# The Global Structure of Moduli Spaces of Polarized $p$-Divisible Groups

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Abstract. We study the global structure of moduli spaces of quasi-isogenies of polarized $p$-divisible groups introduced by Rapoport and Zink. Using the corresponding results for non-polarized $p$-divisible groups from a previous paper, we determine their dimensions and their sets of connected components and of irreducible components.

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## 1 Introduction

Let $k$ be an algebraically closed field of characteristic $p > 2$. Let $W = W(k)$ be its ring of Witt vectors and $L = \mathrm{Quot}(W)$. Let $\sigma$ be the Frobenius automorphism on $k$ as well as on $W$. By $\mathrm{Nilp}_W$ we denote the category of schemes $S$ over $\mathrm{Spec}(W)$ such that $p$ is locally nilpotent on $S$. Let $\overline{S}$ be the closed subscheme of $S$ that is defined by the ideal sheaf $p\mathcal{O}_S$. Let $(\mathbb{X}, \lambda_{\mathbb{X}})$ be a principally polarized $p$-divisible group over $k$. If $X$ is a $p$-divisible group, we denote its dual by $\hat{X}$. Then the polarization $\lambda_{\mathbb{X}}$ is an isomorphism $\mathbb{X} \rightarrow \hat{\mathbb{X}}$.
We consider the functor

$$\mathcal{M} : \mathrm{Nilp}_W \rightarrow \mathrm{Sets},$$

which assigns to $S \in \mathrm{Nilp}_W$ the set of isomorphism classes of pairs $(X, \rho)$, where $X$ is a $p$-divisible group over $S$ and $\rho : \mathbb{X}_{\overline{S}} = \mathbb{X} \times_{\mathrm{Spec}(k)} \overline{S} \rightarrow X \times_S \overline{S}$ is a quasi-isogeny such that the following condition holds. There exists a principal polarization $\lambda : X \rightarrow \hat{X}$ such that $\rho^\vee \circ \lambda_{\overline{S}} \circ \rho$ and $\lambda_{\mathbb{X},\overline{S}}$ coincide up to a scalar. Two pairs $(X_1, \rho_1)$ and $(X_2, \rho_2)$ are isomorphic if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \rightarrow X_1$. This functor is representable by a formal scheme $\mathcal{M}$

which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RaZ], Thm. 3.25). Let $\mathcal{M}_{\mathrm{red}}$ be its underlying reduced subscheme, that is the reduced subscheme of $\mathcal{M}$ defined by the maximal ideal of definition. Then $\mathcal{M}_{\mathrm{red}}$ is a scheme over $\mathrm{Spec}(k)$.
The analogues of these moduli spaces for $p$-divisible groups without polarization have been studied by de Jong and Oort in [JO] for the case that the rational Dieudonné module of $\mathbb{X}$ is simple and in [V1] without making this additional assumption. There, the sets of connected components and of irreducible components, as well as the dimensions, are determined. In the polarized case, the moduli spaces $\mathcal{M}_{\mathrm{red}}$ have been examined in several low-dimensional cases. For example, Kaiser ([Kai]) proves a twisted fundamental lemma for $GSp_4$ by giving a complete description in the case that $\mathbb{X}$ is two-dimensional and supersingular. An independent description of this case is given by Kudla and Rapoport in [KR]. In [Ri], Richartz describes the moduli space in the case of three-dimensional supersingular $\mathbb{X}$. In this paper we derive corresponding results on the global structure of the mo duli space $\mathcal{M}_{\mathrm{red}}$ for arbitrary $\mathbb{X}$.
The first main result of this paper concerns the set of connected components of $\mathcal{M}_{\mathrm{red}}$.

Theorem 1. *Let $\mathbb{X}$ be nontrivial and let $\mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{et}}$ be the decomposition into its multiplicative, bi-infinitesimal, and étale part. Then*

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \cong \big(GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Z}_p)\big) \times \mathbb{Z}.$$

Next we consider the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$. Let $(N, F)$ be the rational Dieudonné module of $\mathbb{X}$. Here, $N$ is an $L$-vector space of dimension $\mathrm{ht}(\mathbb{X})$ and $F : N \to N$ is a $\sigma$-linear isomorphism. The polarization $\lambda_{\mathbb{X}}$ induces an anti-symmetric bilinear perfect pairing $\langle\cdot,\cdot\rangle$ on $N$. Let $G$ be the corresponding general symplectic group of automorphisms of $N$ respecting $\langle\cdot,\cdot\rangle$ up to a scalar. Let

$$J = \{g \in G(L) \mid g \circ F = F \circ g\}.$$

It is the set of $\mathbb{Q}_p$-valued points of an algebraic group over $\mathbb{Q}_p$ (see [RaZ], Prop. 1.12). There is an action of $J$ on $\mathcal{M}_{\mathrm{red}}$.

Theorem 2. *The action of $J$ on the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ is transitive.*

We choose a decomposition $N = \bigoplus_{j=1}^{l} N^j$ with $N^j$ simple of slope $\lambda_j = m_j/(m_j + n_j)$ with $(m_j, n_j) = 1$ and $\lambda_j \leq \lambda_{j'}$ for $j < j'$. Let

$$m = \left\lfloor \frac{1}{2} \sum_j \min\{m_j, n_j\} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less or equal $x$. As $N$ is the isocrystal of a polarized $p$-divisible group, its Newton polygon is symmetric, i. e. $\lambda_{l+1-j} =$

$1-\lambda_j$. Hence we obtain

$$m = \left\lfloor \sum_{\{j \mid m_j < n_j\}} m_j + \frac{1}{2} \mid \{j \mid m_j = n_j = 1\} \mid \right\rfloor . \tag{1.1}$$

Theorem 3. $\mathcal{M}_{\mathrm{red}}$ *is equidimensional of dimension*

$$\dim \mathcal{M}_{\mathrm{red}} = \frac{1}{2}\left( \sum_j \frac{(m_j-1)(n_j-1)}{2} + \sum_{j<j'} m_j n_{j'} + m \right). \tag{1.2}$$

Note that the equidimensionality is already a consequence of Theorem 2. However, it also follows from the proof of the dimension formula without requiring additional work.

Our results on the set of connected components and on the dimension of $\mathcal{M}_{\mathrm{red}}$ are analogous to those for other affine Deligne-Lusztig sets for split groups where a scheme structure is known. We now define these affine Deligne-Lusztig varieties and give a brief overview over the general results in comparison to the results for the case treated in this paper.

Let $\mathcal{O}$ be a finite unramified extension of $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$ and let $G$ be a split connected reductive group over $\mathcal{O}$. Let $\boldsymbol{F}$ be the quotient field of $\mathcal{O}$. Let $K = G(\mathcal{O})$. Let $L$ be the completion of the maximal unramified extension of $\boldsymbol{F}$ and let $\sigma$ be the Frobenius of $L$ over $\boldsymbol{F}$. Let $A$ be a maximal torus and $B$ a Borel subgroup containing $A$. Let $\mu \in X_*(A)$ be dominant and let $b \in G(L)$. Let $\varepsilon^\mu$ be the image of $p$ or $t \in \boldsymbol{F}^\times$ under $\mu$. Let

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1} b\sigma(g) \in K\varepsilon^\mu K\} \tag{1.3}$$

be the generalized affine Deligne-Lusztig set associated to $\mu$ and $b$. We assume that $b \in B(G,\mu)$ to have that $X_\mu(b)$ is nonempty (compare [Ra], 5). There are two cases where it is known that $X_\mu(b)$ is the set of $k$-valued points of a scheme. Here, $k$ denotes the residue field of $\mathcal{O}_L$. The first case is that $\boldsymbol{F} = \mathbb{Q}_p$ and that $X_\mu(b)$ is the set of $k$-valued points of a Rapoport-Zink space of type (EL) or (PEL). In that case $\mu$ is always minuscule. Rapoport-Zink spaces without polarization were considered in [V1], in that case $G = GL_h$. For the moduli spaces considered in this paper let $G = GSp_{2h}$. We choose a basis $\{e_i, f_i \mid 1 \le i \le h\}$ identifying $N$ with $L^{2h}$ and the symplectic form on $N$ with the symplectic form on $L^{2h}$ defined by requiring that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle e_i, f_j\rangle = \delta_{i,h+1-j}$. Let $B$ be the Borel subgroup of $G = GSp_{2h}$ fixing the complete isotropic flag $(e_1) \subset (e_1,e_2) \subset \cdots \subset (e_1,\dots,e_h)$. We choose $A$ to be the diagonal torus. Let $\pi_1(G)$ be the quotient of $X_*(A)$ by the coroot lattice of $G$. Then the multiplier $G \to \mathbb{G}_{\mathrm{m}}$ induces an isomorphism $\pi_1(G) \to \pi_1(\mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}$. Let $\mu \in X_*(A)$ be the unique minuscule element whose image in $\pi_1(G)$ is 1. Then $p^\mu$ is a diagonal matrix with diagonal entries 1 and $p$, each with multiplicity $h$. We write $F = b\sigma$ with $b \in G$. Note that there is a bijection between $\mathcal{M}_{\mathrm{red}}(k)$ and the set of Dieudonné lattices in $N$. Using the above notation, we have the bijection

$$\begin{array}{rcl} X_\mu(b) & \to & \mathcal{M}_{\mathrm{red}}(k) \\ g & \mapsto & g(W(k)^{2h}). \end{array}$$

The second case is that $\boldsymbol{F}$ is a function field. Here $X_\mu(b)$ obtains its scheme structure by considering it as a subset of the affine Grassmannian $G(L)/G(\mathcal{O}_L)$. In this case we do not have to assume $\mu$ to be minuscule. The $X_\mu(b)$ are locally closed subschemes of the affine Grassmannian. The closed affine Deligne-Lusztig varieties $X_{\preceq\mu}(b)$ are defined to be the closed reduced subschemes of $G(L)/G(\mathcal{O}_L)$ given by $X_{\preceq\mu}(b) = \bigcup_{\mu'\preceq\mu} X_{\mu'}(b)$. Here $\mu' \preceq \mu$ if $\mu - \mu'$ is a non-negative linear combination of positive coroots. Note that the two schemes $X_\mu(b)$ and $X_{\preceq\mu}(b)$ coincide if $\mu$ is minuscule.

The sets of connected components of the moduli spaces of non-polarized $p$-divisible groups are given by a formula completely analogous to Theorem 1 (compare [V1], Thm. A). For closed affine Deligne-Lusztig varieties in the function field case, the set of connected components is also given by a generalization of the formula in Theorem 1 (see [V3], Thm. 1). The sets of connected components of the non-closed $X_\mu(b)$ are not known in general. There are examples (compare [V3], Section 3) which show that a result analogous to Theorem 1 cannot hold for all non-closed $X_\mu(b)$.

The only further general case where the set of irreducible components is known are the reduced subspaces of moduli spaces of $p$-divisible groups without polarization. Here, the group $J$ also acts transitively on the set of irreducible components. There are examples of affine Deligne-Lusztig varieties in the function field case associated to non-minuscule $\mu$ where this is no longer true (compare [V2], Ex. 6.2).

To discuss the formula for the dimension let us first reformulate Theorem 3. Let $G = GSp_{2h}$ and $\mu$ be as above. Let $\nu = (\lambda_i) \in \mathbb{Q}^h \cong X_*(A)_{\mathbb{Q}}$ be the (dominant) Newton vector associated to $(N, F)$ as defined by Kottwitz, see [Ko1]. Let $\rho$ be the half-sum of the positive roots of $G$ and $\omega_i$ the fundamental weights of the adjoint group $G_{\mathrm{ad}}$. Then one can reformulate (1.2) as

$$\dim \mathcal{M}_{\mathrm{red}} = \langle 2\rho, \mu - \nu\rangle + \sum_i \lfloor \langle \omega_i, \nu - \mu\rangle \rfloor. \tag{1.4}$$

In this form, the dimension formula proves a special case of a conjecture by Rapoport (see [Ra], Conjecture 5.10) for the dimension of affine Deligne-Lusztig varieties. Denote by $\mathrm{rk}_{\mathbb{Q}_p}$ the dimension of a maximal $\mathbb{Q}_p$-split subtorus and let $\mathrm{def}_G(F) = \mathrm{rk}_{\mathbb{Q}_p} G - \mathrm{rk}_{\mathbb{Q}_p} J$. Note that $\mathrm{def}_G(F)$ only depends on the conjugacy class of $F$ or equivalently on the $\sigma$-conjugacy class of $b$ if we write $F = b\sigma$ for some $b \in G$. In our case, it is equal to $h - \lceil l/2 \rceil$ where $l$ is the number of simple summands of $N$. Using Kottwitz's reformulation of the right hand side of (1.4) in [Ko2], we obtain

$$\dim \mathcal{M}_{\mathrm{red}} = \langle \rho, \mu - \nu\rangle - \frac{1}{2}\mathrm{def}_G(F). \tag{1.5}$$

For the case of moduli of $p$-divisible groups for $G = GL_h$, the analogous formula for the dimension is shown in [V1]. In the function field case, the dimension of the generalized affine Deligne-Lusztig variety has been determined in [V2], [GHKR]. The formula for the dimension is also in this case the analogue of (1.5).

The dimension of the moduli spaces $\mathcal{M}_{\mathrm{red}}$ is also studied by Oort and by Chai using a different approach. In [O2], Oort defines an almost product structure (that is, up to a finite morphism) on Newton strata of moduli spaces of polarized abelian varieties. It is given by an isogeny leaf and a central leaf for the $p$-divisible group. The dimension of the isogeny leaf is the same as that of the corresponding $\mathcal{M}_{\mathrm{red}}$. The dimension of the central leaf is determined by Chai in [C] and also by Oort in [O4]. The dimension of the Newton polygon stratum itself is known from [O1]. Then the dimension of $\mathcal{M}_{\mathrm{red}}$ can also be computed as the difference of the dimensions of the Newton polygon stratum and the central leaf.

We outline the content of the different sections of the paper. In Section 2 we introduce the necessary background and notation, and reduce the problem to the case of bi-infinitesimal groups. In the third and fourth section, we define the open dense subscheme $\mathcal{S}_1$ where the $a$-invariant of the $p$-divisible group is 1 and describe its set of closed points. This description is refined in Sections 5 and 6 to prove the theorems on the set of irreducible components and on the dimension, respectively. In the last section we determine the set of connected components.

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## 2 Notation and preliminary reductions

### 2.1 A decomposition of the rational Dieudonné module

The principal polarization $\lambda_{\mathbb{X}}$ equips the rational Dieudonné module $(N, F)$ of $\mathbb{X}$ with a nondegenerate anti-symmetric bilinear pairing $\langle \cdot, \cdot \rangle$. It satisfies

$$\langle v, Fw \rangle = \sigma(\langle Vv, w \rangle) \tag{2.1}$$

for all $x, y \in N$.

We assumed $k$ to be algebraically closed. Then the classification of isocrystals shows that $N$ has a decomposition into subisocrystals $N_i$ of one of the following types. Let $l$ be the number of supersingular summands in a decomposition of $N$ into simple isocrystals. Then

$$N = \begin{cases} N_0 \oplus N_1 & \text{if } l \text{ is even} \\ N_0 \oplus N_{\frac{1}{2}} \oplus N_1 & \text{otherwise,} \end{cases} \tag{2.2}$$

satisfying the following three properties.

1. The slopes of $N_0$ are smaller or equal to $\frac{1}{2}$.
2. The summand $N_{\frac{1}{2}}$ is simple and supersingular.
3. $N_1$ is the isocrystal dual to $N_0$, i.e.
$$\langle N_0, N_0\rangle = \langle N_1, N_1\rangle = \langle N_0, N_{\frac{1}{2}}\rangle = \langle N_1, N_{\frac{1}{2}}\rangle = 0.$$

Note that if $l > 1$, then this decomposition is not unique and $N_0$ and $N_1$ also contain supersingular summands. For $i \in \{0, \frac{1}{2}, 1\}$ we denote by $p_i$ the canonical projection $N \to N_i$.
The moduli spaces $\mathcal{M}_{\mathrm{red}}$ for different $(\mathbb{X}, \lambda_{\mathbb{X}})$ in the same isogeny class are isomorphic. Replacing $\mathbb{X}$ by an isogenous group we may assume that

$$\mathbb{X} = \begin{cases} \mathbb{X}_0 \times \mathbb{X}_1 & \text{if } l \text{ is even} \\ \mathbb{X}_0 \times \mathbb{X}_{\frac{1}{2}} \times \mathbb{X}_1 & \text{otherwise.} \end{cases} \tag{2.3}$$

Here, $\mathbb{X}_i$ is such that its rational Dieudonné module is $N_i$.
Mapping $(X, \rho) \in \mathcal{M}_{\mathrm{red}}(k)$ to the Dieudonné module of $X$ defines a bijection between $\mathcal{M}_{\mathrm{red}}(k)$ and the set of Dieudonné lattices in $N$ that are self-dual up to a scalar. Here a sublattice $\Lambda$ of $N$ is called a Dieudonné lattice if $\varphi(\Lambda) \subseteq \Lambda$ for all $\varphi$ in the Dieudonné ring of $k$,

$$\mathcal{D} = \mathcal{D}(k) = W(k)[F, V]/(FV = VF = p, aV = V\sigma(a), Fa = \sigma(a)F). \tag{2.4}$$

All lattices considered in this paper are Dieudonné lattices. A lattice $\Lambda \subset N$ is self-dual up to a scalar if the dual lattice $\Lambda^\vee$ satisfies $\Lambda^\vee = c\Lambda$ with $c \in L^\times$. The following notion is introduced by Oort in [O3].

Definition 2.1. Let $X$ be a $p$-divisible group over $k$ and $\Lambda_{\min}$ be its Dieudonné module. Then $X$ is a minimal $p$-divisible group if $\mathrm{End}(\Lambda_{\min})$ is a maximal order in $\mathrm{End}(\Lambda_{\min}) \otimes_W L$.

*Remark* 2.2. By Morita equivalence $X$ is minimal if and only if $\Lambda_{\min}$ is the direct sum of submodules $\Lambda^i_{\min}$ such that $N^i = \Lambda^i_{\min} \otimes_W L$ is simple and that $\mathrm{End}(\Lambda^i_{\min})$ is a maximal order in $\mathrm{End}(N^i)$, which is Oort's original definition. Note that in every isogeny class of $p$-divisible groups over $k$ there is exactly one isomorphism class of minimal $p$-divisible groups (compare [O3], 1.1).

Lemma 2.3. *There is a $k$-valued point $(X, \rho)$ of $\mathcal{M}_{\mathrm{red}}$ such that $X$ is minimal.*

*Proof.* Let $N_0$ and $N_1$ as in the decomposition above. Let $\Lambda_{\min,0} \subset N_0$ be the lattice of a minimal $p$-divisible group and let $\Lambda_{\min,\frac{1}{2}} \subset N_{\frac{1}{2}}$ be the Dieudonné module of $\mathbb{X}_{\frac{1}{2}}$. There is only one isomorphism class of one-dimensional supersingular $p$-divisible groups and it consists of minimal $p$-divisible groups. Let $c \in L^\times$ with $\Lambda^\vee_{\min,\frac{1}{2}} = c\Lambda_{\min,\frac{1}{2}}$. Let

$$\Lambda_{\min,1} = \{x \in N_1 \mid \langle x, cy\rangle \in W \text{ for all } y \in \Lambda_{\min,0}\}.$$

Then $\Lambda_{\min,1}$ is also the Dieudonné module of a minimal $p$-divisible group. Furthermore, $\Lambda_{\min} = \Lambda_{\min,0} \oplus \Lambda_{\min,\frac{1}{2}} \oplus \Lambda_{\min,1}$ satisfies $\Lambda_{\min}^\vee = c\Lambda_{\min}$. Thus $\Lambda_{\min}$ corresponds to an element of $\mathcal{M}_{\mathrm{red}}(k)$ and to a minimal $p$-divisible group. $\square$

*Remark* 2.4. There is the following explicit description of the Dieudonné module of a minimal $p$-divisible group: Let $N = \bigoplus_j N^j$ be a decomposition of $N$ into simple isocrystals. For each $j$ we write the slope of $N^j$ as $m_j/(m_j+n_j)$ with $(m_j, n_j) = 1$. Then there is a basis $e_1^j, \dots, e_{m_j+n_j}^j$ of $N^j$ with $F(e_i^j) = e_{i+m_j}^j$ for all $i, j$. Here we use the notation $e_{i+m_j+n_j}^j = pe_i^j$. For the existence compare for example [V1], 4.1. Furthermore, these bases may be chosen such that $\langle e_i^j, e_{i'}^{j'} \rangle = \delta_{j,l+1-j'} \cdot \delta_{i,m_j+n_j+1-i'}$ for $1 \le i, i' \le m_j + n_j = m_{l+1-j} + n_{l+1-j}$. Then we can take the lattice $\Lambda_{\min}$ to be the lattice generated by these basis elements $e_i^j$.

## 2.2 Moduli of non-polarized $p$-divisible groups

For the moment let $\mathbb{X}$ be a $p$-divisible group without polarization. Then associated to $\mathbb{X}$ there is an analogous moduli problem of quasi-isogenies of $p$-divisible groups without polarization. If $\mathbb{X}$ is polarized, we thus obtain two functors which are closely related. In this section we recall the definition of the moduli spaces of non-polarized $p$-divisible groups and relate them to $\mathcal{M}_{\mathrm{red}}$. Besides, we provide a technical result on lattices in isocrystals which we need in the following section.

Let $\mathcal{M}_{\mathbb{X}}^{\mathrm{np}}$ be the functor associating to a scheme $S \in \mathrm{Nilp}_W$ the set of pairs $(X, \rho)$ where $X$ is a $p$-divisible group over $S$ and $\rho$ a quasi-isogeny $\mathbb{X}_{\overline{S}} \to X_{\overline{S}}$. Two such pairs $(X_1, \rho_1)$ and $(X_2, \rho_2)$ are identified in this set if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$ over $S$. This functor is representable by a formal scheme which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RaZ], Theorem 2.16). Let $\mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}$ be its reduced subscheme. We always include $\mathbb{X}$ in this notation, because we compare $\mathcal{M}_{\mathrm{red}}$ to the two moduli spaces $\mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}$ and $\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}}$.

Let $J^{\mathrm{np}} = \{g \in GL(N) \mid g \circ F = F \circ g\}$. Then $J \subseteq J^{\mathrm{np}}$.

If $\mathbb{X}$ is a principally polarized $p$-divisible group, then forgetting the polarization induces a natural inclusion as a closed subscheme

$$\mathcal{M}_{\mathrm{red}} \hookrightarrow \mathcal{M}_{\mathbb{X},\mathrm{red}}^{\mathrm{np}}.$$

Furthermore, there is a natural inclusion as a closed subscheme

$$\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}} \hookrightarrow \mathcal{M}_{\mathrm{red}} \tag{2.5}$$

mapping an $S$-valued point $(X_0, \rho_0)$ to $(X_0 \times X_0^\vee, (\rho, \rho^\vee))$ if the number of supersingular summands of $N$ is even and to $(X_0 \times X_{\frac{1}{2}} \times X_0^\vee, (\rho, \rho_{\frac{1}{2}}, \rho^\vee))$ otherwise. Here $X_{\frac{1}{2}} = \mathbb{X}_{\frac{1}{2},S}$ is the base-change of the unique one-dimensional supersingular $p$-divisible group over $k$ and $\rho_{\frac{1}{2}} = \mathrm{id}$.

Let $\tilde{v}$ be the valuation on the Dieudonné ring $\mathcal{D}$ determined by

$$\tilde{v}(aF^iV^j) = 2v_p(a) + i + j \tag{2.6}$$

for every $a \in W(k)$.

LEMMA 2.5. *One can decompose each $B \in \mathcal{D}$ uniquely as $B = \mathrm{LT}(B) + B'$ with $\tilde{v}(B') > \tilde{v}(B)$ and*

$$\mathrm{LT}(B) = \sum_{0 \le i \le \tilde{v}(B), 2\alpha + i = \tilde{v}(B)} p^\alpha([a_i]V^i + [b_i]F^i).$$

*Here $[a_i]$ and $[b_i]$ are Teichmüller representatives of elements of $k^\times$ or 0 and $[b_0] = 0$.*

*Proof.* The $V^i$ with $i \ge 0$ and the $F^i$ with $i > 0$ together form a basis of the $W(k)$-module $\mathcal{D}$. Besides, as $k$ is perfect, every element of $W(k)$ can be written in a unique way as $x = \sum_{\alpha \ge 0} p^\alpha [x_\alpha]$. Hence we can write $B = \sum_{i \ge 0} x_i V^i + \sum_{i > 0} y_i F^i = \sum_{i \ge 0} \sum_{\alpha \ge 0} p^\alpha [x_{i,\alpha}] V^i + \sum_{i>0} \sum_{\alpha \ge 0} p^\alpha [y_{i,\alpha}] F^i$ where $x_i, y_i$ are 0 for all but finitely many $i$. By the definition of $\tilde{v}(B)$, all $x_{i,\alpha}, y_{i,\alpha}$ with $i + 2\alpha < \tilde{v}(B)$ vanish. Let $\mathrm{LT}(B)$ be the sum of all terms $p^\alpha[x_{i,\alpha}]V^i$ and $p^\alpha [y_{i,\alpha}] F^i$ on which $\tilde{v}$ takes the value $\tilde{v}(B)$, i.e. those with $2\alpha + i = \tilde{v}(B)$. Then $\mathrm{LT}(B)$ is as in the lemma and $\tilde{v}(B - \mathrm{LT}(B)) > \tilde{v}(B)$. □

LEMMA 2.6. *Let $(N_0, b_0\sigma)$ be the rational Dieudonné module of some $p$-divisible group over $k$. Let $m = v_p(\det b_0)$ and $n = \dim_L(N_0) - m$. Let $v \in N_0$ be not contained in any proper sub-isocrystal of $N_0$.*

1. *$\mathrm{Ann}(v) = \{\varphi \in \mathcal{D} \mid \varphi(v) = 0\}$ is a principal left ideal of $\mathcal{D}$. There is a generating element of the form*

$$A = aF^n + bV^m + \sum_{i=0}^{n-1} a_i F^i + \sum_{i=1}^{m-1} b_i V^i.$$

   *with $a, b \in W^\times$ and $a_i, b_i \in W$.*

2. *If $N_0$ is simple (and thus of slope $m/(m+n)$), we have*

$$\mathrm{LT}(A) = \begin{cases} [a]F^n & \text{if } n < m \\ [b]V^m & \text{if } m < n \\ [a]F + [b]V & \text{if } m = n = 1 \end{cases} \tag{2.7}$$

   *for some $a, b \in k^\times$.*

3. *Let $N_0 = \oplus_j N^j$ be a decomposition of $N_0$ into simple summands. Then $\mathrm{LT}(A) = \mathrm{LT}(\prod_j L_j)$. Here each $L_j$ is of the form (2.7) associated to some nonzero element in $N^j$.*

*Proof.* We use induction on the number of summands in a decomposition of $N_0$ into simple isocrystals. If $N_0$ is simple, the lemma follows immediately from [V1], Lemma 4.12. For the induction step write $N_0 = N' \oplus N''$ where $N'$ is simple. Let $A'$ be as in the lemma and associated to $N'$ and $p_{N'}(v)$ where $p_{N'} : N \to N'$ is the projection. Note that an element of an isocrystal is not contained in any proper sub-isocrystal if and only if the Dieudonné module generated by the element is a lattice. Let $\Lambda$ be the lattice generated by $v$. The Dieudonné module generated by $A'(v)$ is equal to $\Lambda \cap N''$, and hence also a lattice. We may therefore apply the induction hypothesis to $A'(v)$ and $N''$ and obtain some $A''$ generating $\mathrm{Ann}(A'(v))$. Thus $\mathrm{Ann}(v)$ is a principal left ideal generated by $A''A'$. Multiplying the corresponding expressions for $A''$ and $A'$, the lemma follows. $\square$

### 2.3 Reduction to the bi-infinitesimal case

Let $\mathbb{X} = \mathbb{X}_{\mathrm{et}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{m}}$ be the decomposition of $\mathbb{X}$ into its étale, bi-infinitesimal, and multiplicative parts.

Lemma 2.7. *We have*

$$\mathcal{M}_{\mathbb{X},\mathrm{red}} \cong \begin{cases} \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \times \mathcal{M}_{\mathbb{X}_{\mathrm{bi}},\mathrm{red}} & \textit{if } \mathbb{X}_{\mathrm{bi}} \textit{ is nontrivial} \\ \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \times \mathbb{Z} & \textit{otherwise.} \end{cases}$$

*and*

$$\mathcal{M}^{\mathrm{np}}_{\mathbb{X}_{\mathrm{et}},\mathrm{red}} \cong GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Z}_p).$$

*Proof.* Consider the following morphism $\iota$ from the right to the left hand side of the first isomorphism. In the first case, an $S$-valued point $((X_{\mathrm{et}}, \rho_{\mathrm{et}}), (X_{\mathrm{bi}}, \rho_{\mathrm{bi}}))$ is mapped to $(X_{\mathrm{et}} \times X_{\mathrm{bi}} \times X_{\mathrm{m}}, (\rho_{\mathrm{et}}, \rho_{\mathrm{bi}}, \rho_{\mathrm{m}}))$ where $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$. Furthermore, $\rho_{\mathrm{m}}$ is the dual isogeny of $c \cdot \rho_{\mathrm{et}}$ and $c$ is the scalar determined by the duality condition for $\rho_{\mathrm{bi}}$. In the second case $((X_{\mathrm{et}}, \rho_{\mathrm{et}}), l)$ is mapped to $(X_{\mathrm{et}} \times X_{\mathrm{m}}, (\rho_{\mathrm{et}}, \rho_{\mathrm{m}}))$ with $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$ and $\rho_{\mathrm{m}} = (p^l \cdot \rho_{\mathrm{et}})^\vee$. In both cases $\iota$ is a monomorphism, and to check that it is a closed immersion we verify the valuation criterion for properness. Let $(X, \rho)$ be a $k[[t]]$-valued point of $\mathcal{M}_{\mathbb{X},\mathrm{red}}$ such that the generic point is in the image of $\iota$. Let $\pi_X : X \twoheadrightarrow X_{\mathrm{et}}$ with $X_{\mathrm{et}}$ étale over $\mathrm{Spec}(k[[t]])$ and $X$ inf initesimal over $X_{\mathrm{et}}$, as in [M], Lemma II.4.8. Our assumption implies that this map has a right inverse $X_{\mathrm{et},k((t))} \to X_{k((t))}$ after base change to $\mathrm{Spec}(k((t)))$. By [J1], Corollary 1.2, this morphism lifts to a morphism $X_{\mathrm{et}} \to X$ over $k[[t]]$. Together with the inclusion of the kernel of $\pi_X$ in $X$ we obtain a morphism of the product of an étale and an infinitesimal $p$-divisible group over $\mathrm{Spec}(k[[t]])$ to $X$. Its inverse is constructed similarly by lifting the projection morphism of $X_{k((t))}$ to the kernel of $\pi_X$ from $k((t))$ to $k[[t]]$. Hence $X$ can be written as a product of an étale and an infinitesimal $p$-divisible group. As $X$ is selfdual, it is then also the product of an étale, a bi-infinitesimal, and a multiplicative $p$-divisible group, thus of the form $X_{\mathrm{et}} \times X_{\mathrm{bi}} \times X_{\mathrm{m}}$ with $X_{\mathrm{m}} = \hat{X}_{\mathrm{et}}$. The quasi-isogeny $\rho$ is compatible with this decompos ition, and the compatibility with the polarizations shows that the induced quasi-isogenies $(\rho_{\mathrm{et}}, \rho_{\mathrm{bi}}, \rho_{\mathrm{m}})$ have the property that $\rho_{\mathrm{bi}}$ is selfdual up to some scalar $c$ and $\rho_{\mathrm{m}}$ is the dual isogeny of $c \cdot \rho_{\mathrm{et}}$. Hence $(X, \rho)$ is in the image of $\iota$. This finishes the proof that $\iota$ is proper, hence a closed immersion.

To show that $\iota$ is an isomorphism it is thus enough to show that each $k$-valued point of the left hand side is contained in its image. From the Hodge-Newton decomposition (see [Kat], Thm. 1.6.1) we obtain for each $k$-valued point $(X,\rho)$ a decomposition $X = X_{\mathrm{et}} \times X_{\mathrm{bi}} \times X_{\mathrm{m}}$ and $\rho = \rho_{\mathrm{et}} \times \rho_{\mathrm{bi}} \times \rho_{\mathrm{m}}$ into the étale, bi-infinitesimal, and multiplicative parts. The compatibility with the polarization then yields that up to some scalar $p^l$, the quasi- isogenies $\rho_{\mathrm{m}}$ and $\rho_{\mathrm{et}}$ are dual. From this the first isomorphism follows. The second isomorphism is shown by an easy calculation (compare [V1], Section 3). $\square$

The lemma reduces the questions after the global structure of $\mathcal{M}_{\mathrm{red}}$ to the same questions for $\mathcal{M}_{\mathbb{X}_{\mathrm{bi}},\mathrm{red}}$. Thus from now on we assume that $\mathbb{X}$ is bi-infinitesimal.

## 3 The dense subscheme $\mathcal{S}_1$

In [V1], 4.2 we define an open dense subscheme $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ of $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$. Let $\Lambda \subset N$ be the lattice associated to $x \in \mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}(k)$. Then $x \in \mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ if and only if $a(\Lambda) = \dim_k(\Lambda/(F\Lambda + V\Lambda)) = 1$. As $F$ and $V$ are topologically nilpotent on $\Lambda$, this is equivalent to the existence of some $v \in \Lambda$ such that $\Lambda$ is the Dieudonné submodule of $N$ generated by $v$. Note that $a(\Lambda)$ can also be defined as $\dim_k(\mathrm{Hom}(\alpha_p, X))$ where $X$ is the $p$-divisible group associated to $\Lambda$.
Let
$$\mathcal{S}_1 = \mathcal{S}^{\mathrm{np}}_{\mathbb{X},1} \cap \mathcal{M}_{\mathrm{red}} \subseteq \mathcal{M}_{\mathrm{red}}.$$
Then $\mathcal{S}_1$ is open in $\mathcal{M}_{\mathrm{red}}$.

Lemma 3.1. *The open subscheme $\mathcal{S}_1$ is dense in $\mathcal{M}_{\mathrm{red}}$.*

*Proof.* Recall that we assume $\mathbb{X}$ to be bi-infinitesimal. Let $(\overline{X},\overline{\rho}) \in \mathcal{M}_{\mathrm{red}}(k)$ and let $\overline{\lambda}$ be a corresponding polarization of $\overline{X}$. Note that by [M], Lemma II.4.16 (or by [J2], Lemma 2.4.4) there is an equivalence of categories between $p$-divisible groups over an adic, locally noetherian affine formal scheme $\mathrm{Spf}(A)$ and over $\mathrm{Spec}(A)$. From [O1], Corollary 3.11 we obtain a deformation $(X,\lambda)$ of $(\overline{X},\overline{\lambda})$ over $\mathrm{Spec}(k[[t]])$ such that the generic fiber satisfies $a = 1$. Next we show that after a base change we may also deform $\overline{\rho}$ to a quasi-isogeny $\rho$ between $(X,\lambda)$ and the constant $p$-divisible group $(\mathbb{X},\lambda_{\mathbb{X}})$ that is compatible with the polarizations. From [OZ], Corollary 3.2 we obtain a deformation of $\overline{\rho}$ to a quasi-isogeny between $X$ and a constant $p$-divisible group $Y$ after a base-change to the perfect hull of $k[[t]]$. As $Y$ is constant it is quasi-isogenous to the base change $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\mathrm{perf}})}$ of $\mathbb{X}$. After composing the deformation of $\overline{\rho}$ with a quasi-isogeny between $Y$ and $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\mathrm{perf}})}$ we may assume that $Y$ is already equal to $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\mathrm{perf}})}$. Let $x$ be the point of $\mathrm{Spec}(k[[t]]^{\mathrm{perf}})$ over the generic point of $\mathrm{Spec}(k[[t]])$. Then we may further compose the quasi-isogeny with a self-quasi-isogeny of $\mathbb{X}_{\mathrm{Spec}(k[[t]]^{\mathrm{perf}})}$ such that in $x$ it is compatible with the polarizations of the two groups in this point. Thus we obtain a $k[[t]]^{\mathrm{perf}}$-valued point of $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ such that the image of $x$ is in $\mathcal{M}_{\mathrm{red}}$. As $\mathcal{M}_{\mathrm{red}}$ is closed, this has to be a $k[[t]]^{\mathrm{perf}}$-valued point of $\mathcal{M}_{\mathrm{red}}$. Modifying the point by a suitable elemen t of $J$, we may assume that the special point is mapped to $(\overline{X}, \overline{\rho})$. In the generic point, the $a$-invariant of the $p$-divisible group $X$ is 1. Thus this provides the desired deformation of $(\overline{X}, \overline{\rho})$ to a point of $\mathcal{S}_1$. □

To determine the dimension and the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ it is thus sufficient to consider $\mathcal{S}_1$. We proceed in the same way as for the moduli spaces of $p$-divisible groups without polarization. In contrast to the non-polarized case it turns out to be useful to use two slightly different systems of coordinates to prove the assertions on the dimension and on the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$.

Let us briefly recall the main steps for the moduli spaces $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ of non-polarized $p$-divisible groups. Their sets of irreducible components and their dimension are determined by studying $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$. In [V1], 4 it is shown that the connected components of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ are irreducible and that $J^{\mathrm{np}} = \{j \in GL(N) | j \circ F = F \circ j\}$ acts transitively on them. The first step to prove this is to give a description of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}(k)$. One uses that each such point corresponds to a lattice in $N$ with $a$-invariant 1. As Dieudonné modules, these lattices are generated by a single element and the description of the set of points is given by classifying these elements generating the lattices. The second step consists in the construction of a family in $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ to show that a set of points which seems to parametrize an irreducible compo nent of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X},1}$ indeed comes from an irreducible subscheme. More precisely, a slight reformulation of the results in [V1], Section 4 yields the following proposition.

Proposition 3.2. *Let $(N, F)$ be the isocrystal of a $p$-divisible group $\mathbb{X}$ over $k$. Let $m = v_p(\det F)$. Let $S = \mathrm{Spec}(R) \in \mathrm{Nilp}_W$ be a reduced affine scheme with $pR = 0$ and let $j \in J^{\mathrm{np}}$. Let $v \in N_R = N \otimes_L W(R)[\frac{1}{p}]$ such that in every $x \in S(k)$, the reduction $v_x$ of $v$ in $x$ satisfies that*

$$v_x \in j\Lambda_{\min}$$

*and*

$$v_p(\det j) = \max\{v_p(\det j') \mid j' \in J^{\mathrm{np}} \text{ and } v_x \in j'\Lambda_{\min}\}.$$

*Here, $\Lambda_{\min} \subset N$ is the lattice of the minimal $p$-divisible group in Remark 2.4. Let $\tilde{R} = \sigma^{-m}(R)$ be the unique reduced extension of $R$ such that $\sigma^m : \tilde{R} \to \tilde{R}$ has image $R$. Let $\tilde{v} \in N_{\tilde{R}}$ with $\sigma^m(\tilde{v}) = v$. Then there is a morphism $\varphi : \mathrm{Spec}(\tilde{R}) \to \mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$ such that for every $x \in \mathrm{Spec}(\tilde{R})(k)$, the image $\varphi(x)$ corresponds to the Dieudonné module $\Lambda_x$ in $N$ generated by $v_x$.*
*Assume in addition that $\mathbb{X}$ is principally polarized and that for every $x$, the Dieudonné module $\Lambda_x$ corresponds to a point of $\mathcal{M}_{\mathrm{red}}$. Then $\varphi$ factors through $\mathcal{M}_{\mathrm{red}}$.*

Note that the second condition on $v_x$ (or more precisely the existence of the maximum) implies that the Dieudonné submodule of $N$ generated by $v_x$ is a lattice.

*Proof.* To prove the first assertion we may assume that $j = \mathrm{id}$. Note that $\tilde{v}$ satisfies the same conditions as $v$. The conditions on the $\tilde{v}_x$ are reformulated in [V1], Lemma 4.8. The condition given there is exactly the condition needed in [V1], Section 4.4 to construct a display over $S$ leading to the claimed morphism $\varphi$. It maps $x$ to the Dieudonné lattice generated by $\sigma^m(\tilde{v}_x) = v_x$. The second assertion is trivial as $S$ is reduced and $\mathcal{M}_{\mathrm{red}}$ a closed subscheme of $\mathcal{M}^{\mathrm{np}}_{\mathbb{X},\mathrm{red}}$. $\square$

*Remark* 3.3. We use the same notation as in the proposition. From [V1], 4.4 we also obtain that under the conditions of Proposition 3.2, the elements

$$v, Vv, \dots, V^{v_p(\det F)}v, Fv, \dots, F^{\dim N - v_p(\det F)-1}v$$

are a basis of the free $W(\tilde{R})[\frac{1}{p}]$-module $N_{\tilde{R}}$. They are the images of the standard basis of $N$ under some element of $GL(N_{\tilde{R}})$.
We apply the preceding to the situation of an isocrystal $N = N_0$ and its dual, $N_1$. Then $GL(N_0) \times \mathbb{G}_{\mathrm{m}}$ is isomorphic to the Siegel Levi subgroup of $GSp(N_0 \oplus N_1)$. Let $v \in N_0 = N$ be as in Proposition 3.2. Then there are elements $y_i \in (N_1)_{\tilde{R}}$ which form a basis of $(N_1)_{\tilde{R}}$ which is dual to the basis

$$(x_1, \dots, x_{\dim N_0}) = (v, Vv, \dots, V^{v_p(\det F)}v, Fv, \dots, F^{\dim N_0 - v_p(\det F)-1}v)$$

with respect to $\langle \cdot,\cdot \rangle$. In other words, the $y_i \in (N_1)_{\tilde{R}}$ are such that $\langle x_i, y_j \rangle = \delta_{ij}$.

# 4 GEOMETRIC POINTS OF $\mathcal{S}_1$

## 4.1 $N$ WITH AN EVEN NUMBER OF SUPERSINGULAR SUMMANDS

In this subsection we consider the case that $N$ has an even number of supersingular summands. By (2.2) we have a decomposition $N = N_0 \oplus N_1$.
Recall that by a lattice we always mean a Dieudonné lattice. Let $\Lambda \subset N$ be the lattice corresponding to a $k$-valued point of $\mathcal{M}_{\mathrm{red}}$. Then $\Lambda^\vee = c\Lambda$ for some $c \in L^\times$. Let $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$. For a subset $M$ of $N$ and $\delta \in \{0,1\}$ let

$$(M)^\vee_\delta = \{x \in N_\delta \mid \langle x, x' \rangle \in W \text{ for all } x' \in M\}. \tag{4.1}$$

Then $c\Lambda_1 = (\Lambda_0)^\vee_1$. Hence $\Lambda_0$ and $\Lambda_1$ correspond to dual $p$-divisible groups, which implies $a(\Lambda_0) = a(\Lambda_1)$.
The geometric points of $\mathcal{S}_1$ correspond to lattices $\Lambda$ that in addition satisfy $a(\Lambda) = 1$. Especially, $a(\Lambda_0) = a(\Lambda_1) = 1$. In this subsection we classify a slightly larger class of lattices. We fix a lattice $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$ and $c \in L^\times$. Then we consider all lattices $\Lambda \subset N$ with

$$p_0(\Lambda) = \Lambda_0 \text{ and } \Lambda^\vee = c\Lambda. \tag{4.2}$$

Note that we have a description of the set of lattices $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$ from [V1], see also Section 2.2.
The considerations above show that $\Lambda \cap N_1 = \Lambda_1 = c^{-1}(\Lambda_0)^\vee_1$ is determined by $\Lambda_0$ and $c$. Let $v_0$ be an element generating $\Lambda_0$ as a Dieudonné module. If

$v \in \Lambda$ with $p_0(v) = v_0$, then $\Lambda$ is generated by $v$ and $\Lambda_1$. Let $A$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. We write $v = v_0 + v_1$ for some $v_1 \in N_1$. Then $Av = Av_1 \in \Lambda_1$.

*Remark* 4.1. Let $\Lambda_0$ and $c$ be as above, and let $\Lambda_1 = c^{-1}(\Lambda_0)_1^\vee$. Let $\Lambda$ be a Dieudonné lattice with $p_0(\Lambda) = \Lambda_0$, $\Lambda \cap N_1 \supseteq \Lambda_1$ and

$$\Lambda^\vee \supseteq c\Lambda. \tag{4.3}$$

Let $\mathrm{vol}(\cdot)$ denote the volume of a lattice, normalized in such a way that the lattice corresponding to the basepoint $(\mathbb{X}, \mathrm{id})$ of $\mathcal{M}_{\mathrm{red}}$ has volume 0. The conditions imply above that $\mathrm{vol}(\Lambda^\vee) \leq \mathrm{vol}(c\Lambda) \leq \mathrm{vol}(c(\Lambda_0 \oplus \Lambda_1)) = \mathrm{vol}((\Lambda_0 \oplus \Lambda_1)^\vee)$. Dualizing the inequality for the first and last term, we see that all terms must be equal. Thus $\Lambda$ satisfies (4.2) and $\Lambda \cap N_1 = \Lambda_1$.

The next step in the description of lattices with (4.2) is to reformulate (4.3). To do so, we fix a generator $v_0$ of $\Lambda_0$ and describe the set of all $v_1 \in N_1$ such that the lattice $\Lambda$ generated by $v = v_0 + v_1$ and $\Lambda_1$ as a Dieudonné lattice satisfies (4.3) and $\Lambda \cap N_1 = \Lambda_1$. Generators for $\Lambda$ as a $W$-module are given by $\Lambda_1$, and all $F^i v$ with $i \geq 0$ and $V^i v$ with $i > 0$. Let $m, n$ be as in Lemma 2.6 (associated to the given $N_0$). Note that as $N_0$ contains all simple summands of $N$ with slope $< 1/2$ and half of the supersingular summands, $m$ is the same as in (1.1) and $n = h - m \geq m$. By Lemma 2.6, 1. applied to $v_0 \in N_0$, the $F^i v$ with $i > n$, and the $V^i v$ with $i \geq m$ can be written as a linear combination of the $F^i v$ with $i \leq n$ and the $V^i v$ with $i < m$, and a summand in $\mathrm{Ann}(v_0) \cdot v \subset \Lambda \cap N_1 = \Lambda_1$. Hence $\Lambda$ is already generated by $\Lambda_1$, the $F^i v$ with $0 \leq i \leq n$ and the $V^i v$ with $0 < i < m$. The inclusion (4.3) is equivalent to $\langle x, y \rangle \in c^{-1} W$ for all $x, y \in \Lambda$. This is equivalent to the same condition for pairs $(x, y)$ where $x$ and $y$ are among the generators of $\Lambda$ described above. From the definition of $\Lambda_1$ we see that the values on pairs of elements of $\Lambda$ automatically satisfy this if one of the elements is in $\Lambda_1$. By (2.1) it is enough to consider the products of $v$ with all other generators. Thus (4.3) is equivalent to

$$\langle v, F^i v \rangle \in c^{-1} W$$

and

$$\langle v, V^i v \rangle \in c^{-1} W \tag{4.4}$$

for $n \geq i > 0$. Furthermore, the equations for $V^i$ together with (2.1) imply those for $F^i$.

If $x$ and $y$ are elements of the same of the summands $N_0$ or $N_1$, then $\langle x, y \rangle = 0$. Hence the decomposition of $v$ together with (2.1) shows that (4.4) is equivalent to

$$\langle v_0, V^i v_1 \rangle - \langle V^i v_0, v_1 \rangle = \langle F^i v_0, v_1 \rangle^{\sigma^{-i}} - \langle V^i v_0, v_1 \rangle \in c^{-1} W. \tag{4.5}$$

For $\phi \in \mathcal{D}$ let

$$\xi_{v_1}(\phi) = \langle \phi v_0, v_1 \rangle. \tag{4.6}$$

Then $\xi_{v_1}$ is left-$W$-linear in $\phi$. Let $A$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then

$$\xi_{v_1}(\psi A) = 0 \tag{4.7}$$

for all $\psi \in \mathcal{D}$. Note that an element $v_1 \in N_1$ is uniquely determined by $\langle v_1, F^i v_0\rangle$ for $i \in \{0, \dots, n-1\}$ and $\langle v_1, V^i v_0\rangle$ for $i \in \{1, \dots, m\}$. We are looking for the set of $v_1$ satisfying (4.5). In terms of $\xi_{v_1}$, this is

$$\xi_{v_1}(F^i)^{\sigma^{-i}} - \xi_{v_1}(V^i) \in c^{-1}W. \tag{4.8}$$

Lemma 4.2. *1. Let $M$ be the set of $W$-linear functions $\xi : \mathcal{D} \to L$ with (4.7) and (4.8) for $i \leq n$. Then (4.6) defines a bijection between $M$ and the set of elements $v_1 \in N_1$ as above.*

*2. Let $\overline{M}$ be the set of functions $\xi : \mathcal{D} \to L/c^{-1}W$ with the same properties as in 1. Then (4.6) defines a bijection between $\overline{M}$ and the set of equivalence classes of elements $v_1$ as above. Here two such elements are called equivalent if their difference is in $c^{-1}(\Lambda_0)_1^{\vee}$.*

*Proof.* Let $\xi : \mathcal{D} \to L$ be given. An element $v_1$ of $N_1$ is uniquely determined by the value of $\langle \cdot, v_1\rangle$ on $v_0, Fv_0, \dots, F^{h-m-1}v_0$, $Vv_0, \dots, V^m v_0$. These $h$ values may be chosen arbitrarily. For the values of $\langle \cdot, v_1\rangle$ on the other elements of $\mathcal{D}v_0$, a complete set of relations is given by $\langle \psi A v_0, v_1\rangle = 0$ for all $\psi \in \mathcal{D}$. This is equivalent to (4.7). Furthermore, (4.8) is equivalent to the condition that the lattice generated by $\Lambda_1$ and $v_0 + v_1$ satisfies all required duality properties. To prove 2., we want to lift $\xi : \mathcal{D} \to L/c^{-1}W$ to a function with values in $L$. We lift the values of $\xi$ at $\phi \in \{V^m, V^{m-1}, \dots, 1, \dots, F^{h-m-1}\}$ arbitrarily. Then the lifts of the remaining values are uniquely determined by (4.7). As (4.8) was satisfied before, it still holds (as a relation modulo $c^{-1}W$) for the lifted functions. Then 1. implies the existence of $v_1$. Let now $w_1$ be a second element inducing $\xi \pmod{c^{-1}W}$. Then $\langle \phi v_0, w_1 - v_1\rangle \in c^{-1}W$ for all $\phi \in \mathcal{D}$. Hence $w_1 - v_1 \in c^{-1}(\Lambda_0)_1^{\vee}$. □

### 4.2 $N$ with an odd number of supersingular summands

As parts of this case are similar to the previous one, we mainly describe the differences. By (2.2) we have a decomposition $N = N_0 \oplus N_{\frac{1}{2}} \oplus N_1$.
We want to classify the lattices $\Lambda \subset N$ corresponding to $k$-valued points of $\mathcal{S}_1$. As before let $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$. Let $c \in L^{\times}$ with $\Lambda^{\vee} = c\Lambda$. Then $c\Lambda_1 = (\Lambda_0)_1^{\vee}$. Besides,

$$c\Lambda \cap N_{\frac{1}{2}} = (p_{\frac{1}{2}}(\Lambda))_{\frac{1}{2}}^{\vee}. \tag{4.9}$$

Here we use $(\cdot)_{\frac{1}{2}}^{\vee}$ analogously to (4.1).
Again we use the description of the Dieudonné lattices $\Lambda_0 \subset N_0$ with $a(\Lambda_0) = 1$. We have to classify the $\Lambda$ corresponding to some fixed $\Lambda_0$ and $c$, and begin by describing and normalizing the possible images under the projection to $N_0 \oplus N_{\frac{1}{2}}$. Let $v \in \Lambda$ with $\mathcal{D}v = \Lambda$ and write $v = v_0 + v_{\frac{1}{2}} + v_1$ with $v_i \in N_i$. Let $A$ with $\tilde{v}(A) = m$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6.

Proposition 4.3. *1. There is a $j\in J$ such that $j(v)$ is of the form $v_0+\tilde v_{\frac12}+\tilde v_1$ with $\tilde v_i\in N_i$ and $A\mathcal{D}\tilde v_{\frac12}=\mathcal{D}A\tilde v_{\frac12}$.*

*2. Let $j$ be as in the previous statement. Then $p_{\frac12}(j\Lambda)$ is the unique Dieudonné lattice in $N_{\frac12}$ with $p_{\frac12}(j\Lambda)^\vee=(cp^m)p_{\frac12}(j\Lambda)$. Besides, $(j\Lambda)\cap N_{\frac12}=p^m p_{\frac12}(j\Lambda)$.*

*Proof.* To prove 1. let $\tilde v_{\frac12}\in N_{\frac12}$ be such that $v'_{\frac12}=v_{\frac12}-\tilde v_{\frac12}$ is in the kernel of $A$ and $A\mathcal{D}\tilde v_{\frac12}=\mathcal{D}A\tilde v_{\frac12}$.
We first reduce the assertion of 1. to the case where $N_0$ and $N_1$ are simple of slope $\frac12$. Let $A_{\frac12}$ be a generator of $\mathrm{Ann}(v'_{\frac12})$ as in Lemma 2.6. As $A\in\mathrm{Ann}(v'_{\frac12})$, we can write $A=\tilde AA_{\frac12}$ with $\tilde A\in\mathcal{D}$. Then $\tilde A$ generates $\mathrm{Ann}(A_{\frac12}v_0)$. From the description of annihilators of elements of $N_0$ in Lemma 2.6 we see that we may write $v_0=v'_0+\tilde v_0$ with $A_{\frac12}v'_0=0$ and $\tilde v_0$ lying in a proper subisocrystal $\tilde N_0$ of $N_0$. Then $v'_0$ generates a simple subisocrystal $N'_0$ of $N_0$ of slope $\frac12$ and $N_0=N'_0\oplus\tilde N_0$. Let $N'_1$ be the subisocrystal of $N_1$ which is dual to $N'_0$. Then we want to show that the assertion of the proposition holds for some $j\in J\cap\mathrm{End}(N'_0\oplus N'_{\frac12}\oplus N'_1)$. To simplify the notation, we may assume that $N$ only consists of these three summa nds.
We construct the inverse of the claimed element $j\in J$. Let $\tilde j\in\{g\in GL(N_0\oplus N_1)\mid g\circ F=F\circ g\}$ be in the unipotent radical of the parabolic subgroup stabilizing the subspace $N_1$. We assume that $\tilde j\notin J$, i. e. that $\tilde j$ is not compatible with the pairing. Let $v_0+v_1$ with $v_1\in N_1$ be the image of $v_0$. Then $\mathrm{Ann}(v_1)=\mathrm{Ann}(v_0)$ and $f=\langle v_0+v_1,F(v_0+v_1)\rangle\neq 0$. Let $A$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then $A=aF+a_0+bV$ for some $a,b\in W^\times$ and $a_0\in W$. We obtain

$$0=\langle v_0+v_1,A(v_0+v_1)\rangle^\sigma=a^\sigma f^\sigma-bf.$$

This is a $\mathbb{Q}_p$-linear equation of degree $p$, thus the set of solutions is a one-dimensional $\mathbb{Q}_p$-vector space in $L$. As $A$ also generates $\mathrm{Ann}(v'_{\frac12})$, the number $\langle v'_{\frac12},F(v'_{\frac12})\rangle$ is also in this vector space. Hence there is an $\alpha\in\mathbb{Q}_p^\times$ wit h $\alpha f=\langle v'_{\frac12},F(v'_{\frac12})\rangle$. By multiplying $v_1$ by a suitable factor, we may assume that $\alpha=-1$. Note that this does not change $\mathrm{Ann}(v_1)$. This implies that

$$\langle v_0+v'_{\frac12}+v_1,F(v_0+v'_{\frac12}+v_1)\rangle=0.$$

Besides, we have $\mathrm{Ann}(v_0+v'_{\frac12}+v_1)=\mathrm{Ann}(v_0)$. The element $j^{-1}$ we are constructing will map $v_0$ to $v_0+v'_{\frac12}+v_1$. Let $\tilde N_0=\mathcal{D}(v_0+v'_{\frac12}+v_1)$. Then we can extend $j^{-1}$ uniquely to a linear map from $N_0$ to $\tilde N_0$ which is compatible with $F$. On $N_1$, we define $j^{-1}$ to be the identity. Then one easily checks that $j^{-1}:N_0\oplus N_1\to\tilde N_0\oplus N_1$ respects the pairing. It remains to define $j^{-1}$ on $N_{\frac12}$. Let $\tilde N_{\frac12}$ be the orthogonal complement of $\tilde N_0\oplus N_1$. Then $\tilde N_{\frac12}\subseteq N_{\frac12}\oplus N_1$

Let $j^{-1}(v'_{\frac{1}{2}}) = u$ where $u \in \tilde{N}_{\frac{1}{2}}$ is such that $p_{N_{\frac{1}{2}}}(u) = v'_{\frac{1}{2}}$. Then $\mathrm{Ann}(u) \subseteq \mathrm{Ann}(v'_{\frac{1}{2}})$. As $u$ is contained in a simple isocrystal of slope $\frac{1}{2}$, this inclusion has to be an equality. As $\langle N_{\frac{1}{2}} \oplus N_1, N_1\rangle = 0$, we have $\langle u, Fu\rangle = \langle v'_{\frac{1}{2}}, Fv'_{\frac{1}{2}}\rangle$. Hence we can extend $j^{-1}$ to an element of $J$. Then $p_{\frac{1}{2}}(j(v_0+v_{\frac{1}{2}}+v_1)) = v_{\frac{1}{2}} - v'_{\frac{1}{2}} = \tilde{v}_{\frac{1}{2}}$. Thus $j$ satisfies all properties of 1.

It remains to prove 2. We may assume that $j = 1$. Note that there is exactly one Dieudonné lattice of each volume in $N_{\frac{1}{2}}$. Equivalently, for each $\alpha \in L^\times$ there is exactly one $\Lambda \subset N_{\frac{1}{2}}$ with $\Lambda^\vee = \alpha\Lambda$. (For the rest of the proof all dual lattices are the dual lattices inside the selfdual isocrystal $N_{\frac{1}{2}}$.) We have $\Lambda \cap N_{\frac{1}{2}} = c^{-1}(p_{\frac{1}{2}}(\Lambda))^\vee \subseteq p_{\frac{1}{2}}(\Lambda)$. Let $\Lambda_{\frac{1}{2}}$ be the lattice with $c^{-1}(\Lambda_{\frac{1}{2}})^\vee = \Lambda_{\frac{1}{2}}$. Then

$$\Lambda \cap N_{\frac{1}{2}} \subseteq \Lambda_{\frac{1}{2}} \subseteq p_{\frac{1}{2}}(\Lambda) \tag{4.10}$$

and the lengths of the two inclusions are equal. We have to show that the length of the inclusions are both equal to $m$. The lattice $p_{\frac{1}{2}}(\Lambda)$ also contains $A(p_{\frac{1}{2}}(\Lambda))$. As $\tilde{v}(A) = m$, the length of this inclusion is $m$. Furthermore,

$$\Lambda \cap N_{\frac{1}{2}} = \mathrm{Ann}(Av_1)Av = \mathrm{Ann}(Av_1)Av_{\frac{1}{2}} \subseteq \mathcal{D}Av_{\frac{1}{2}} = A\mathcal{D}v_{\frac{1}{2}} = Ap_{\frac{1}{2}}(\Lambda).$$

Note that here we only know that the length of the inclusion is $\geq m = \tilde{v}(A_1)$ where $A_1$ is a generator of $\mathrm{Ann}(Av_1)$. Thus we obtain a second chain of inclusions

$$\Lambda \cap N_{\frac{1}{2}} \subseteq Ap_{\frac{1}{2}}(\Lambda) \subseteq p_{\frac{1}{2}}(\Lambda).$$

We compare this to (4.10). To show that the length of the first inclusion of this chain is not greater than the length of the second inclusion, we have to show that $Ap_{\frac{1}{2}}(\Lambda) \subseteq \Lambda_{\frac{1}{2}}$. By definition of $\Lambda_{\frac{1}{2}}$ this is equivalent to $Ap_{\frac{1}{2}}(\Lambda) \subseteq c^{-1}(Ap_{\frac{1}{2}}(\Lambda))^\vee$. To prove this last inclusion we use again the duality relation for $\Lambda$. Note that $Ap_{\frac{1}{2}}(\Lambda) = \mathcal{D}Av_{\frac{1}{2}} = p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1))$. Let $x, y \in N_{\frac{1}{2}} \oplus N_1$. Then $\langle x, y\rangle = \langle p_{\frac{1}{2}}(x), p_{\frac{1}{2}}(y)\rangle$. Thus the duality relation for $\Lambda$ implies that $Ap_{\frac{1}{2}}(\Lambda) \subseteq c^{-1}(Ap_{\frac{1}{2}}(\Lambda))^\vee$. □

For both Theorem 2 and Theorem 3 it is enough to describe a locally closed subset of $\mathcal{S}_1$ whose image under the action of $J$ is all of $\mathcal{S}_1$. Thus we may assume that $j = 1$ and that $v$ itself already satisfies the property of the proposition. Especially, $p_{\frac{1}{2}}(\Lambda)$ is then determined by $c$.

The element $v_{\frac{1}{2}}$ may be modified by arbitrary elements in $p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1))$ without changing $\Lambda$. Indeed, for each such element there is an element in $\Lambda$ whose projection to $N_0 \oplus N_{\frac{1}{2}}$ is the given element. Thus for fixed $v_0$, the projection of $\Lambda$ to $N_0 \oplus N_{\frac{1}{2}}$ is described by the element $v_{\frac{1}{2}}$ varying in the $W$-module

$$p_{\frac{1}{2}}(\Lambda)/p_{\frac{1}{2}}(\Lambda \cap (N_{\frac{1}{2}} \oplus N_1)) = p_{\frac{1}{2}}(\Lambda)/A(p_{\frac{1}{2}}(\Lambda))$$

of length $m$ which is independent of $\Lambda$. To choose coordinates for $v_{\frac{1}{2}}$ we use that this module is isomorphic to $W/p^{\lfloor m/2\rfloor}W \oplus W/p^{\lceil m/2\rceil}W$. Under this isomorphism, the element $v_{\frac{1}{2}}$ is mapped to an element of the form

$$\sum_{i=1}^{\lfloor m/2\rfloor}[y_i]p^{i-1} \oplus \sum_{i=\lfloor m/2\rfloor+1}^{m}[y_i]p^{i-\lfloor m/2\rfloor-1}. \tag{4.11}$$

Here we use that $k$ is perfect, and $[y_i]$ is the Teichmüller representative of an element $y_i$ of $k$.

Note that $a(\Lambda)=1$ (or the condition that $j=1$) implies that $A(v_{\frac{1}{2}})$ is a generator of $\Lambda\cap N_{\frac{1}{2}}$ and not only an arbitrary element. This is an open condition on $p_{\frac{1}{2}}(\Lambda)/p_{\frac{1}{2}}(\Lambda\cap(N_{\frac{1}{2}}\oplus N_1))$. More precisely, it excludes a finite number of hyperplanes (compare [V1], Lemma 4.8).

Let now $v_{\frac{1}{2}}$ also be fixed. It remains to determine the set of possible $v_1$ such that $\Lambda=\mathcal{D}(v_0+v_{\frac{1}{2}}+v_1)$ is a lattice with $\Lambda^\vee=c\Lambda$. The same arguments as in the previous case show that $v_1$ can be chosen in an open subset of the set of $v_1$ with

$$\langle v_0,\phi v_1\rangle+\langle v_1,\phi v_0\rangle \equiv -\langle v_{\frac{1}{2}},\phi v_{\frac{1}{2}}\rangle \pmod{c^{-1}W}. \tag{4.12}$$

for all $\phi\in\mathcal{D}$.

*Remark* 4.4. Let $\phi\in\mathcal{D}$ with $\tilde{v}(\phi)=2m$. Then $\phi v_{\frac{1}{2}}\in p^m p_{\frac{1}{2}}(\Lambda)\subset c^{-1}\Lambda^\vee$. Especially, $\langle v_{\frac{1}{2}},\phi v_{\frac{1}{2}}\rangle$ is in $c^{-1}W$. This is later used in the form that $a_i=-\langle v_{\frac{1}{2}},F^i v_{\frac{1}{2}}\rangle$ satisfies (6.3).

Analogously to the previous case we use (4.6) to define $\xi_v$. Then we also obtain the analogue of Lemma 4.2.

Lemma 4.5. *1. Let $M$ be the set of $W$-linear functions $\xi:\mathcal{D}\to L$ with (4.7) and (4.12) for $i\leq n$. Then (4.6) defines a bijection between $M$ and the set of elements $v_1\in N_1$ as above.*

*2. Let $\overline{M}$ be the set of functions $\xi:\mathcal{D}\to L/c^{-1}W$ with the same properties as in 1. Then (4.6) defines a bijection between $\overline{M}$ and the set of equivalence classes of elements $v_1$ as above. Here two such elements are called equivalent if their difference is in $c^{-1}(\Lambda_0)_1^\vee$.*

## 5 The set of irreducible components

Lemma 5.1. *Let $\Lambda\subset N_0\oplus N_1$ be a lattice generated by a sublattice $\Lambda_1\subset N_1$ and an element $v$ with $v=v_0+v_1$ for some $v_0\in N_0$ and $v_1\in N_1$. Let $\tilde{\Lambda}$ be generated by $\Lambda_1$ and $v_0+\tilde{v}_1$ for some $\tilde{v}_1\in N_1$. If $\xi_{\tilde{v}_1}(F^i)^{\sigma^{-i}}-\xi_{\tilde{v}_1}(V^i)=\xi_{v_1}(F^i)^{\sigma^{-i}}-\xi_{v_1}(V^i)$ for every $i\in\{1,\dots,h\}$ then there is a $j\in J$ with $j(\Lambda)=\tilde{\Lambda}$.*

*Proof.* The assumption implies that $\langle v_0+\tilde{v}_1-v_1,\varphi(v_0+\tilde{v}_1-v_1)\rangle=0$ for $\varphi\in\{1,V,\dots,V^h\}$ (see the reformulation of (4.4) in Section 4.1). By (2.1), the same holds for $\varphi \in \{F, \dots, F^h\}$. As $\dim N = 2h$, the $\varphi(v_0 + \tilde{v}_1 - v_1)$ for these elements $\varphi \in \mathcal{D}$ generate $N' = (\mathcal{D}(v_0 + \tilde{v}_1 - v_1))[1/p] \subseteq N$ as an $L$-vector space. Especially,

$$\langle v_0 + \tilde{v}_1 - v_1, \varphi(v_0 + \tilde{v}_1 - v_1)\rangle = 0 \tag{5.1}$$

for all $\varphi \in \mathcal{D}$. Let $A$ be a generator of $\mathrm{Ann}(v_0)$ as in Lemma 2.6. Then (5.1) for $\varphi = \varphi' A$ implies that $\langle v_0, \varphi' A(\tilde{v}_1 - v_1)\rangle = 0$ for all $\varphi' \in \mathcal{D}$. Thus $A(\tilde{v}_1 - v_1) = 0$. Let $j \in GL(N)$ be defined by $v_0 \mapsto v_0 + \tilde{v}_1 - v_1$, $j|_{N_1} = \mathrm{id}$, and $j \circ F = F \circ j$. To check that this is well-defined we have to verify that $Aj(v_0) = j(Av_0) = 0$. But $A(j(v_0)) = A(v_0 + \tilde{v}_1 - v_1) = 0$. By definition $j$ commutes with $F$. Furthermore, (5.1) implies that $j \in G(L)$. Hence $j \in J$. $\square$

For $v_1$ as above and $i \in \{1, \dots, h\}$ let

$$\psi_i(v_1) = \xi_{v_1}(V^i) - \xi_{v_1}(F^i)^{\sigma^{-i}}. \tag{5.2}$$

Then the lemma yields the following corollary.

Corollary 5.2. *Let $\Lambda$ and $\tilde{\Lambda}$ be two extensions of $\Lambda_0$ and $\Lambda_1$ as described in the previous section (or, in the case of an odd number of supersingular summands, two extensions of $\Lambda_0$ and $\Lambda_1$ associated to the same $v_{\frac{1}{2}}$) and let $v = v_0 + v_1$ and $\tilde{v} = v_0 + \tilde{v}_1$ (resp. $v = v_0 + v_{\frac{1}{2}} + v_1$ and $\tilde{v} = v_0 + v_{\frac{1}{2}} + \tilde{v}_1$) be the generators. Then $\psi_i(v_1) = \psi_i(\tilde{v}_1)$ for all $i$ implies that $\Lambda$ and $\tilde{\Lambda}$ are in one $J$-orbit.*

Let $v_0 \in N_0$ such that $\mathcal{D}v_0$ is a lattice in $N_0$. Then the next proposition implies that for each $(c_1, \dots, c_h) \in L^h$ there is a $v_1 \in N_1$ with $\psi_i(v_1) = c_i$ for all $i$.
We fix an irreducible component of $\mathcal{S}^{\mathrm{np}}_{\mathbb{X}_0,1}$. Then [V1], 4 describes a morphism from a complement of hyperplanes in an affine space to this irreducible component that is a bijection on $k$-valued points. Let $\mathrm{Spec}(R_0)$ be this open subscheme of the affine space. One first defines a suitable element $v_{0,R_0} \in N_0 \otimes_W W(R_0)$. The morphism is then constructed in such a way that each $k$-valued point $x$ of $\mathrm{Spec}(R_0)$ is mapped to the lattice in $N_0$ generated by the reduction of $\sigma^m(v_{0,R_0})$ at $x$.

Proposition 5.3. *Let $R$ be a reduced $k$-algebra containing $\sigma^m(R_0)$. Let $c_1, \dots, c_h \in W(R)[1/p]$. Then there is a morphism $R \to R'$ where $R'$ is a limit of étale extensions of $R$ and a $v_1 \in N_{1,R'}$ with $\psi_i(v_1) = c_i$ for all $i$. Here, the $\psi_i$ are defined with respect to the universal element $\sigma^m(v_{0,R_0}) \in (N_0)_{\sigma^m(R_0)}$.*

For the proof we need the following lemma to simplify the occurring system of equations.

Lemma 5.4. *Let $R$ be an $\mathbb{F}_p$-algebra and let $m, n \in \mathbb{N}$ with $m \leq n$. For $0 \leq i \leq m$ and $0 \leq j \leq n$ let $P_{ij}(x) \in (W(R)[\frac{1}{p}])[x]$ be a linear combination of* *the* $\sigma^l(x) = x^{p^l}$ *with* $l \geq 0$. *Assume that the coefficient of* $x$ *is zero for* $j < i$ *and in* $W(R)^\times$ *for* $i = j$. *Consider the system of equations*

$$\sum_{j=0}^{n} P_{ij}(x_j) = a_i$$

*with* $a_i \in W(R)[\frac{1}{p}]$ *and* $i = 0, \dots, m$. *It is equivalent to a system of equations of the form* $\sum_j Q_{ij}(x_j) = b_i$ *with* $b_i \in W(R)[\frac{1}{p}]$ *such that the* $Q_{ij}$ *satisfy the same conditions as the* $P_{ij}$ *and in addition* $Q_{ij} = 0$ *if* $j < i$.

*Proof.* We use a modification of the Gauss algorithm to show by induction on $\lambda$ that the system is equivalent to a system of relations of the form $\sum_j Q^\lambda_{ij}(x_j) = b^\lambda_i$ with $b^\lambda_i \in L$ such that the $Q^\lambda_{ij}$ satisfy the same conditions as the $P_{ij}$ and in addition $Q^\lambda_{ij} = 0$ if $j < i$ and $j \leq \lambda$. For the induction step we have to carry out the following set of modifications for $j = \lambda + 1$ and each $i > \lambda + 1$. If $Q^\lambda_{ij}$ vanishes, we do not make any modification. We now assume $Q^\lambda_{ij}$ to be nontrivial. Let $\sigma^{l_i}(x)$ and $\sigma^{l_j}(x)$ be the highest powers of $x$ occurring in $Q^\lambda_{ii}$ and $Q^\lambda_{ij}$. If $l_i < l_j$, we modify the $j$th equation by a suitable multiple of $\sigma^{l_j - l_i}$ applied to the $i$th equation to lower $l_j$. Else we modify the $i$th equation by a suitable multiple of $\sigma^{l_i - l_j}$ applied to the $j$th equation to lower $l_i$. We proceed in this way as long as none of the two polynomials $Q^\lambda_{ii}$ and $Q^\lambda_{ij}$ becomes trivial. Note that the defining properties of the $P_{ij}$ are preserved by these modifications. As (by induction) $Q^\lambda_{ij}$ does not have a linear term, the linear term of $Q^\lambda_{ii}$ remains unchanged. Thus this process of modifications ends after a finite number of steps with equations $\sum_j Q^{\lambda+1}_{ij}(x_j) = b^{\lambda+1}_i$ which satisfy $Q^{\lambda+1}_{ij} = 0$ for $j < i$ and $j \leq \lambda + 1$. For $\lambda + 1 = n$, this is what we wanted. □

*Proof of Proposition 5.3.* An element $v_1 \in N_{1,R'}$ is determined by the values of $\xi_{v_1}$ at any $h$ consecutive elements of $\dots, F^2, F, 1, V, V^2, \dots$. The other values of $\xi$ are then determined by $\xi_{v_1}(\phi A) = 0$ for all $\phi \in \mathcal{D}$. Here $A \in \mathrm{Ann}(v_0)$ is as in Lemma 2.6. Indeed, each of these equations for $\phi = F^i$ or $V^i$ for some $i$ gives a linear equation with coefficients in $L$ between the values of $\xi_{v_1}$ at $h+1$ consecutive elements of $\dots, F^2, F, 1, V, V^2, \dots$. For the proof of the proposition we take the values $\xi_{v_1}(F^i)$ for $i \in \{1, \dots, h\}$ as values determining $v_1$. Then all other values are linear combinations of these $\xi_{v_1}(F^i)$.
The definition of $\psi_{v_1}$ in (5.2) yields

$$\xi_{v_1}(V^i)^{\sigma^i} = \xi_{v_1}(F^i) + \psi_i(v_1)^{\sigma^i}$$

for $i \in \{1, \dots, h\}$. On the other hand, $\xi_{v_1}(V^i)^{\sigma^i}$ is a linear combination of the $\xi_{v_1}(F^j)^{\sigma^i}$ for $j \in \{1, \dots, h\}$. From this we obtain a system of $h$ equations for the $\xi_{v_1}(F^i)$ with $1 \leq i \leq h$ of the same form as in Lemma 5.4. The resulting equations $\sum_j Q_{ij}(\xi_{v_1}(F^j)) = b_i$ may be reformulated as $Q_{ii}(\xi_{v_1}(F^i)) = c_i$ where $c_i$ also contains the summands corresponding to powers of $F$ larger than $i$. We can then consider these equations by decreasing induction on $i$. For each $i$, the polynomial $Q_{ii}(x)$ is a linear combination of powers of $x$ of the form $x^{\sigma^l}$, and its linear term does not vanish. Thus there is a limit $R'$ of étale extensions of $R$ and $\xi_{v_1}(F^i) \in W(R') \otimes \mathbb{Q}$ with $v_p(\xi_{v_1}(F^i)) \geq v_p(c_i)$ satisfying these equations. Note that $R'$ is in general an infinite extension of $R$, because the equations are between elements of $W(R) \otimes \mathbb{Q}$ and not over $R$ itself. Given $\xi_{v_1}$, Remark 3.3 shows that there is an element $v_1 \in (N_1)_{R'}$ which induces $\xi_{v_1}$. Indeed, choose $v_1$ to be a suitable linear combination of the dual basis defined there. $\square$

### 5.1 Proof of Theorem 2

We begin by constructing an irreducible subscheme of the subscheme of $\mathcal{S}_1$ where the height of the universal quasi-isogeny is 0. The $k$-valued points of this subscheme correspond to lattices $\Lambda$ with $a(\Lambda) = 1$ and $\Lambda^\vee = \Lambda$. There is a $d \in \mathbb{N}$ such that for each $(c_1, \dots, c_h) \in (p^d W)^h$, the $v_1$ constructed in Proposition 5.3 lies in the lattice $\Lambda_1 \subset N_1$. Let $R_0$ as above. In the case of an even number of supersingular summands let $R_1 = \sigma^m(R_0)$. Otherwise let $\sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} \in N_{\sigma^m(R_0)[y_1,\dots,y_m]}$ where $v_{0,R_0}$ is as above and where $v_{\frac{1}{2}} \in (N_{\frac{1}{2}})_{\sigma^m(R_0)[y_1,\dots,y_m]}$ is identified with the element in (4.11). The open condition on $\mathrm{Spec}(\sigma^m(R_0)[y_1, \dots, y_m])$ that $Av_{\frac{1}{2}}$ is a generator and not only an element of $p_{\frac{1}{2}}(\Lambda \cap N_{\frac{1}{2}} \oplus N_1)$ is equivalent to $\mathcal{D}Av_{\frac{1}{2}} = A\mathcal{D}v_{\frac{1}{2}}$. This condition is satisfied by all $y_1$ that do not lie in some finite-dimensional $\mathbb{Q}_p$-subvector space of $k$ determined by the kernel of $A$ (compare the proof of Proposition 4.3 1.). In this case let $R_1$ be the extension of $\sigma^m(R_0)$ corresponding to this affine open subscheme. Let in both cases

$$R = R_1[x_{i,j} \mid i \in \{1, \dots, h\}, j \in \{0, \dots, d-1\}].$$

For $i \in \{1, \dots, h\}$ let $c_i = \sum_{j=0}^{d-1} [x_{i,j}]p^j \in W(R)$. Let $\mathrm{Spec}(R')$ and $v_1 \in N_{R'}$ be as in Proposition 5.3. Let $v = \sigma^m(v_{0,R_0}) + v_1$, resp. $v = \sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} + v_1$. Let $S = \mathrm{Spec}(R)$ be an irreducible component of the affine open subscheme of $\mathrm{Spec}(R')$ consisting of the points $x$ with $v_{1,x} \in (\mathcal{D}v_x)_1^\vee \setminus (F(\mathcal{D}v_x)_1^\vee + V(\mathcal{D}v_x)_1^\vee)$. We denote the image of $v$ in $N_R$ also by $v$. A s we already know that $\mathcal{S}_1$ is dense, this open subset is nonempty. Let $\tilde{R}$ be the inverse image of $R$ under $\sigma^h$ as in Proposition 3.2. Note that $v_p(\det F) = h$, whereas $v_p(\det F|_{N_0}) = m$. Let $\tilde{S} = \mathrm{Spec}(\tilde{R})$. The next step is to define an associated morphism $\varphi : \tilde{S} \to \mathcal{M}_{\mathrm{red}}$ such that in each $k$-valued point $x$ of $\tilde{S}$, the image in $\mathcal{M}_{\mathrm{red}}(k)$ corresponds to the lattice generated by the reduction $v_x$ of $v$ at $x$. By Proposition 3.2 it is enough to show that there is a $j \in J$ such that for each $x \in \tilde{S}(k)$, we have $v_x \in j\Lambda_{\min}$ and $v_p(\det j) = \max\{v_p(\det j') \mid v_x \in j'\Lambda_{\min}\}$. Let $\eta$ be the generic point of $\tilde{S}$ and let $j_\eta \in J$ be such a maximizing element for $\eta$. Then the same holds for each $k$-valued point in an open and thus dense subscheme of $\tilde{S}$. As the property $v_x \in j_\eta\Lambda_{\min}$ is closed, it is true for each $x \in \tilde{S}(k)$. In [V1], 4 it is shown that for lattices $\Lambda \subset N$ with $a(\Lambda) = 1$, the difference $\mathrm{vol}(\Lambda) - \max\{v_p(\det j') \mid \Lambda \subset eqj'\Lambda_{\min}\}$ is a constant only depending on $N$. In

our case, the duality condition shows that $\mathrm{vol}(\mathcal{D}v_x)$ is constant on $\tilde{S}$ and only depending on $c$ and $N$. Thus the maximum is also constant. Hence in every $k$-valued point, $v_p(\det j_\eta)$ is equal to this maximum, which is what we wanted for the existence of $\varphi : \tilde{S} \to \mathcal{M}_{\mathrm{red}}$. We obtain an irreducible subscheme $\varphi(\tilde{S})$ of $\mathcal{S}_1 \subseteq \mathcal{M}_{\mathrm{red}}$.
To show that $J$ acts transitively on the set of irreducible components we have to show that for each $x \in \mathcal{S}_1(k)$ there is an element $j \in J$ such that $jx$ lies in the image of $\varphi$. Let $\Lambda \subset N$ be the lattice corresponding to $x$. The first step is to show that there is a $j \in J$ such that $j(\Lambda)$ is selfdual (and not only up to a scalar $c(\Lambda)$). It is enough to show that there is a $j \in J$ such that $v_p(c(\Lambda)) = v_p(c(j\Lambda)) + 1$. Such an element is given by taking the identity on $N_1$, multiplication by $p$ on $N_0$, and the map $e_i^{\frac{1}{2}} \mapsto e_{i+1}^{\frac{1}{2}}$ on $N_{\frac{1}{2}}$. Here we use the notation of Remark 2.4 for the basis of $N$. Next we want to apply an element of $J$ modifying $\Lambda_0$. We have $a(p_0(\Lambda)) = a(\Lambda \cap N_1) = 1$. From the classification of lattices with $a = 1$ in [V1], 4 we obtain that $J_{\mathbb{X}_0}^{\mathrm{np}}$ (which may be considered as a subgroup of $J$ by mapping $j \in J_{\mathbb{X}_0}^{\mathrm{np}}$ to the map consisting of $j$ and its dual on $N_1$) is acting transitively on the set of irreducible components of $\mathcal{M}_{\mathbb{X}_0,\mathrm{red}}^{\mathrm{np}}$. Thus by possibly multiplying with such an element we assume that $\Lambda_0$ lies in the fixed irreducible component chosen for Proposition 5.3. Recall from Section 4.2 that in the case of an odd number of supersingular summands, there is a $j \in J$ mapping the element $v_{\frac{1}{2}}$ to the irreducible family described there. It remains to study the possible extensions of the lattices $\Lambda_0$ and $\Lambda_1$ (or in the second case of the sublattice of $N_0 \oplus N_{\frac{1}{2}}$ determined by $\Lambda_0$ and $v_{\frac{1}{2}}$ and of $\Lambda_1$). They are given by the associated elements $v_1$. Fix a generating element $\sigma^m(v_0)$ of $\Lambda_0$ (in the second case also an element $v_{\frac{1}{2}}$) and let $v_1$ be an element associated to the e xtension $\Lambda$ with $a(\Lambda) = 1$. Then Lemma 5.1 and the construction of $S$ show that there is an element of $J$ mapping $\Lambda$ to a lattice associated to a point of $S$ inducing the same $\psi_i$ as $\Lambda$. Thus the image of $\varphi(\tilde{S})$ under $J$ is $\mathcal{S}_1$, which proves the theorem. $\square$

## 6 Dimension

We use the same notation as before, namely $\Lambda$ is the lattice corresponding to a point of $\mathcal{S}_1$, generated by an element $v = v_0 + v_{\frac{1}{2}} + v_1$ with $v_i \in N_i$. Again, $A$ is a generator of $\mathrm{Ann}(v_0)$ and $\Lambda_0 = p_0(\Lambda)$ and $\Lambda_1 = \Lambda \cap N_1$.
To determine the dimension of $\mathcal{S}_1$ and of $\mathcal{M}_{\mathrm{red}}$ we have to classify the elements $v_1$ of Section 4 up to elements in $c^{-1}\Lambda_1$ and not up to the (locally finite) action of $J$ which we used in Section 5. To do so, it is more useful to use the values of $\xi_{v_1}$ as coordinates instead of the values of $\psi_{v_1}$.
We investigate the set of possible values $\xi(\phi) \in L/c^{-1}W$ for $\phi \in \mathcal{D}$ using decreasing induction on $\tilde{v}(\phi) \geq 0$. Here, $\tilde{v}$ is as in (2.6). Recall from Lemma 4.2 2. that the use of functions $\xi$ with values in $L/c^{-1}W$ instead of $L$ corresponds to considering $v_1$ as an element of $N_1/\Lambda_1$. But as $v_1$ and $v_1 + \delta$ with $\delta \in \Lambda_1$ lead to the same lattice $\Lambda$, this is sufficient to determine the set of possible

extensions of $\Lambda_0$ and $\Lambda_1$.

Instead of equations (4.7) and (4.8) we consider the following slightly more general problem to treat at the same time the case of an odd number of supersingular summands. There, (4.8) is replaced by (4.12). We want to consider $W$-linear functions $\xi : \mathcal{D} \to L/c^{-1}W$ with

$$\xi(F^i) - \xi(V^i)^{\sigma^i} \equiv a_i \pmod{c^{-1}W} \qquad (6.1)$$

$$\xi(\psi A) \equiv 0 \pmod{c^{-1}W} \qquad (6.2)$$

for all $\psi \in \mathcal{D}$. Here $a_i \in L$ are given elements satisfying

$$a_i p^{j_i} \in c^{-1}W \quad \text{if} \quad 2j_i + i \geq 2m. \qquad (6.3)$$

Let $\mathcal{D}^i = \{\phi \in \mathcal{D} \mid \tilde{v}(\phi) \geq i\}$. We call a $W$-linear function

$$\xi^{i_0} : \mathcal{D}^{i_0} \to L/(c^{-1}W)$$

satisfying (6.1) and (6.2) a partial solution of level $i_0$. Then the induction step consists in determining the possible partial solutions $\xi^{i_0}$ of level $i_0$ leading to a fixed solution of level $i_0 + 1$. Note that the assumption on $a_i$ implies that there exists the trivial partial solution $\xi^{2m} \equiv 0$ of level $2m$ inducing partial solutions of all higher levels. Recall that we assumed $F$ and $V$ to be elementwise topologically nilpotent on $N$. Thus for each function $\xi$ with (6.1) and (6.2) there is a level $i$ such that $\xi$ induces the trivial partial solution of level $i$.

Assume that we already know the $\xi(\phi)$ for $\tilde{v}(\phi) > i_0$ and want to determine its possible values for $\tilde{v}(\phi) = i_0$. Then we know in particular $\xi(p\phi) = p\xi(\phi) \in L/c^{-1}W$, or $\xi(\phi) \in L/p^{-1}c^{-1}W$. We want to determine the possible liftings modulo $c^{-1}W$.

A basis of the $k$-vector space $\mathcal{D}^{i_0}/\mathcal{D}^{i_0+1}$ is given by the $i_0 + 1$ monomials

$$F^{i_0}, VF^{i_0-1} = pF^{i_0-2}, \dots, V^{i_0-1}F = pV^{i_0-2}, V^{i_0}.$$

Equation (6.1) leads to $\lfloor i_0/2 \rfloor$ relations between the values of $\xi$ on these monomials. Recall that $\tilde{v}(A) = m$. Thus if $\tilde{v}(\phi) = i_0 - m$ for some $\phi \in \mathcal{D}$, (6.2) leads to a relation between the value of $\xi$ on $\mathrm{LT}(\phi A) \in \mathcal{D}^{i_0}$ and values on $\mathcal{D}^{i_0+1}$. As the $\xi$ are linear, it is sufficient to consider the $\max\{0, i_0 - m + 1\}$ relations for $\phi \in \{F^{i_0-m}, pF^{i_0-m-2}, \dots, V^{i_0-m}\} \cap \mathcal{D}^{i_0-m}$. This count of relations leads to the notation

$$r(i_0) = \lfloor i_0/2 \rfloor + \max\{0, i_0 - m + 1\}.$$

Then $i_0 + 1 \leq r(i_0)$ is equivalent to $i_0 \geq 2m$.

The following proposition is the main tool to prove Theorem 3 on the dimension of the moduli spaces.

Proposition 6.1. *1. Let $i_0 \geq 2m$. Then there is a partial solution $\xi^{i_0}$ of (6.1) and (6.2) of level $i_0$. If we fix $\xi^{i_0}$ and an $l \in \mathbb{N}$ with $l \geq i_0$, there are only finitely many other partial solutions $\tilde{\xi}^{i_0}$ of level $i_0$ such that $\xi^{i_0} - \tilde{\xi}^{i_0}$ induces the trivial partial solution of level $l$ of the associated homogenous system of equations.*

2. *Let $i_0 + 1 > r(i_0)$ and let $\xi^{i_0+1}$ be a partial solution of (6.1) and (6.2) of level $i_0 + 1$. Then to obtain a partial solution $\xi^{i_0}$ of level $i_0$ inducing $\xi^{i_0+1}$, one may choose the lifts to $L/c^{-1}W$ of the values of $\xi^{i_0}$ at the first $i_0+1-r(i_0)$ monomials $p^\alpha V^\beta$ with $2\alpha+\beta = i_0$ and $\beta \leq 2(i_0-r(i_0))+1$ arbitrarily. Each of the remaining values lies in some finite nonempty set depending polynomially on the values on the previous monomials.*

*Proof.* Note that the existence statement in the first assertion is satisfied as the condition on the $a_i$ yields that there is the trivial solution of level $2m$. We show the two assertions simultaneously. Let $\xi^{i_0+1}$ be a fixed partial solution of level $i_0 + 1$ for any $i_0$. It is enough to show that for a lift $\xi^{i_0}$, the values of the first $\max\{0, i_0 + 1 - r(i_0)\}$ variables can be chosen arbitrarily, and that the remaining values then lie in some finite set depending polynomially on the values on the previous variables. If $i_0 + 1 > r(i_0)$, we have to show that this finite set is nonempty. We investigate the relations (6.1) and (6.2) more closely. The first set of relations shows that $\xi^{i_0}(p^aF^b)$ with $2a + b = i_0$ is determined by $\xi^{i_0}(p^aV^b)$. Thus it is sufficient to consider this latter set of values. Besides, we have to consider (6.2) for $\psi \in \{V^{i_0-m}, pV^{i_0-m-2}, \dots, F^{i_0-m}\}$. For $B \in \mathcal{D}$ let $\mathrm{LT}(B)$ as in Lemma 2.6. Then the equations for the values of $\xi^{i_0}$ relate $\xi^{i_0}(\mathrm{LT}(\psi A))$ to something which is known by the induction hypothesis. Let us recall the description of $\mathrm{LT}(A)$ from Lemma 2.6 3. Let $h'$ be the number of supersingular summands of $N_0$. Let $j \geq 0$ with $i_0 - m - j \geq 0$. Then $\mathrm{LT}(V^{i_0-m-j}F^jA)$ is a linear combination of $V^{i_0-j}F^j, \dots, V^{i_0-j-h'}F^{j+h'}$ whose coefficients are Teichmüller representatives of elements of $k$. Furthermore, the coefficients of $\xi^{i_0}(V^{i_0-j}F^j)$ and $\xi^{i_0}(V^{i_0-j-h'}F^{j+h'})$ are units in $W$. Using (6.1) we may replace values of $\xi^{i_0}$ at monomials in $F$ by $\sigma$-powers of the values of the corresponding monomials in $V$. We thus obtain a relation between a polynomial in the remaining $\lceil (i_0+1)/2 \rceil$ values of $\xi^{i_0}$ and an expression which is known by induction. For $2j \leq i_0$, the first summand $\xi^{i_0}(V^{i_0-j}F^j)$ remains the variable associated to the highest power of $V$ which occurs linearly in this polynomial. In the following we ignore all equations for $2j > i_0$. They only occur for $i_0 > 2m$, a case where we only want to prove the finiteness of the set of solutions. The system of equations with $2j \leq i_0$ is of the form considered in Lemma 5.4. The proof of this Lemma for coefficients in $L/c^{-1}W$ is the same as for coefficients in $L$. Thus we obtain that the lifts of the values at the $i_0+1-r(i_0)$ variables associated to the largest values of $j$ can be chosen freely and the other ones have to satisfy some relation of the form $Q_{ii}(x) \equiv b_i$ for some given $b_i$. As the $Q_{ii}$ have a linear term they are nontrivial. This implies that the set of solutions of these equations is nonempty and finite and dep ends polynomially on the previous values. □

### 6.1 PROOF OF THEOREM 3

By Lemma 3.1 it is enough to show that $\mathcal{S}_1$ is equidimensional of the claimed dimension. From [V1], 4 we obtain that the connected components of $\mathcal{S}_{\mathbb{X},1}^{\mathrm{np}}$ are irreducible. The discrete invariant with values in $J^{\mathrm{np}}/(J^{\mathrm{np}} \cap \mathrm{Stab}(\Lambda_{\min}))$

distinguishing the components is given by $\Lambda \mapsto j_\Lambda$ with $\Lambda \subseteq j_\Lambda \Lambda_{\min}$ and

$$v_p(\det j_\Lambda) = \max\{v_p(\det j) \mid j \in J^{\mathrm{np}}, \Lambda \subseteq j\Lambda_{\min}\}.$$

Especially, $j_\Lambda$ is constant on each connected component of $\mathcal{S}_1 \subseteq \mathcal{S}_{\mathbb{X},1}^{\mathrm{np}}$. Besides, $p_0(j_\Lambda \Lambda_{\min})$ determines the connected component of $p_0(\Lambda) = \Lambda_0$ inside $\mathcal{S}_{\mathbb{X}_0,1}$. Thus we may fix an irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$ and determine the dimension of the union of connected components of $\mathcal{S}_1$ such that $\Lambda_0$ is in this fixed component. Let $R_0$ and $R_1$, $v_0$ and $v_{\frac{1}{2}}$ be as in the proof of Theorem 2. Again we use the functions $\xi$ defined with respect to $\sigma^m(v_0)$ instead of $v_0$. Fix an arbitrary partial solution $\xi^{2m}$ of (6.1) and (6.2) of level $2m$. Let

$$R_2 = R_1[x_{i\beta} \mid i \geq 0, 1 \leq \beta \leq i + 1 - r(i)].$$

We use decreasing induction on $i$ to lift $\xi^{2m}$ to a partial solution of level $i$ over an étale extension $R_2^i$ of $R_2$. Let $R_2^{2m} = R_2$. Assume that a lift $\xi^{i+1}$ is given. Then Proposition 6.1 shows that the values at $i + 1 - r(i)$ monomials with $\tilde{v} = i$ may be lifted arbitrarily to a value of $\xi^i$. If $p^\alpha V^\beta$ with $2\alpha + \beta = i$ and $\beta \leq i + 1 - r(i)$ is such a monomial we write (using the induction hypothesis) $\xi^{i+1}(p^{\alpha+1}V^\beta) = \sum_{i < v_p(c^{-1})}[a_i]p^i$ with $a_i \in R_2^{i+1}$. Then we choose

$$\xi^i(p^\alpha V^\beta) = \sum_{i < v_p(c^{-1})} [a_i]p^{i-1} + [x_{i\beta}]p^{v_p(c^{-1})-1}.$$

Let $R_2^i$ be the extension of $R_2^{i+1}$ given by adjoining further variables $x_{i\beta}$ for larger $\beta$ parametrizing the other values of the lift of $\xi^{i+1}$ to $\xi^i$ and with relations as in Proposition 6.1, 2. and its proof. More precisely, $R_2^i$ is obtained from $R_2^{i+1}$ by a finite number of extensions given by equations of the form $Q_{jj}(x) \equiv b_j \pmod{c^{-1}W}$ where $Q_{jj}(x)$ is a polynomial that is a finite linear combination of the monomials $x^{p^l}$ with $l \geq 0$ such that the coefficient of $x$ is in $W(R_2^{i+1})^\times$. This implies that $R_2^i$ is a finite étale extension of $R_2^{i+1}$. Let $R_3 = R_2^0$. Let $v_{1,R_3} \in N_{1,R_3}$ be such that $\xi_{v_{1,R_3}} = \xi^0$. Its existence follows again from the existence of the dual basis in Remark 3.3. Let $v = \sigma^m(v_{0,R_0}) + v_{1,R_3}$, or $v = \sigma^m(v_{0,R_0}) + v_{\frac{1}{2}} + v_{1,R_3}$. As in the proof of Theorem 2 let $S = \S pec(R)$ be an irreducible component of the affine open subscheme of $\mathrm{Spec}(R_3)$ over which $\mathcal{D}v$ contains $(\mathcal{D}v)_1^\vee$. As we want to compute the dimension of $\mathcal{S}_1$, we only have to consider these subschemes. Let $\tilde{R} = \sigma^{-m}(R)$ as in Proposition 3.2. The same argument as in the proof of Theorem 2 shows that there is a morphism $\varphi : \mathrm{Spec}(\tilde{R}) \to \mathcal{M}_{\mathrm{red}}$ mapping $x \in \mathrm{Spec}(\tilde{R})(k)$ to the lattice generated by $v_x$. The finiteness statements in Proposition 6.1 imply that for each given $y \in \mathcal{S}_1$ (and thus given $\xi$) there is an open neighborhood in $\mathcal{S}_1$ which only contains points of $\varphi(\mathrm{Spec}(\tilde{R}))$ for a finite number of choices of an irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$ and a corresponding component $S$. Besides, the construction of $R_3$ together with the description of the $k$-valued points of $\mathcal{S}_1$ shows that for each $y \in \mathcal{S}_1(k)$ there is exactly one irreducible component of $\mathcal{S}_{\mathbb{X}_0,1}$, one corresponding component $S$, and one point $x \in \mathrm{Spec}(\tilde{R})(k)$ such that $\varphi(x) = y$. Thus $\dim \mathcal{M}_{\mathrm{red}} = \dim \mathcal{S}_1$ is the maximum of $\dim \operatorname{Spec}(\tilde{R})$ for all irreducible components $S$. It remains to show that this is equal to the right hand side of (1.2). Note that $R_i$ is equidimensional for $i = 0, 1, 2, 3$. From the construction of $S$ we see that in case of an even number of supersingular summands,

$$\begin{aligned} \dim \operatorname{Spec}(\tilde{R}) &= \dim S = \dim R_3 = \dim R_2 \\ &= \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} + \sum_{i \geq 0} \max\{0, i+1-r(i)\}. \end{aligned} \tag{6.4}$$

In the other case,

$$\begin{aligned} \dim \operatorname{Spec}(\tilde{R}) &= \dim S = \dim R_3 = \dim R_2 \\ &= \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} + \sum_{i \geq 0} \max\{0, i+1-r(i)\} + m. \end{aligned} \tag{6.5}$$

The last summand corresponds to the choice of $v_{\frac{1}{2}}$.

From the decomposition of $N$ into $l$ simple summands $N^j$ we obtain a decomposition $N_0 = \bigoplus_{j=1}^{l_0} N^j$ with $l_0 = \lfloor l/2 \rfloor$. Let again $\lambda_j = m_j/(m_j + n_j)$ be the slope of $N^j$. Recall from [V1], Theorem B that

$$\dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} = \sum_{j=1}^{l_0} \frac{(m_j - 1)(n_j - 1)}{2} + \sum_{\{j,j' | j < j' \leq l_0\}} m_j n_{j'}.$$

We denote the right hand side of (1.2) by $D$. Let us first consider the case of an even number of supersingular summands. Then by the symmetry of the Newton polygon we obtain

$$D - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} = \frac{1}{2} \sum_{j < j' \leq l} m_j n_{j'} + \frac{m}{2} - \sum_{j < j' \leq l_0} m_j n_{j'}.$$

Again by the symmetry of the Newton polygon this is equal to

$$\begin{aligned} &= \sum_{j=1}^{l_0} \sum_{j'=l_0+1}^{l} \frac{m_j n_{j'}}{2} + \frac{m}{2} \\ &= \sum_{j,j'=1}^{l_0} \frac{m_j m_{j'}}{2} + \frac{m}{2} \\ &= \frac{m(m+1)}{2}. \end{aligned}$$

In the other case, the same calculation shows that

$$\begin{aligned} D - \dim \mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} &= \frac{m(m+1)}{2} + 2 \sum_{j=1}^{l_0} \frac{m_j n_{l_0+1}}{2} \\ &= \frac{m(m+1)}{2} + m. \end{aligned}$$

In the last step we used that $N^{l_0+1}$ is supersingular, hence $n_{l_0+1}=1$.
On the other hand (6.4) implies that in the case of an even number of supersingular summands,

$$\begin{aligned}\dim\mathcal{M}_{\mathrm{red}}-\dim\mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}} &= \sum_{i\geq 0}\max\{0,i+1-r(i)\}\\ &= \sum_{i=0}^{m-1}\left(\left\lfloor\frac{i}{2}\right\rfloor+1\right)+\sum_{i=m}^{2m-1}\left(\left\lfloor\frac{i}{2}\right\rfloor-i+m\right)\\ &= m+\sum_{i=0}^{2m-1}\left\lfloor\frac{i}{2}\right\rfloor-\sum_{i=0}^{m-1}i\\ &= \frac{m(m+1)}{2}.\end{aligned}$$

The same calculation with (6.5) shows that for an odd number of supersingular summands

$$\dim\mathcal{M}_{\mathrm{red}}-\dim\mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}}=\frac{m(m+1)}{2}+m.$$

Together with the calculation of $D-\dim\mathcal{M}^{\mathrm{np}}_{\mathbb{X}_0,\mathrm{red}}$, this implies Theorem 3. $\square$

## 7 Connected components

In this section we determine the set of connected components of $\mathcal{M}_{\mathrm{red}}$. The reduction to the bi-infinitesimal case in Section 2.3 shows that Theorem 1 follows from the next theorem.

Theorem 7.1. *Let $\mathbb{X}$ be bi-infinitesimal and non-trivial. Then*

$$\begin{aligned}\kappa:\mathcal{M}_{\mathrm{red}}(k) &\to \mathbb{Z}\\ \Lambda &\mapsto v_p(c(\Lambda)),\end{aligned}$$

*where $\Lambda^\vee=c(\Lambda)\cdot\Lambda$, induces a bijection*

$$\pi_0(\mathcal{M}_{\mathrm{red}})\cong\mathbb{Z}.$$

*Proof.* From Theorem 2 we obtain a $J$-equivariant surjection $\pi:J\twoheadrightarrow\pi_0(\mathcal{M}_{\mathrm{red}})$. Besides, the map $\kappa$ induces a surjection $\pi_0(\mathcal{M}_{\mathrm{red}})\to\mathbb{Z}$. We choose the base point of $\mathcal{M}$ to be a minimal $p$-divisible group. Let $\Lambda_{\min}$ be the corresponding lattice in $N$. An element $j\Lambda_{\min}$ with $j\in J$ is in the kernel of $\kappa$ if and only if $(j\Lambda_{\min})^\vee=j\Lambda_{\min}$. This is equivalent to $j\Lambda_{\min}=j'\Lambda_{\min}$ for some $j'\in J\cap Sp_{2h}(L)$. Thus we have to show that $J\cap Sp_{2h}(L)$ is mapped to a single connected component of $\mathcal{M}_{\mathrm{red}}$. Our choice of the base point implies that $\mathrm{Stab}(\Lambda_{\min})=K$. Thus the surjection $\pi$ maps $J\cap K$ to the component of the identity. Note that $J\cap Sp_{2h}(L)=J_{\mathrm{der}}(\mathbb{Q}_p)$ where $J_{\mathrm{der}}$ is the derived group of $J$. Hence the elements of $(J\cap Sp_{2h}(L))/(J\cap K)$ correspond to vertices in the building of $J_{\mathrm{der}}$. The building of $J_{\mathrm{der}}$ is connected. To show that all vertices correspond to points in one connected component of $\mathcal{M}_{\mathrm{red}}$, it is thus enough to show that if $\Lambda_0, \Lambda_1$ are the lattices corresponding to two such vertices such that $\Lambda_0 \cap \Lambda_1 = \Lambda$ is of colength 1 in $\Lambda_0$ and $\Lambda_1$, then the two lattices correspond to points in the same connected component of $\mathcal{M}_{\mathrm{red}}$. As a $W$-module $\Lambda_0$ is generated by $\Lambda$ and $v_0$ for some $v_0 \in N$. As the slopes of $F$ are in $(0,1)$ we have $Fv_0, Vv_0 \in \Lambda$. Similarly $\Lambda_1$ is generated by $\Lambda$ and some $v_1$ with $Fv_1, Vv_1 \in \Lambda$. For $a \in \mathbb{A}^1(k)$ let $\Lambda_a = \langle \Lambda, v_0 + a(v_1 - v_0)\rangle$. As $\Lambda_0$ and $\Lambda_1$ are selfdual one easily sees that $\Lambda_a$ is selfdual for each $a$. By [V1], Lemma 3.4 there is a morphism $\mathbb{A}^1 \rightarrow \mathcal{M}^{\mathrm{np}}_{\mathrm{red}}(\mathbb{X})$ mapping each point $a$ as above to the point of $\mathcal{M}^{\mathrm{np}}_{\mathrm{red}}(\mathbb{X})$ corresponding to $\Lambda_a$. As all $\Lambda_a$ are selfdual, this induces a corresponding morphism $f : \mathbb{A}^1 \rightarrow \mathcal{M}_{\mathrm{red}}$. Hence $f(0)$ and $f(1)$, the points corresponding to $\Lambda_0$ and $\Lambda_1$, are in the same connected component of $\mathcal{M}_{\mathrm{red}}$. □

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