A Criterion for Flatness of Sections of Adjoint Bundle of a Holomorphic Principal Bundle over a Riemann Surface

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Received: August 8, 2013

Communicated by Edward Frenkel

Abstract. Let E_G be a holomorphic principal G–bundle over a compact connected Riemann surface, where G is a connected reductive affine algebraic group defined over $\mathbb{C}$, such that E_G admits a holomorphic connection. Take any $\beta \in H^0(X, \text{ad}(E_G))$, where $\text{ad}(E_G)$ is the adjoint vector bundle for E_G, such that the conjugacy class $\beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x. We give a sufficient condition for the existence of a holomorphic connection on E_G such that β is flat with respect to the induced connection on $\text{ad}(E_G)$.

2010 Mathematics Subject Classification: 14H60, 14F05, 53C07
Keywords and Phrases: Holomorphic connection, adjoint bundle, flatness, canonical connection

1. Introduction

A holomorphic vector bundle E over a compact connected Riemann surface X admits a holomorphic connection if and only if every indecomposable component of E is of degree zero [We], [At]. This criterion generalizes to the holomorphic principal G–bundles over X, where G is a connected reductive affine algebraic group defined over $\mathbb{C}$ [AB].

Let E_G be a holomorphic principal G–bundle over X, where X and G are as above. Let $\mathfrak{g}$ denote the Lie algebra of G. Let β be a holomorphic section of the adjoint vector bundle $\text{ad}(E_G) = E_G \times^G \mathfrak{g}$. Our aim here is to find a criterion for the existence of a holomorphic connection on E_G such that β is flat with respect to the induced connection on $\text{ad}(E_G)$. A sufficient condition is obtained in Theorem 3.4.

For $G = \text{GL}(r, \mathbb{C})$, Theorem 3.4 says the following:

Let E be a holomorphic vector bundle of rank r on X, and $\beta \in H^0(X, End(E))$. Let

$$E = \bigoplus_{i=1}^{\ell} E_i$$

be the generalized eigen-bundle decomposition for β. So $\beta|_{E_i} = \lambda_i \cdot \mathrm{Id}_{E_i} + N_i$, where $\lambda_i \in \mathbb{C}$, and either $N_i = 0$ or N_i is nilpotent. If $N_i \neq 0$, then assume that the section N^{r_i-1} is nowhere vanishing, where r_i is the rank of the vector bundle E_i. Also, assume that E admits a holomorphic connection. Then Theorem 3.4 says that E admits a holomorphic connection D such that β is flat with respect to the connection on $End(E)$ induced by D.
One may ask whether the above condition that N^{r_i-1} is nowhere vanishing whenever $N_i \neq 0$ can be replaced by the weaker condition that the conjugacy class of $\beta(x)$, $x \in X$, is independent of x. As example constructed by the referee shows that this cannot be done (see Example 3.6).

2. Flat sections of the adjoint bundle

Let X be a compact connected Riemann surface. Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. The set of all conjugacy classes in $\mathfrak{g}$ will be denoted by $\mathfrak{g}/G$.
Let

$$f : E_G \longrightarrow X \tag{2.1}$$

be a holomorphic principal G–bundle. Define the adjoint vector bundle

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g}\,.$$

In other words, $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$ where any $(z\,, v) \in E_G \times \mathfrak{g}$ is identified with $(zg\,, \mathrm{Ad}(g)(v))$, $g \in G$; here $\mathrm{Ad}(g)$ is the automorphism of $\mathfrak{g}$ corresponding to the automorphism of G defined by $g' \longmapsto g^{-1}g'g$. Therefore, we have a set-theoretic map

$$\phi : \mathrm{ad}(E_G) \longrightarrow \mathfrak{g}/G \tag{2.2}$$

that sends any $(z\,, v) \in E_G \times \mathfrak{g}$ to the conjugacy class of v.
Let

$$\mathrm{At}(E_G) := (f_* TE_G)^G \subset f_* TE_G$$

be the Atiyah bundle for E_G, where f is the projection in (2.1), and TE_G is the holomorphic tangent bundle of E_G (the action of G on E_G produces an action of G on the direct image $f_* TE_G$). The Atiyah bundle fits in a short exact sequence of vector bundles

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \longrightarrow TX \longrightarrow 0\,; \tag{2.3}$$

the above projection $\mathrm{At}(E_G) \longrightarrow TX$, where TX is the holomorphic tangent bundle of X, is defined by the differential $df : TE_G \longrightarrow f^*TX$ of f. A *holomorphic connection* on E_G is a holomorphic splitting of the sequence in (2.3) [At].

A holomorphic connection D on E_G induces a holomorphic connection on each holomorphic fiber bundle associated to E_G. In particular, the vector bundle $\mathrm{ad}(E_G)$ gets a holomorphic connection from D. A section β of $\mathrm{ad}(E_G)$ is said to be *flat* with respect to D if β is flat with respect to the connection on $\mathrm{ad}(E_G)$ induced by D.

Lemma 2.1. *Take a holomorphic connection D on E_G, and let $\beta \in H^0(X,\, \mathrm{ad}(E_G))$ be flat with respect to D. Then the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, where $x \in X$, is independent of x.*

Proof. Any holomorphic connection on X is flat because $\Omega_X^2 = 0$. Using the flat connection D, we may holomorphically trivialize E_G on any connected simply connected open subset of X. With respect to such a trivialization, the section β is a constant one because it is flat with respect to D. This immediately implies that $\phi \circ \beta(x) \in \mathfrak{g}/G$ is independent of $x \in X$. □

3. Holomorphic connections on Principal G-bundles

A nilpotent element v of the Lie algebra of a complex semisimple group H is called *regular nilpotent* if the dimension of the centralizer of v in H coincides with the rank of H [Hu, p. 53].
As before, G is a connected reductive affine algebraic group defined over $\mathbb{C}$. Take E_G as in (2.1).

Proposition 3.1. *Take any $\beta \in H^0(X,\, \mathrm{ad}(E_G))$. Assume that*

- *E_G admits a holomorphic connection,*
- *the element $\phi \circ \beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in (2.2), and*
- *for every adjoint type simple quotient H of G, the section of the adjoint bundle $\mathrm{ad}(E_H)$ given by β, where $E_H := E_G \times^G H$ is the principal H–bundle over X associated to E_G for the projection $G \longrightarrow H$, has the property that it is either zero or it is regular nilpotent at some point of X.*

Then the principal G–bundle E_G admits a holomorphic connection for which β is flat.

Proof. Let $Z := G/[G\,,G]$ be the abelian quotient. It is a product of copies of $\mathbb{C}^*$. There are quotients $H_1\,, \cdots\,, H_\ell$ of G such that

(1) each H_i is simple of adjoint type (the center is trivial), and
(2) the natural homomorphism

$$\varphi\,:\, G \longrightarrow Z \times \prod_{i=1}^{\ell} H_i \tag{3.1}$$

is surjective, and the kernel of φ is a finite group.

Let

$$E_Z \,:=\, E_G \times^G Z \quad \text{and} \quad E_{H_i} \,:=\, E_G \times^G H_i\,,\ i \in [1\,,\ell]\,,$$

be the holomorphic principal Z–bundle and principal H_i–bundle associated to E_G for the quotient Z and H_i respectively. Let $\mathrm{ad}(E_Z)$ and $\mathrm{ad}(E_{H_i})$ be the adjoint vector bundles for E_Z and E_{H_i} respectively. Since the homomorphism φ in (3.1) induces an isomorphism of Lie algebras, we have

$$\mathrm{ad}(E_G) \,=\, \mathrm{ad}(E_Z) \oplus (\bigoplus_{i=1}^{\ell} \mathrm{ad}(E_{H_i}))\,. \tag{3.2}$$

Let β_Z (respectively, β_i) be the holomorphic section of $\mathrm{ad}(E_Z)$ (respectively, $\mathrm{ad}(E_{H_i})$) given by β using the decomposition in (3.2). Since the conjugacy class of $\beta(x)$ is independent of $x \in X$ (the second condition in the proposition), we conclude that the conjugacy class of $\beta_i(x)$ is also independent of $x \in X$.

A holomorphic connection on E_G induces a holomorphic connection on E_Z. Since E_Z admits a holomorphic connection, and Z is a product of copies of $\mathbb{C}^*$, there is a unique holomorphic connection D^Z on E_Z whose monodromy lies inside the maximal compact subgroup of Z. The connection on $\mathrm{ad}(E_Z)$ induced by this connection D^Z has the property that any holomorphic section of $\mathrm{ad}(E_Z)$ is flat with respect to it. In particular, the section β_Z is flat with respect to this induced connection on $\mathrm{ad}(E_Z)$.

Now take any $i \in [1\,,\ell]$. A holomorphic connection on E_G induces a holomorphic connection on E_{H_i}. If the section β_i is zero at some point, then β_i is identically zero because the conjugacy class of $\beta_i(x)$ is independent of x. Hence, in that case β_i is flat with respect to any connection on $\mathrm{ad}(E_{H_i})$. Therefore, assume that β_i is not zero at any point of X.

By the assumption in the proposition, β_i is regularly nilpotent over some point of X. Since the conjugacy class of $\beta_i(x)$, $x \in X$, is independent of x, we conclude that β_i is regular nilpotent over every point of X. We will now show that the holomorphic principal H_i–bundle E_{H_i} is semistable.

For each point $x \in X$, from the fact that $\beta_i(x)$ is regular nilpotent we conclude that there is a unique Borel subalgebra $\widetilde{\mathfrak{b}}_x$ of $\mathrm{ad}(E_{H_i})_x$ such that $\beta_i(x) \in \widetilde{\mathfrak{b}}_x$ [Hu, p. 62, Theorem]. Let

$$\widetilde{\mathfrak{b}} \subset \mathrm{ad}(E_{H_i})$$

be the Borel subalgebra bundle such that for every point x the fiber $(\widetilde{\mathfrak{b}})_x$ is $\widetilde{\mathfrak{b}}_x$. Fix a Borel subgroup $B \subset H_i$. Using $\widetilde{\mathfrak{b}}$, we will construct a holomorphic reduction of structure group of E_{H_i} to the subgroup B.

Let $\mathfrak{b}$ be the Lie algebra of B. The Lie algebra of H_i will be denoted by $\mathfrak{h}_i$. We recall that $\mathrm{ad}(E_{H_i})$ is the quotient of $E_{H_i} \times \mathfrak{h}_i$ where two points $(z_1\,,v_1)$ and $(z_2\,,v_2)$ of $E_{H_i} \times \mathfrak{h}_i$ are identified if there is an element $h \in H_i$ such that $z_2 = z_1 h$ and $v_2 = \mathrm{Ad}(h)(v_1)$, where $\mathrm{Ad}(h)$ is the automorphism of $\mathfrak{h}_i$ corresponding to the automorphism $y \longmapsto h^{-1}yh$ of H_i. For any point $x \in X$, let $E_{B,x} \subset (E_{H_i})_x$ be the complex submanifold consisting of all $z \in (E_{H_i})_x$ such that for all $v \in \mathfrak{b}$, the image of $(z\,,v)$ in $\mathrm{ad}(E_{H_i})_x$ lies in $\widetilde{\mathfrak{b}}_x$. Since any two Borel subalgebras of $\mathfrak{h}_i$ are conjugate, it follows that $E_{B,x}$ is nonempty. The normalizer of $\mathfrak{b}$ in H_i coincides with B. From this it follows that $E_{B,x}$ is preserved by the action of B on $(E_{H_i})_x$, with the action of B on $E_{B,x}$ being transitive. Let

$$E_B \subset E_{H_i}$$

be the complex submanifold such that $E_B \bigcap (E_{H_i})_x = E_{B,x}$ for every $x \in X$. From the above properties of $E_{B,x}$ it follows immediately that E_B is a holomorphic reduction of structure group of the principal H_i–bundle E_{H_i} to the subgroup B.

Consider the adjoint action of B on $\mathfrak{b}_1 := \mathfrak{b}/[\mathfrak{b}\,,\mathfrak{b}]$. Let

$$E_B(\mathfrak{b}_1) := E_B \times^B \mathfrak{b}_1 \longrightarrow X$$

be the holomorphic vector bundle associated to E_B for the B–module $\mathfrak{b}_1$. Since β_i is everywhere regular nilpotent, it follows that the vector bundle $E_B(\mathfrak{b}_1)$ is trivial. Consequently, for any character χ of B which is a nonnegative integral combination of simple roots, the line bundle $E_B(\chi) \longrightarrow X$ associated to E_B for the character χ is trivial [AAB, p. 708, Theorem 5]. Therefore, for any character χ of B, the line bundle $E_B(\chi)$ associated to E_B for χ is trivial.

Let d be the complex dimension of $\mathfrak{h}_i$. Consider the adjoint action on B on $\mathfrak{h}_i$. Note that $\mathrm{ad}(E_{H_i})$ is identified with the vector bundle associated to the principal B–bundle E_B for this B–module $\mathfrak{h}_i$. Since B is solvable, there is a filtration of B–modules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = \mathfrak{h}_i$$

such that $\dim V_j = j$ for all $j \in [1\,,d]$. The corresponding filtration of vector bundles associated to E_B is a filtration of $\mathrm{ad}(E_{H_i})$ such that the successive quotients are the line bundles $E_B(V_j/V_{j-1})$, $i \in [1\,,d]$, associated to E_B for the B–modules V_j/V_{j-1}. We noted above that the line bundles associated to E_B for the characters of B are trivial.

Therefore, we get a filtration of $\mathrm{ad}(E_{H_i})$ such that each successive quotient is the trivial line bundle. This immediately implies that the vector bundle $\mathrm{ad}(E_{H_i})$ is semistable. Hence the holomorphic principal H_i–bundle E_{H_i} is semistable [AAB, p. 698, Lemma 3].

Since H_i is simple, and E_{H_i} is semistable, there is a natural holomorphic connection on E_{H_i} [BG, p. 20, Theorem 1.1] (set the Higgs field in [BG, Theorem 1.1] to be zero). Let D^{H_i} denote this connection. The vector bundle $\mathrm{ad}(E_{H_i})$ being semistable of degree zero has a natural holomorphic connection [Si, p. 36, Lemma 3.5]. See also [BG, p. 20, Theorem 1.1]. (In both [Si, Lemma 3.5] and [BG, Theorem 1.1] set the Higgs field to be zero.) Let D^{ad} denote this holomorphic connection on $\mathrm{ad}(E_{H_i})$. This connection D^{ad} coincides with the one induced by D^{H_i} (see the construction of the connection in [BG]).

Any holomorphic section of $\mathrm{ad}(E_{H_i})$ is flat with respect to D^{ad}. To see this, let

$$\phi : \mathcal{O}_X \longrightarrow \mathrm{ad}(E_{H_i})$$

be the homomorphism given by a nonzero holomorphic section of $\mathrm{ad}(E_{H_i})$. Since $\mathrm{image}(\phi)$ is a semistable subbundle of $\mathrm{ad}(E_{H_i})$ of degree zero, the connection D^{ad} preserves $\mathrm{image}(\phi)$, and, moreover, the restriction of D^{ad} to $\mathrm{image}(\phi)$ coincides with the canonical connection of $\mathrm{image}(\phi)$ [Si, p. 36, Lemma 3.5].

The canonical connection on the trivial holomorphic line bundle image(ϕ) is the trivial connection (the monodromy is trivial).
In particular, the connection on $\mathrm{ad}(E_{H_i})$ induced by D^{H_i} has the property that the section β_i is flat with respect to it.
Since the homomorphism of Lie algebras corresponding to φ (in (3.1)) is an isomorphism, if we have holomorphic connections on E_Z and E_{H_i}, $[1\,,\ell]$, then we get a holomorphic connection on E_G; simply pullback the connection form using the map

$$E_G \longrightarrow E_Z \times_X E_{H_1} \times_X \cdots \times_X E_{H_\ell}\,.$$

The connection on E_G given by the connections on E_Z and E_{H_i}, $[1\,,\ell]$, constructed above satisfies the condition that β is flat with respect to it. This completes the proof of the proposition. □

Lemma 3.2. *Take any semisimple section $\beta_s \in H^0(X,\,\mathrm{ad}(E_G))$ such that the element $\phi\circ\beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in* (2.2)*. Then β_s produces a holomorphic reduction of structure group of E_G to a Levi subgroup of a parabolic subgroup of G. The conjugacy class of the Levi subgroup is determined by $\phi\circ\beta_s(x) \in \mathfrak{g}/G$.*

Proof. Fix an element

$$v_0 \in \mathfrak{g}$$

such that the image of v_0 in $\mathfrak{g}/G$ coincides with $\phi\circ\beta_s(x)$. Let $\mathbb{L} \subset G$ be the centralizer of v_0. It is known that $\mathbb{L}$ is a Levi subgroup of some parabolic subgroup of G [DM, p. 26, Proposition 1.22] (note that $\mathbb{L}$ is the centralizer of the torus in G generated by v_0). In particular, $\mathbb{L}$ is connected and reductive.
For any point $x \in X$, let $F_x \subset (E_G)_x$ be the complex submanifold consisting of all points z such that the image of $(z\,,v_0)$ in $\mathrm{ad}(E_G)_x$ coincides with $\beta_s(x)$. (Recall that $\mathrm{ad}(E_G)$ is a quotient of $E_G\times\mathfrak{g}$.) Let

$$F_{\mathbb{L}} \subset E_G$$

be the complex submanifold such that $F_{\mathbb{L}}\bigcap(E_G)_x = F_x$ for all $x \in X$. It is straightforward to check that $F_{\mathbb{L}}$ is a holomorphic reduction of structure group of the principal G–bundle E_G to the subgroup $\mathbb{L}$. □

Remark 3.3. If $\beta_s \in H^0(X,\,\mathrm{ad}(E_G))$ is such that $\beta_s(x)$ is semisimple for every $x \in X$, then the conjugacy class of $\beta_s(x)$ is in fact independent of x. But we do not need this here.

From the Jordan decomposition of a complex reductive Lie algebra we know that for any holomorphic section θ of $\mathrm{ad}(E_G)$, there is a naturally associated semisimple (respectively, nilpotent) section θ_s (respectively, θ_n) such that $\theta = \theta_s+\theta_n$.
Take any $\beta \in H^0(X,\,\mathrm{ad}(E_G))$. Let

$$\beta = \beta_s+\beta_n$$

be the Jordan decomposition. Assume that the element $\phi\circ\beta(x) \in \mathfrak{g}/G$, $x \in X$, is independent of x, where ϕ is defined in (2.2). This implies that

$\phi \circ \beta_s(x) \in \mathfrak{g}/G$, $x \in X$, is also independent of x. Let $(\mathbb{L}\,, F_{\mathbb{L}})$ be the principal bundle constructed in Lemma 3.2 from β_s. Let H be an adjoint type simple quotient of $\mathbb{L}$. Let

$$E_H \,:=\, F_{\mathbb{L}} \times^{\mathbb{L}} H \,\longrightarrow\, X$$

be the holomorphic principal H–bundle associated to $F_{\mathbb{L}}$ for the projection $\mathbb{L} \longrightarrow H$.

Since $[\beta_s\,, \beta_n] \,=\, 0$, from the construction of $F_{\mathbb{L}}$ it follows that

$$\beta_n \,\in\, H^0(X,\, \mathrm{ad}(F_{\mathbb{L}})) \,\subset\, H^0(X,\, \mathrm{ad}(E_G))\,.$$

Therefore, using the natural projection $\mathrm{ad}(F_{\mathbb{L}}) \,\longrightarrow\, \mathrm{ad}(E_H)$, given by the projection of the Lie algebra $\mathrm{Lie}(\mathbb{L}) \,\longrightarrow\, \mathrm{Lie}(H)$, the above section β_n produces a holomorphic section of $\mathrm{ad}(E_H)$. Let

$$\widetilde{\beta}_n \,\in\, H^0(X,\, \mathrm{ad}(E_H)) \tag{3.3}$$

be the section constructed from β_n.

Theorem 3.4. *Take any $\beta \,\in\, H^0(X,\, \mathrm{ad}(E_G))$. Let $\beta \,=\, \beta_s + \beta_n$ be the Jordan decomposition. Assume that*

- *E_G admits a holomorphic connection,*
- *the element $\phi \circ \beta(x) \,\in\, \mathfrak{g}/G$, $x \,\in\, X$, is independent of x, where ϕ is defined in* (2.2)*, and*
- *for every adjoint type simple quotient H of $\mathbb{L}$, the section $\widetilde{\beta}_n$ in* (3.3) *of $\mathrm{ad}(E_H)$ has the property that it is either zero or it is regular nilpotent at some point of X.*

Then the principal G–bundle E_G admits a holomorphic connection for which β is flat.

Proof. Note that

$$\beta_s \,\in\, H^0(X,\, \mathrm{ad}(F_{\mathbb{L}})) \,\subset\, H^0(X,\, \mathrm{ad}(E_G))\,.$$

In fact, for each point $x \,\in\, X$, the element $\beta_s(x) \,\in\, \mathrm{ad}(F_{\mathbb{L}}))_x$ is in the center of $\mathrm{ad}(F_{\mathbb{L}}))_x$. Consider the abelian quotient

$$Z_{\mathbb{L}} \,=\, \mathbb{L}/[\mathbb{L}\,, \mathbb{L}]\,.$$

Let $F_{Z_{\mathbb{L}}}$ be the holomorphic principal $Z_{\mathbb{L}}$–bundle over X obtained by extending the structure group of the principal $\mathbb{L}$–bundle $F_{\mathbb{L}}$ using the quotient map $\mathbb{L} \longrightarrow Z_{\mathbb{L}}$. The adjoint vector bundle $\mathrm{ad}(F_{Z_{\mathbb{L}}})$ is a direct summand of $\mathrm{ad}(F_{\mathbb{L}})$. In fact, for each $x \,\in\, X$, the subspace $\mathrm{ad}(F_{Z_{\mathbb{L}}})_x \,\subset\, \mathrm{ad}(F_{\mathbb{L}})_x$ is the center of the Lie algebra $\mathrm{ad}(F_{\mathbb{L}})_x$.

A holomorphic connection on $F_{\mathbb{L}}$ induces a holomorphic connection on E_G. We can now apply Proposition 3.1 to $F_{\mathbb{L}}$ to complete the proof of the theorem. But for that we need to show that $F_{\mathbb{L}}$ admits a holomorphic connection.

Let $\mathfrak{l}$ be the Lie algebra of $\mathbb{L}$. Consider the inclusion of $\mathbb{L}$–modules $\mathfrak{l} \,\hookrightarrow\, \mathfrak{g}$ given by the inclusion of $\mathbb{L}$ in G. Since $\mathbb{L}$ is reductive, there is a sub $\mathbb{L}$–module $S \,\subset\, \mathfrak{g}$ such that the natural homomorphism

$$\mathfrak{l} \oplus S \,\longrightarrow\, \mathfrak{g}$$

is an isomorphism (so S is a complement of $\mathfrak{l}$). Let

$$p : \mathfrak{g} \longrightarrow \mathfrak{l} \tag{3.4}$$

be the projection given by the above decomposition of $\mathfrak{g}$.
Let D be a holomorphic connection on E_G. So D is a holomorphic 1–form on the total space of E_G with values in the Lie algebra $\mathfrak{g}$. Let D' be the restriction of this 1–form to the complex submanifold $F_{\mathbb{L}} \subset E_G$. Consider the $\mathfrak{l}$–valued 1–form $p \circ D'$ on $E_{\mathbb{L}}$, where p is the projection in (3.4). This $\mathfrak{l}$–valued 1–form on $F_{\mathbb{L}}$ defines a holomorphic connection of the principal $\mathbb{L}$–bundle $F_{\mathbb{L}}$. Now Proposition 3.1 completes the proof of the theorem. □

We recall that a holomorphic vector bundle W on X has a holomorphic connection if and only if each indecomposable component of W is of degree zero [We], [At, p. 203, Theorem 10]. This criterion generalizes to holomorphic principal G–bundles on X (see [AB] for details).
We now set $G = \mathrm{GL}(r, \mathbb{C})$ in Theorem 3.4. Let E be a holomorphic vector bundle of rank r on X. Take any

$$\beta \in H^0(X, End(E)).$$

Let

$$E = \bigoplus_{i=1}^{\ell} E_i \tag{3.5}$$

be the generalized eigen-bundle decomposition of E for β. Therefore,

$$\beta|_{E_i} = \lambda_i \cdot \mathrm{Id}_{E_i} + N_i ,$$

where $\lambda \in \mathbb{C}$, and either $N_i = 0$ or N_i is nilpotent.
Then Theorem 3.4 has the following corollary:

COROLLARY 3.5. *For every $N_i \neq 0$, assume that the section N^{r_i-1} of $End(E_i)$ is nowhere vanishing, where r_i is the rank of the vector bundle E_i in* (3.5). *If the holomorphic vector bundle E admits a holomorphic connection, then it admits a holomorphic connection D such that the section β is flat with respect to the connection on $End(E)$ induced by D.*

Consider the condition on β in Corollary 3.5 which says that N^{r_i-1} is nowhere vanishing whenever $N_i \neq 0$. This condition implies that the image of $\beta(x)$ in $\mathrm{M}(r, \mathbb{C})/\mathrm{GL}(r, \mathbb{C})$ is independent of $x \in X$ (here $\mathrm{GL}(r, \mathbb{C})$ acts on its Lie algebra $\mathrm{M}(r, \mathbb{C})$ via conjugation). Therefore, one may ask whether the above mentioned condition in Corollary 3.5 can be replaced by the weaker condition that the conjugacy class of $\beta(x)$ is independent of $x \in X$. Note that if this can be done, then the sufficient condition in Corollary 3.5 for the existence of a connection on E such that β is flat with respect to it actually becomes a necessary and sufficient condition. The following construction of the referee shows that the condition in Corollary 3.5 cannot be replaced by the weaker condition that the conjugacy class of $\beta(x)$ is independent of $x \in X$.

Example 3.6 (Referee). Let X be of sufficiently high genus. Let L and M be holomorphic line bundles on X of degree 1 and degree -2 respectively. Then there exists an indecomposable holomorphic vector bundle E of rank three on X satisfying the following condition: it admits a filtration of holomorphic subbundles

$$L = E_1 \subset E_2 \subset E$$

such that $E_2/L = M$ and $E/E_2 = L$. We omit the detailed arguments given by the referee showing that such a vector bundle E exists. Let β denote the composition

$$E \longrightarrow E/E_2 = L = E_1 \hookrightarrow E\,.$$

Clearly, the conjugacy class of $\beta(x)$ is independent of $x \in X$. The vector bundle E admits a holomorphic connection because it is indecomposable of degree zero. If D is a holomorphic connection on E such that β is flat with respect to the connection on $End(E)$ induced by D, then the subsheaf image$(\beta) \subset E$ is flat with respect to D. But image$(\beta) = L$ does not admit a holomorphic connection because it is of nonzero degree.

Acknowledgements

The referee pointed out an error in an earlier version of the paper, and also made detailed comments that included Example 3.6. This helped the author to find the right formulation of Theorem 3.4. The author is immensely grateful to the referee for this.

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