a 7-dimensional orthogonal vector space

In this section we want to show how the method of subsection 3.3 can be adapted -using theorem 3.2.7 on quadric bundles- to produce sets of vector bundles that generate the derived categories of coherent sheaves on orthogonal Grassmannians (with the ultimate goal to obtain (strong) complete exceptional sequences on them by appropriately modifying these sets of bundles). Since the computations are more involved than in the symplectic case, we will restrict ourselves to illustrating the method by means of a specific example:
Let V be a 7-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. $\mathrm{IFlag}(k_1,\dots,k_t;V)$ denotes the flag variety of isotropic flags of type $(k_1,\dots,k_t)$, $1\le k_1<\dots<k_t\le 3$, in V and $\mathrm{IGrass}(k,V)$, $1\le k\le 3$, the Grassmannian of isotropic k-planes in V; again in this setting we have the tautological flag of subbundles

$$\mathcal{R}_{k_1}\subset\dots\subset\mathcal{R}_{k_t}\subset V\otimes\mathcal{O}_{\mathrm{IFlag}(k_1,\dots,k_t;V)}$$

on $\mathrm{IFlag}(k_1,\dots,k_t;V)$ and the tautological subbundle $\mathcal{R}$ on $\mathrm{IGrass}(k,V)$.
Now consider $\mathrm{IGrass}(3,V)$ which sits in the diagram (D)

$$\begin{tikzcd}
\mathbb{P}(\mathcal{E}_2)\supset\mathcal{Q}_2\simeq\mathrm{IFlag}(1,2,3;V)\simeq\mathrm{Flag}_{\mathrm{IGrass}(3,V)}(1,2,3;\mathcal{R}) \arrow[d, "\pi_2"] \arrow[dr, "\pi"] & \\
\mathbb{P}(\mathcal{E}_1)\supset\mathcal{Q}_1\simeq\mathrm{IFlag}(1,2;V) \arrow[d, "\pi_1"] & \\
\mathbb{P}^6\supset Q\simeq\mathrm{IFlag}(1;V) & \mathrm{IGrass}(3,V)
\end{tikzcd}\qquad\text{(D)}$$

The rank i tautological subbundle on $\mathrm{IFlag}(1,\dots,j;V)$ pulls back to the rank i tautological subbundle on $\mathrm{IFlag}(1,\dots,j+1;V)$ under π_j, $1\le i\le j\le 2$, and for ease of notation it will be denoted by $\mathcal{R}_i$ with the respective base spaces being tacitly understood in each case.
The choice of an isotropic line L_1 in V amounts to picking a point in the

quadric hypersurface $Q = \{[v] \in \mathbb{P}(V) \,|\, \langle v, v\rangle = 0\} \subset \mathbb{P}^6$. An isotropic plane L_2 containing L_1 is nothing but an isotropic line L_2/L_1 in the orthogonal vector space $L_1^\perp/L_1$. Thus $\mathrm{IFlag}(1,2;V)$ is a quadric bundle $\mathcal{Q}_1$ over $\mathrm{IFlag}(1;V)$ inside the projective bundle $\mathbb{P}(\mathcal{E}_1)$ of the orthogonal vector bundle $\mathcal{E}_1 := \mathcal{R}_1^\perp/\mathcal{R}_1$ on $\mathrm{IFlag}(1;V)$. Similarly, $\mathrm{IFlag}(1,2,3;V)$ is a quadric bundle $\mathcal{Q}_2 \subset \mathbb{P}(\mathcal{E}_2)$ over $\mathrm{IFlag}(1,2;V)$ where $\mathcal{E}_2 := \mathcal{R}_2^\perp/\mathcal{R}_2$, and at the same time $\mathrm{IFlag}(1,2,3;V)$ is isomorphic to the relative variety of complete flags $\mathrm{Flag}_{\mathrm{IGrass}(k,V)}(1,2,3;\mathcal{R})$ in the fibres of the tautological subbundle $\mathcal{R}$ on $\mathrm{IGrass}(3,V)$.
Moreover, $\mathcal{O}_Q(-1) \simeq \mathcal{R}_1$, $\mathcal{O}_{\mathcal{Q}_1}(-1) \simeq \mathcal{R}_2/\mathcal{R}_1$, $\mathcal{O}_{\mathcal{Q}_2}(-1) \simeq \mathcal{R}_3/\mathcal{R}_2$. We want to switch to a more representation-theoretic picture. For this, put $G := \mathrm{Spin}_7\,\mathbb{C}$ and turning to the notation and set-up introduced at the beginning of subsection 2.2, rewrite diagram (D) as

$$\begin{tikzcd}
G/P(\alpha_1,\alpha_2,\alpha_3) = G/B \arrow[d, "\pi_2"] \arrow[ddr, "\pi"] & \\
G/P(\alpha_1,\alpha_2) \arrow[d, "\pi_1"] & \\
G/P(\alpha_1) & G/P(\alpha_3)
\end{tikzcd} \qquad \text{(D')}$$

The orthogonal vector bundles $\mathcal{R}_1^\perp/\mathcal{R}_1$ on $Q = G/P(\alpha_1)$ resp. $\mathcal{R}_2^\perp/\mathcal{R}_2$ on $\mathrm{IFlag}(1,2;V) = G/P(\alpha_1,\alpha_2)$ admit spinor bundles in the sense of assumption (A) in subsection 3.2:
In fact, on $G/P(\alpha_1)$ we will use the homogeneous vector bundle $S(\mathcal{R}_1^\perp/\mathcal{R}_1)$ which is associated to the irreducible representation r_1 of $P(\alpha_1)$ with highest weight the fundamental weight ω_3, and on $G/P(\alpha_1,\alpha_2)$ we will use the homogeneous vector bundle $S(\mathcal{R}_2^\perp/\mathcal{R}_2)$ which is the pull-back under the projection $G/P(\alpha_1,\alpha_2) \to G/P(\alpha_2)$ of the homogeneous vector bundle defined by the irreducible representation r_2 of $P(\alpha_2)$ with highest weight ω_3.
Therefore we can apply theorem 3.2.7 (or rather its corollary 3.2.8) iteratively to obtain the following assertion:
The bundles on $\mathrm{IFlag}(1,2,3;V)$

$$(\heartsuit) \qquad \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$$

where $\mathcal{A}$ runs through the set

$$A := \{\Sigma(\mathcal{R}_1^\perp/\mathcal{R}_1) \otimes \mathcal{R}_1^{\otimes 5},\ \mathcal{R}_1^{\otimes 4},\ \mathcal{R}_1^{\otimes 3},\ \mathcal{R}_1^{\otimes 2},\ \mathcal{R}_1,\ \mathcal{O}\}$$

and $\mathcal{B}$ runs through

$$B := \{\Sigma(\mathcal{R}_2^\perp/\mathcal{R}_2) \otimes (\mathcal{R}_2/\mathcal{R}_1)^{\otimes 3},\ (\mathcal{R}_2/\mathcal{R}_1)^{\otimes 2},\ \mathcal{R}_2/\mathcal{R}_1,\ \mathcal{O}\}$$

and $\mathcal{C}$ runs through

$$C := \{\Sigma(\mathcal{R}_3^{\perp}/\mathcal{R}_3) \otimes (\mathcal{R}_3/\mathcal{R}_2),\ \mathcal{O}\}$$

generate $D^b(Coh\,\mathrm{IFlag}(1,2,3;V))$, and in fact form a complete exceptional sequence when appropriately ordered.
Here $\Sigma(\mathcal{R}_i^{\perp}/\mathcal{R}_i)$, $i = 1,\ 2,\ 3$, denote the bundles on $\mathrm{IFlag}(1,2,3;V)$ which are the pull-backs under the projections $G/B \to G/P(\alpha_i)$ of the vector bundles on $G/P(\alpha_i)$ which are *the duals* of the homogeneous vector bundles associated to the irreducible representations r_i of $P(\alpha_i)$ with highest weight the fundamental weight ω_3.
We know that the full direct images under π of the bundles in $(\heartsuit)$ will generate $D^b(Coh\,\mathrm{IGrass}(3,V))$ downstairs. When one wants to apply Bott's theorem to calculate direct images the trouble is that $\Sigma(\mathcal{R}_1^{\perp}/\mathcal{R}_1)$ and $\Sigma(\mathcal{R}_2^{\perp}/\mathcal{R}_2)$, though homogeneous vector bundles on $\mathrm{IFlag}(1,2,3;V) = \mathrm{Spin}_7\mathbb{C}/B$, are not defined by irreducible representations, i.e. characters of, the Borel subgroup B. Therefore, one has to find Jordan-Hölder series for these, i.e. filtrations

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_M = \Sigma(\mathcal{R}_1^{\perp}/\mathcal{R}_1)$$

and

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_N = \Sigma(\mathcal{R}_2^{\perp}/\mathcal{R}_2)$$

by homogeneous vector subbundles $\mathcal{V}_i$ resp. $\mathcal{W}_j$ such that the quotients $\mathcal{V}_{i+1}/\mathcal{V}_i$, $i = 0, \dots, M-1$, resp. $\mathcal{W}_{j+1}/\mathcal{W}_j$, $j = 0, \dots, N-1$, are line bundles defined by characters of B.
Recall that in terms of an orthonormal basis $\epsilon_1, \dots, \epsilon_r$ of $\mathfrak{h}^*$ we can write the fundamental weights for $\mathfrak{so}_{2r+1}\mathbb{C}$ as $\omega_i = \epsilon_1 + \dots \epsilon_i$, $1 \le i < r$, $\omega_r = (1/2)(\epsilon_1 + \dots + \epsilon_r)$, and simple roots as $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $1 \le i < r$, $\alpha_r = \epsilon_r$, and that (cf. [FuHa], §20.1) the weights of the spin representation of $\mathfrak{so}_{2r+1}\mathbb{C}$ are just given by

$$\frac{1}{2}(\pm\epsilon_1 \pm \dots \pm \epsilon_r)$$

(all possible 2^r sign combinations).
Therefore, on the level of Lie algebras, the weights of dr_1, dr_2, and dr_3 are given by:

$$dr_1: \quad \frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3), \quad dr_2: \quad \frac{1}{2}(\epsilon_1 + \epsilon_2 \pm \epsilon_3),$$
$$dr_3: \quad \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3).$$

(Indeed, if v_{ω_3} is a highest weight vector in the irreducible G-module of highest weight ω_3, then the span of $P(\alpha_i) \cdot v_{\omega_3}$, $i = 1, \dots, 3$, is the irreducible $P(\alpha_i)$-module of highest weights ω_3, and its weights are therefore those weights of the ambient irreducible G-module wich can be written as $\omega_3 - \sum_{j \neq i} c_j\alpha_j$, $c_j \in \mathbb{Z}^+$).

Therefore, the spinor bundle $\Sigma(\mathcal{R}_3^{\perp}/\mathcal{R}_3)$ on G/B is just the line bundle $\mathcal{L}(\omega_3) = \mathcal{L}(1/2,1/2,1/2)$ associated to ω_3 (viewed as a character of B), $\Sigma(\mathcal{R}_2^{\perp}/\mathcal{R}_2)$ has a Jordan-Hölder filtration of length 2 with quotients
$\mathcal{L}(1/2,1/2,\pm 1/2)$, and $\Sigma(\mathcal{R}_1^{\perp}/\mathcal{R}_1)$ has a Jordan-Hölder filtration of length 4 with quotients the line bundles $\mathcal{L}(1/2,\pm 1/2,\pm 1/2)$.
In conclusion we get that $D^b(Coh\, G/B)$ is generated by the line bundles

$$(\heartsuit') \qquad \mathcal{A}' \otimes \mathcal{B}' \otimes \mathcal{C}'$$

where $\mathcal{A}'$ runs through the set

$$\begin{gathered} A' := \Big\{ \mathcal{L}(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \otimes \mathcal{L}(-5,0,0),\ \mathcal{L}(\frac{1}{2},-\frac{1}{2},-\frac{1}{2}) \otimes \mathcal{L}(-5,0,0), \\ \mathcal{L}(\frac{1}{2},-\frac{1}{2},\frac{1}{2}) \otimes \mathcal{L}(-5,0,0),\ \mathcal{L}(\frac{1}{2},\frac{1}{2},-\frac{1}{2}) \otimes \mathcal{L}(-5,0,0),\ \mathcal{L}(-4,0,0), \\ \mathcal{L}(-3,0,0),\ \mathcal{L}(-2,0,0),\ \mathcal{L}(-1,0,0),\ \mathcal{L}(0,0,0)\} \end{gathered}$$

and $\mathcal{B}'$ runs through

$$\begin{gathered} B' := \Big\{ \mathcal{L}(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \otimes \mathcal{L}(0,-3,0),\ \mathcal{L}(\frac{1}{2},\frac{1}{2},-\frac{1}{2}) \otimes \mathcal{L}(0,-3,0), \\ \mathcal{L}(0,-2,0),\ \mathcal{L}(0,-1,0),\ \mathcal{L}(0,0,0)\} \end{gathered}$$

and $\mathcal{C}'$ runs through

$$C' := \Big\{ \mathcal{L}(\frac{1}{2},\frac{1}{2},\frac{1}{2}) \otimes \mathcal{L}(0,0,-1),\ \mathcal{L}(0,0,0) \Big\}.$$

Then we can calculate $R\pi_*(\mathcal{A}' \otimes \mathcal{B}' \otimes \mathcal{C}')$ by applying the relative version of Bott's theorem as explained in subsection 3.3 to each of the 90 bundles $\mathcal{A}' \otimes \mathcal{B}' \otimes \mathcal{C}'$; here of course one takes into account that $\mathcal{L}(1/2,1/2,1/2) = \pi^* L$, where for simplicity we denote by L the line bundle on $G/P(\alpha_3)$ defined by the one-dimensional representation of $P(\alpha_3)$ with weight $-\omega_3$ and one uses the projection formula. After a lengthy calculation one thus arrives at the following

Theorem 3.4.1. *The derived category $D^b(Coh\, \mathrm{IGrass}(3,V))$ is generated as triangulated category by the following 22 vector bundles:*

$$\begin{gathered} \bigwedge^2 \mathcal{R}(-1),\ \mathcal{O}(-2),\ \mathcal{R}(-2)\otimes L,\ \mathrm{Sym}^2\mathcal{R}(-1)\otimes L,\ \mathcal{O}(-3)\otimes L, \\ \bigwedge^2 \mathcal{R}(-2)\otimes L,\ \Sigma^{2,1}\mathcal{R}(-1)\otimes L,\ \mathcal{R}(-1),\ \mathcal{O}(-2)\otimes L,\ \mathcal{O}(-1), \\ \mathcal{R}(-1)\otimes L,\ \bigwedge^2 \mathcal{R}(-1)\otimes L,\ \Sigma^{2,1}\mathcal{R}\otimes L,\ \mathrm{Sym}^2\mathcal{R}^\vee(-2)\otimes L,\ \bigwedge^2 \mathcal{R},\ \mathcal{O}, \\ \Sigma^{2,1}\mathcal{R},\ \mathrm{Sym}^2\mathcal{R}^\vee(-2),\ \mathcal{O}(-1)\otimes L,\ \mathrm{Sym}^2\mathcal{R}^\vee(-1),\ \bigwedge^2 \mathcal{R}\otimes L,\ \mathcal{R}\otimes L. \end{gathered}$$

One should remark that the expected number of vector bundles in a complete exceptional sequence is 8 in this case since there are 8 Schubert varieties in $\mathrm{IGrass}(3,V)$ (cf. [BiLa], §3).

4 Degeneration techniques

Whereas in the preceding section a strategy for proving existence of complete exceptional sequences on rational homogeneous varieties was exposed which was based on the method of fibering them into simpler ones of the same type, here we propose to explain an idea for a possibly alternative approach to tackle this problem. It relies on a theorem due to M. Brion that provides a degeneration of the diagonal $\Delta \subset X \times X$, X rational homogeneous, into a union (over the Schubert varieties in X) of the products of a Schubert variety with its opposite Schubert variety.
We will exclusively consider the example of $\mathbb{P}^n$ and the main goal will be to compare resolutions of the structure sheaves of the diagonal and its degeneration product in this case. This gives a way of proving Beilinson's theorem on $\mathbb{P}^n$ without using a resolution of $\mathcal{O}_\Delta$ but only of the structure sheaf of the degeneration.

4.1 A theorem of Brion

The notation concerning rational homogeneous varieties introduced at the beginning of subsection 2.2 is retained.
The following theorem was proven by M. Brion (cf. [Bri], thm. 2).

Theorem 4.1.1. *Regard the simple roots $\alpha_1, \dots, \alpha_r$ as characters of the maximal torus H and put*

$$\mathfrak{X} := \text{closure of } \{(hx,\, x,\, \alpha_1(h), \dots, \alpha_r(h)) \,|\, x \in X = G/P,\ h \in H\} \\ \text{in } X \times X \times \mathbb{A}^r$$

with its projection $\mathfrak{X} \xrightarrow{\pi} \mathbb{A}^r$. If H acts on $\mathfrak{X}$ via its action on the ambient $X \times X \times \mathbb{A}^r$ given by

$$h \cdot (x_1,\, x_2,\, t_1, \dots, t_r) := (hx_1,\, x_2,\, \alpha_1(h)t_1, \dots, \alpha_r(h)t_r)$$

and acts in $\mathbb{A}^r$ with weights $\alpha_1, \dots, \alpha_r$, then π is equivariant, surjective, flat with reduced fibres such that

$$\mathfrak{X}_0 := \pi^{-1}((0, \dots, 0)) \simeq \bigcup_{w \in W^P} X_w \times X^w\,,$$

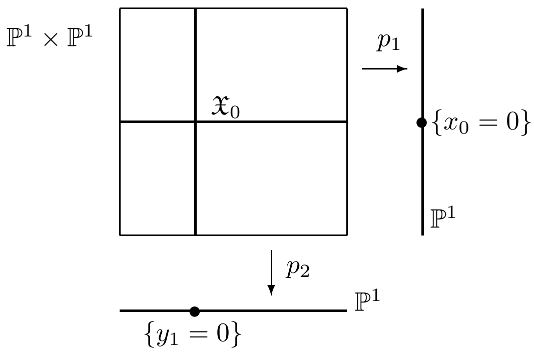
and is a trivial fibration over $H \cdot (1, \dots, 1)$, the complement of the union of all coordinate hyperplanes, with fibre the diagonal $\Delta = \Delta_X \subset X \times X$.

Now the idea to use this result for our purpose is as follows: In [Bei], Beilinson proved his theorem using an explicit resolution of $\mathcal{O}_{\Delta_{\mathbb{P}^n}}$. However, on a general rational homogeneous variety X a resolution of the structure sheaf of the diagonal is hard to come up with. The hope may be therefore that a resolution of

$\mathfrak{X}_0$ is easier to manufacture (by combinatorial methods) than one for $\mathcal{O}_\Delta$, and that one could afterwards lift the resolution of $\mathcal{O}_{\mathfrak{X}_0}$ to one of $\mathcal{O}_\Delta$ by flatness. If we denote by p_1 resp. p_2 the projections of $X \times X$ to the first resp. second factor, the preceding hope is closely connected to the problem of comparing the functors $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ and $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_\Delta) \simeq \mathrm{id}_{D^b(Coh\, X)}$. In the next subsection we will present the computations to clarify these issues for projective space.

4.2 Analysis of the degeneration of the Beilinson functor on $\mathbb{P}^n$

Look at two copies of $\mathbb{P}^n$, one with homogeneous coordinates $x_0, \dots, x_n$, the other with homogeneous coordinates $y_0, \dots, y_n$. In this case $\mathfrak{X}_0 = \bigcup_{i=0}^n \mathbb{P}^i \times \mathbb{P}^{n-i}$, and $\mathfrak{X}_0$ is defined by the ideal $J = (x_i y_j)_{0 \le i < j \le n}$ and the diagonal by the ideal $I = (x_i y_j - x_j y_i)_{0 \le i < j \le n}$.
Consider the case of $\mathbb{P}^1$. The first point that should be noticed is that $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ is no longer isomorphic to the identity: By Orlov's representability theorem (cf. [Or2], thm. 3.2.1) the identity functor is represented uniquely by the structure sheaf of the diagonal on the product (this is valid for any smooth projective variety and not only for $\mathbb{P}^1$). Here one can also see this in an easier way as follows. For $d >> 0$ the sheaf $p_1^*\mathcal{O}(d) \otimes \mathcal{O}_{\mathfrak{X}_0}$ is p_{2*}-acyclic and p_{2*} commutes with base extension whence $\dim_{\mathbb{C}} (p_{2*}(p_1^*\mathcal{O}(d) \otimes \mathcal{O}_{\mathfrak{X}_0}) \otimes \mathbb{C}_P) = d + 1$ if P is the point $\{y_1 = 0\}$ and $= 1$ otherwise:



Thus $p_{2*}(p_1^*\mathcal{O}(d) \otimes \mathcal{O}_{\mathfrak{X}_0}) = Rp_{2*}(p_1^*\mathcal{O}(d) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ is not locally free in this case. We will give a complete description of the functor $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ below for $\mathbb{P}^n$. If one compares the resolutions of $\mathcal{O}_{\mathfrak{X}_0}$ and $\mathcal{O}_\Delta$ on $\mathbb{P}^1$:

$$0 \longrightarrow \mathcal{O}(-1,-1) \xrightarrow{(x_0 y_1)} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathfrak{X}_0} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}(-1,-1) \xrightarrow{(x_0 y_1 - x_1 y_0)} \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

and on $\mathbb{P}^2$:

$$0 \to \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2) \xrightarrow{A'} \mathcal{O}(-1,-1)^{\oplus 3} \xrightarrow{B'} \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2} \to \mathcal{O}_{\mathfrak{X}_0} \to 0$$
$$0 \to \mathcal{O}(-2,-1) \oplus \mathcal{O}(-1,-2) \xrightarrow{A} \mathcal{O}(-1,-1)^{\oplus 3} \xrightarrow{B} \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2} \to \mathcal{O}_{\Delta} \to 0$$

where

$$A = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \quad A' = \begin{pmatrix} x_0 & 0 \\ x_1 & y_1 \\ 0 & y_2 \end{pmatrix}$$
$$B = (x_2y_1 - x_1y_2,\ x_0y_2 - x_2y_0,\ x_1y_0 - x_0y_1) \quad B = (-x_1y_2,\ x_0y_2,\ -x_0y_1)$$

(these being Hilbert-Burch type resolutions; here $\mathfrak{X}_0$ is no longer a local complete intersection!) one may wonder if on $\mathbb{P}^n$ there exist resolutions of $\mathcal{O}_{\mathfrak{X}_0}$ and $\mathcal{O}_\Delta$ displaying an analogous similarity. This is indeed the case, but will require some work.

Consider the matrix

$$\begin{pmatrix} x_0 & \dots & x_n \\ y_0 & \dots & y_n \end{pmatrix}$$

as giving rise to a map between free bigraded modules F and G over $\mathbb{C}[x_0,\dots,x_n;\, y_0,\dots,y_n]$ of rank $n+1$ and 2 respectively. Put $K_h := \bigwedge^{h+2} F \otimes \mathrm{Sym}^h\, G^\vee$ for $h = 0,\dots,n-1$. Choose bases $f_0,\dots,f_n$ resp. ξ, η for F resp. $G^\vee$. Define maps $d_h : K_h \to K_{h-1}$, $h = 1,\dots,n-1$ by

$$d_h\left(f_{j_1} \wedge \dots \wedge f_{j_{h+2}} \otimes \xi^{\mu_1}\eta^{\mu_2}\right) := \sum_{l=1}^{h+2} (-1)^{l+1} x_{j_l} f_{j_1} \wedge \dots \wedge \hat{f}_{j_l} \wedge \dots \wedge f_{j_{h+2}}$$
$$\otimes \xi^{-1}(\xi^{\mu_1}\eta^{\mu_2}) + \sum_{l=1}^{h+2} (-1)^{l+1} y_{j_l} f_{j_1} \wedge \dots \wedge \hat{f}_{j_l} \wedge \dots \wedge f_{j_{h+2}} \otimes \eta^{-1}(\xi^{\mu_1}\eta^{\mu_2})$$

where $0 \le j_1 < \dots < j_{h+2} \le n$, $\mu_1 + \mu_2 = h$ and the homomorphism ξ^{-1} (resp. η^{-1}) is defined by

$$\xi^{-1}(\xi^{\mu_1}\eta^{\mu_2}) := \begin{cases} \xi^{\mu_1 - 1}\eta^{\mu_2} & \text{if } \mu_1 \ge 1 \\ 0 & \text{if } \mu_1 = 0 \end{cases}$$

(resp. analogously for η^{-1}). Then

$$0 \longrightarrow K_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_1} K_0 \longrightarrow I \longrightarrow 0$$

is a resolution of I which is the Eagon-Northcott complex in our special case (cf. [Nor], appendix C).

PROPOSITION 4.2.1. *The ideal J has a resolution*

$$0 \longrightarrow K_{n-1} \xrightarrow{d'_{n-1}} \dots \xrightarrow{d'_1} K_0 \longrightarrow J \longrightarrow 0$$

where the differential $d'_h : K_h \to K_{h-1}$ is defined by

$$d'_h \left(f_{j_1} \wedge \dots \wedge f_{j_{h+2}} \otimes \xi^{\mu_1}\eta^{\mu_2} \right) := \sum_{l=1}^{h-\mu_2+1} (-1)^{l+1} x_{j_l} f_{j_1} \wedge \dots \wedge \hat{f}_{j_l} \wedge \dots \wedge f_{j_{h+2}}$$

$$\otimes \xi^{-1}(\xi^{\mu_1}\eta^{\mu_2}) + \sum_{l=\mu_1+2}^{h+2} (-1)^{l+1} y_{j_l} f_{j_1} \wedge \dots \wedge \hat{f}_{j_l} \wedge \dots \wedge f_{j_{h+2}} \otimes \eta^{-1}(\xi^{\mu_1}\eta^{\mu_2})\,.$$

Intuitively the differentials d'_h are gotten by degenerating the differentials d_h. To prove proposition 4.2.1 we will use the fact that J is a monomial ideal. There is a combinatorial method for sometimes writing down resolutions for these by looking at simplicial or more general cell complexes from topology. The method can be found in [B-S]. We will recall the results we need in the following. Unfortunately the resolution of proposition 4.2.1 is not supported on a simplicial complex, one needs a more general cell complex.
Let X be a finite regular cell complex. This is a non-empty topological space X with a finite set Γ of subsets of X (the *cells* of X) such that

(a) $X = \bigcup_{e\in\Gamma} e$,

(b) the $e \in \Gamma$ are pairwise disjoint,

(c) $\emptyset \in \Gamma$,

(d) for each non-empty $e \in \Gamma$ there is a homeomorphism between a closed i-dimensional ball and the closure $\bar{e}$ which maps the interior of the ball onto e (i.e. e is an *open i-cell*).

We will also call the $e \in \Gamma$ *faces*. We will say that $e' \in \Gamma$ is a *face of* $e \in \Gamma$, $e \neq e'$, or that e contains e' if $e' \subset \bar{e}$. The maximal faces of e under containment are called its *facets*. 0- and 1-dimensional faces will be called *vertices* and *edges* respectively. The set of vertices is denoted $\mathfrak{V}$. A subset $\Gamma' \subset \Gamma$ such that for each $e \in \Gamma'$ all the faces of e are in Γ' determines a *subcomplex* $X_{\Gamma'} = \bigcup_{e\in\Gamma'} e$ of X. Moreover we assume in addition

(e) If e' is a codimension 2 face of e there are exactly two facets e_1, e_2 of e containing e'.

The prototypical example of a finite regular cell complex is the set of faces of a convex polytope for which property (e) is fulfilled. In general (e) is added as a kind of regularity assumption.
Choose an *incidence function* $\epsilon(e, e')$ on pairs of faces of e, e'. This means that ϵ takes values in $\{0, +1, -1\}$, $\epsilon(e, e') = 0$ unless e' is a facet of e, $\epsilon(v, \emptyset) = 1$ for all vertices $v \in \mathfrak{V}$ and moreover

$$\epsilon(e, e_1)\epsilon(e_1, e') + \epsilon(e, e_2)\epsilon(e_2, e') = 0$$

for e, e_1, e_2, e' as in (e).
Let now $M = (m_v)_{v\in\mathfrak{V}}$ be a monomial ideal (m_v monomials) in the polynomial ring $k[T_1,\dots,T_N]$, k some field. For multi-indices $\underline{a}$, $\underline{b} \in \mathbb{Z}^N$ we write $\underline{a} \le \underline{b}$ to denote $a_i \le b_i$ for all $i = 1,\dots,N$. $T^{\underline{a}}$ denotes $T_1^{a_1}\cdot\dots\cdot T_N^{a_N}$.
The oriented chain complex $\tilde{C}(X,k) = \bigoplus_{e\in\Gamma} ke$ (the homological grading is given by dimension of faces) with differential

$$\partial e := \sum_{e'\in\Gamma} \epsilon(e,e')e'$$

computes the *reduced cellular homology groups* $\tilde{H}^i(X,k)$ of X.
Think of the vertices $v \in \mathfrak{V}$ as labelled by the corresponding monomials m_v. Each non-empty face $e \in \Gamma$ will be identified with its set of vertices and will be labelled by the least common multiple m_e of its vertex labels. The *cellular complex* $F_{X,M}$ associated to (X,M) is the $\mathbb{Z}^N$-graded $k[T_1,\dots,T_N]$-module $\bigoplus_{e\in\Gamma, e\neq\emptyset} k[T_1,\dots,T_N]e$ with differential

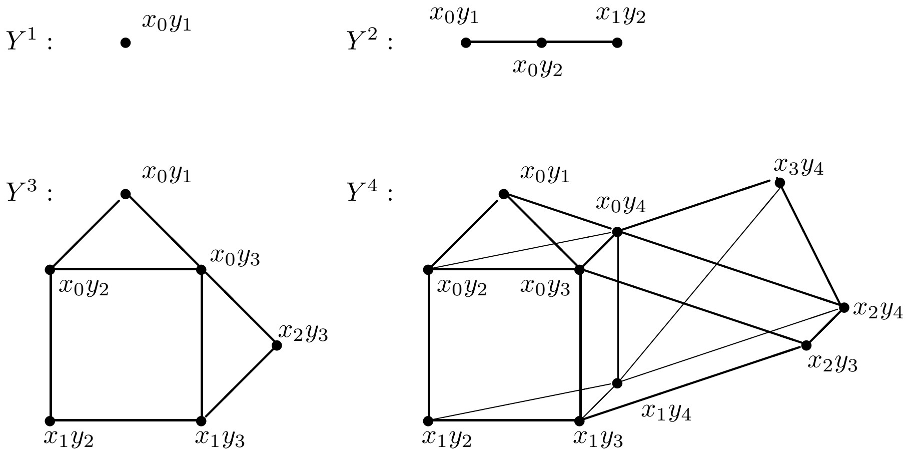
$$\partial e := \sum_{e'\in\Gamma, e'\neq\emptyset} \epsilon(e,e')\frac{m_e}{m_{e'}}e'$$

(where again the homological grading is given by the dimension of the faces).
For each multi-index $\underline{b}\in\mathbb{Z}^N$ let $X_{\le\underline{b}}$ be the subcomplex of X consisting of all the faces e whose labels m_e divide $T^{\underline{b}}$. We have

Proposition 4.2.2. *$F_{X,M}$ is a free resolution of M if and only if $X_{\le\underline{b}}$ is acyclic over k for all $\underline{b}\in\mathbb{Z}^N$ (i.e.$\tilde{H}_i(X_{\le\underline{b}},k)=0$ for all i).*

We refer to [B-S], prop. 1.2, for a proof.

Next we will construct appropriate cell complexes Y^n, $n = 1,\ 2,\dots,$ that via the procedure described above give resolutions of $J = (x_iy_j)_{0\le i<j\le n}$. We will apply proposition 4.2.2 by showing that for all $\underline{b}\in\mathbb{Z}^{2n+2}$ the subcomplexes $Y^n_{\le\underline{b}}$ are contractible.
It is instructive to look at the pictures of Y^1, Y^2, Y^3, Y^4 with their labellings first:



The general procedure for constructing Y^n geometrically is as follows: In $\mathbb{R}^{n-1}$ take the standard $(n-1)$-simplex P^1 on the vertex set $\{x_0y_1, x_0y_2, \dots, x_0y_n\}$. Then take an $(n-2)$-simplex on the vertex set $\{x_1y_2, \dots, x_1y_n\}$, viewed as embedded in the same $\mathbb{R}^{n-1}$, and join the vertices $x_1y_2, \dots, x_1y_n$, respectively, to the vertices $x_0y_2, \dots, x_0y_n$,, respectively, of P^1 by drawing an edge between x_0y_i and x_1y_i for $i = 2, \dots, n$. This describes the process of attaching a new $(n-1)$-dimensional polytope P^2 to the facet of P^1 on the vertex set $\{x_0y_2, \dots, x_0y_n\}$.

Assume that we have constructed inductively the $(n-1)$-dimensional polytope P^i, $2 \le i \le n-1$, with one facet on the vertex set $\{x_\mu y_\nu\}_{\substack{0\le\mu\le i-1\\ i+1\le\nu\le n}}$. Then take an $(n-i-1)$-simplex on the vertex set $\{x_iy_{i+1}, \dots, x_iy_n\}$, viewed as embedded in the same $\mathbb{R}^{n-1}$, and for every α with $1 \le \alpha \le n-i$ join the vertex $x_iy_{i+\alpha}$ of this simplex to the vertices $x_\mu y_{i+\alpha}$, $0 \le \mu \le i-1$, of P^i by an edge. This corresponds to attaching a new $(n-1)$-dimensional polytope P^{i+1} to the facet of P^i on the vertex set $\{x_\mu y_\nu\}_{\substack{0\le\mu\le i-1\\ i+1\le\nu\le n}}$.

In the end we get $(n-1)$-dimensional polytopes $P_1, \dots, P_n$ in $\mathbb{R}^{n-1}$ where P_j and P_{j+1}, $j = 1, \dots, n-1$, are glued along a common facet. These will make up our labelled cell complex Y^n.

The h-dimensional faces of Y^n will be called *h-faces* for short. We need a more convenient description for them:

Lemma 4.2.3. *There are natural bijections between the following sets:*

(i) $\{$*h-faces of* $Y^n\}$

(ii) matrices

$$\begin{pmatrix} x_{i_1}y_{i_{\mu_1+2}} & \cdots & x_{i_1}y_{i_{h+2}} \\ \vdots & \ddots & \vdots \\ x_{i_{\mu_1+1}}y_{i_{\mu_1+2}} & \cdots & x_{i_{\mu_1+1}}y_{i_{h+2}} \end{pmatrix}$$

with vertex labels of Y^n *as entries, where* $0 \le i_1 < i_2 < \ldots < i_{\mu_1+1} < i_{\mu_1+2} < \ldots < i_{h+2} \le n$ *and* $0 \le \mu_1 \le h$ *are integers. The* (κ, λ)*-entry of the above matrix is thus* $x_{i_\kappa} y_{i_{\mu_1+\lambda+1}}$.

(iii) standard basis vectors

$$f_{i_1} \wedge \ldots \wedge f_{i_{\mu_1+1}} \wedge f_{i_{\mu_1+2}} \wedge \ldots \wedge f_{i_{h+2}} \otimes \xi^{\mu_1}\eta^{\mu_2}, \quad 0 \le \mu_1 \le h,$$
$$0 \le i_1 < i_2 < \ldots < i_{\mu_1+1} < i_{\mu_1+2} < \ldots < i_{h+2} \le n$$

of $\bigwedge^{h+2} F \otimes \mathrm{Sym}^h G^\vee$.

Proof. The bijection between the sets in *(ii)* and *(iii)* is obvious: To $f_{i_1} \wedge \ldots \wedge f_{i_{h+2}} \otimes \xi^{\mu_1}\eta^{\mu_2}$ in $\bigwedge^{h+2} F \otimes \mathrm{Sym}^h G^\vee$ one associates the matrix

$$\begin{pmatrix} x_{i_1}y_{i_{\mu_1+2}} & \cdots & x_{i_1}y_{i_{h+2}} \\ \vdots & \ddots & \vdots \\ x_{i_{\mu_1+1}}y_{i_{\mu_1+2}} & \cdots & x_{i_{\mu_1+1}}y_{i_{h+2}} \end{pmatrix}.$$

To set up a bijection between the sets under (i) and (ii) the idea is to identify an h-face e of Y^n with its vertex labels and collect the vertex labels in a matrix of the form given in (ii). We will prove by induction on j that the h-faces e contained in the polytopes $P_1, \ldots, P_j$ are exactly those whose vertex labels may be collected in a matrix of the form written in (ii) satisfying the additional property $i_{\mu_1+1} \le j-1$. This will prove the lemma.
P_1 is an $(n-1)$-simplex on the vertex set $\{x_0y_1, \ldots, x_0y_n\}$ and its h-faces e can be identified with the subsets of cardinality $h+1$ of $\{x_0y_1, \ldots, x_0y_n\}$. We can write such a subset in matrix form

$$\begin{pmatrix} x_0y_{i_2} & x_0y_{i_3} & \cdots & x_0y_{i_{h+2}} \end{pmatrix}$$

with $0 \le i_2 < i_3 < \ldots < i_{h+2} \le n$. This shows that the preceding claim is true for $j = 1$.
For the induction step assume that the h-faces of Y^n contained in $P_1, \ldots, P_j$ are exactly those whose vertex labels may be collected in a matrix as in (ii) with $i_{\mu_1+1} \le j-1$. Look at the h-faces e contained in $P_1, \ldots, P_{j+1}$. If e is contained in $P_1, \ldots, P_j$ (which is equivalent to saying that none of the vertex labels of e involves the indeterminate x_j) then there is nothing to show. Now there are two types of h-faces contained in $P_1, \ldots, P_{j+1}$ but not in $P_1, \ldots, P_j$: The first type corresponds to h-faces e entirely contained in the simplex on the vertex set $\{x_jy_{j+1}, \ldots, x_jy_n\}$. These correspond to matrices

$$\begin{pmatrix} x_jy_{i_2} & \cdots & x_jy_{i_{h+2}} \end{pmatrix},$$

$0 \le i_2 < i_3 < \ldots < i_{h+2} \le n$, of the form given in (ii) which involve the indeterminate x_j and have only one row.

The second type of h-faces e is obtained as follows: We take an $(h-1)$-face e' contained in the facet on the vertex set $\{x_a y_b\}_{\substack{0\le a\le j-1\\ j+1\le b\le n}}$ which P_j and P_{j+1} have in common; by induction e' corresponds to a matrix

$$\begin{pmatrix} x_{i_1}y_{i_{\mu_1+2}} & \cdots & x_{i_1}y_{i_{h+1}} \\ \vdots & \ddots & \vdots \\ x_{i_{\mu_1+1}}y_{i_{\mu_1+2}} & \cdots & x_{i_{\mu_1+1}}y_{i_{h+1}} \end{pmatrix}$$

with $0 \le i_1 < i_2 < \ldots < i_{\mu_1+1} \le j-1$ and $j+1 \le i_{\mu_1+2} < \ldots < i_{h+1} \le n$, $0 \le \mu_1 \le h-1$. Then by construction of P_{j+1} there is a unique h-face e in $P_1, \ldots, P_{j+1}$, but not in $P_1, \ldots, P_j$, which contains the $(h-1)$-face e': It is the h-face whose vertex labels are the entries of the preceding matrix together with $\{x_j y_{i_{\mu_1+2}},\, x_j y_{i_{\mu_1+3}}, \ldots,\, x_j y_{i_{h+1}}\}$. Thus e corresponds to the matrix

$$\begin{pmatrix} x_{i_1}y_{i_{\mu_1+2}} & \cdots & x_{i_1}y_{i_{h+1}} \\ \vdots & \ddots & \vdots \\ x_{i_{\mu_1+1}}y_{i_{\mu_1+2}} & \cdots & x_{i_{\mu_1+1}}y_{i_{h+1}} \\ x_j y_{i_{\mu_1+2}} & \cdots & x_j y_{i_{h+1}} \end{pmatrix}.$$

This proves the lemma. $\square$

Now we want to define an incidence function $\epsilon(e, e')$ on pairs of faces e, e' of Y^n. Of course if e' is not a facet of e , we put $\epsilon(e, e') = 0$ and likewise put $\epsilon(v, \emptyset) := 1$ for all vertices v of Y^n. Let now e be an h-face. Using lemma 4.2.3 it corresponds to a matrix

$$M(e) = \begin{pmatrix} x_{i_1}y_{i_{\mu_1+2}} & \cdots & x_{i_1}y_{i_{h+2}} \\ \vdots & \ddots & \vdots \\ x_{i_{\mu_1+1}}y_{i_{\mu_1+2}} & \cdots & x_{i_{\mu_1+1}}y_{i_{h+2}} \end{pmatrix}.$$

A facet e' of e corresponds to a submatrix $M(e')$ of $M(e)$ obtained from $M(e)$ by erasing either a row or a column. We define $\epsilon(e, e') := (-1)^l$ if $M(e')$ is obtained from $M(e)$ by erasing the lth row; we define $\epsilon(e, e') := (-1)^{\mu_1 + j}$ if $M(e')$ is obtained from $M(e)$ by erasing the jth column.

One must check that then $\epsilon(e, e_1)\epsilon(e_1, e'') + \epsilon(e, e_2)\epsilon(e_2, e'') = 0$ for a codimension 2 face e'' of e and e_1, e_2 the two facets of e containing e''. This is now a straightforward computation. There are 3 cases: The matrix $M(e'')$ is obtained from $M(e)$ by (i) deleting two rows, (ii) deleting two columns, (iii) erasing one row and one column:

(i) Let $l_1 < l_2$ and assume that $M(e_1)$ is $M(e)$ with l_1th row erased and $M(e_2)$ is $M(e)$ with l_2th row erased. Then

$$\epsilon(e, e_1) = (-1)^{l_1}, \quad \epsilon(e, e_2) = (-1)^{l_2}, \quad \epsilon(e_1, e'') = (-1)^{l_2 - 1}$$
$$\epsilon(e_2, e'') = (-1)^{l_1}.$$

(ii) This is the same computation as for (i) with the roles of rows and columns interchanged.

(iii) Assume that $M(e_1)$ is $M(e)$ with lth row erased and $M(e_2)$ is $M(e)$ with jth column erased. Then

$$\epsilon(e,e_1)=(-1)^l,\quad \epsilon(e,e_2)=(-1)^{\mu_1+j},\quad \epsilon(e_1,e'')=(-1)^{\mu_1-1+j}$$
$$\epsilon(e_2,e'')=(-1)^l.$$

Thus ϵ is an incidence function on Y^n. Now one has to compute the cellular complex $F_{Y^n,J}$: Indeed by lemma 4.2.3 we know that its term in homological degree h identifies with $\bigwedge^{h+2}F\otimes \mathrm{Sym}^h G^\vee$. If e is an h-face recall that the differential ∂ of $F_{Y^n,J}$ is given by

$$\partial e=\sum_{e' \text{ a facet of } e,\ e'\neq\emptyset}\epsilon(e,e')\frac{m_e}{m_{e'}}e'$$

and if e corresponds to $f_{i_1}\wedge\ldots\wedge f_{i_{h+2}}\otimes\xi^{\mu_1}\eta^{\mu_2}\in\bigwedge^{h+2}F\otimes\mathrm{Sym}^h G^\vee$ we find

$$\partial\left(f_{j_1}\wedge\cdots\wedge f_{j_{h+2}}\otimes\xi^{\mu_1}\eta^{\mu_2}\right)=\sum_{l=1}^{h-\mu_2+1}(-1)^{l+1}x_{j_l}f_{j_1}\wedge\cdots\wedge\hat{f}_{j_l}\wedge\cdots\wedge f_{j_{h+2}}$$
$$\otimes\xi^{-1}(\xi^{\mu_1}\eta^{\mu_2})+\sum_{l=\mu_1+2}^{h+2}(-1)^{l+1}y_{j_l}f_{j_1}\wedge\cdots\wedge\hat{f}_{j_l}\wedge\cdots\wedge f_{j_{h+2}}\otimes\eta^{-1}(\xi^{\mu_1}\eta^{\mu_2})\,.$$

Thus the complex $F_{Y^n,J}$ is nothing but the complex in proposition 4.2.1. Thus to prove proposition 4.2.1 it is sufficient in view of proposition 4.2.2 to prove the following

LEMMA 4.2.4. *For all $\underline{b}\in\mathbb{Z}^{2n+2}$ the subcomplexes $Y^n_{\preceq\underline{b}}$ of Y^n are contractible.*

Proof. Notice that it suffices to prove the following: If

$$x_{i_1}\ldots x_{i_k}y_{j_1}\ldots y_{j_l}\quad 0\le i_1<\ldots i_k\le n,\quad 0\le j_1<\cdots<j_l\le n$$

is a monomial that is the least common multiple of some subset of the vertex labels of Y^n then the subcomplex $\tilde{Y}^n$ of Y^n that consists of all the faces e whose label divides $x_{i_1}\ldots x_{i_k}y_{j_1}\ldots y_{j_l}$ is contractible. This can be done as follows:

Put $\kappa(i_d):=\min\{t\,:\,j_t>i_d\}$ for $d=1,\ldots,k$. Note that we have $\kappa(i_1)=1$ and $\kappa(i_1)\le\kappa(i_2)\le\cdots\le\kappa(i_k)$. Choose a retraction of the face e^0 of $\tilde{Y}^n$ corresponding to the matrix

$$\begin{pmatrix} x_{i_1}y_{j_{\kappa(i_k)}} & \cdots & x_{i_1}y_{j_l}\\ \vdots & \ddots & \vdots\\ x_{i_k}y_{j_{\kappa(i_k)}} & \cdots & x_{i_k}y_{j_l}\end{pmatrix}$$

onto its facet $e^{0'}$ corresponding to

$$\begin{pmatrix} x_{i_1}y_{j_{\kappa(i_k)}} & \dots & x_{i_1}y_{j_l} \\ \vdots & \ddots & \vdots \\ x_{i_{k-1}}y_{j_{\kappa(i_k)}} & \dots & x_{i_{k-1}}y_{j_l} \end{pmatrix}.$$

Then choose a retraction of the face e^1 corresponding to

$$\begin{pmatrix} x_{i_1}y_{j_{\kappa(i_{k-1})}} & \dots & x_{i_1}y_{j_l} \\ \vdots & \ddots & \vdots \\ x_{i_{k-1}}y_{j_{\kappa(i_{k-1})}} & \dots & x_{i_{k-1}}y_{j_l} \end{pmatrix}$$

onto its facet $e^{1'}$ corresponding to

$$\begin{pmatrix} x_{i_1}y_{j_{\kappa(i_{k-1})}} & \dots & x_{i_1}y_{j_l} \\ \vdots & \ddots & \vdots \\ x_{i_{k-2}}y_{j_{\kappa(i_{k-1})}} & \dots & x_{i_{k-2}}y_{j_l} \end{pmatrix}.$$

Notice that $e^{0'}$ is contained in e^1. Continuing this pattern, one can finally retract the face corresponding to

$$\begin{pmatrix} x_{i_1}y_{j_{\kappa(i_1)}} & x_{i_1}y_{j_{\kappa(i_1)+1}} & \dots & x_{i_1}y_{j_l} \end{pmatrix},$$

i.e. a simplex, onto one of its vertices. Composing these retractions, one gets a retraction of $\tilde{Y}^n$ onto a point. $\square$

In conclusion what we get from proposition 4.2.1 is that on $\mathbb{P}^n \times \mathbb{P}^n$ the sheaf $\mathcal{O}_{\mathfrak{X}_0}$ has a resolution

$$\begin{aligned} (*) \quad 0 &\longrightarrow \bigoplus_{\substack{i+j=n-1\\ i,j\geq 0}} \mathcal{O}(-i-1,-j-1) \xrightarrow{d'_{n-1}} \dots \\ \dots &\xrightarrow{d'_{h+1}} \left(\bigoplus_{\substack{i+j=h\\ i,j\geq 0}} \mathcal{O}(-i-1,-j-1) \right)^{\oplus \binom{n+1}{h+2}} \xrightarrow{d'_h} \dots \\ \dots &\xrightarrow{d'_1} \mathcal{O}(-1,-1)^{\oplus \binom{n+1}{2}} \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathfrak{X}_0} \longrightarrow 0 \end{aligned}$$

where the differentials can be identified with the differentials in the complex of proposition 4.2.1, and $\mathcal{O}_\Delta$ has a resolution

$$\begin{aligned} (**) \quad 0 &\longrightarrow \bigoplus_{\substack{i+j=n-1\\ i,j\geq 0}} \mathcal{O}(-i-1,-j-1) \xrightarrow{d_{n-1}} \dots \\ \dots &\xrightarrow{d_{h+1}} \left(\bigoplus_{\substack{i+j=h\\ i,j\geq 0}} \mathcal{O}(-i-1,-j-1) \right)^{\oplus \binom{n+1}{h+2}} \xrightarrow{d_h} \dots \\ \dots &\xrightarrow{d_1} \mathcal{O}(-1,-1)^{\oplus \binom{n+1}{2}} \longrightarrow \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0 \end{aligned}$$

which is an Eagon-Northcott complex.
The next theorem gives a complete description of the functor $Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_{\mathfrak{X}_0}) : D^b(Coh\,\mathbb{P}^n) \to D^b(Coh\,\mathbb{P}^n)$ (recall that in $D^b(Coh\,\mathbb{P}^n)$ one has the strong complete exceptional sequence $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$).

THEOREM 4.2.5. *Let $\{pt\} = L_0 \subset L_1 \subset \dots \subset L_n = \mathbb{P}^n$ be a full flag of projective linear subspaces of $\mathbb{P}^n$ (the Schubert varieties in $\mathbb{P}^n$) and let $L^0 = \mathbb{P}^n \supset \dots \supset L^n = \{pt\}$ form a complete flag in general position with respect to the L_j ("L^j is the Schubert variety opposite to L_j").*
For $d \ge 0$ one has in $D^b(Coh\,\mathbb{P}^n)$

$$Rp_{2*}(p_1^*(\mathcal{O}(d)) \otimes^L \mathcal{O}_{\mathfrak{X}_0}) \simeq \bigoplus_{j=0}^{n} \mathcal{O}_{L_j} \otimes H^0(L^j, \mathcal{O}(d))^\vee / H^0(L^{j+1}, \mathcal{O}(d))^\vee .$$

In terms of the coordinates $x_0, \dots, x_n$, $y_0, \dots, y_n$ introduced above:

$$\begin{aligned} Rp_{2*}(p_1^*(\mathcal{O}(d)) \otimes^L \mathcal{O}_{\mathfrak{X}_0}) \simeq \mathcal{O} \oplus (\mathcal{O}/(y_n))^{\oplus d} \oplus (\mathcal{O}/(y_n, y_{n-1}))^{\oplus \frac{d(d+1)}{2}} \oplus \dots \\ \dots \oplus (\mathcal{O}/(y_n, \dots, y_{n-i}))^{\oplus \binom{d+i}{d-1}} \oplus \dots \oplus (\mathcal{O}/(y_n, \dots, y_1))^{\oplus \binom{d+n-1}{d-1}}. \end{aligned}$$

Moreover for the map $\mathcal{O}(e) \xrightarrow{\cdot x_k} \mathcal{O}(e+1)$ ($e \ge 0$, $0 \le k \le n$) one can describe the induced map $Rp_{2*}(p_1^*(\cdot x_k) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ as follows:
For each $d \ge 0$ and each $i = -1, \dots, n-1$ choose a bijection between the set of monomials M_i^d in the variables $x_{n-1-i}, x_{n-i}, \dots, x_n$ of the form $x_{n-i-1}^{\alpha_1} x_{n-i}^{\alpha_2} \dots x_n^{\alpha_{i+2}}$ with $\alpha_1 > 0$ and $\sum \alpha_j = d$, and the set of copies of $\mathcal{O}/(y_n, \dots, y_{n-i})$ occuring in the above expression for $Rp_{2*}(p_1^*(\mathcal{O}(d)) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$. Then the copy of $\mathcal{O}/(y_n, \dots, y_{n-i})$ corresponding to a monomial $m \in M_i^e$ is mapped under $Rp_{2*}(p_1^*(\cdot x_k) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ identically to the copy of $\mathcal{O}/(y_n, \dots, y_{n-i})$ corresponding to the monomial $x_k m$ iff x_k occurs in m. If x_k does not occur in m then the copy of $\mathcal{O}/(y_n, \dots, y_{n-i})$ corresponding to the monomial $m \in M_i^e$ is mapped to the copy of $\mathcal{O}/(y_n, \dots, y_{k+1})$ corresponding to $x_k m$ via the natural surjection

$$\mathcal{O}/(y_n, \dots, y_{n-i}) \to \mathcal{O}/(y_n, \dots, y_{k+1}).$$

Proof. For $d \ge 0$ one tensors the resolution $(*)$ of $\mathcal{O}_{\mathfrak{X}_0}$ by $p_1^*\mathcal{O}(d)$ and notes that then all the bundles occuring in the terms of $(*) \otimes p_1^*\mathcal{O}(d)$ are p_{2*}-acyclic whence $Rp_{2*}(p_1^*(\mathcal{O}(d)) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ is (as a complex concentrated in degree 0) the cokernel of the map

$$\Phi : \left(H^0(\mathbb{P}^n, \mathcal{O}(d-1)) \otimes \mathcal{O}(-1)\right)^{\oplus \binom{n+1}{2}} \to H^0(\mathbb{P}^n, \mathcal{O}(d)) \otimes \mathcal{O}$$

which on the various summands $H^0(\mathbb{P}^n, \mathcal{O}(d-1)) \otimes \mathcal{O}(-1)$ of the domain is given by the maps

$$\begin{aligned} H^0(\mathbb{P}^n, \mathcal{O}(d-1)) \otimes \mathcal{O}(-1) \to H^0(\mathbb{P}^n, \mathcal{O}(d)) \otimes \mathcal{O} \\ m \otimes \sigma \mapsto x_i m \otimes y_j \sigma \end{aligned}$$

for $0 \le i < j \le n$. For $i = -1, \dots, n-1$ let M_i^d be as above the set of monomials in $x_{n-1-i}, x_{n-i}, \dots, x_n$ of the form $x_{n-i-1}^{\alpha_1} x_{n-i}^{\alpha_2} \dots x_n^{\alpha_{i+2}}$ with $\alpha_1 > 0$ and $\sum \alpha_j = d$. Then we have the identification

$$H^0(\mathbb{P}^n, \mathcal{O}(d)) \otimes \mathcal{O} \simeq \bigoplus_{i=-1}^{n-1} \left(\bigoplus_{m \in M_i^d} \mathcal{O} \right);$$

For given $m \in M_i^d$ write $\mathrm{cont}(m)$ for the subset of the variables $x_0, \dots, x_n$ that occur in m. Then the map Φ above is the direct sum of maps

$$\bigoplus_{x_i \in \mathrm{cont}(m)} \bigoplus_{n \ge j > i} \mathcal{O}(-1) \to \mathcal{O} \quad \forall i = -1, \dots, n-1 \, \forall m \in M_i^d$$

which on the summand $\mathcal{O}(-1)$ on the left side of the arrow corresponding to $x_{i_0} \in \mathrm{cont}(m)$ and $j_0 > i_0$ are multiplication by y_{j_0}. Since M_i^d has cardinality

$$\binom{d+i}{d-1} = \binom{d+i+1}{d} - \binom{d+i}{d}$$

one finds that the cokernel of Φ is indeed

$$\mathcal{O} \oplus (\mathcal{O}/(y_n))^{\oplus d} \oplus \dots \oplus (\mathcal{O}/(y_n, \dots, y_{n-i}))^{\oplus \binom{d+i}{d-1}} \oplus \\ \dots \oplus (\mathcal{O}/(y_n, \dots, y_1))^{\oplus \binom{d+n-1}{d-1}}$$

as claimed.
The second statement of the theorem is now clear because $Rp_{2*}(p_1^*(\cdot x_k) \otimes^L \mathcal{O}_{\mathfrak{X}_0})$ is induced by the map $H^0(\mathbb{P}^n, \mathcal{O}(e)) \otimes \mathcal{O} \to H^0(\mathbb{P}^n, \mathcal{O}(e+1)) \otimes \mathcal{O}$ which is multiplication by x_k. $\square$

Remark 4.2.6. It is possible to prove Beilinson's theorem on $\mathbb{P}^n$ using only knowledge of the resolution $(*)$ of $\mathcal{O}_{\mathfrak{X}_0}$: Indeed by theorem 4.1.1 one knows a priori that one can lift the resolution $(*)$ of $\mathcal{O}_{\mathfrak{X}_0}$ to a resolution of $\mathcal{O}_\Delta$ of the form $(**)$ by flatness (cf. e.g. [Ar], part I, rem. 3.1). Since the terms in the resolution $(*)$ are direct sums of bundles $\mathcal{O}(-k, -l)$, $0 \le k, l \le n$, we find by the standard argument from [Bei](i.e., the decomposition $\mathrm{id} \simeq Rp_{2*}(p_1^*(-) \otimes^L \mathcal{O}_\Delta)$) that $D^b(Coh\, \mathbb{P}^n)$ is generated by $(\mathcal{O}(-n), \dots, \mathcal{O})$.

Finally it would be interesting to know if one could find a resolution of $\mathcal{O}_{\mathfrak{X}_0}$ on $X \times X$ for any rational homogeneous $X = G/P$ along the same lines as in this subsection, i.e. by first finding a "monomial description" of $\mathfrak{X}_0$ inside $X \times X$ (e.g. using standard monomial theory, cf. [BiLa]) and then using the method of cellular resolutions from [B-S]. Thereafter it would be even more important to see if one could obtain valuable information about $D^b(Coh\, X)$ by lifting the resolution of $\mathcal{O}_{\mathfrak{X}_0}$ to one of $\mathcal{O}_\Delta$.

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