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# Log Homogeneous Compactifications of Some Classical Groups

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Abstract. We generalize in positive characteristics some results of Bien and Brion on log homogeneous compactifications of a homogeneous space under the action of a connected reductive group. We also construct an explicit smooth log homogeneous compactification of the general linear group by successive blow-ups starting from a grassmannian. By taking fixed points of certain involutions on this compactification, we obtain smooth log homogeneous compactifications of the special orthogonal and the symplectic groups.

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## Introduction

Let $k$ be an algebraically closed field and $G$ a connected reductive group defined over $k$. Given a homogeneous space $\Omega$ under the action of the group $G$ it is natural to consider equivariant compactifications or partial equivariant compactifications of it. Embeddings are normal irreducible varieties equipped with an action of $G$ and containing $\Omega$ as a dense orbit, and compactifications are complete embeddings. Compactifications have shown to be powerful tools to produce interesting representations of the group $G$ or to solve enumerative problems. In the influent paper [21], Luna and Vust developed a classification theory of embeddings of the homogeneous space $\Omega$ assuming that the field $k$ is of characteristic zero. Their theory can be made very explicit and extended to all characteristics, see for instance [15], in the spherical case, that is, when a Borel subgroup of $G$ possesses a dense orbit in the homogeneous space $\Omega$. In this case, the embeddings of $\Omega$ are classified by combinatorial objects called colored fans. If the homogeneous space is a torus acting on itself by multiplication then one recovers the classification of torus embeddings or toric varieties in terms of fans, see for instance [14].

In the first part of the paper we focus on a certain category of "good" compactifications of the homogeneous space $\Omega$. For example, these compactifications are smooth and the boundaries are strict normal crossing divisors. There are several notions of "good" compactifications in the literature. Some of them are defined by geometric conditions, as for example the TOROIDAL compactifications of Mumford [14], the REGULAR compactifications of Bifet De Concini and Procesi [3], the LOG HOMOGENEOUS compactifications of Brion [5] and some of them are defined by conditions from the embedding theory of Luna and Vust, as for example the COLORLESS compactifications. As it was shown by Bien and Brion [5], if the base field $k$ is of characteristic zero then the homogeneous space $\Omega$ admits a log homogeneous compactification if and only if it is spherical, and in that case the four different notions of "good" compactifications mentioned above coincide. We generalize their results in positive characteristics in Section 1. We prove that a homogeneous space admitting a log homogeneous compactification is necessarily SEPARABLY SPHERICAL in the sense of Proposition-Definition 1.7. In that case, we relate the log homogeneous compactifications to the regular and the colorless one, see Theorem 1.8 for a precise statement. We do not know whether the condition of being separably spherical is sufficient for a homogeneous space to have a log homogeneous compactification. Along the way we prove Theorem 1.4, which is of independent interest, on the local structure of colorless compactifications of spherical homogeneous spaces, generalizing a result of Brion, Luna and Vust, see [7].

In Section 2, we focus on the explicit construction of equivariant compactifications of a connected reductive group. That is, the homogeneous space $\Omega$ is a connected reductive group $G$ acted upon by $G\times G$ by left and right translations. The construction of "good" compactifications of a reductive group is a very old problem, with roots in the 19th century in the work of Chasles, Schubert, who were motivated by questions from enumerative geometry. When the group $G$ is semi-simple there is a particular compactification $\overline{G}$ called CANONICAL which possesses interesting properties, making it particularly convenient to work with. For example, the boundary is a divisor whose irreducible component intersect properly and the closure of the $G\times G$-orbits are exactly the partial intersections of these prime divisors. Also, there is a unique closed orbit of $G \times G$ in the canonical compactification of $G$. Moreover, every toroidal compactification of $G$ has a dominant equivariant morphism to $\overline{G}$. If the canonical compactification $\overline{G}$ is smooth, then it is WONDERFUL in sense of Luna [20]. When the group $G$ is of adjoint type, its canonical compactification is smooth, and there are many known constructions of this wonderful compactification, see for example [29], [17], [18], [19], [30], [27], [26] for the case of the projective linear group $\mathrm{PGL}(n)$ and [8], [24], [4] for the general case. In general the canonical compactification is not smooth, as it can be seen for example when $G$ is the special orthogonal group $\mathrm{SO}(2n)$.

One way to construct a compactification of $G$ is by considering a linear representation $V$ of $G$ and taking the closure of $G$ in the projective space $\mathbb{P}(\mathrm{End}(V))$. The compactifications arising in this way are called linear. It was shown by De Concini and Procesi [8] that the linear compactifications of a semi-simple group of adjoint type are of particular interest. Recently, Timashev [28], Gandini and Ruzzi [11], found combinatorial criterions for certain linear compactifications to be normal, or smooth. In [10], Gandini classifies the linear compactifications of the odd special orthogonal group having one closed orbit. By a very new and elegant approach, Martens and Thaddeus [22] recently discovered a general construction of the toroidal compactifications of a connected reductive group $G$ as the coarse moduli spaces of certain algebraic stacks parametrizing objects called "framed principal $G$-bundles over chain of lines".

Our approach is much more classical. In Section 2, we construct a log homogeneous compactification $\mathcal{G}_n$ of the general linear group $\mathrm{GL}(n)$ by successive blow-ups, starting from a grassmannian. The compactification $\mathcal{G}_n$ is defined over an arbitrary base scheme. We then identify the compactifications of the special orthogonal group or the symplectic group obtained by taking the fixed points of certain involutions on the compactification $\mathcal{G}_n$. This provides a new construction of the wonderful compactification of the odd orthogonal group $\mathrm{SO}(2n+1)$, which is of adjoint type, of the symplectic group $\mathrm{Sp}(2n)$, which is not of adjoint type, and of a toroidal desingularization of the canonical compactification of the even orthogonal group $\mathrm{SO}(2n)$ having only two closed orbits. This is the minimal number of closed orbits on a smooth log homogeneous compactification, as the canonical compactification of $\mathrm{SO}(2n)$ is not smooth.

Our procedure is similar to that used by Vainsencher, see [30], to construct the wonderful compactification of the projective linear group $\mathrm{PGL}(n)$ or that of Kausz, see [13], to construct his compactification of the general linear group $\mathrm{GL}(n)$. However, unlike Kausz, we are not able to describe the functor of points of our compactification $\mathcal{G}_n$. In that direction, we obtained a partial result in [12], where we describe the set $\mathcal{G}_n(K)$ for every field $K$. We decided not to include this description in the present paper, as it is long and technical. The functor of points of the wonderful compactification of the projective linear group is described in [27] and that of the symplectic group is described in [1].

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## 1 Log homogeneous compactifications

First we fix some notations. Let $k$ be an algebraically closed field of arbitrary characteristic $p$. By a variety over $k$ we mean a separated integral $k$-scheme of finite type. If $X$ is a variety over $k$ and $x$ is a point of $X$, we denote by $T_{X,x}$ the tangent space of $X$ at $x$. If $Y$ is a subvariety of $X$ containing $x$, we denote by $N_{Y/X,x}$ the normal space to $Y$ in $X$ at $x$.

For an algebraic group $G, H, P \ldots$ we denote by the corresponding gothic letter $\mathfrak{g}, \mathfrak{h}, \mathfrak{p} \ldots$ its Lie algebra. Let $G$ be a connected reductive group defined over $k$. A $G$-variety is a variety equipped with an action of $G$. Let $X$ be a $G$-variety. For each point $x \in X$ we denote by $G_x$ the isotropy group scheme of $x$. We also denote by $orb_x$ the morphism

$$orb_x : G \to X, \quad g \mapsto g \cdot x.$$

The orbit of $x$ under the action of $G$ is called separable if the morphism $orb_x$ is, that is, if its differential is surjective, or, equivalently, if the group scheme $G_x$ is reduced.

We fix a homogeneous space $\Omega$ under the action of $G$. Let $X$ be a smooth compactification of $\Omega$, that is, a complete smooth $G$-variety containing $\Omega$ as an open dense orbit. We suppose that the complement $D$ of $\Omega$ in $X$ is a strict normal crossing divisor.

In [3], Bifet, De Concini and Procesi introduce and study the regular compactifications of a homogeneous space over an algebraically closed field of characteristic zero. We generalize their definition in two different ways :

Definition 1.1. *The compactification $X$ is* regular *(resp.* strongly regular*) if the orbits of $G$ in $X$ are separable, the partial intersections of the irreducible components of $D$ are precisely the closures of the $G$-orbits in $X$ and, for each point $x \in X$, the isotropy group $G_x$ possesses an open (resp. open and separable) orbit in the normal space $N_{Gx/X,x}$ to the orbit $Gx$ in $X$ at the point $x$.*

If the characteristic of the base field $k$ is zero, then the notion of regular and strongly regular coincide with the original notion of [3]. This is no longer true in positive characteristic, as we shall see at the end of Section 1.2.

In [5], Brion defines the log homogeneous compactifications over an algebraically closed field - throughout his paper the base field is also of characteristic zero, but the definition makes sense in arbitrary characteristic. Recall that the logarithmic tangent bundle $T_X(-\log D)$ is the vector bundle over $X$ whose sheaf of section is the subsheaf of the tangent sheaf of $X$ consisting of the derivations that preserve the ideal sheaf $\mathcal{O}_X(-D)$ of $D$. As $G$ acts on $X$ and $D$ is stable under the action of $G$, it is easily seen that the infinitesimal action of the Lie algebra $\mathfrak{g}$ on $X$ gives rise to a natural vector bundle morphism:

$$X \times \mathfrak{g} \to T_X(-\log D).$$

We refer the reader to [5] for further details.

Definition 1.2. *The compactification $X$ is called* LOG HOMOGENEOUS *if the morphism of vector bundles on $X$:*

$$X \times \mathfrak{g} \to T_X(-\log D)$$

*is surjective.*

Assuming that the characteristic of the base field is zero, Bien and Brion prove in [2] that the homogeneous space $\Omega$ possesses a log homogeneous compactification if and only if it is spherical. In this case, they also prove that it is equivalent for a smooth compactification $X$ of $\Omega$ to be log homogeneous, regular or to have no color - as an embedding of a spherical homogeneous space, see [15]. Their proof relies heavily on a local structure theorem for spherical varieties in characteristic zero established by Brion, Luna and Vust in [7].

A generalization of the local structure theorem was obtained by Knop in [16]; essentially, one has to replace in the statement of that theorem an isomorphism by a finite surjective morphism. In Section 1.1 we shall prove that under a separability assumption, the finite surjective morphism in Knop's theorem is an isomorphism. Then, in Section 1.2 we prove that the smooth compactification $X$ of $\Omega$ is regular if and only if the homogeneous space $\Omega$ is spherical, the embedding $X$ has no color and each closed orbit of $G$ in $X$ is separable (Theorem 1.5). We also prove that the smooth compactification $X$ of $\Omega$ is strongly regular if and only if it is log homogeneous (Theorem 1.6). Finally, we exhibit a class of spherical homogeneous spaces for which the notion of regular and strongly regular compactifications coincide. In Section 1.3 we show that log homogeneity is preserved under taking fixed points by an automorphism of finite order prime to the characteristic of the base field $k$. In Section 1.4 we recall the classification of Luna and Vust in the setting of compactification of reductive groups, as this will be useful in Section 2.

## 1.1 A local structure theorem

Let $X$ be a smooth $G$-variety. We assume that there is a unique closed orbit $\omega$ of $G$ in $X$ and that this orbit is complete and separable. We fix a point $x$ on $\omega$. The isotropy group $G_x$ is a parabolic subgroup of $G$. We fix a Borel subgroup $B$ of $G$ such that $BG_x$ is open in $G$. We fix a maximal torus $T$ of $G$ contained in $G_x$ and $B$ and we denote by $P$ the opposite parabolic subgroup to $G_x$ containing $B$. We also denote by $L$ the Levi subgroup of $P$ containing $T$ and by $R_u(P)$ the unipotent radical of $P$. With these notations we have the following proposition, which relies on a result of Knop [16, Theorem 1.2].

Proposition 1.3. *There exists an affine open subvariety $X_s$ of $X$ which is stable under the action of $P$ and a closed subvariety $Z$ of $X_s$ stable under the action of $T$, containing $x$ such that:*

(1) *The variety $Z$ is smooth at $x$ and the vector space $T_{Z,x}$ endowed with the action of $T$ is isomorphic to the vector space $N_{\omega/X,x}$ endowed with the action of $T$.*

(2) *The morphism:*

$$\mu : R_u(P) \times Z \to X_s, \quad (p, z) \mapsto p \cdot z$$

*is finite, surjective, étale at $(e, x)$, and the fiber $\mu^{-1}(x)$ is reduced to the single point $\{(e, x)\}$.*

*Proof.* As the smooth $G$-variety $X$ has a unique closed orbit, it is quasi-projective by a famous result of Sumihiro, see [25]. We fix a very ample line bundle $\mathcal{L}$ on $X$. We fix a $G$-linearization of this line bundle. By [16, Theorem 2.10], there exists an integer $N$ and a global section $s$ of $\mathcal{L}^N$ such that the nonzero locus $X_s$ of $s$ is an affine open subvariety containing the point $x$ and the stabilizer of the line spanned by $s$ in the vector space $H^0(X, \mathcal{L}^N)$ is $P$. The open subvariety $X_s$ is therefore affine, contains the point $x$ and is stable under the action of the parabolic subgroup $P$. Using the line bundle $\mathcal{L}^N$, we embed $X$ into a projective space $\mathbb{P}(V)$ on which $G$ acts linearly. We choose a $T$-stable complement $S$ to $T_{\omega,x}$ in the tangent space $T_{\mathbb{P}(V),x}$, such that $S$ is the direct sum of a $T$-stable complement of $T_{\omega,x}$ in $T_{X,x}$ and a $T$-stable complement of $T_{X,x}$ in $T_{\mathbb{P}(V),x}$. This is possible because $T$ is a linearly reductive group.

We consider now the linear subspace $S'$ of $\mathbb{P}(V)$ containing $x$ and whose tangent space at $x$ is $S$. It is a $T$-stable subvariety of $\mathbb{P}(V)$. By [16, Theorem 1.2], there is an irreducible component $Z$ of $X_s \cap S'$ containing $x$ and such that the morphisms

$$\mu : R_u(P) \times Z \to X_s, \quad (p, z) \mapsto p \cdot z$$
$$\nu : Z \to X_s/R_u(P), \quad z \mapsto zR_u(P)$$

are finite and surjective. Moreover, the fiber $\mu^{-1}(x)$ is reduced to the single point $(e, x)$. We observe now that $S'$ intersects $X_s$ transversally at $x$. This implies that the subvariety $Z$ is smooth at $x$. It is also $T$-stable, as an irreducible component of $X_s \cap S'$. By definition, the parabolic subgroup $P$ contains the Borel subgroup $B$, therefore the orbit $Px = R_u(P)x$ is open in $\omega$. Moreover, we have the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{p}_u$, where $\mathfrak{p}_u$ is the Lie algebra of the unipotent radical $R_u(P)$ of $P$. The morphism

$$d_e orb_x : \mathfrak{g} \to T_{\omega,x}$$

is surjective and identically zero on $\mathfrak{g}_x$. This proves that the restriction of this morphism to $\mathfrak{p}_u$ is an isomorphism. The morphism

$$\mu : R_u(P) \times Z \to X_s, \quad (p, z) \mapsto p \cdot z$$

is therefore étale at $(e, x)$. Indeed, its differential at this point is:

$$\mathfrak{p}_u \times T_{Z,x} \to T_{X,x}, \quad (h, k) \mapsto d_e orb_x(h) + k.$$

We also see that the spaces $T_{Z,x}$ and $N_{\omega/X,x}$ endowed with their action of the torus $T$ are isomorphic, completing the proof of the proposition. $\square$

We now suppose further that $X$ is an embedding of the homogeneous space $\Omega$. With this additional assumption we have:

THEOREM 1.4. *The following three properties are equivalent:*

*(1) The homogeneous space $\Omega$ is spherical and the embedding $X$ has no color.*

*(2) The torus $T$ possesses an open orbit in the normal space $N_{\omega/X,x}$. Moreover, the complement $D$ of $\Omega$ in $X$ is a strict normal crossing divisor and the partial intersections of the irreducible components of $D$ are the closure of the $G$-orbits in $X$.*

*(3) The set $X_0 = \{y \in X, \quad x \in \overline{By}\}$ is an affine open subvariety of $X$ which is stable by $P$. Moreover, there exists a closed subvariety $Z$ of $X_0$ which is smooth, stable by $L$, on which the derived subgroup $[L, L]$ acts trivially and containing an open orbit of the torus $L/[L, L]$, such that the morphism:*

$$R_u(P) \times Z \to X_0, \quad (p, z) \mapsto p \cdot z$$

*is an isomorphism. Finally, each orbit of $G$ in $X$ intersects $Z$ along a unique orbit of $T$.*

*Proof.* (3) $\Rightarrow$ (1) As $T$ possesses an open orbit in $Z$, we see that the Borel subgroup $B$ has an open orbit in $X$, and the homogeneous space $\Omega$ is spherical. Moreover, let $D$ be a $B$-stable prime divisor on $X$ containing $\omega$. Using the isomorphism in (3) we can write

$$D \cap X_0 = R_u(P) \times (D \cap Z).$$

As $D \cap Z$ is a closed irreducible $T$-stable subvariety of $Z$, it is the closure of a $T$-orbit in $Z$. As the $T$-orbits in $Z$ correponds bijectively to the $G$-orbits in $X$, we see that $D$ is the closure of a $G$-orbit in $X$ and is therefore stable under the action of $G$. This proves that the embedding $X$ of $\Omega$ has no color.

(3) $\Rightarrow$ (2) The isomorphism in (3) proves that the spaces $T_{Z,x}$ and $N_{\omega/X,x}$ endowed with their actions of the torus $T$ are isomorphic. As $T$ possesses an open dense orbit in the first one, it also has an open dense orbit in the latter. As $Z$ is smooth toric variety, we see that the complement of the open orbit of $T$ in $Z$ is a strict normal crossing divisor whose associated strata are the $T$-orbits in $Z$. Using the isomorphism given by (3), we see that the complement of the open orbit of the parabolic subgroup $P$ in $X_0$ is a strict normal crossing divisor whose associated strata are the products $R_u(P) \times \Omega'$, where $\Omega'$ runs over the set of $T$-orbits in $Z$. To complete the proof that property (2) is satisfied, we translate the open subvariety $X_0$ by elements of $G$ and we use the fact that each $G$-orbit in $X$ intersects $Z$ along a unique $T$-orbit.

(1) $\Rightarrow$ (3) We use the notations of Proposition 1.3. By [15, Lemma 6.5] the fact that the embedding $X$ has no color implies that the parabolic subgroup $P$ is the stabilizer of the open $B$-orbit $\Omega$ in $X$. Using this fact and [16, Theorem 2.8] we obtain that the derived subgroup of $P$, and therefore the derived group of $L$, acts trivially on $X_s/R_u(P)$. Moreover, as the homogeneous space $\Omega$ is spherical, the Levi subgroup $L$ has an open orbit in $X_s/R_u(P)$. The torus $T$ has therefore an open orbit in $X_s/R_u(P)$, as the derived group of $L$ acts trivially. Using the finite surjective morphism $\nu$ appearing in the proof of Proposition 1.3, we see that $T$ has an open orbit in $Z$. $Z$ is therefore a smooth affine toric variety with a fixed point under the action of a quotient of $T$. Moreover, as the subvariety $Z$ is left stable under the action of $T$ and the derived group $[L,L]$ acts trivially on $Z$, we see that the Levi subgroup $L$ leaves the subvariety $Z$ invariant.

We observe now that the locus of points of $R_u(P)\times Z$ where $\mu$ is not étale is closed and stable under the actions of $R_u(P)$ and $T$. The unique closed orbit of $R_u(P)\rtimes T$ in $R_u(P)\times Z$ is $R_u(P)x$ and $\mu$ is étale at $(e,x)$, therefore we obtain that $\mu$ is an étale morphism. As the morphism $\mu$ is also finite of degree 1 (the fiber of $\{x\}$ being reduced to a single point), it is an isomorphism.

We prove now that each $G$-orbit in $X$ intersects $Z$ along a unique $T$-orbit. First, we observe that, as $\omega$ is the unique closed orbit of $G$ in $X$, the open subvariety $X_s$ intersects every $G$-orbit. We shall prove that the closures of the $G$-orbits in $X$ corresponds bijectively to the closures of the $T$-orbits in $Z$. Let $X'$ be the closure of a $G$-orbit in $X$. As $X'$ is the closure of $X'\cap X_s$, it is also equal, using the isomorphism $\mu$, to the closure of $R_u(P)(X'\cap Z)$. The closed subvariety $X'\cap Z$ of $Z$ is therefore a closed irreducible $T$-stable subvariety. We can conclude that it is the closure of a $T$-orbit in $Z$. Conversely let $Z'$ be the closure of a $T$-orbit in $Z$. As $Z$ is a smooth toric variety, we can write

$$Z' = D_1' \cap D_2' \cap ... \cap D_r',$$

where the $D_i'$s are $T$-stable prime divisors on $Z$. We observe that the prime divisors

$$\overline{R_u(P)D_1'}, ..., \overline{R_u(P)D_r'}$$

on $X$ are stable under the action of $P$. Indeed, the orbits of $P$ in $X_s$ are exactly the orbits of $R_u(P)\rtimes T$ in $X_s$. As $X$ has no color, the fact that these divisors contain the closed orbit $\omega$ proves that they are stable under the action of $G$. Their intersection $\overline{R_u(P)Z}$ is also $G$-stable. As it is irreducible, we can conclude that it is the closure of a $G$-orbit in $X$.

In order to complete the proof that (1) $\Rightarrow$ (3), it remains to show that

$$X_s = \{y \in X, \quad x \in \overline{By}\}.$$

Let $y$ be a point on $X$ such that $x$ belongs to $\overline{By}$. The intersection $X_s \cap \overline{By}$ is a non empty open subset of $\overline{By}$ which is stable under the action of $B$. Therefore it contains $y$, that is, $y$ belongs to $X_s$. Now let $y$ be a point on $X_s$. The closed subvariety $\overline{By}$ contains a closed $B$-orbit in $X_s$. As the unique closed orbit of $B$ in $X_s$ is the orbit of $x$, we see that $x$ belongs to $\overline{By}$, completing the argument.

($2 \Rightarrow 3$) We use the notations introduced in Proposition 1.3. By assumption, the torus $T$ possesses an open orbit in the normal space $N_{\omega/X,x}$. Moreover, by Proposition 1.3, the spaces $T_{Z,x}$ and $N_{\omega/X,x}$ endowed with their actions of $T$ are isomorphic. Therefore, the torus $T$ possesses an open dense orbit in $T_{Z,x}$. It is then an easy exercise left to the reader to prove that the variety $Z$ is a smooth toric variety for a quotient of $T$. The same arguments as above prove that the morphism $\mu$ is an isomorphism.

We prove now that each $G$-orbit in $X$ intersects $Z$ along a unique orbit of $T$. Let $D$ be the complement of $\Omega$ in $X$. By assumption, it is a strict normal crossing divisor whose associated strata are the $G$-orbits in $X$. We denote by $D_1, \ldots, D_r$ the irreducible component of $D$. As there is a unique closed orbit of $G$ on $X$ each partial intersection $\bigcap_{i \in I} D_i$ is non empty and irreducible or, in other words, it is a stratum of $D$. The integer $r$ is the codimension of the closed orbit $\omega$ in $X$, and there are exactly $2^r$ $G$-orbits in $X$. As the variety $Z$ is a smooth affine toric variety of dimension $r$ with a fixed point, we see that there are exactly $2^r$ orbits of $T$ on $Z$. As each orbit of $G$ in $X$ intersect $Z$ we see that the intersection of a $G$-orbit with $Z$ is a single $T$-orbit.

Finally, we prove that the open subvariety $X_s$ is equal to $X_0$ by the same argument as in the proof of $(1) \Rightarrow (3)$, completing the proof of the theorem. □

## 1.2 Regular, strongly regular and log homogeneous compactifications

In this section we use the following notation. Let $X$ be a $G$-variety with a finite number of orbits (for example, a spherical variety). Let $\omega$ be an orbit of $G$ in $X$. We denote by

$$X_{\omega,G} = \{y \in X, \omega \subseteq \overline{Gy}\}.$$

It is an open $G$-stable subvariety of $X$ in which $\omega$ is the unique closed orbit.

Theorem 1.5. *Let $X$ be a smooth compactification of the homogeneous space $\Omega$. The following two properties are equivalent:*

*(1) $X$ is regular.*

*(2) The homogeneous space $\Omega$ is spherical, the embedding $X$ has no color and the orbits of $G$ in $X$ are separable.*

*Proof.* Suppose that $X$ is regular. Let $D$ be the complement of $\Omega$ in $X$. It is a strict normal crossing divisor. Let $\omega$ be a closed, and therefore complete and separable, orbit of $G$ in $X$. We use the notations introduced at the beginning of Section 1.4 with $X_{\omega,G}$ in place of $X$. The normal space $N_{\omega/X,x}$ is the normal space to a stratum of the divisor $D$ and therefore possesses a natural direct sum decomposition into a sum of lines, each of them being stable under the action of $G_x$ (which is connected, as it is a parabolic subgroup of $G$). Therefore the representation of $G_x$ in $N_{\omega/X,x}$ factors through the action of a torus.This proves that the derived group of $L$ acts trivially in this space, proving that the torus $T$ has a dense orbit in $N_{\omega/X,x}$. By Theorem 1.4 (applied to $X_{\omega,G}$) the homogeneous space $\Omega$ is spherical and the embedding $X_{\omega,G}$ has no color. As this is true for each closed orbit $\omega$ of $G$ in $X$, we see that the embedding $X$ has no color.

We assume now that $\Omega$ is spherical, $X$ has no color and that each orbit of $G$ in $X$ is separable. By applying Theorem 1.4 to each open subvariety $X_{\omega,G}$, where $\omega$ runs over the set of closed orbits of $X$, we see that the complement $D$ of $\Omega$ in $X$ is a strict normal crossing divisor and that, for each point $x$ in $X$, the isotropy group $G_x$ has an open orbit in the normal space $N_{Gx/X,x}$. Moreover, by assumption, the $G$-orbits in $X$ are separable. To complete the proof of the theorem, it remains to show that the partial intersections of the irreducible components of $D$ are irreducible. But this is true on every colorless embedding of a spherical homogeneous space, due to the combinatorial description of these embeddings, see [15, Section 3]. □

Theorem 1.6. *Let $X$ be a smooth compactification of $\Omega$. The following two properties are equivalent:*

*(1) $X$ is a log homogeneous compactification.*

*(2) $X$ is strongly regular.*

*Proof.* We suppose first that the compactification $X$ is log homogeneous. We denote by $D$ the complement of $\Omega$ in $X$. It is a strict normal crossing divisor. Following the argument given in [5, Proposition 2.1.2] we prove that each stratum of the strict normal crossing divisor $D$ is a single orbit under the action of $G$ which is separable and that for each point $x \in X$, the isotropy group $G_x$ possesses an open and separable orbit in the normal space $N_{Gx/X,x}$. In order to conclude, it remains to prove that the partial intersection of the irreducible components of $D$ are irreducible. But the same argument as in the proof of Theorem 1.5 prove that $\Omega$ is spherical and $X$ has no color, which is sufficient to complete the proof.

Conversely, if $X$ is supposed to be strongly regular, the proof of [5, Proposition 2.1.2] adapts without change and shows that $X$ is a log homogeneous compactification of $\Omega$.

□

Proposition-Definition 1.7. *If the homogeneous space $\Omega$ possesses a log homogeneous compactification, then it satisfies the following equivalent conditions:*

*(1) The homogeneous space $\Omega$ is spherical and there exists a Borel subgroup of $G$ whose open orbit in $\Omega$ is separable.*

*(2) The homogeneous space $\Omega$ is spherical and the open orbit of each Borel subgroup of $G$ in $\Omega$ is separable.*

*(3) The homogeneous space $\Omega$ is separable under the action of $G$, and there exists a point $x$ in $X$ and a Borel subgroup $B$ of $G$ such that : $\mathfrak{b} + \mathfrak{g}_x = \mathfrak{g}$.*

*A homogeneous space satisfying one of these properties is said to be* SEPARABLY SPHERICAL.

*Proof.* We suppose first that the homogeneous space $\Omega$ possesses a log homogeneous compactification $X$ and we prove that it satisfies the first condition. By Theorem 1.6 and 1.5, the homogeneous space $\Omega$ is spherical. Let $\omega$ be a closed, and therefore complete and separable, orbit of $G$ in $X$. We apply Theorem 1.4 to the open subvariety $X_{\omega,G}$. We use the notations introduced for this theorem. As $X$ is strongly regular, the maximal torus $T$ has an open and separable orbit in $T_{Z,x} = N_{\omega/X,x}$. As this space endowed with its action of $T$ is isomorphic to $Z$ endowed with its action of $T$, because $Z$ is an affine smooth toric variety with fixed point for a quotient of $T$, we see that the open orbit of $T$ in $Z$ is separable. Consequently, the open orbit of $R_u(P) \rtimes T$ in $R_u(P) \times Z$ is separable, and the open orbit of $B$ in $\Omega$ is separable.

We prove now that the three conditions in the statement of the proposition-definition are equivalent. As the Borel subgroups of $G$ are conjugated, condition (1) and (2) are equivalent. Suppose now that condition (1) is satisfied. Let $B$ be a Borel subgroup of $G$ and $x$ a point in the open and separable orbit of $B$ in $\Omega$. The linear map $d_e orb_x : \mathfrak{b} \to T_{Bx,x}$ is surjective. As the orbit $Bx$ is open in $\Omega$ we see that the homogeneous space $\Omega$ is separable under the action of $G$ and that

$$\mathfrak{b} + \mathfrak{g}_x = \mathfrak{g}.$$

Conversely, we suppose that condition (3) is satisfied. As the homogeneous space $\Omega$ is separable, the linear map

$$d_e orb_x : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_x$$

is the natural projection. As we have $\mathfrak{b} + \mathfrak{g}_x = \mathfrak{g}$, we see that the linear map

$$d_e orb_x : \mathfrak{b} \to \mathfrak{g}/\mathfrak{g}_x$$

is surjective. This means precisely that the orbit $Bx$ is open in $\Omega$ and separable. $\square$

Here are some example of separably spherical homogeneous spaces: separable quotients of tori, partial flag varieties, symmetric spaces in characteristic not 2 (Vust proves in [31] that symmetric spaces in characteristic zero are spherical; his proof extends to characteristic not 2 to show that symmetric spaces are separably spherical).

Theorem 1.8. *We assume that the homogeneous space $\Omega$ is separably spherical. Let $X$ be a smooth compactification of $\Omega$. The following conditions are equivalent:*

*(1) $X$ has no color and the closed orbits of $G$ in $X$ are separable.*

*(2) $X$ is regular.*

*(3) $X$ is strongly regular.*

*(4) $X$ is log homogeneous under the action of $G$.*

*Proof.* In view of Theorem 1.5 and 1.6 it suffices to show that (1) $\Rightarrow$ (3). We assume that condition (1) is satisfied. Let $\omega$ be a closed, and therefore separable orbit of $G$ in $X$. We apply Theorem 1.4 to the open subvariety $X_{\omega,G}$ of $X$ introduced in the proof of Theorem 1.5. We use the notations introduced for Theorem 1.4. As the open orbit of $B$ in $\Omega$ is separable, we see that the quotient of $T$ acting on $Z$ is separable. As $Z$ is a smooth affine toric variety with fixed point under this quotient, we see that the orbits of $T$ in $Z$ are all separable and that for each point $z \in Z$, the stabilizer $T_z$ has an open and separable orbit in the normal space $N_{Tz/Z,z}$. From this we get readily that the embedding $X_{\omega,G}$ of $\Omega$ satisfies the conditions defining a strongly regular embedding. As this is true for each closed orbit $\omega$, we see that $X$ is a strongly regular compactification of $\Omega$. $\square$

We end this section with an example of a regular compactification of a homogeneous space which is not strongly regular. We suppose that the base field $k$ has characteristic 2. Let $G$ be the group $\mathrm{SL}(2)$ acting on $X := \mathbb{P}^1 \times \mathbb{P}^1$. There are two orbits: the open orbit $\Omega$ of pairs of distinct points and the closed orbit $\omega$, the diagonal, which has codimension one in $X$. These orbits are separable under the action of $G$. Moreover, the complement of the open orbit, that is, the closed orbit $\omega$, is a strict normal crossing divisor and the partial intersections of its irreducible components are the closure of $G$-orbits in $X$. A quick computation shows that for each point on the closed orbit $\omega$, the isotropy group has an open non separable orbit in the normal space to the closed orbit at that point. Therefore the compactification $X$ of $\Omega$ is regular and not strongly regular. By Theorem 1.8 the homogeneous space cannot be separably spherical. This can be seen directly as follows. The homogeneous space $\Omega$ is the quotient of $G$ by a maximal torus $T$. A Borel subgroup $B$ of $G$ has an open orbit in $\Omega$ if and only if it does not contain $T$. But in that case the intersection $B \cap T$ is the center of $G$, which is not reduced because the characteristic of the base field is 2.

### 1.3 Log homogeneous compactifications and fixed points

Let $X$ be a smooth variety over the field $k$ and $\sigma$ an automorphism of $X$ which has finite order $r$ prime to the characteristic $p$ of $k$. Fogarty proves in [9] that the fixed point subscheme $X^\sigma$ is smooth and that, for each fixed point $x$ of $\sigma$ in $X$, the tangent space to $X^\sigma$ at $x$ is $T^\sigma_{X,x}$.

We suppose now that $X$ is a smooth log homogeneous compactification of the homogeneous space $\Omega$. We also assume that the automorphism $\sigma$ leaves $\Omega$ stable and is $G$-equivariant, in the sense that there exists an automorphism $\sigma$ of the group $G$ satisfying

$$\forall g \in G, \quad \forall x \in X, \quad \sigma(gx) = \sigma(g)\sigma(x).$$

By [23, Proposition 10.1.5], the neutral component $G'$ of the group $G^\sigma$ is a reductive group. Moreover, each connected component of the variety $\Omega^\sigma$ is a homogeneous space under the action of $G'$. We let $\Omega'$ be such a component and $X'$ be the connected component of $X^\sigma$ containing $\Omega'$.

Proposition 1.9. *$X'$ is a log homogeneous compactification of $\Omega'$ under the action of $G'$.*

*Proof.* Let $D$ be the complement of $\Omega$ in $X$. Let $x$ be a point in $X'$. Let $D_1, \dots, D_s$ be the irreducible components of $D$ containing $x$. First we prove that the intersection $D' := D \cap X'$ is a strict normal crossing divisor. For each index $i$, the intersection $D'_i := D_i \cap X'$ is a divisor on $X'$. Indeed, $X'$ is not contained in $D_i$ as it contains $\Omega'$. As $x$ is fixed by the automorphism $\sigma$, we can assume that the components $D_i$s are ordered in such a way that

$$\sigma(D_2) = D_1 \ \dots \ \sigma(D_{i_1}) = D_{i_1-1}, \ \sigma(D_1) = D_{i_1}$$

$$\dots$$

$$\sigma(D_{i_{t-1}+2}) = D_{i_{t-1}+1} \ \dots \ \sigma(D_{i_t}) = \sigma(D_s) = D_{i_t-1}, \ \sigma(D_{i_{t-1}+1}) = D_{i_t}.$$

By convention we define $i_0 = 0$. For each integer $j$ from 1 to $t$, and each integer $i$ from $i_{j-1}+1$ to $i_j$ we have $D'_i = D'_{i_j}$. Therefore we see that $D'_{i_j}$ is the connected component of the smooth variety $(D_{i_{j-1}+1} \cap \dots \cap D_{i_j})^\sigma$ containing $x$. Consequently, it is smooth. For the moment, we have proved that $D'$ is a divisor on $X'$ whose irreducible components are smooth.

We prove now that the divisor $D'_{i_1}, \dots, D'_{i_t}$ intersect transversally at the point $x$. Let $U_x$ be an open neighborhood of $x$ in $X$ which is stable by the automorphism $\sigma$ and on which the equation of $D$ is $u_1 \dots u_s = 0$, where $u_1 \dots u_s \in \mathcal{O}_X(U_x)$ are part of a regular local parameter system at $x$ and satisfy:

$$\sigma(u_2) = u_1 \ \dots \ \sigma(u_{i_1}) = u_{i_1-1}$$

$$\dots$$

$$\sigma(u_{i_{t-1}+2}) = u_{i_{t-1}+1} \ \dots \ \sigma(u_{i_t}) = u_{i_t-1}.$$

We aim to prove that the images of the differential $d_x u_{i_j}$ by the natural projection

$$(T_{X,x})^* \to (T_{X',x})^*$$

are linearly independent, where $j$ run from 1 to $t$. As the point $x$ is fixed by $\sigma$, $\sigma$ acts by the differential on the tangent space $T_{X,x}$ and by the dual action on $(T_{X,x})^*$. As the order of the automorphism $\sigma$ is prime to the characteristic $p$, we have a direct sum decomposition:

$$(T_{X,x})^* = ((T_{X,x})^*)^\sigma \oplus \mathrm{Ker}(id + \sigma + \cdots + \sigma^{r-1})$$

where the projection on the first factor is given by

$$l \mapsto \frac{1}{r}(l + \sigma(l) + \cdots + \sigma^{r-1}(l)).$$

Moreover, as $T_{X',x}$ is equal to $(T_{X,x})^\sigma$, the second factor in this decomposition is easily seen to be $(T_{X',x})^\perp$, so that the natural projection

$$(T_{X,x})^* \to (T_{X',x})^*$$

gives an isomorphism

$$((T_{X,x})^*)^\sigma \to (T_{X',x})^*.$$

Finally, the images of the differential $d_x u_{i_j}$ in $(T_{X',x})^*$ are linearly independent, because the differentials $d_x u_i$ are linearly independent in $(T_{X,x})^*$.

We have proved that the divisor $D'$ is a strict normal crossing divisor. We leave it as an exercise to the reader to prove that there exists a natural exact sequence of vector bundle on $X'$

$$0 \to T_{X'}(-\log D') \to T_X(-\log D)_{|X'} \to N_{X'/X} \to 0,$$

and that the space $T_{X'}(-\log D')_x$ is the subspace of fixed point by $\sigma$ in the space $T_X(-\log D)_x$. Now, the compactification $X$ is log homogeneous, therefore the linear map

$$\mathfrak{g} \to T_X(-\log D)_x$$

is surjective. As $r$ and $p$ are relatively prime, this linear map is still surjective at the level of fixed points. That is, the linear map

$$\mathfrak{g}^\sigma \to T_X(-\log D)_x^\sigma = T_{X'}(-\log D')_x$$

is surjective. This complete the proof of the proposition. □

## 1.4 The example of reductive groups

In this section the homogeneous space $\Omega$ is a connected reductive group $G$ acted upon by the group $G \times G$ by the following formula:

$$\forall (g,h) \in G \times G, \quad \forall x \in G, \quad (g,h) \cdot x = gxh^{-1}$$

We would like to explain here the classification of smooth log homogeneous compactifications of $G$. Observe that the homogeneous space $G$ under the action of $G \times G$ is actually separably spherical. By Theorem 1.8, its smooth log homogeneous compactifications are the smooth colorless compactifications with separable closed orbits. The last condition is actually superfluous : by [6, Chapter 6], the closed orbits of $G \times G$ in a colorless compactification of $G$ are isomorphic to $G/B \times G/B$, where $B$ is a Borel subgroup of $G$. The log homogeneous compactifications of $G$ are therefore the smooth colorless one.

We now recall the combinatorial description of the smooth colorless compactifications of $G$. Let $T$ be a maximal torus of $G$ and $B$ a Borel subgroup of $G$ containing $T$. We denote by $V$ the $\mathbb{Q}$-vector space spanned by the one-parameter subgroups of $T$ and by $\mathcal{W}$ the Weyl chamber corresponding to $B$. Let $X$ be a smooth colorless embedding of $G$. We let the torus $T$ act "on the left" on $X$. For this action, the closure of $T$ in $X$ is a smooth complete toric variety. We associate to $X$ the fan consisting of those cones in the fan of the toric variety $\overline{T}$ which are included in $-\mathcal{W}$. This sets a map from the set of smooth colorless compactifications of $G$ to the set of fans in $V$ with support $-\mathcal{W}$ and which are smooth with respect to the lattice of one parameter subgroups in $V$. This map is actually a bijection, see for instance [6, Chapter 6].

## 2 Explicit compactifications of classical groups

We construct a log homogeneous compactification $\mathcal{G}_n$ of the general linear group $\mathrm{GL}(n)$ by successive blow-ups, starting from a grassmannian. The precise procedure is explained in Section 2.1. The compactification $\mathcal{G}_n$ is defined over an arbitrary base scheme. In Section 2.2 we study the local structure of the action of $\mathrm{GL}(n) \times \mathrm{GL}(n)$ on $\mathcal{G}_n$, still over an arbitrary base scheme. This enables us to compute the colored fan of $\mathcal{G}_n$ over an algebraically closed field in Section 2.4. Using this computation, we are able to identify the compactifications of the special orthogonal group or the symplectic group obtained by taking the fixed points of certain involutions on the compactification $\mathcal{G}_n$. In the odd orthogonal and symplectic case we obtain the wonderful compactification. In the even orthogonal case we obtain a log homogeneous compactification with two closed orbits. This is the minimal number of closed orbits on a smooth log homogeneous compactification, as the canonical compactification of $\mathrm{SO}(2n)$ is not smooth.

### 2.1 The compactifications $\mathcal{G}_m$

As we mentioned above our construction works over an arbitrary base scheme: until the end of Section 2.3 we work over a base scheme $S$. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two free modules of constant finite rank $n$ on $S$. We denote by $\mathcal{V}$ the direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$. We denote by $p_1$ and $p_2$ the projections respectively on the first and the second factor of this direct sum. We denote by $G$ the group scheme $\mathrm{GL}(\mathcal{V}_1) \times \mathrm{GL}(\mathcal{V}_2)$ which is a subgroup scheme of $\mathrm{GL}(\mathcal{V})$.

Definition 2.1. *We denote by* $\Omega := \mathrm{Iso}(\mathcal{V}_2, \mathcal{V}_1)$ *the scheme over* $S$ *parametrizing the isomorphisms from* $\mathcal{V}_2$ *to* $\mathcal{V}_1$.

There is a natural action of the group scheme $G$ on $\Omega$, via the following formulas

$$\forall (g_1, g_2) \in G, \quad \forall x \in \Omega, \quad (g_1, g_2) \cdot x = g_1 x g_2^{-1}$$

For this action, $\Omega$ is a homogeneous space under the action of $G$.

Definition 2.2. *We denote by $\mathcal{G}$ the grassmannian*

$$\pi : \mathcal{G}r_S(n, \mathcal{V}) \to S$$

*parametrizing the submodules of $\mathcal{V}$ which are locally direct summands of rank $n$. We denote by $\mathcal{T}$ the tautological module on $\mathcal{G}$.*

The module $\mathcal{T}$ is a submodule of $\pi^*\mathcal{V}$ which is locally a direct summand of finite constant rank $n$. There is a natural action of the group scheme $\mathrm{GL}(\mathcal{V})$, and therefore of the group scheme $G$, on the grassmannian $\mathcal{G}$. Moreover, $\Omega$ is contained in $\mathcal{G}$ as a $G$-stable open subscheme via the graph

$$\Omega \to \mathcal{G}, \quad x \mapsto \mathrm{Graph}(x).$$

Definition 2.3. *We denote by $p$ the following morphism of modules on the grassmannian $\mathcal{G}$ :*

$$p = \pi^* p_1 \oplus \pi^* p_2 : \mathcal{T}^{\oplus 2} \to \pi^* \mathcal{V}.$$

Definition 2.4. *For $d \in [\![0, n]\!]$, we denote by $\mathcal{H}_d$ the locally free module*

$$\mathcal{H}om\Big(\bigwedge^{n+d}(\mathcal{T}^{\oplus 2}), \bigwedge^{n+d}(\pi^*\mathcal{V})\Big).$$

*on the grassmannian $\mathcal{G}$.*

Definition 2.5. *For $d \in [\![0, n]\!]$, the exterior power $\wedge^{n+d} p$ is a global section of $\mathcal{H}_d$. We denote by $\mathcal{Z}_d$ the zero locus of $\wedge^{n+d} p$ on the grassmannian $\mathcal{G}$.*

We define in this way a sequence of $G$-stable closed subschemes on the grassmannian $\mathcal{G}$

$$\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \cdots \subset \mathcal{Z}_n \subset \mathcal{G}.$$

Observe that the closed subscheme $\mathcal{Z}_0$ is actually empty. Moreover, it is easy to prove that the open subscheme $\Omega$ is the complement of $\mathcal{Z}_n$ in $\mathcal{G}$.

We will now define a sequence of blow-ups

$$\mathcal{G}_n \xrightarrow{b_n} \mathcal{G}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_1 \xrightarrow{b_1} \mathcal{G}_0$$

and, for each integer $m$ between 0 and $n$, a family of closed subschemes $\mathcal{Z}_{m,d}$ of $\mathcal{G}_m$, where $d$ runs from $m$ to $n$.

Definition 2.6. *Let $m \in [\![0, n]\!]$ and $d \in [\![m, n]\!]$. The definition is by induction:*

- *For $m = 0$, we set $\mathcal{G}_0 := \mathcal{G}$ and $\mathcal{Z}_{0,d} := \mathcal{Z}_d$.*
- *Assuming that the scheme $\mathcal{G}_{m-1}$ and its subschemes $\mathcal{Z}_{m-1,d}$ are defined, we define*
$$b_m : \mathcal{G}_m \to \mathcal{G}_{m-1}$$
*to be the blow-up centered at $\mathcal{Z}_{m-1,m}$ and, for each integer $d$ from $m$ to $n$, we define $\mathcal{Z}_{m,d}$ to be the strict transform of $\mathcal{Z}_{m-1,d}$ that is, the schematic closure of*
$$b_m^{-1}(\mathcal{Z}_{m-1,d} \setminus \mathcal{Z}_{m-1,m})$$
*in $\mathcal{G}_m$.*

*Moreover, we denote by $\mathcal{I}_{m,d}$ the ideal sheaf on $\mathcal{G}_m$ defining $\mathcal{Z}_{m,d}$.*

The group scheme $G$ acts on the schemes $\mathcal{G}_m$ and leaves the subschemes $\mathcal{Z}_{m,d}$ globally invariant. Modulo Proposition 2.17 below, we prove now:

Theorem 2.7. *For each integer $m$ from $0$ to $n-1$, the $S$-scheme $\mathcal{G}_m$ is a smooth projective compactification of $\Omega$.*

*Proof.* By Proposition 2.17 the scheme $\mathcal{G}_m$ is covered by a collection of open subschemes isomorphic to affine spaces over $S$. In particular, the $S$-scheme $\mathcal{G}_m$ is smooth. It is a classical fact that the grassmannian $\mathcal{G}$ is projective over $S$. As the blow-up of a projective scheme over $S$ along a closed subscheme is projective over $S$, we see that $\mathcal{G}_m$ is projective over $S$. Finally, observe that the open subscheme $\Omega$ of $\mathcal{G}$ is disjoint from the closed subscheme $\mathcal{Z}_n$ and therefore from each of the closed subscheme $\mathcal{Z}_d$. As a consequence, $\Omega$ is an open subscheme of each of the $\mathcal{G}_m$.

□

### 2.2 An atlas of affine charts for $\mathcal{G}_m$

Let $V$ be the set $[\![1, n]\!] \times \{1, 2\}$. We denote by $V_1$ the subset $[\![1, n]\!] \times \{1\}$ and by $V_2$ the subset $[\![1, n]\!] \times \{2\}$. We shall refer to elements of $V_1$ as elements of $V$ of type 1 and elements of $V_2$ as elements of type 2. We fix a basis $v_i$, $i \in V$, of the free module $\mathcal{V}$. We suppose that $v_i$, $i \in V_1$ is a basis for $\mathcal{V}_1$ and $v_i$, $i \in V_2$ is a basis for $\mathcal{V}_2$. Moreover, for each subset $I$ of $V$, we denote by $\mathcal{V}_I$ the free submodule of $\mathcal{V}$ spanned by the $(v_i)_{i\in I}$. For every integer $m$ from 1 to $n$, we denote by $V^{>m}$ the set $[\![m+1, n]\!] \times \{1, 2\}$. We define the sets $V^{\geqslant m}$, $V^{<m}$ and $V^{\leqslant m}$ similarly. We also have, with obvious notations, the sets $V_1^{>m}$, $V_2^{>m}$, $V_1^{\geqslant m}$, $V_2^{\geqslant m}$... If $I$ is a subset of $V$ we denote by $I_1$ the set $I \cap V_1$ and by $I_2$ the set $I \cap V_2$.

One word on terminology. If $X$ is an $S$-scheme, by a point $x$ of $X$ we mean an $S$-scheme $S'$ and a point $x$ of the set $X(S')$. However, as it is usually unnecessary, we do not mention the $S$-scheme $S'$ and simply write: let $x$ be a point of $X$.

DEFINITION 2.8. *We denote by $R$ the set of permutations $f$ of $V$ such that, for each integer $m$ from 1 to $n$, the elements $f(m,1)$ and $f(m,2)$ of $V$ have different types.*

DEFINITION 2.9. *Let $f \in R$. We denote by $\mathcal{U}_f$ the affine space*

$$\mathrm{Spec}(\mathcal{O}_S[x_{i,j},(i,j) \in f(V_1) \times f(V_2)])$$

*over $S$. It is equipped with a structural morphism $\pi_f$ to $S$. Denote by $\mathcal{F}_f$ the closed subscheme*

$$\mathrm{Spec}(\mathcal{O}_S[x_{i,j},(i,j) \in (f(V_1)_1 \times f(V_2)_2) \sqcup (f(V_1)_2 \times f(V_2)_1)])$$

We think of a point $x$ of $\mathcal{U}_f$ as a matrix indexed by the set $f(V_1) \times f(V_2)$. For a subset $I_1$ of $f(V_1)$ and $I_2$ of $f(V_2)$, we denote by $x_{I_1,I_2}$ the submatrix of $x$ indexed by $I_1 \times I_2$. For example, the closed subscheme $\mathcal{F}_f$ is defined by the vanishing of the two matrices $x_{f(V_1)_1 \times f(V_2)_1}$ and $x_{f(V_1)_2 \times f(V_2)_2}$.

PROPOSITION-DEFINITION 2.10. *Let $f \in R$. There exists a unique morphism*

$$\iota_f : \mathcal{U}_f \to \mathcal{G}$$

*such that $\mathcal{T}_f := \iota_f^* \mathcal{T}$ is the submodule of $\pi_f^* \mathcal{V}$ spanned by*

$$\pi_f^* v_j + \sum_{i \in f(V_1)} x_{i,j} \pi_f^* v_i$$

*where $j$ runs over the set $f(V_2)$. The morphism $\iota_f$ is an open immersion. We denote by $\mathcal{G}_f$ the image of the open immersion $\iota_f$. The open subschemes $\mathcal{G}_f$ cover the grassmannian $\mathcal{G}$ as $f$ runs over the set $R$.*

*Proof.* This is classical. The open subscheme $\mathcal{G}_f$ of the grassmannian parametrizes the complementary submodules of $\mathcal{V}_{f(V_1)}$ in $\mathcal{V}$. $\square$

DEFINITION 2.11. *Let $f \in R$. We denote by $P_{f,0}$ the subgroup scheme*

$$\mathrm{Stab}_G(\mathcal{V}_{f(V_1)})$$

*of $G$. It is a parabolic subgroup. We also denote by $L_{f,0}$ its Levi subgroup*

$$L_{f,0} := \mathrm{Stab}_G(\mathcal{V}_{f(V_1)}, \mathcal{V}_{f(V_2)}) = \prod_{i,j \in \{1,2\}} \mathrm{GL}(\mathcal{V}_{f(V_i)_j})$$

In the next proposition we describe the local structure of the action of the group scheme $G$ on $\mathcal{G}$. This is analogous to Proposition 1.3.

PROPOSITION 2.12. *Let $f \in R$. The open subscheme $\mathcal{G}_f$ of $\mathcal{G}$ is left stable under the action of $P_{f,0}$. For the corresponding action of $P_{f,0}$ on $\mathcal{U}_f$ through* *the isomorphism $\iota_f$, the closed subscheme $\mathcal{F}_f$ is left stable under the action of $L_{f,0}$ and we have the following formulas*

$\forall g \in L_{f,0}, \quad \forall x \in \mathcal{F}_f,$

$$x' = g \cdot x \text{ where } \begin{cases} x'_{f(V_1)_1,f(V_2)_2} = g_{f(V_1)_1} x_{f(V_1)_1,f(V_2)_2} g^{-1}_{f(V_2)_2} \\ x'_{f(V_1)_2,f(V_2)_1} = g_{f(V_1)_2} x_{f(V_1)_2,f(V_2)_1} g^{-1}_{f(V_2)_1} \end{cases}$$

*Finally, the natural morphism*

$$m_{f,0} : R_u(P_{f,0}) \times \mathcal{F}_f \to \mathcal{U}_f, \quad (r,x) \mapsto r \cdot x$$

*is an isomorphism.*

*Proof.* The open subscheme $\mathcal{G}_f$ of the grassmannian parametrizes the complementary submodules of $\mathcal{V}_{f(V_1)}$ in $\mathcal{V}$. It follows that it is stable under the action of the stabilizer $P$ of $\mathcal{V}_{f(V_1)}$ in $\mathrm{GL}(\mathcal{V})$ and therefore under the action of its subgroup $P_{f,0}$.

Let $x$ be a point of $\mathcal{U}_f$ and $g$ a point of $P$. By definition, the point $\iota_f(x)$ is the graph of $x$. Therefore, the point $g \cdot \iota_f(x)$ is the module consisting of elements of type

$$g(v + xv) = g_{f(V_2)}v + (g_{f(V_1),f(V_2)} + g_{f(V_1)}x)v, \quad v \in \mathcal{V}_{f(V_2)}.$$

It is thus equal to the point

$$\iota_f((g_{f(V_1),f(V_2)} + g_{f(V_1)}x)g^{-1}_{f(V_2)}).$$

In other words, the action of $P$ on $\mathcal{U}_f$ is given by

$$P \times \mathcal{U}_f \to \mathcal{U}_f, \quad (g,x) \mapsto (g_{f(V_1),f(V_2)} + g_{f(V_1)}x)g^{-1}_{f(V_2)}.$$

By specializing this action to the subgroup $P_{f,0}$ of $P$, we immediately see that $\mathcal{F}_f$ is left stable under the action of $L_{f,0}$ we obtain the formulas in the statement of the proposition.

Moreover, still using the description of the action of $P$ on $\mathcal{U}_f$ found above, we see that if $g$ is a point of $R_u(P_{f,0})$ and $x$ a point of $\mathcal{F}_f$, then the point $x' = g \cdot x$ of $\mathcal{U}_f$ is given by :

$$\begin{cases} x'_{f(V_1)_1,f(V_2)_1} = g_{f(V_1)_1,f(V_2)_1} \\ x'_{f(V_1)_1,f(V_2)_2} = x_{f(V_1)_1,f(V_2)_2} \\ x'_{f(V_1)_2,f(V_2)_1} = x_{f(V_1)_2,f(V_2)_1} \\ x'_{f(V_1)_2,f(V_2)_2} = g_{f(V_1)_2,f(V_2)_2}. \end{cases}$$

This proves that the natural $P_{f,0}$-equivariant morphism :

$$m_{f,0} : R_u(P_{f,0}) \times \mathcal{F}_f \to \mathcal{U}_f$$

is indeed an isomorphism.

□

Definition 2.13. *Let $f \in R$ and $d \in [\![0, n]\!]$. We denote by $\mathcal{I}_{f,0,d}$ the ideal sheaf on $\mathcal{F}_f$ spanned by the minors of size $d$ of the matrix*

$$\begin{pmatrix} 0 & x_{f(V_1)_1, f(V_2)_2} \\ x_{f(V_1)_2, f(V_2)_1} & 0 \end{pmatrix}.$$

*We denote by $\mathcal{Z}_{f,0,d}$ the closed subscheme of $\mathcal{F}_f$ defined by the ideal sheaf $\mathcal{I}_{f,0,d}$.*

Proposition 2.14. *Let $f \in R$ and $d \in [\![0, n-1]\!]$. Through the isomorphism*

$$m_{f,0} : R_u(P_{f,0}) \times \mathcal{F}_f \to \mathcal{U}_f$$

*of Proposition 2.12 the closed subscheme $\iota_f^{-1}(\mathcal{Z}_{0,d})$ is equal to $R_u(P_{f,0}) \times \mathcal{Z}_{f,0,d}$.*

*Proof.* Due to the formula in the proof of Proposition 2.12, it suffices to show that the defining ideal of $\iota_f^{-1}(\mathcal{Z}_{0,d})$ on $\mathcal{U}_f$ is spanned by the minors of size $d$ of the matrix

$$\begin{pmatrix} 0 & x_{f(V_1)_1, f(V_2)_2} \\ x_{f(V_1)_2, f(V_2)_1} & 0 \end{pmatrix}.$$

To prove this, we express the matrix of the homomorphism

$$\iota_f^* p : \mathcal{T}_f^{\oplus 2} \to \pi_f^* \mathcal{V}$$

in appropriate basis. We choose the basis of $\mathcal{T}_f$ described in Proposition-Definition 2.10. This basis is indexed by the set $f(V_2)$, which is the disjoint union of $f(V_2)_1$ and $f(V_2)_2$. We also choose the basis

$$\pi_f^*(v_i), \quad i \in f(V_1)_1, \quad \pi_f^*(v_j) + \sum_{i \in f(V_1)_1} x_{i,j} \pi_f^*(v_i), \quad j \in f(V_2)_1$$

for $\pi_f^* \mathcal{V}_1$ and

$$\pi_f^*(v_i), \quad i \in f(V_1)_2, \quad \pi_f^*(v_j) + \sum_{i \in f(V_1)_2} x_{i,j} \pi_f^*(v_i), \quad j \in f(V_2)_2$$

for $\pi_f^* \mathcal{V}_2$. The matrix of $\iota_f^* p$ in these basis can be expressed in blocks as follows:

$$\begin{pmatrix} 0 & x_{f(V_1)_1, f(V_2)_2} & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & x_{f(V_1)_2, f(V_2)_1} & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}.$$

By definition, the defining ideal of $\iota_f^{-1}(\mathcal{Z}_{0,d})$ is generated by the minors of size $n + d$ of this matrix. By reordering the vector in the basis, we get the block diagonal square matrix with blocks $I_n$ and

$$\begin{pmatrix} 0 & x_{f(V_1)_1, f(V_2)_2} \\ x_{f(V_1)_2, f(V_2)_1} & 0 \end{pmatrix}.$$

We see therefore that the defining ideal of $\iota_f^{-1}(\mathcal{Z}_{0,d})$ is generated by the minors of size $d$ of the last matrix, as we wanted. $\square$

Definition 2.15. *Let $f \in R$, $m \in [\![0, n]\!]$ and $d \in [\![m, n]\!]$.*

- *We define a parabolic subgroup scheme $P_{f,m}$ of $G$ by induction $m$. For $m$ equals $0$, we have already defined $P_{f,0}$. Then, assuming that $P_{f,m-1}$ has been defined, we set*

$$P_{f,m} = \begin{cases} \mathrm{Stab}_{P_{f,m-1}}(\mathcal{V}_{f(V_1^{>m})\cap V_1}, \mathcal{V}_{\{f(m,2)\}}) \\ \qquad \text{if } f(m,1) \in V_1 \text{ and } f(m,2) \in V_2 \\ \mathrm{Stab}_{P_{f,m-1}}(\mathcal{V}_{f(V_1^{>m})\cap V_2}, \mathcal{V}_{\{f(m,2)\}}) \\ \qquad \text{if } f(m,1) \in V_2 \text{ and } f(m,2) \in V_1. \end{cases}$$

- *We denote by $L_{f,m}$ the following Levi subgroup of $P_{f,m}$:*

$$\prod_{i=1}^{m} (\mathrm{GL}(\mathcal{V}_{f(i,1)}) \times \mathrm{GL}(\mathcal{V}_{f(i,2)})) \times \prod_{i,j\in\{1,2\}} \mathrm{GL}(\mathcal{V}_{f(V_i^{>m})\cap V_j})$$

- *We denote by $\mathcal{F}_{f,m}$ the affine space over $S$ on the indeterminates $x_{i,j}$ where $(i,j)$ runs over the union of the sets*

$$\{(f(1,1), f(1,2)), \ldots, (f(m,1), f(m,2))\}$$

*and*

$$((f(V_1^{>m})_1) \times (f(V_2^{>m})_2)) \cup ((f(V_1^{>m})_2) \times (f(V_2^{>m})_1)).$$

- *We let the group scheme $L_{f,m}$ act on $\mathcal{F}_{f,m}$ by the following formulas*

$$\begin{cases} x'_{f(1,1),f(1,2)} = g_{f(1,1)} g^{-1}_{f(1,2)} x_{f(1,1),f(1,2)} \\ x'_{f(i,1),f(i,2)} = g_{f(i,1)} g_{f(i-1,2)} g^{-1}_{f(i-1,1)} g^{-1}_{f(i,2)} x_{f(i,1),f(i,2)} \text{ for } i \in [\![2, m]\!] \\ x'_{f(V_1^{>m})_1, f(V_2^{>m})_2} = g^{-1}_{f(m,1)} g_{f(m,2)} g_{f(V_1^{>m})_1} x_{f(V_1^{>m})_1, f(V_2^{>m})_2} g^{-1}_{f(V_2^{>m})_2} \\ x'_{f(V_1^{>m})_2, f(V_2^{>m})_1} = g^{-1}_{f(m,1)} g_{f(m,2)} g_{f(V_1^{>m})_2} x_{f(V_1^{>m})_2, f(V_2^{>m})_1} g^{-1}_{f(V_2^{>m})_1} \end{cases}$$

*where $g$ is a point of $L_{f,m}$, $x$ a point of $\mathcal{F}_{f,m}$ and $x' := g \cdot x$.*

- *We denote by $\mathcal{U}_{f,m}$ the product*

$$R_u(P_{f,m}) \times \mathcal{F}_{f,m}$$

*acted upon by the group scheme $P_{f,m} = R_u(P_{f,m}) \rtimes L_{f,m}$ via the formula*

$$\forall (r,l) \in P_{f,m}, \quad \forall (r',x) \in \mathcal{U}_{f,m}, \quad (r,l)\cdot(r',x) = (rlr'l^{-1}, l\cdot x).$$

- *We denote by $\mathcal{I}_{f,m,d}$ the ideal sheaf on $\mathcal{F}_{f,m}$ spanned by the minors of size $d-m$ of the matrix*

$$\begin{pmatrix} 0 & x_{f(V_1^{>m})_1, f(V_2^{>m})_2} \\ x_{f(V_1^{>m})_2, f(V_2^{>m})_1} & 0 \end{pmatrix}.$$

- *We denote by $\mathcal{Z}'_{f,m,d}$ the closed subscheme of $\mathcal{F}_{f,m}$ defined by the ideal sheaf $\mathcal{I}_{f,m,d}$ and by $\mathcal{Z}_{f,m,d}$ the closed subscheme $R_u(P_{f,m,d}) \times \mathcal{Z}'_{f,m,d}$ of $\mathcal{U}_{f,m}$.*

- *We denote by $\mathcal{B}_{f,m}$ the blow-up of $\mathcal{U}_{f,m}$ along the closed subscheme $\mathcal{Z}_{f,m,m+1}$.*

Let $f \in R$, $m \in [\![1,n]\!]$ and $d \in [\![m,n]\!]$. The blow-up $\mathcal{B}_{f,m-1}$ is the closed subscheme of

$$\mathcal{U}_{f,m-1} \times \mathrm{Proj}(\mathcal{O}_S[X_{i,j},(i,j) \in (f(V_1^{\geqslant m})_1 \times f(V_2^{\geqslant m})_2) \sqcup (f(V_1^{\geqslant m})_2 \times f(V_2^{\geqslant m})_1)])$$

defined by the equations

$$\forall (i,j),(i',j') \in (f(V_1^{\geqslant m})_1 \times f(V_2^{\geqslant m})_2) \sqcup (f(V_1^{\geqslant m})_2 \times f(V_2^{\geqslant m})_1)$$

$$x_{i,j}X_{i',j'} - X_{i,j}x_{i',j'} = 0.$$

PROPOSITION 2.16. *With these notations, the open subscheme $\{X_{f(m,1),f(m,2)} \neq 0\}$ of $\mathcal{B}_{f,m-1}$ is left stable under the action of $P_{f,m}$ and is isomorphic, as a $P_{f,m}$-scheme, to $\mathcal{U}_{f,m}$. Moreover, via this isomorphism, the strict transform of $\mathcal{Z}_{f,m-1,d}$ in $\mathcal{U}_{f,m}$ is $\mathcal{Z}_{f,m,d}$.*

*Proof.* We prove analogous statement for the the blow-up $\mathcal{B}'_{f,m-1}$ of $\mathcal{F}_{f,m-1}$ along the closed subscheme $\mathcal{Z}'_{f,m-1,m}$ from which the proposition is easily derived.

The scheme $\mathcal{B}'_{f,m-1}$ is the closed subscheme of

$$\mathcal{F}_{f,m-1} \times \mathrm{Proj}(\mathcal{O}_S[X_{i,j},(i,j) \in (f(V_1^{\geqslant m})_1 \times f(V_2^{\geqslant m})_2) \sqcup (f(V_1^{\geqslant m})_2 \times f(V_2^{\geqslant m})_1)])$$

defined by the equations

$$\forall (i,j),(i',j') \in (f(V_1^{\geqslant m})_1 \times f(V_2^{\geqslant m})_2) \sqcup (f(V_1^{\geqslant m})_2 \times f(V_2^{\geqslant m})_1)$$

$$x_{i,j}X_{i',j'} - X_{i,j}x_{i',j'} = 0.$$

Observe that the open subscheme $\mathcal{U}'_{f,m}$ of $\mathcal{B}'_{f,m-1}$ defined by $X_{f(m,1),f(m,2)} \neq 0$ is isomorphic, as a scheme over $\mathcal{F}_{f,m-1}$ to

$$b : \mathcal{F}_{f,m-1} \to \mathcal{F}_{f,m-1}, \quad x_{i,j} \mapsto \begin{cases} x_{f(m,1),f(m,2)}x_{i,j} \\ \qquad \text{if } (i,j) \in f(V_1^{>m}) \times f(V_2^{>m}) \\ \qquad \text{and } i \text{ and } j \text{ have different types} \\ x_{i,j} \;\; \text{otherwise.} \end{cases}$$

In the following we shall make use of this remark and use the coordinates $x_{i,j}$ to describe the points of $\mathcal{U}'_{f,m}$.

Observe that the center of the blow-up $\mathcal{B}'_{f,m-1} \to \mathcal{F}_{f,m-1}$ is stable under the action of $L_{f,m-1}$ and therefore the group scheme $L_{f,m-1}$ acts on $\mathcal{B}'_{f,m-1}$. This action can be described as follows.

$$\forall g \in L_{f,m-1}, \quad \forall (x,X) \in \mathcal{B}'_{f,m}, \quad (x',X') = g\cdot(x,X) \text{ where } x' = g\cdot x \text{ and}$$

$$\begin{cases} X'_{f(V_1^{\geqslant m})_1, f(V_2^{\geqslant m})_2} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(V_1^{\geqslant m})_1} X_{f(V_1^{\geqslant m})_1, f(V_2^{\geqslant m})_2} g^{-1}_{f(V_2^{\geqslant m})_2} \\ X'_{f(V_1^{\geqslant m})_2, f(V_2^{\geqslant m})_1} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(V_1^{\geqslant m})_2} X_{f(V_1^{\geqslant m})_2, f(V_2^{\geqslant m})_1} g^{-1}_{f(V_2^{\geqslant m})_1} \end{cases}$$

where we denote by $X$ the matrix formed by the $X_{i,j}$. Observe now that the open subscheme $\mathcal{U}'_{f,m}$ is the locus of points $(x,X)$ of $\mathcal{B}'_{f,m-1}$ such that $X_{f(m,1),f(m,2)}$ does not vanish. Therefore, it is left stable under the action of the parabolic subgroup of $L_{f,m-1}$

$$P = \begin{cases} \operatorname{Stab}_{L_{f,m-1}}(\mathcal{V}_{f(V_1^{>m})}, \mathcal{V}_{f(m,2)}) \\ \qquad \text{if } f(m,1) \text{ belongs to } V_1 \text{ and } f(m,2) \text{ to } V_2 \\ \operatorname{Stab}_{L_{f,m-1}}(\mathcal{V}_{f(V_2^{>m})}, \mathcal{V}_{f(m,2)}) \\ \qquad \text{if } f(m,1) \text{ belongs to } V_2 \text{ and } f(m,2) \text{ to } V_1 \end{cases}$$

The group scheme $L_{f,m}$ is a Levi subgroup scheme of $P$. Let $g$ be a point of $L_{f,m}$ and $x$ a point of $\mathcal{F}_{f,m}$. The point $x$ corresponds to the couple $(b(x),X)$ in $\mathcal{B}'_{f,m-1}$, where

$$X_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \neq (f(m,1),f(m,2)) \\ 1 & \text{if } (i,j) = (f(m,1),f(m,2)). \end{cases}$$

Let $(x',X') = g\cdot(x,X)$. By a quick computation we get

$$\begin{cases} x'_{f(m,1),f(m,2)} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(m,1)} g^{-1}_{f(m,2)} x_{f(m,1),f(m,2)} \\ X'_{f(m,1),f(m,2)} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(m,1)} g^{-1}_{f(m,2)} \\ X'_{f(V_1^{>m})_1, f(V_2^{>m})_2} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(V_1^{>m})_1} X_{f(V_1^{>m})_1, f(V_2^{>m})_2} g^{-1}_{f(V_2^{>m})_2} \\ X'_{f(V_1^{>m})_2, f(V_2^{>m})_1} = g^{-1}_{f(m-1,1)} g_{f(m-1,2)} g_{f(V_1^{>m})_2} X_{f(V_1^{>m})_2, f(V_2^{>m})_1} g^{-1}_{f(V_2^{>m})_1} \\ X'_{i,j} = 0 \text{ otherwise} \end{cases}$$

Therefore we see that the closed subscheme $\mathcal{F}_{f,m}$ of $\mathcal{U}'_{f,m}$ is left stable under the action of the Levi subgroup $L_{f,m}$ and, moreover, the action of $L_{f,m}$ on $\mathcal{F}_{f,m}$ is given by the formulas in Definition 2.15. In a similar way, we prove that the natural morphism

$$R_u(P) \times \mathcal{F}_{f,m} \to \mathcal{U}'_{f,m}, \quad (g,x) \mapsto g\cdot x = x'$$

is an isomorphism, given by the following formulas

$$\begin{cases} x'_{f(l,1),f(l,2)} = x_{f(l,1),f(l,2)} \text{ for all } l \in [\![1,m]\!] \\ x'_{f(V_1^{>m})_1,f(m,2)} = g_{f(V_1^{>m})_1,f(m,1)} \\ x'_{f(m,1),f(V_2^{>m})_2} = -g_{f(m,2),f(V_2^{>m})_2} \\ x'_{f(V_1^{>m})_1,f(V_2^{>m})_2} = x_{f(V_1^{>m})_1,f(V_2^{>m})_2} - g_{f(V_1^{>m})_1,f(m,1)} g_{f(m,2),f(V_2^{>m})_2} \\ x'_{f(V_1^{>m})_2,f(V_2^{>m})_1} = x_{f(V_1^{>m})_2,f(V_2^{>m})_1} \end{cases}$$

Now we compute the strict transform of $\mathcal{Z}'_{f,m,d}$ in $\mathcal{U}'_{f,m}$. By definition, the strict transform of $\mathcal{Z}'_{f,m,d}$ is the schematic closure of

$$\mathcal{Z} := b^{-1}(\mathcal{Z}'_{f,m-1,d} \cap \{x_{f(m,1),f(m,2)} \neq 0\})$$

in $\mathcal{U}'_{f,m}$. Let $x$ be a point of $\mathcal{U}'_{f,m}$. We denote $(r, y)$ its components through the isomorphism

$$R_u(P) \times \mathcal{F}_{f,m} \to \mathcal{U}'_{f,m}.$$

By definition, the point $x$ belongs to $\mathcal{Z}$ if and only $x_{f(m,1),f(m,2)}$ is invertible and all the square $d-m+1$ minors extracted from the following matrix

$$\begin{pmatrix} 0 & b(x_{f(V_1^{\geqslant m})_1,f(V_2^{\geqslant m})_2}) \\ b(x_{f(V_1^{\geqslant m})_2,f(V_2^{\geqslant m})_1}) & 0 \end{pmatrix}$$

are zero. By definition of the morphism $b$, each coefficient in this matrix is a multiple of $x_{f(m,1),f(m,2)}$ and the coefficient in place $(f(m,1), f(m,2))$ is exactly $x_{f(m,1),f(m,2)}$. As this indeterminate is invertible on the open subscheme $\{x_{f(m,1),f(m,2)} \neq 0\}$ we see that the point $x$ belongs to $\mathcal{Z}$ if and only $x_{f(m,1),f(m,2)}$ is invertible and all the minors of size $d-m+1$ extracted from the matrix

$$\begin{pmatrix} 0 & x'_{f(V_1^{\geqslant m})_1,f(V_2^{\geqslant m})_2} \\ x'_{f(V_1^{\geqslant m})_2,f(V_2^{\geqslant m})_1} & 0 \end{pmatrix}$$

are zero, where $x'_{f(m,1),f(m,2)} = 1$ and $x'_{i,j} = x_{i,j}$ otherwise. By operating standard row and column operations on this matrix we see now that $x$ belongs to $\mathcal{Z}$ if and only if $x_{f(m,1),f(m,2)}$ is invertible and all the minors of size $d-m$ extracted from the matrix

$$\begin{pmatrix} 0 & x_{f(V_1^{>m})_1,f(V_2^{>m})_2} \\ x_{f(V_1^{>m})_2,f(V_2^{>m})_1} & 0 \end{pmatrix}$$

are zero. This proves that $\mathcal{Z}$ is the intersection of $R_u(P) \times \mathcal{Z}'_{f,m,d}$ with the open subscheme $\{x_{f(m,1),f(m,2)} \neq 0\}$.

In order to complete the proof, we now show that the schematic closure of $\mathcal{Z}$ in $\mathcal{U}'_{f,m}$ is equal to $R_u(P) \times \mathcal{Z}'_{f,m,d}$. We can check this Zariski locally on the base $S$ and therefore assume that $S$ is affine. First of all, it is obvious that the closure of $\mathcal{Z}$ is contained in $R_u(P) \times \mathcal{Z}'_{f,m,d}$. Conversely, let $\varphi$ be a global function on $\mathcal{U}'_{f,m}$ which vanishes on $\mathcal{Z}$. This means that there exists an integer $q$ such that $x^q_{f(m,1),f(m,2)}\varphi$ belongs to the ideal spanned by the minors of size $d-m$ of the following matrix:

$$\begin{pmatrix} 0 & x_{f(V_1^{>m})_1, f(V_2^{>m})_2} \\ x_{f(V_1^{>m})_2, f(V_2^{>m})_1} & 0 \end{pmatrix}.$$

As the indeterminate $x_{f(m,1),f(m,2)}$ does not appear in this matrix, we can conclude that $\varphi$ itself belongs to this ideal, completing the proof of the proposition. $\square$

PROPOSITION 2.17. *Let $f \in R$. There exists a unique collection of open immersions $\iota_{f,m}$, for $m$ from 1 to $n$ such that each of the squares below are commutative :*

$$\begin{tikzcd}
\mathcal{U}_{f,0} \arrow[d, "\iota_{f,0}"'] & \mathcal{U}_{f,1} \arrow[l, "b_{f,1}"'] \arrow[d, "\iota_{f,1}"] & \cdots \arrow[l] & \mathcal{U}_{f,n-1} \arrow[l] \arrow[d, "\iota_{f,n-1}"'] & \mathcal{U}_{f,n} \arrow[l, "b_{f,n}"'] \arrow[d, "\iota_{f,n}"] \\
\mathcal{G}_0 & \mathcal{G}_1 \arrow[l, "b_1"] & \cdots \arrow[l] & \mathcal{G}_{n-1} \arrow[l] & \mathcal{G}_n \arrow[l, "b_n"]
\end{tikzcd}$$

*The open immersion $\iota_{f,m}$ is equivariant for the action of $P_{f,m}$. We denote by $\mathcal{G}_{f,m}$ the image of the open immersion $\iota_{f,m}$. The open subschemes $\mathcal{G}_{f,m}$ cover $\mathcal{G}_m$ as $f$ run over $R$.*

*Proof.* The existence and $P_{f,m}$-equivariance of the $\iota_{f,m}$ follows directly from Proposition 2.16. The uniqueness comes from the fact that for each index $m$, the intersection $\Omega \cap \mathcal{G}_{f,m}$ is dense in $\mathcal{G}_{f,m}$. The last assertion is proved as follows. By Proposition 2.10, the open subschemes $\mathcal{G}_{f,0}$ cover the scheme $\mathcal{G}_0$. Moreover, it follows from Proposition 2.16 that the blow-up $\mathcal{B}_{f,m-1}$ is covered by the open subschemes $\mathcal{U}_{f',m}$ where $f'$ runs over the elements of $R$ having the same restriction as $f$ to $[\![1,k]\!] \times \{1,2\}$ and satisfying $f(V_1) = f'(V_1)$ and $f(V_2) = f'(V_2)$. $\square$

## 2.3 AN ALTERNATIVE CONSTRUCTION

In this section we provide an alternative construction for the schemes $\mathcal{G}_m$. Recall from Section 2.1 that the section $\wedge^{n+d}p$ of the locally free module $\mathcal{H}_d$ does not vanish on $\Omega$. Therefore it defines an invertible submodule of $\mathcal{H}_d$ on $\Omega$ that is locally a direct summand. In other words, it defines a morphism from $\Omega$ to the projective bundle $\mathbb{P}(\mathcal{H}_d)$ over $\Omega$.

Definition 2.18. *Let $d \in [\![0, n]\!]$. We denote by $\varphi_d$ the morphism*

$$\varphi_d : \Omega \to \mathbb{P}(\mathcal{H}_d).$$

*defined by the global section $\wedge^{n+d} p$ of $\mathcal{H}_d$.*

Proposition 2.19. *Let $d \in [\![0, n]\!]$ and $m \in [\![d, n]\!]$. The morphism $\varphi_d$ extends to $\mathcal{G}_m$ in a unique way.*

*Proof.* We first observe that $\Omega$ is dense in each of the schemes $\mathcal{G}_m$. If the morphism $\varphi_d$ extends to $\mathcal{G}_m$ it is therefore in a unique way. Moreover, it suffices to show that $\varphi_d$ extends to $\mathcal{G}_d$, because then the composite

$$\mathcal{G}_m \xrightarrow{b_m} \mathcal{G}_{m-1} \longrightarrow \dots \longrightarrow \mathcal{G}_d \xrightarrow{\varphi_d} \mathbb{P}(\mathcal{H}_d)$$

is an extension of $\varphi_d$ to $\mathcal{G}_m$.

By the same density argument as above it suffices to show that the morphism $\varphi_d$ extends from $\Omega \cap \mathcal{G}_{f,d}$ to $\mathcal{G}_{f,d}$ for each element $f$ of the set $R$. We identify the scheme $\mathcal{G}_{f,d}$ with $\mathcal{U}_{f,d}$ via the isomorphism $\iota_{f,d}$. Observe that the scheme $\mathbb{P}(\mathcal{H}_d)$ is equipped with an action of the group scheme $G$ and the morphism $\varphi_d$ is equivariant with respect to this action. By using this remark, we see that it suffices to extend the morphism $\varphi_d$ from $\Omega \cap \mathcal{F}_{f,d}$ to $\mathcal{F}_{f,d}$.

We denote by $c$ the composite $b_{f,1} \circ \cdots \circ b_{f,d}$. We use the trivialization of $\mathcal{H}_d$ on $\mathcal{G}_{f,0}$ as in the proof of Proposition 2.14. For this trivialization, the coordinates are indexed by the product $(V_2 \sqcup V_2) \times V$. The morphism $\varphi_d$ is given over $\Omega \cap \mathcal{F}_{f,d}$ in this coordinates by the $n + d$ minors of the matrix

$$\begin{pmatrix} 0 & c(x)_{f(V_1)_1, f(V_2)_2} & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & c(x)_{f(V_1)_2, f(V_2)_1} & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}.$$

It follows from the definition of $c$ that all these minors are multiples of

$$x^d_{f(1,1),f(1,2)} x^{d-1}_{f(1,2),f(2,2)} \cdots x_{f(d,1),f(d,2)}$$

and that one of them, namely the one indexed by the product of

$$f(V_1)_2 \sqcup f(V_2^{\leqslant d})_2 \sqcup (f(V_1^{\leqslant d})_2) \sqcup f(V_2)_2$$

and

$$f(V_1^{\leqslant d})_1 \sqcup f(V_2)_1 \sqcup f(V_1^{\leqslant d})_2 \sqcup f(V_2)_2$$

is exactly equal to this product or its opposite. By dividing each coordinate by this product we therefore extend the morphism $\varphi_d$ to $\mathcal{F}_{f,d}$. ☐

Proposition 2.20. *Let $m \in [\![0, n]\!]$. The morphism*

$$\psi_m := \varphi_0 \times \varphi_1 \times ... \times \varphi_m : \mathcal{G}_m \to \mathbb{P}(\mathcal{H}_0) \times_{\mathcal{G}} \mathbb{P}(\mathcal{H}_1) \times_{\mathcal{G}} ... \times_{\mathcal{G}} \mathbb{P}(\mathcal{H}_m)$$

*is a closed immersion.*

*Proof.* We prove this by induction on $m$. For $m = 0$ the morphism $\varphi_0$ is defined by the nowhere vanishing section $\wedge^n p$ of $\mathcal{H}_0$ and is therefore a closed immersion. We suppose now that $\psi_{m-1}$ is a closed immersion and prove that $\psi_m$ is also a closed immersion. As this morphism is proper, it suffices to check that it is a monomorphism in order to prove that it is a closed immersion. Let $q_1$ and $q_2$ be two points of $\mathcal{G}_m$ which are mapped to the same point by $\psi_m$. We want to show that they are equal.

First, we suppose that $q_1$ and $q_2$ are points of $\mathcal{G}_{f,m}$, where $f$ is an element of $R$. We identify $\mathcal{G}_{f,m}$ and $\mathcal{U}_{f,m}$ via the isomorphism $\iota_{f,m}$. We use the notations introduced in the proof of Proposition 2.16. The scheme $\mathcal{U}_{f,m}$ is isomorphic to the product $R_u(P_{f,m-1}) \times \mathcal{U}'_{f,m}$. Using the induction hypothesis, we can assume that $q_1$ and $q_2$ are actually points of $\mathcal{U}'_{f,m}$. Viewed as points of $\mathcal{U}'_{f,m}$ (which is isomorphic to $\mathcal{F}_{f,m-1}$ via the morphism $b$) we denote the coordinates of $q_1$ by $(x_{i,j,1})$ and and the coordinates of $q_2$ by $(x_{i,j,2})$. Still by the induction hypothesis, we have $x_{i,j,1} = x_{i,j,2}$ for $(i,j)$ in the following set

$$\{(f(1,1), f(1,2)), \ldots, (f(m,1), f(m,2))\}$$

as it follows from the definition of the morphism $b$. Observe now that the morphism $\varphi_m$ is defined over $\mathcal{U}'_{f,m}$ by exactly the same process as explained in the proof of Proposition 2.19. By this we mean that the coordinates of $\varphi_d$ are obtained by computing the $n+d$ minors of the matrix

$$\begin{pmatrix} 0 & c(x)_{f(V_1)_1, f(V_2)_2} & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & c(x)_{f(V_1)_2, f(V_2)_1} & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}$$

and dividing by the product

$$x^m_{f(1,1),f(1,2)} x^{m-1}_{f(1,2),f(2,2)} \cdots x_{f(m,1),f(m,2)}.$$

Indeed this process makes sense over $\mathcal{U}'_{f,m}$ and extend $\varphi_d$ over $\Omega \cap \mathcal{U}'_{f,m}$. Moreover, such an extension is unique. Let $(i,j)$ be an element of

$$(f(V_1^{\geqslant m})_1 \times f(V_2^{\geqslant m})_2) \sqcup (f(V_1^{\geqslant m})_2 \times f(V_2^{\geqslant m})_1)$$

different from $(f(m,1), f(m,2))$. We suppose that $i$ is of type 1 and $j$ of type 2, the other case being entirely similar. The coordinate of $\varphi_m$ corresponding to the minor indexed by the product of

$$f(V_1)_2 \sqcup f(V_2^{<d})_2 \sqcup \{j\} \sqcup (f(V_1^{\leqslant d})_2) \sqcup f(V_2)_2$$

and

$$f(V_1^{\leqslant d})_1 \sqcup f(V_2)_1 \sqcup f(V_1^{<d})_2 \sqcup \{i\} \sqcup f(V_2)_2.$$

is $x_{i,j}$ or its opposite. We can therefore conclude that $x_{i,j,1} = x_{i,j,2}$. Finally we have proved that $q_1 = q_2$.

We go back to the general case. Let $f$ be an element of $R$. We prove now that the open subschemes $q_1^{-1}(\mathcal{G}_{f,m})$ and $q_2^{-1}(\mathcal{G}_{f,m})$ are equal. This is sufficient to complete the proof of the proposition. By the induction hypothesis, we already know that the open subschemes $q_1^{-1}(\mathcal{G}_{f,m-1})$ and $q_2^{-1}(\mathcal{G}_{f,m-1})$ are equal. Observe now that the computations above actually prove the following: through $\varphi_m$, the non zero locus on $\mathcal{B}_{f,m}$ of the coordinate of $\mathbb{P}(\mathcal{H}_m)$ corresponding to the minor indexed by the product of

$$f(V_1)_2 \sqcup f(V_2^{\leqslant m})_2 \sqcup (f(V_1^{\leqslant m})_2) \sqcup f(V_2)_2$$

and

$$f(V_1^{\leqslant m})_1 \sqcup f(V_2)_1 \sqcup f(V_1^{\leqslant m})_2 \sqcup f(V_2)_2$$

is $\mathcal{U}_{f,m}$. This implies the result.

□

## 2.4 The colored fan of $\mathcal{G}_n$

In this section we assume that the base scheme $S$ is the spectrum of an algebraically closed field $k$ of arbitrary characteristic. The scheme $\mathcal{G}_n$ is an equivariant compactification of the homogeneous space $\mathrm{Iso}(\mathcal{V}_2, \mathcal{V}_1)$ under the action of the group $\mathrm{GL}(\mathcal{V}_1) \times \mathrm{GL}(\mathcal{V}_2)$. Through the fixed trivializations of the free modules $\mathcal{V}_1$ and $\mathcal{V}_2$ we see that $\mathcal{G}_n$ is an equivariant compactification of the general linear group $\mathrm{GL}(n)$ under the action of $\mathrm{GL}(n) \times \mathrm{GL}(n)$. The aim of this section is to compute the colored fan of this compactification, as explained in Section 1.4.

But first, we say a word about the blow-up procedure explained in Section 2.1 in this setting. By definition, the set $\mathcal{Z}_d(k)$ is the set of $n$-dimensional subspaces of $\mathcal{V}(k)$ such that the sum of the ranks of $p_1(k)$ and $p_2(k)$ is strictly less than $n+d$ at every point. Another way to state this is that $\mathcal{Z}_d(k)$ is the set

$$\{F \in \mathcal{G}(k), \quad \dim(F \cap \mathcal{V}_1(k)) + \dim(F \cap \mathcal{V}_2(k)) > n - d\}.$$

For example, the set $\mathcal{Z}_1(k)$ is the set of $n$-dimensional subspaces of $\mathcal{V}(k)$ which are direct sum of a subspace of $\mathcal{V}_1(k)$ and a subspace of $\mathcal{V}_2(k)$. Using this description, it is not difficult to prove that $\mathcal{Z}_1(k)$ is the union of the closed orbits of $G(k)$ in $\mathcal{G}(k)$. We leave it as an exercise to the reader to prove that, for $d$ from 1 to $n-1$, an orbit $\omega$ of $G(k)$ in $\mathcal{G}(k)$ is contained in $\mathcal{Z}_d(k)$ if and only if its closure is the union of $\omega$ and some orbits contained in $\mathcal{Z}_{d-1}(k)$.

We use the notations introduced in Section 1.4. We choose for $T$ the diagonal torus in $\mathrm{GL}(n)$ and for $B$ the Borel subgroup of upper triangular matrices. The torus $T$ is naturally isomorphic to the torus $\mathbb{G}_m^n$. The vector space $V$ is therefore naturally isomorphic to $\mathbb{Q}^n$. The Weyl chamber $\mathcal{W}$ corresponding to the chosen Borel subgroup $B$ of $\mathrm{GL}(n)$ is given by

$$\mathcal{W} = \{(a_1, \ldots, a_n) \in V, \quad a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n\}.$$

Proposition-Definition 2.21. *We denote by $Q$ the set of permutations $g$ of $[\![1,n]\!]$ such that*

$$\exists m \in [\![0,n]\!], \quad g_{|g^{-1}([\![1,m]\!])} \text{ is decreasing and } g_{|g^{-1}([\![m,n]\!])} \text{ is increasing.}$$

*If such an integer $m$ exists, it is unique. We call it the integer associated to $g$ and denote it by $m_g$. Also, we denote by $\varepsilon_g$ the function*

$$\varepsilon_g : [\![1,n]\!] \to \{+1,-1\}, \quad x \mapsto \begin{cases} -1 & \text{if } g(x) \in [\![1,m_g]\!] \\ \\ 1 & \text{if } g(x) \in [\![m_g+1,n]\!] \end{cases}$$

*Finally, we denote by $C_g$ the following cone in $V$ :*

$$C_g := \{(a_1,\ldots,a_n) \in V, \quad 0 \leqslant \varepsilon_g(1)a_{g(1)} \leqslant \cdots \leqslant \varepsilon_g(n)a_{g(n)}\}$$

Proposition 2.22. *The compactification $\mathcal{G}_n$ of $\mathrm{GL}(n)$ is log homogeneous and its colored fan consists of the cones $C_g$ and their faces, where $g$ runs over the set $Q$.*

*Proof.* Let us first prove that the compactification $\mathcal{G}_n$ is log homogeneous. Let $f$ be an element of $R$. We claim that the complement of $\Omega$ in $\mathcal{F}_{f,n}$ is the union of the coordinate hyperplanes $x_{f(d,1),f(d,2)} = 0$, where $d$ runs from 1 to $n$. Indeed, a point $x$ of $\mathcal{F}_{f,n}$ belongs to $\Omega$ if and only if the point $x' = (b_{f,n} \circ \cdots \circ b_{f,1})(x)$ belongs to $\Omega$. Moreover, we have

$$\begin{cases} x'_{f(d,1),f(d,2)} = x_{f(1,1),f(1,2)} \cdots x_{f(d,1),f(d,2)} \text{ for all } d \in [\![1,n]\!] \\ x'_{i,j} = 0 \text{ if } (i,j) \notin \{(f(1,1),f(1,2)),\ldots,(f(n,1),f(n,2))\}. \end{cases}$$

By Proposition 2.14, the point $x'$ belongs to $\Omega$ if and only if the product

$$x'_{f(1,1),f(1,2)} \cdots x'_{f(n,1),f(n,2)} = x^n_{f(1,1),f(1,2)} x^{n-1}_{f(2,1),f(2,2)} \cdots x_{f(n,1),f(n,2)}$$

does not vanish, that is, if and only if each of the $x_{f(d,1),f(d,2)}$ does not vanish. This proves the claim. In particular, the complement of $\Omega$ in $\mathcal{G}_n$ is a strict normal crossing divisor. By Definition 2.15, the variety $\mathcal{F}_{f,n}$ is a smooth toric variety for a quotient of the torus $T \times T = L_{f,n}$. From this we see that it is log homogeneous. It is now straightforward to check that the $P_{f,n}$-variety $\mathcal{U}_{f,n}$ is log homogeneous. It readily follows that the $G$-variety $\mathcal{G}_n$ is log homogeneous.

We let $T$ acts on $\mathcal{G}_n$ on the left. By Section 1.4, the closure $\overline{T}$ of $T$ in $\mathcal{G}_n$ is a toric variety under the action of $T$ and we can use the fan of this toric variety to compute the colored fan of $\mathcal{G}_n$. We shall now identify some of the cones in the fan of $\overline{T}$. Let $g$ be an element of $Q$. We fix an element $f$ of $R$ such that, for each integer $d$ from 1 to $n$, $f(d,1) = (g(d),1)$ if $\varepsilon_g(d) = 1$ and $f(d,2) = (g(d),1)$ if $\varepsilon_g(d) = -1$ The variety $\mathcal{F}_{f,n}$ is an open affine toric subvariety of $\overline{T}$. It corresponds to a cone in the fan of $\overline{T}$, namely the cone spanned by the one-parameter subgroups having a limit in $\mathcal{F}_{f,n}$ at 0. Let

$$\lambda : \mathbb{G}_m \to T, \quad t \to (t^{a_1}, \dots, t^{a_n})$$

be a one-parameter subgroup of $T$. By the formulas in Definition 2.15, we see that the one-parameter subgroup $\lambda$ has a limit in $\mathcal{F}_{f,n}$ at 0 if and only if

$$0 \leqslant \varepsilon_g(1)a_{g(1)} \leqslant \ldots \leqslant \varepsilon_g(n)a_{g(n)}$$

that is, if and only if $\lambda$ belongs to $C_g$. This proves that, for each element $g$ of $Q$, the cone $C_g$ belongs to the fan of the toric variety $\overline{T}$.

To complete the proof, we show now that the cone $-\mathcal{W}$ is equal to the union of the cones $C_g$, where $g$ runs over the set $Q$. Let $g$ be an element of $Q$ and let $(a_1, \dots, a_n)$ be a point in $C_g$. Let also $i$ be an integer between 1 and $n-1$. If $i < m_g$, then there are two integers $j > j'$ such that $g(j) = i$ and $g(j') = i+1$. These integers satisfy $\varepsilon(j) = -1$ and $\varepsilon(j') = -1$. By definition of the cone $C_g$, we have $\varepsilon(j')a_{g(j')} \leqslant \varepsilon(j)a_{g(j)}$, that is, $a_i \leqslant a_{i+1}$. The same kind of argument prove that $a_i \leqslant a_{i+1}$ for $i = m_g$ and for $i > m_g$. This proves that the cone $C_g$ is contained in the cone $-\mathcal{W}$. We consider now an element $(a_1, \dots, a_n)$ of $-\mathcal{W}$. By definition it satisfies $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. Let $m$ be an integer such that $a_m \leqslant 0$ and $a_{m+1} \geqslant 0$. The rational numbers $-a_1, \dots, -a_m$ and $a_{m+1}, \dots, a_n$ are nonnegative. By ordering them in increasing order, we construct an element $g$ of $Q$ such that the point $(a_1, \dots, a_n)$ belongs to $C_g$.

□

### 2.5 Fixed points

In this section, the scheme $S$ is the spectrum of an algebraically closed field $k$ of characteristic not 2. We apply the results obtained in Section 1.3 for some involutions on the log homogeneous compactification $\mathcal{G}_n$ of $\mathrm{GL}(n)$. We denote by $J_r$ the antidiagonal square matrix of size $r$ with all coefficients equal to one on the antidiagonal.

Let $b$ be a nondegenerate symmetric or antisymmetric bilinear form on $k^n$. Via the fixed trivializations of $\mathcal{V}_1$ and $\mathcal{V}_2$ we obtain nondegenerate symmetric or antisymmetric bilinear forms $b_1$ and $b_2$ on the $k$-vector spaces $\mathcal{V}_1(k)$ and $\mathcal{V}_2(k)$. We equip the direct sum $\mathcal{V}(k)$ of $\mathcal{V}_1(k)$ and $\mathcal{V}_2(k)$ with the nondegenerate symmetric or antisymmetric bilinear form $b_1 \oplus b_2$. We let $\sigma$ be the involution of $\mathcal{G}$ mapping a $n$-dimensional $k$-vector subspace $F$ of $\mathcal{V}(k)$ to its orthogonal. It is an easy exercise to check that

$$\dim(F^\perp \cap \mathcal{V}_1(k)) = \dim(F \cap \mathcal{V}_2(k)) \text{ and } \dim(F^\perp \cap \mathcal{V}_2(k)) = \dim(F \cap \mathcal{V}_1(k)).$$

By the description of $\mathcal{Z}_d$ given in Section 2.4, we see that the involution $\sigma$ leaves each of the closed subvarieties $\mathcal{Z}_d$ of $\mathcal{G}$ invariant. Therefore it extends to an involution, still denoted $\sigma$, of each of the varieties $\mathcal{G}_m$. To prove that we are in the setting of Section 1.3, it remains to observe that there is an involution $\sigma$ of $\mathrm{GL}(n)$, namely the one associated to $b$, such that

$$\forall g \in \mathrm{GL}(n) \times \mathrm{GL}(n), \quad \forall x \in \mathcal{G}_m, \quad \sigma((g_1, g_2) \cdot x) = \sigma(g_1) \cdot \sigma(x) \cdot \sigma(g_2)^{-1}.$$

As in Section 1.3, we denote by $G'$ the neutral component of $G^\sigma$ and by $\mathcal{G}'_n$ the connected component of $\mathcal{G}_n^\sigma$ containing $G'$.

The odd orthogonal case. We suppose that $n = 2r+1$ is odd. We let $b$ be the scalar product with respect to the matrix $J_{2r+1}$. We have $G' := \mathrm{SO}(2r+1)$. We let $T'$ be the intersection of $T$ with $G'$ and $B'$ the intersection of $B$ with $G'$. The maximal torus $T'$ is naturally isomorphic to the split torus $\mathbb{G}_m^r$ via the following morphism

$$\mathbb{G}_m^r \to T', \quad (t_1, \ldots, t_r) \mapsto \mathrm{diag}(t_1, \ldots, t_r, 1, t_r^{-1}, \ldots, t_1^{-1}).$$

The space $V'$ is therefore naturally isomorphic to $\mathbb{Q}^r$. It is contained in $V$ via the following linear map

$$V' \to V, \quad (a'_1, \ldots, a'_r, 0, -a'_r, \ldots, -a'_1).$$

The Weyl chamber with respect to $B'$ is given by

$$\mathcal{W}' = \{(a'_1, \ldots, a'_r) \in V, \quad a'_1 \geqslant a'_2 \geqslant \ldots \geqslant a'_r \geqslant 0\}.$$

Proposition 2.23. *The compactification $\mathcal{G}'_n$ of $G'$ is the wonderful compactification.*

*Proof.* First of all, by Proposition 1.9, the compactification $\mathcal{G}'_n$ is log homogeneous. As explained in Section 1.4, we use the closure of $T'$ in $\mathcal{G}'_n$ to compute the colored fan of $\mathcal{G}'_n$. Observe that $T'$ is a subtorus of $T$ and therefore the fan of the toric variety $\overline{T'}$ is the trace on $V'$ of the fan of the toric variety $\overline{T}$. By Proposition 2.22, the cone

$$\{(a_1, \ldots, a_{2r+1}) \in V, \quad 0 \leqslant a_{r+1} \leqslant -a_r \leqslant a_{r+2} \leqslant \cdots \leqslant -a_1 \leqslant a_{2r+1}\}$$

belongs to the fan of $\overline{T}$. The trace of this cone on $V'$ is $-\mathcal{W}'$, proving that the cone $-\mathcal{W}'$ belongs to the colored fan of $\mathcal{G}'_n$. But the only fan in $V'$ with support $-\mathcal{W}'$ containing $-\mathcal{W}'$ is the fan formed by $-\mathcal{W}'$ and its faces. This completes the proof.

□

The even orthogonal case. We suppose that $n = 2r$ is odd. We let $b$ be the scalar product with respect to the matrix $J_{2r}$. We have $G' := \mathrm{SO}(2r)$. We let $T'$ be the intersection of $T$ with $G'$ and $B'$ the intersection of $B$ with $G'$. The maximal torus $T'$ is naturally isomorphic to the split torus $\mathbb{G}_m^r$ via the following morphism

$$\mathbb{G}_m^r \to T', \quad (t_1, \ldots, t_r) \mapsto \mathrm{diag}(t_1, \ldots, t_r, t_r^{-1}, \ldots, t_1^{-1}).$$

The space $V'$ is therefore naturally isomorphic to $\mathbb{Q}^r$. It is contained in $V$ via the following linear map

$$V' \to V, \quad (a'_1, \ldots, a'_r, -a'_r, \ldots, -a'_1).$$

The Weyl chamber with respect to $B'$ is given by

$$\mathcal{W}' = \{(a'_1, \ldots, a'_r) \in V, \quad a'_1 \geqslant a'_2 \geqslant \ldots \geqslant a'_{r-1} \geqslant |a'_r|\}.$$

Proposition 2.24. *The compactification $\mathcal{G}'_n$ of $G'$ is log homogeneous and its fan consists of the cones*

$$C_+ := \{(a'_1, \ldots, a'_r) \in V, \quad a'_1 \leqslant a'_2 \leqslant \ldots \leqslant a'_{r-1} \leqslant a'_r \leqslant 0\}$$

*and*

$$C_- := \{(a'_1, \ldots, a'_r) \in V, \quad a'_1 \leqslant a'_2 \leqslant \ldots \leqslant a'_{r-1} \leqslant -a'_r \leqslant 0\}$$

*and their faces.*

*Proof.* The proof is similar to that of Proposition 2.23. With the arguments given in this proof it suffices to observe that the trace of the following cone in $V$:

$$\{(a_1, \ldots, a_{2r}) \in V, \quad 0 \leqslant -a_r \leqslant a_{r+1} \leqslant \cdots \leqslant -a_1 \leqslant a_{2r}\}$$

on $V'$ is $C_+$, the trace of

$$\{(a_1, \ldots, a_{2r}) \in V, \quad 0 \leqslant a_r \leqslant -a_{r+1} \leqslant -a_{r-1} \leqslant a_{r+2} \leqslant \cdots \leqslant -a_1 \leqslant a_{2r}\}$$

is $C_-$ and that $-\mathcal{W}'$ is the union of $C_+$ and $C_-$. □

Observe that the Weyl chamber $\mathcal{W}'$ is not smooth with respect to the lattice of one-parameter subgroups of $T'$. Therefore the canonical compactification of $G'$ is not smooth, and the compactification $\mathcal{G}'_n$ is a minimal log homogeneous compactification, in the sense that it has a minimal number of closed orbits.

The symplectic case. We suppose that $n = 2r$ is even. We let $b$ be the scalar product with respect to the block antidiagonal matrix

$$\begin{pmatrix} 0 & -J_r \\ J_r & 0 \end{pmatrix}.$$

We have $G' := \mathrm{Sp}(2r)$. We let $T'$ be the intersection of $T$ with $G'$ and $B'$ the intersection of $B$ with $G'$. The maximal torus $T'$ is naturally isomorphic to the split torus $\mathbb{G}_m^r$ via the following morphism

$$\mathbb{G}_m^r \to T', \quad (t_1, \ldots, t_r) \mapsto \mathrm{diag}(t_1, \ldots, t_r, t_r^{-1}, \ldots, t_1^{-1}).$$

The space $V'$ is therefore naturally isomorphic to $\mathbb{Q}^r$. It is contained in $V$ via the following linear map

$$V' \to V, \quad (a'_1, \ldots, a'_r, -a'_r, \ldots, -a'_1).$$

The Weyl chamber with respect to $B'$ is given by

$$\mathcal{W}' = \{(a'_1, \ldots, a'_r) \in V, \quad a'_1 \geqslant a'_2 \geqslant \ldots \geqslant a'_r \geqslant 0\}.$$

Proposition 2.25. *The compactification $\mathcal{G}'_n$ of $G'$ is the wonderful compactification.*

*Proof.* The proof is similar to that of Proposition 2.23. With the arguments given in this proof it suffices to observe that the trace of the following cone in $V$:

$$\{(a_1, \ldots, a_{2r}) \in V, \quad 0 \leqslant -a_r \leqslant a_{r+1} \cdots \leqslant -a_1 \leqslant a_{2r}\}$$

on $V'$ is $-\mathcal{W}'$. □

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# The Dirac Operator with Mass $m_0 \geq 0$ : Non-Existence of Zero Modes and of Threshold Eigenvalues

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Abstract. A simple global condition on the potential is given which excludes zero modes of the massless Dirac operator. As far as local conditions at infinity are concerned, it is shown that at energy zero the Dirac equation without mass term has no non-trivial $L^2$-solutions at infinity for potentials which are either very slowly varying or decaying at most like $r^{-s}$ with $s \in (0, 1)$. When a mass term is present, it is proved that at the thresholds there are again no such solutions when the potential decays at most like $r^{-s}$ with $s \in (0, 2)$. In both situations the decay rate is optimal.

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## 1 Introduction

In their 1986 study of the stability of matter in the relativistic setting of the Pauli operator J. Fröhlich, E.H. Lieb and M. Loss recognised that there was a restriction on the nuclear charge if and only if the three-dimensional Dirac operator with mass zero has a non-trivial kernel (see [LS], Chapters 8, 9 and the references there). An example of such a zero mode was first given by M. Loss and H.T. Yau [LY]; for many more examples see [LS], p.167. Later, the Loss-Yau example was used in a completely different setting, viz. to show the necessity of certain restrictions in analogues of Hardy and Sobolev inequalities [BEU].

An observation with remarkable technological consequences is that in certain situations of non-relativistic quantum mechanics the dynamics of wave packets in crystals can be modelled by the two-dimensional massless Dirac operator (see [FW] and the references there). When the potential is spherically symmetric, a detailed spectral analysis of the Dirac operator with mass zero was given in two and three dimensions by K.M. Schmidt [S]. In particular he showed that a variety of potentials with compact support can give rise to zero modes.
In Theorem 2.1 of the present paper we give a simple global condition on the potential which rules out zero modes of the massless Dirac operator in any dimension. Theorem 2.7 deals with a fairly large class of massless Dirac operators under conditions on the potential solely at infinity. It is shown, for example, that for energy zero there is no non-trivial solution of integrable square at infinity if the potential is very slowly varying or decaying like $r^{-s}$ with $s \in (0,1)$. This decay rate is in a certain sense the best possible one (see Appendix B). To rule out non-trivial $L^2$-solutions at infinity for the threshold energies $\pm m_0$ turns out to be more complicated. Here the asymptotic analysis of Appendix B suggests $1/r^2$-behaviour as the borderline case and Theorem 2.10 indeed permits potentials which tend to zero with a rate at most like $r^{-s}$ with $s \in (0,2)$. Theorem 2.10 relies on a transformation of the solutions which is intimately connected with the block-structure that can be given to the Dirac matrices. In connection with this theorem we should like to draw attention to the very interesting paper [BG] where global conditions are used.
In broad outline the proof of our Theorems 2.7 and 2.10 follows from the virial technique which was developed by Vogelsang [V] for the Dirac operator and later extended in [KOY], but basic differences in the assumptions on the potential are required in the situations considered here. At the beginning of §4 the general strategy of proof is outlined and comparison with [KOY] made, but the present paper is self-contained.

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## 2 Results and Examples

Given $n \geq 2$ and $N := 2^{[(n+1)/2]}$, there exist $n+1$ anti-commuting Hermitian $N \times N$ matrices $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta := \alpha_{n+1}$ with square one. No specific representation of these matrices will be needed except in Appendix B. Using the notation

$$\alpha \cdot p = \sum_{j=1}^{n} \alpha_j(-i\partial_j),$$

we start by considering the differential expression

$$\tau := \alpha \cdot p + Q$$

in $L^2(\boldsymbol{R}^n)^N$. Here $Q : \boldsymbol{R}^n \to \boldsymbol{C}^{N\times N}$ is a matrix-valued function with measurable entries.
Let $\phi \in \boldsymbol{C}^N$. Then

$$u(x) := \frac{1 + i\alpha \cdot x}{(1 + |x|^2)^{n/2}}\,\phi$$

satisfies

$$\tau u = 0 \quad \text{with} \quad Q(x) = -\frac{n}{1 + r^2}. \tag{2.1}$$

This is the example of Loss and Yau [LY] mentioned in the Introduction. (Note that $u$ is in $L^2$ iff $n \geq 3$.) For $n = 3$ the potential in (2.1) is in fact the first in a hierarchy of potentials with constants larger than 3, all giving rise to zero modes; see [SU].
In contrast, we have the following result.

THEOREM 2.1. *Let $Q : \boldsymbol{R}^n \to \boldsymbol{C}^{N\times N}$ be measurable with*

$$\sup_{x\in\boldsymbol{R}^n} |x||Q(x)| \leq C \quad \textit{for some} \quad 0 < C < \frac{n-1}{2}.$$

*Then any solution $u \in H^1_{\mathrm{loc}}(\boldsymbol{R}^n)^N \cap L^2(\boldsymbol{R}^n, r^{-1}dx)^N$ of $\tau u = 0$ is identically zero.*

REMARK 2.2. *a) In case $Q$ is Hermitian, Theorem 2.1 can be rephrased as follows: The self-adjoint realisation $H$ of $\tau$ with*

$$\int \frac{|u|^2}{r}\,dx < \infty \quad (u \in D(H)) \tag{2.2}$$

*does not have the eigenvalue zero. (For the existence of such a self-adjoint realisation see [Ar] and the references therein.)*
*b) Suppose that $V$ is a real-valued scalar potential with $\sup_{x\in\boldsymbol{R}^n} |x||V(x)| < \frac{n-1}{2}$. Then it needs but a small additional assumption to show by means of the virial theorem that the self-adjoint realisation $H$ of $\alpha \cdot p + V + m_0\beta$ $(m_0 \neq 0)$ with property (2.2) does not have the eigenvalues $\pm m_0$ (see, e.g., [L], Theorem 2.4 in conjunction with [We], p.335).*
*c) It seems to be difficult to compare our $C$ with the size of the compactly supported potentials in [S] which produce zero modes for $n \geq 2$.*

Now we replace the whole space with the exterior domain

$$E_R := \{x \in \boldsymbol{R}^n \mid |x| > R\},$$

where $R > 0$ can be arbitrarily large. On $E_R$ we consider the Dirac equation

$$(\alpha \cdot D + Q)u = 0 \quad \text{with} \quad D := p - b. \tag{2.3}$$

We assume for simplicity that the vector potential $b$ is in $C^1(E_R, \boldsymbol{R}^n)$. Further conditions will not be imposed on $b$ but on the magnetic field

$$B := (\partial_i b_k - \partial_k b_i).$$

Note that $B$ can be identified with the scalar function $\partial_1 b_2 - \partial_2 b_1$ if $n = 2$ and the vector-valued function $\operatorname{curl} b$ if $n = 3$. Solutions of (2.3) will be functions in $H^1_{\mathrm{loc}}(E_R)^N$ which satisfy (2.3), if $Q$ is locally bounded in $E_R$.
The following result and its remark are essentially contained in [KOY], Example 6.1 and final remark on p.40.

THEOREM 2.3. *Let $m_0 \geq 0$, $\lambda \in \boldsymbol{R}$ and $Q := V + m_0\beta - \lambda + W$, where*

*(I) $V = V^* \in C^1(E_R, \boldsymbol{C}^{N\times N})$,*
*(II) $W$ is a measurable and bounded matrix function (not necessarily Hermitian).*

*Suppose $V$, $W$ and the magnetic field $B$ satisfy the following conditions:*
*a) $r^{1/2}V = o(1) = r\partial_r V$ uniformly with respect to directions;*
*b) there exist numbers $K \in (0, 1/2)$ and $M > 0$ such that*

$$r|W| \leq K, \quad |Bx| \leq M \quad \text{on} \quad E_R.$$

*Assertion: If $u \in L^2(E_R)^N$ is a solution of (2.3) for*

$$|\lambda| > \sqrt{m_0^2 + M^2}/(1 - 2K),$$

*then $u = 0$ on $E_{R_1}$ for some $R_1 > R$. If $r|W| = o(1)$ or $|Bx| = o(1)$ uniformly w.r.t. directions, then $K = 0$ or $M = 0$ is permitted.*

REMARK 2.4. *If $V \in C^2(R, \infty)$ is a real-valued (scalar) function, condition a) can be replaced by*

$$V(r) = o(1) = rV'(r), \quad rV''(r) = o(1) \quad \text{as} \quad r \to \infty.$$

REMARK 2.5. *Using a unique continuation result, e.g., the simple one [HP] or the more sophisticated one in [DO], one can conclude that $u = 0$ on $\boldsymbol{R}^n$.*

REMARK 2.6. *It follows immediately from Theorem 2.3 and Remark 2.5 that the potential in (2.1) cannot create a non-zero eigenvalue.*

Theorem 2.3 will now be supplemented by Theorems 2.7 and 2.10.

THEOREM 2.7. *Let $m_0 = 0, \lambda \leq 0$ and $V$, $W$ as in (I), (II) of Theorem 2.3. Let $q \in C^2(R, \infty)$ be a positive bounded function with the properties*
*(i) $[r(q - \lambda)]' \geq \delta_0(q - \lambda)$ for some $\delta_0 \in (0, 1)$,*

*(ii)* $\dfrac{q'}{q^2} = o(1) = r\dfrac{q''}{q^2}$.

*Suppose $V$, $W$ and $B$ satisfy the following conditions:*

*(H.1)* $r(V-q) = O(1), \quad \partial_r V - q' = o\left(\frac{q}{r}\right)$;

*(H.2)* $r|W| \leq K$ *for some* $K \in [0, \delta_0/2)$;

*(H.3) there is a function* $a(r)$ *with* $|Bx| \leq a(r)$ *and* $\dfrac{a}{q} = o(1)$.

*Assertion: Any solution* $u \in L^2(E_R)^N$ *of (2.3) with* $Q = V + W - \lambda$ *vanishes identically on* $E_{R_1}$ *for some* $R_1 > R$.

Remark 2.8. *a) To prove Theorem 2.7, it will be important to observe that condition (i) implies*

$$r(q-\lambda) \geq \text{const.}\, r^{\delta_0} \quad (r > R).$$

*b) If $q$ is a negative bounded function with property (ii) and which satisfies* $[r(\lambda - q)]' \geq \delta_0(\lambda - q)$ *for some* $\delta_0 \in (0,1)$*, then Theorem 2.7 holds for* $\lambda \geq 0$.
*c) In case $V$ decays at infinity, hypothesis (H.3) demands a corresponding stronger decay of $B$ to prevent the existence of eigenvalues. (The contrasting situation that $V$ and $B$ become large at infinity is considered in [MS].)*

Examples. For simplicity we assume $V = q$ and $W = 0$. Let $q_0$ be a positive number. Then the functions

$$q = q_0\,[2 + \sin(\log\log r)], \tag{2.4}$$

$$q = q_0\,(\log r)^{-s} \quad (s > 0), \tag{2.5}$$

$$q = q_0\, r^{-s} \quad (0 < s < 1), \tag{2.6}$$

have the required properties (i), (ii). In addition, a magnetic field with the decay property (H.3) is allowed. As far as (2.5) and (2.6) are concerned, Remarks 2.4–2.5 already rule out any eigenvalue $\lambda \neq 0$. In case there is no vector potential, it follows from [S], Corollary 1 that the self-adjoint operator associated with $\tau = \alpha \cdot p + q$ in $L^2(\boldsymbol{R}^n)^N$ has purely absolutely continuous spectrum outside $[q_0, 3q_0]$.

Remark 2.9. *More realistic potentials than (2.4) will have the property*

$$\lim_{r \to \infty} q(r) =: q_\infty \neq 0.$$

*In such situations, however, it may be possible to use Theorem 2.3 or Remark 2.4 to show that*

$$(\alpha \cdot D + Q - q_\infty)u = \lambda u$$

*has no non-trivial solution of integrable square at infinity if* $\lambda = -q_\infty$*. A case in point is the potential*

$$q = \frac{r}{1+r},$$

*which does not obey condition (i). Assuming $|Bx| = o(1)$ uniformly w.r.t. directions, it follows from Theorem 2.3 that*

$$(\alpha \cdot D + q - 1)u = \lambda u$$

*has no solution $u \neq 0$ in $L^2$ at infinity if $\lambda \neq 0$. In particular, $\alpha \cdot D + q$ has no zero mode.*

For the equation

$$(\alpha \cdot D + V + m_0\beta)u = -m_0 u, \quad D := p - b \tag{2.7}$$

our result is as follows.

THEOREM 2.10. *Let $m_0 > 0$ and $\mu := \sqrt{q(q+2m_0)}$. Let $q \in C^2(R, \infty)$ be a positive function with $q = o(1)$ and the following properties:*

*(i)* $\dfrac{(r\mu)'}{\mu} \geq \delta_0$ *for some* $\delta_0 \in (0,1)$*;*

*(ii)* $r\dfrac{q'}{q} = O(1), \quad r\dfrac{q''}{q^{3/2}} = o(1).$

*Suppose $V \in C^1(E_R, \boldsymbol{R})$ and $B$ satisfy the following conditions:*

*(H.1)* $\quad r^2(V - q) = O(1), \quad \partial_r V - q' = o\Big(\dfrac{q}{r}\Big)$*;*

*(H.2)* $\quad$ *there is a function $a(r)$ with $|Bx| \leq a(r)$ and* $\dfrac{a}{\sqrt{q}} = o(1)$*.*

*Assertion: Any solution $u \in L^2(E_R)^N$ of (2.7) vanishes identically on $E_{R_1}$ for some $R_1 > R$.*

REMARK 2.11. *a) The function $\mu$ originates from a transformation in Appendix A (see (A.13)–(A.18)). Theorem 2.10 holds good for solutions of*

$$(\alpha \cdot D + V + m_0\beta)u = m_0 u,$$

*if $q = o(1)$ is a negative function and $\mu := \sqrt{q(q-2m_0)}$.*
*b) Since*

$$\frac{(r\mu)'}{\mu} = 1 + \frac{rq'}{2q} + o(1),$$

*$q$ may decay like $r^{-s}$ with $0 < s < 2$.*

## 3 PROOF OF THEOREM 2.1

Since

$$\int r|\alpha \cdot pu|^2 = \int r|Qu|^2 \leq C^2 \int \frac{|u|^2}{r} < \infty,$$

we can find a sequence of functions $\{u_j\}$ in $C_0^\infty(\boldsymbol{R}^n)^N$ with

$$r^{-1/2}u_j \to r^{-1/2}u, \quad \sqrt{r}\,(\alpha \cdot p)u_j \to \sqrt{r}\,(\alpha \cdot p)u$$

in $L^2(\boldsymbol{R}^n)^N$. Let $U_j = r^{(n-1)/2}u_j$. We write $\|U_j\|$ rather than $\|U_j(r\cdot)\|$ for the norm in $L^2(S^{n-1})^N$ and similarly for the scalar product. We use the decomposition in (A.3) of Appendix A and note that the symmetric operator $S$ with $b = 0$ has a purely discrete spectrum with

$$-\left(\boldsymbol{N}_0 + \frac{n-1}{2}\right) \cup \left(\boldsymbol{N}_0 + \frac{n-1}{2}\right)$$

as eigenvalues. Hence

$$\begin{aligned}
\int r|\alpha \cdot pu_j|^2 &= \int_0^\infty r \left\| \alpha_r \left( -i\partial_r U_j + \frac{i}{r} SU_j \right) \right\|^2 \\
&= \int_0^\infty r \left\langle -i\partial_r U_j + \frac{i}{r} SU_j, -i\partial_r U_j + \frac{i}{r} SU_j \right\rangle \\
&= \int_0^\infty r \left( \|\partial_r U_j\|^2 + \frac{1}{r^2} \|SU_j\|^2 \right) + \int_0^\infty \partial_r \langle -U_j, SU_j \rangle \\
&= \int_0^\infty r \left( \|\partial_r U_j\|^2 + \frac{1}{r^2} \|SU_j\|^2 \right) \\
&\geq \left( \frac{n-1}{2} \right)^2 \int_0^\infty \frac{\|U_j\|^2}{r},
\end{aligned}$$

and the assertion follows in the limit $j \to \infty$. $\square$

## 4 Preliminaries to the proof of Theorem 2.7

To explain the general strategy of the proof of Theorem 2.7, let $u$ be a solution of (2.3). We multiply $U := r^{(n-1)/2}u$ by functions $e^\varphi$, $\varphi = \varphi(|\cdot|)$ real-valued, and $\chi = \chi(|\cdot|)$ with support in $E_R$ and

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } [s, t_k], \quad \chi = 0 \text{ outside } [s-1, t_{k+1}],$$

where $\{t_k\}$ is a sequence tending to infinity as $k \to \infty$. Then $\xi := \chi e^\varphi U$ satisfies

$$\left[ -i\alpha_r \mathcal{D}_r + i\alpha_r \left( \frac{S}{r} + \varphi' \right) + Q \right] \xi = g := -i\alpha_r \chi' e^\varphi U, \tag{4.1}$$

where $\mathcal{D}_r = \partial_r - i(x/r) \cdot b$. As in proofs of unique continuation results by means of Carleman inequalities, the idea is (as it was in the earlier papers ([V], [KOY]) to prove the existence of a constant $C > 0$ such that for large $s$ and in the limit $t_k \to \infty$ the inequality

$$\int_s^\infty e^{2\varphi} \|U\|^2 \leq C \int_{s-1}^s e^{2\varphi} \|U\|^2 \tag{4.2}$$

holds. (For the precise inequality see (5.3) below.) If, for example $\varphi = (\ell/2)r^b$ is permitted for some $b > 0$, (4.2) implies

$$e^{\ell(s+1)^b} \int_{s+1}^\infty \|U\|^2 \leq C e^{\ell s^b} \int_{s-1}^s \|U\|^2, \tag{4.3}$$

and in the limit $\ell \to \infty$ the desired conclusion $U = 0$ on $E_{s+1}$ follows.
The present paper differs from [KOY] in three important respects. Firstly, the function $q$ in Theorem 2.7 and 2.10 is allowed to tend to zero at infinity, while it was absolutely necessary to require $\pm q \geq \text{const.} > 0$ in [KOY]. Secondly, in contrast to [KOY], Proposition 3.1, the virial relation (A.8) from which we set out here

$$\begin{aligned}&\int \langle [\partial_r r(V-\lambda)]\xi, \xi\rangle \\ &= \underbrace{-\int \langle (\alpha\cdot Bx)\xi,\xi\rangle}_{I_1} + \underbrace{\int 2\mathrm{Re}\langle r\,W\xi, \mathcal{D}_r\xi\rangle}_{I_2} + \underbrace{\int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi,\xi\rangle}_{I_3} \\ &\quad \underbrace{-\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle}_{T_1}, \end{aligned} \tag{4.4}$$

does not contain a term involving $q'/q$. Such a term arose in [KOY] as it was necessary to divide $\xi$ by $(\pm q)^{1/2}$ in order to cope with the case that the potential (and possibly a variable mass) became large at infinity. Thirdly, we use a more refined cutoff function.
Given $t_k := 2^k$ and $s < t_k$, there exists a function $\chi \in C^\infty(0,\infty)$ with

$$\chi(r) = 1 \text{ for } s \leq r \leq t_k \quad\text{and}\quad \chi(r) = 0 \text{ for } r \geq t_{k+1}$$

such that

$$0 \leq -r\chi'(r) \leq \text{const.}\frac{r}{t_{k+1}-t_k} \leq \text{const.}\frac{2^{k+1}}{2^k}$$

for $r \in [t_k, t_{k+1}]$ and all $k \in \boldsymbol{N}$. Moreover,

$$r^\ell|\chi^{(\ell)}(r)| \leq \text{const.} \quad (r \in [t_k, t_{k+1}], k \in \boldsymbol{N})$$

for $\ell = 2$ and $\ell = 3$.
Estimates of the five terms in (4.4) will lead us to inequality (5.1) below, from which an inequality of type (4.3) will eventually emerge with the help of a bootstrap argument.
We start with the left-hand side of (4.4) and write

$$\begin{aligned}\partial_r[r(V-\lambda)] &= V - q + r(\partial_r V - q') + [r(q-\lambda)]' \\ &\geq \left[\frac{O(1/r)}{q-\lambda} + o(1)\frac{q}{q-\lambda} + \delta_0\right](q-\lambda)\end{aligned}$$

by means of (i) and (H.1) in Theorem 2.7. Since $0 < q/(q-\lambda) \leq 1$ and $(q-\lambda)^{-1} \leq O(r^{1-\delta_0})$ at infinity in view of Remark 2.8, we have

$$\int \langle [\partial_r r(V-\lambda)]\xi,\xi\rangle \geq \int [\delta_0 + o(1)](q-\lambda)\|\xi\|^2. \tag{4.5}$$

The four terms $I_1$, $I_2$, $I_3$ and $T_1$ on the right-hand side of (4.4) are estimated as follows.

Lemma 4.1.

$$a)\ \ I_1 := -\int \langle (\alpha\cdot Bx)\xi,\xi\rangle \le \int o(1)(q-\lambda)\|\xi\|^2,$$
$$b)\ \ I_2 := \int 2\mathrm{Re}\langle r\,W\xi, \mathcal{D}_r\xi\rangle$$
$$\le \int \left[2K - \frac{K}{(q-\lambda)^2}\left(\frac{\varphi'}{r}+\varphi''\right) + o(1)\right](q-\lambda)\|\xi\|^2 + T_2,$$

*where*

$$T_2 \;:=\; K\int\left\{\frac{(\chi')^2}{q-\lambda} + |\chi'|\left[\frac{\text{const.}}{r(q-\lambda)} + o(1)\right]\right\}\|e^{\varphi}U\|^2 \tag{4.6}$$
$$\le\; \text{const.}\left\{\int_{s-1}^{s} r\|e^{\varphi}U\|^2 + \int_{t_k}^{t_{k+1}} r^{-\delta_0}(q-\lambda)\|e^{\varphi}U\|^2\right\}. \tag{4.7}$$

*Proof.* a) follows immediately from assumption (H.3) since $\alpha\cdot Bx$ is an Hermitian matrix with square $|Bx|^2$.
To prove b), we observe

$$I_2 = \int 2\mathrm{Re}\langle r\,W\xi,\mathcal{D}_r\xi\rangle \le K\left(\int (q-\lambda)\|\xi\|^2 + \int \frac{1}{q-\lambda}\|\mathcal{D}_r\xi\|^2\right)$$

and use relations (A.9), (A.11) with

$$h = \frac{1}{q-\lambda},\quad j = r\left(\frac{h}{r}\right)' = h' - \frac{h}{r}.$$

Then

$$\int \frac{1}{q-\lambda}\|\mathcal{D}_r\xi\|^2 \;\le\; \int \frac{1}{q-\lambda}\|g-Q\xi\|^2 - \int \frac{1}{r(q-\lambda)}\langle A\xi,\xi\rangle$$
$$+\int j\,\mathrm{Im}\langle Q\xi,\alpha_r\xi\rangle + \int\left[\frac{j'}{2} - h\left(\frac{\varphi'}{r}+\varphi''\right)\right]\|\xi\|^2$$
$$+\int j\chi\chi'\,\|e^{\varphi}U\|^2, \tag{4.8}$$

where $Q = V+W-\lambda$, $A = -\alpha_r(\alpha\cdot Bx)$. Before turning to the first term on the right-hand side of (4.8) we note

$$\mathrm{Im}\langle Q\xi,\alpha_r\xi\rangle = \mathrm{Im}\,\langle (V-q)\xi,\alpha_r\xi\rangle + \mathrm{Im}\,\langle W\xi,\alpha_r\xi\rangle$$

and

$$|\mathrm{Im}\,\langle (V-q)\xi,\alpha_r\xi\rangle| \le \frac{\text{const.}}{r(q-\lambda)}(q-\lambda)\|\xi\|^2 \le \frac{\text{const.}}{r^{\delta_0}}(q-\lambda)\|\xi\|^2 \tag{4.9}$$

as well as

$$|\mathrm{Im}\,\langle W\xi, \alpha_r\xi\rangle| \le \frac{\text{const.}}{r(q-\lambda)}(q-\lambda)\|\xi\|^2 \le \frac{\text{const.}}{r^{\delta_0}}(q-\lambda)\|\xi\|^2 \tag{4.10}$$

(see hypotheses (H.1), (H.2) and Remark 2.8 a). Similarly, the term

$$\begin{aligned}-2r\mathrm{Re}\langle g, Q\xi\rangle &= 2r\mathrm{Re}\,\langle i\alpha_r\chi' e^{\varphi}U, Q\chi e^{\varphi}U\rangle \\ &= 2r\mathrm{Re}\,\langle i\alpha_r\chi' e^{\varphi}U, (V-q+W)\chi e^{\varphi}U\rangle\end{aligned}$$

can be estimated by

$$|-2r\mathrm{Re}\langle g, Q\xi\rangle| \le \text{const.}|\chi'|\|e^{\varphi}U\|^2.$$

Next,

$$\begin{aligned}&\|g-Q\xi\|^2 = \|(q-\lambda+V-q+W)\xi\|^2 + \|g\|^2 - 2\mathrm{Re}\,\langle g, Q\xi\rangle \\ &= (q-\lambda)^2\|\xi\|^2 + \|(V-q+W)\xi\|^2 + 2(q-\lambda)\mathrm{Re}\,\langle \xi, (V-q+W)\xi\rangle \\ &\quad +\|g\|^2 - 2\mathrm{Re}\,\langle g, Q\xi\rangle \\ &\le [1+o(1)](q-\lambda)^2\|\xi\|^2 + \left[(\chi')^2 + \frac{\text{const.}}{r}|\chi'|\right]\|e^{\varphi}U\|^2. \end{aligned} \tag{4.11}$$

Using Remark 2.8 a) again, the second term in (4.8) can be majorised by

$$\text{const.}\int \frac{a}{r^{\delta_0}}\|\xi\|^2.$$

With hypothesis (H.2) we see that it is

$$\int o(1)(q-\lambda)\|\xi\|^2.$$

The same is true of the third term in (4.8), since (4.9) and (4.10) hold and $j = o(1)$ by Remark 2.8 a) and the first part of assumption (ii). This leaves us with

$$\frac{j'}{q-\lambda} = \frac{1}{r^2(q-\lambda)^2} + \frac{q'}{r(q-\lambda)^3} - \frac{q''}{(q-\lambda)^3} + \frac{2(q')^2}{(q-\lambda)^4},$$

which, by assumptions (ii) and Remark 2.8 a), is again $o(1)$. Collecting terms, we finally obtain (4.6) from which (4.7) follows, employing the properties of our cutoff function and Remark 2.8 a). □

Lemma 4.2. *Let $\varphi' \ge 0$ and*

$$k_\varphi := -r\varphi'\varphi'' - (\varphi')^2 + \frac{1}{2}(r\varphi'')', \tag{4.12}$$

$$c := \frac{a}{q-\lambda} + \frac{\text{const.}}{r(q-\lambda)} + \frac{1}{2}\frac{q'}{(q-\lambda)^2} + \frac{1}{2}\frac{rq''}{(q-\lambda)^2} \tag{4.13}$$

*Then,*

$$\begin{aligned} I_3 &:= \int 2r\varphi' \mathrm{Re}\langle -i\alpha_r \mathcal{D}_r \xi, \xi\rangle \\ &\leq \int \left[\frac{k_\varphi}{(q-\lambda)^2} + \frac{\varphi'}{q-\lambda}c + \text{const.}\frac{|\varphi''|}{(q-\lambda)^2}\right](q-\lambda)\|\xi\|^2 \\ &\quad + T_3, \end{aligned} \tag{4.14}$$

*where*

$$\begin{aligned} T_3 &:= \int \frac{r}{q-\lambda}[\,\varphi'(\chi')^2 + |\varphi''||\chi''|\,]\,\|e^\varphi U\|^2 \\ &\leq \text{const.}\left\{\int_{s-1}^{s} r^2(\varphi' + |\varphi''|)\,\|e^\varphi U\|^2\right. \\ &\quad \left. + \int_{t_k}^{t_{k+1}} r^{2-2\delta_0}\left(\frac{\varphi'}{r} + |\varphi''|\right)(q-\lambda)\|e^\varphi U\|^2\right\}. \end{aligned} \tag{4.15}$$

*Proof*. By setting $\xi = \sqrt{q-\lambda}\, w$, $I_3$ can be written as

$$\begin{aligned} I_3 &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi, \xi\rangle = \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r \mathcal{D}_r w, (q-\lambda)w\rangle \\ &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, Qw\rangle - \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, (V-q+W)w\rangle. \end{aligned}$$

Using (A.10), (A.12) with $h = r\varphi'$, $j = r\varphi''$, we obtain

$$\begin{aligned} I_3 = &\int r\varphi'(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r\mathcal{D}_r w\|^2) - \int \varphi'\langle Aw, w\rangle \\ &+ \int \left[r\varphi'\varphi'' + \frac{1}{2}(r\varphi'')' - [r(\varphi')^2]' + \frac{1}{2}\left(\frac{r\varphi'}{q-\lambda}q'\right)' - \frac{r\varphi''}{2(q-\lambda)}q'\right]\|w\|^2 \\ &+ \int r\varphi''\mathrm{Im}\langle Qw, \alpha_r w\rangle + \int \frac{r\varphi''}{q-\lambda}\chi\chi'\|e^\varphi U\|^2 \\ &- \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r w, (V-q+W)w\rangle. \end{aligned} \tag{4.16}$$

The last term can simply be estimated by

$$\int r\varphi'\left(\|\mathcal{D}_r w\|^2 + \frac{\text{const.}}{r^2}\|w\|^2\right).$$

Replacing $\xi$ by $w$ in (4.9) and (4.10), we have

$$\begin{aligned} I_3 \leq &\int \left\{k_\varphi + \varphi'\left[a + \frac{\text{const.}}{r} + \frac{1}{2}\left(\frac{rq'}{q-\lambda}\right)'\right] + \text{const.}|\varphi''|\right\}\|w\|^2 \\ &+ \int \frac{r}{q-\lambda}\,[\varphi'(\chi')^2 + |\varphi''||\chi'|]\,\|e^\varphi U\|^2, \end{aligned}$$

which leads to (4.12)–(4.14). The estimate (4.15) is again a consequence of the properties of $\chi$ and of Remark 2.8 a). $\Box$

Lemma 4.3. *Let $\varphi' \geq 0$. Then*

$$\begin{aligned} T_1 &:= -\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle \\ &\leq \text{const.}\left\{\int_{s-1}^{s} r\left[(1+\varphi')\|e^{\varphi}U\|^2 + e^{2\varphi}\|\mathcal{D}_rU\|^2\right]\right. \qquad (4.17) \\ &\quad \left. + \int_{t_k}^{t_{k+1}} r^{1-\delta_0}\left(1+|\varphi''|\right)(q-\lambda)\|e^{\varphi}U\|^2\right\}. \end{aligned}$$

Proof. Let $\eta := e^{\varphi}U$. Since

$$\mathrm{Re}\langle -i\alpha_r\chi'\eta, \partial_r(\chi\eta)\rangle = \chi\chi'\,\mathrm{Re}\langle -i\alpha_r\eta, \partial_r\eta\rangle,$$

we have

$$\begin{aligned} T_1 &= -\int 2r\mathrm{Re}\langle g, \mathcal{D}_r\xi\rangle = \int 2r\mathrm{Re}\langle i\alpha_r\chi' e^{\varphi}U, \mathcal{D}_r\xi\rangle \\ &= \int 2r\chi\chi'\mathrm{Re}\langle i\alpha_r\eta, \mathcal{D}_r\eta\rangle. \end{aligned}$$

On the interval $[s-1, s]$ the integral can simply be estimated by

$$\int_{s-1}^{s} 2r\chi'\|\eta\|\|\mathcal{D}_r\eta\| \leq \int_{s-1}^{s} re^{2\varphi}\|U\|\|\mathcal{D}_rU + \varphi'U\|.$$

On $[t_k, t_{k+1}]$ we use the estimate

$$\int_{s-1}^{s} r(-\chi')(\|\eta\|^2 + \|\mathcal{D}_r\eta\|^2)$$

and observe that (A.9) and (A.11) hold with $\chi = 1$ (i.e., $g = 0$) and $\xi = \eta$, provided $h$ and $j$ have compact support. Hence, with $h = r(-\chi')$, $j = r(h/r)' = -r\chi''$ we have

$$\begin{aligned} &\int r(-\chi')\left[\|\mathcal{D}_r\eta\|^2 + \|\left(\frac{S}{r} + \varphi'\right)\eta\|^2\right] \\ &= \int r(-\chi')\|Q\eta\|^2 + \int \chi'\langle A\eta, \eta\rangle - \int r\chi''\mathrm{Im}\langle (V - q + W)\eta, \alpha_r\eta\rangle \\ &+ \int\left[-\frac{1}{2}(\chi'' + r\chi''') + \chi'\varphi' + r\chi'\varphi''\right]\|\eta\|^2. \end{aligned}$$

The integration extends over $[t_k, t_{k+1}]$ only where $\chi'\varphi' \leq 0$. Using Remark 2.8 a) and the estimates of the derivatives of $\chi$, the assertion follows. $\Box$

Remark 4.4. *The constant in (4.6), (4.13), and (4.14) is the sum of the constants which occur in assumptions (H.1) and (H.2). In view of the hypotheses of Theorem 2.7 the function $c$ in (4.13) is $o(1)$ at infinity. It is important that the constants in (4.7), (4.15) and (4.17) are independent of $\varphi$ and $t_k := 2^k$.*

## 5 Proof of Theorem 2.7

Before proving Theorem 2.7 we prepare the following

Proposition 5.1. *Suppose $f > 0$, $g \geq 0$ are functions on $(0, \infty)$ with*

$$\int^{\infty} \frac{1}{f} = \infty, \quad \int^{\infty} g < \infty,$$

*and $f$ is continuous and non-decreasing. Let $t_k := 2^k$ $(k \in \boldsymbol{N})$. Then we have*

$$\liminf_{k \to \infty} \int_{t_k}^{t_{k+1}} fg = 0$$

*Proof*. Assume to the contrary that there are numbers $\varepsilon_0 > 0$ and $N \in \boldsymbol{N}$ such that

$$\int_{t_k}^{t_{k+1}} fg \geq \varepsilon_0$$

for $N \leq k \in \boldsymbol{N}$. Then

$$\int_{t_N}^{\infty} g = \sum_{k=N+1}^{\infty} \int_{t_{k-1}}^{t_k} (fg)\, \frac{1}{f} \geq \varepsilon_0 \sum_{k=N+1}^{\infty} \frac{1}{f(t_k)} \geq \varepsilon_0 \int_{t_{N+1}}^{\infty} \frac{1}{f} = \infty$$

gives the desired contradiction.

*Proof of Theorem 2.7*

From (4.4) and (4.5) and Lemma 4.1–4.3 we see

$$\begin{aligned} & \int [\delta_0 - 2K + o(1)](q - \lambda) e^{2\varphi} \|\chi U\|^2 \qquad (5.1) \\ \leq \quad & \int \Big[ \frac{k_\varphi}{(q-\lambda)^2} + \frac{\varphi'}{q - \lambda} c + \text{const.} \frac{|\varphi''|}{(q-\lambda)^2} \\ & \quad - \frac{K}{(q-\lambda)^2} \left( \frac{\varphi'}{r} + \varphi'' \right) \Big] (q - \lambda) e^{2\varphi} \|\chi U\|^2 \\ & \quad + \text{const.} \int_{s-1}^{s} r^2 e^{2\varphi} [(1 + \varphi' + |\varphi''|) \|U\|^2 + \|\mathcal{D}_r U\|^2] \\ & \quad + \text{const.} \int_{t_k}^{t_{k+1}} r^{1-\delta_0} e^{2\varphi} (1 + \varphi' + r|\varphi''|) |(q - \lambda) \|U\|^2, \end{aligned}$$

where $t_k := 2^k$.

a) We claim

$$\int_s^{\infty} r^{\ell} (q - \lambda) \|U\|^2 < \infty$$

for all $s > R$ and $\ell > 0$. Let $j \in \boldsymbol{N}$. We choose $\varphi = (j/2)\log\log r$ in (5.1) and note

$$\varphi' = \frac{j}{2r\log r}, \quad \varphi'' = -\frac{j}{4r^2\log r}\left(1 + \frac{1}{\log r}\right),$$
$$k_\varphi = \frac{j}{4r^2\log r}\left[1 + \frac{2}{\log r} + \frac{j+2}{(\log r)^2}\right] \le \frac{j(j-1)}{4r^2(\log r)^3} + \frac{2j}{4r^2\log r}$$

for $r > R_0$ if $R_0$ is sufficiently large. Using Remark 2.8 a), the first integral on the right-hand side of (5.1) can be majorised by

$$\text{const.}\int\left\{\frac{1}{r^{2\delta_0}}\left[\frac{j(j-1)}{(\log r)^3} + \frac{j}{(\log r)^2} + \frac{j}{(\log r)}\right]\right.$$
$$\left.+\frac{1}{r^{\delta_0}}\frac{j}{(\log r)}o(1)\right\}(\log r)^j(q-\lambda)\|\chi U\|^2.$$

The last integral on the right-hand side of (5.1) can be estimated by

$$\int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + \text{const.}\, j)(\log r)^j(q-\lambda)\|U\|^2. \tag{5.2}$$

With $(q-\lambda)$ being bounded, $(q-\lambda)\|U\|^2$ is in $L^1(R_0,\infty)$. Thus there is a sub-sequence $\{t_{k_\ell}\}_{\ell=1}^\infty$ on which (5.2) tends to zero in view of Proposition 5.1. This proves

$$\int_{R_0}^\infty (\log r)^j(q-\lambda)\|U\|^2 < \infty.$$

Moreover, for some $c_0 > 0$ the inequality (5.1) implies

$$c_0\int_{R_0}^\infty \sum_{j=0}^{L}\frac{(\ell\log r)^j}{j!}(q-\lambda)\|U\|^2$$
$$\le \text{const.}\int_{s-1}^\infty\left[\frac{\ell^2}{r^{2\delta_0}\log r}\sum_{j=2}^{L}\frac{(\ell\log r)^{j-2}}{(j-2)!} + \frac{\ell}{r^{\delta_0}}\sum_{j=1}^{L}\frac{(\ell\log r)^{j-1}}{(j-1)!}\right]$$
$$\cdot(q-\lambda)\|U\|^2 + \text{const.}\int_{s-1}^{s} r^2\sum_{j=0}^{L}\frac{(\ell\log r)^j}{j!}[(1+j)\|U\|^2 + \|\mathcal{D}_r U\|^2].$$

Since we can let $L \to \infty$, this establishes the claim.

b) Next we assert

$$\int_s^\infty e^{\ell r^b}(q-\lambda)\|U\|^2 < \infty$$

for all $s > R$, $\ell > 0$ and $b \in (0, \delta_0)$.

We insert $\varphi = (jb/2)\log r$ into (5.1) and observe

$$\varphi' = \frac{jb}{2r}, \quad \varphi'' = -\frac{jb}{2r^2}, \quad k_\varphi = \frac{jb}{4r^2}.$$

In view of Part a) there is a sub-sequence $\{t_{k_\ell}\}$ with

$$\lim_{\ell\to\infty} \int_{t_{k_\ell}}^{t_{k_\ell}+1} r^{1-\delta_0}(1+jb)r^{jb}(q-\lambda)\|U\|^2 = 0.$$

Using Remark 2.8 a) again, we see that (5.1) implies

$$c_0 \int_s^\infty r^{jb}(q-\lambda)\|U\|^2 \le \text{const.} \int_{s-1}^\infty \left(\frac{1}{r^{2\delta_0}} + \frac{1}{r^{\delta_0}}\right) jbr^{jb}(q-\lambda)\|U\|^2$$
$$+ \text{const.} \int_{s-1}^s r^{2+jb}[(1+jb)\|U\|^2 + \|\mathcal{D}_r U\|^2]$$

or

$$c_0 \int_s^\infty \sum_{j=0}^L \frac{(\ell r^b)^j}{j!}(q-\lambda)\|U\|^2$$
$$\le \text{const.} \int_{s-1}^\infty \frac{\ell b}{r^{\delta_0 - b}} \sum_{j=1}^L \frac{(\ell r^b)^{j-1}}{(j-1)!}(q-\lambda)\|U\|^2$$
$$+ \text{const.} \int_{s-1}^s r^2 \sum_{j=0}^L \frac{(\ell r^b)^j}{j!}[(1+jb)\|U\|^2 + \|\mathcal{D}_r U\|^2].$$

For $b \in (0, \delta_0)$ we can again move the first term of the right-hand side to the left and let $L \to \infty$.

c) In order to show that $U$ vanishes a.e. on $E_{R_1}$ for some $R_1 > R$, we choose $\varphi = (\ell/2)r^b$ where $\ell > 0$ and $b \in (0, \delta_0)$. From

$$\varphi' = \frac{\ell b}{2} r^{b-1}, \quad \varphi'' = -\frac{\ell b}{2}(1-b)r^{b-2}$$

we observe $(\varphi'/r) + \varphi'' > 0$, so that the last part of the first integral on the right-hand side of (5.1) can be discarded. On account of Part b) there is a sequence $\{t_{k_\ell}\}$ on which the last integral vanishes. Finally we note

$$k_\varphi = -\frac{\ell b}{4}\left[\ell b^2 - \frac{(1-b)^2}{r^b}\right] r^{2(b-1)}.$$

With

$$X := \frac{\ell b}{2(q-\lambda)} r^{b-1}$$

we therefore have

$$-\frac{k_\varphi}{(q-\lambda)^2} - \frac{\varphi'}{q-\lambda}c(r) - \text{const.}\frac{|\varphi''|}{(q-\lambda)^2} = bX^2 - d(r)X,$$

where

$$d(r) := \frac{1-b}{r(q-\lambda)}\left(\frac{1-b}{2} + \text{const.}\right) + c(r) = o(1).$$

Hence

$$\int_s^\infty \left[\delta_0 - 2K + o(1) - \frac{d(r)^2}{4b}\right](q-\lambda)e^{\ell r^b}\|U\|^2$$
$$\leq \text{const.}\int_{s-1}^s r^2 e^{\ell r^b}[(1+\ell b r^b)\|U\|^2 + \|\mathcal{D}_r U\|^2]. \tag{5.3}$$

Now there is an $R_1 > R$ with the property that the left-hand side of (5.3) can be estimated from below by

$$\text{const.}\, e^{\ell(s+1)^b}\int_{s+1}^\infty \|U\|^2$$

for $s > R_1$. The assertion therefore follows in the limit $\ell \to \infty$.

## 6 Proof of Theorem 2.10

From (A.16)-(A.17) with $\lambda = -m_0$ we see

$$\mu := [(q+m_0)^2 - m_0^2]^{1/2} = \sqrt{q}\sqrt{q+2m_0}, \quad F := \left(\frac{q+2m_0}{q}\right)^{1/4}, \tag{6.1}$$

and

$$\sqrt{\mu}F = \sqrt{q+2m_0}, \quad \frac{\sqrt{\mu}}{F} = \sqrt{q}.$$

As a consequence

$$\xi := \chi e^\varphi \begin{pmatrix} FU_1 \\ (1/F)U_2 \end{pmatrix} = \mu^{-1/2}\chi e^\varphi \zeta \quad \text{with} \quad \zeta := \begin{pmatrix} \sqrt{q+2m_0}\,U_1 \\ \sqrt{q}\,U_2 \end{pmatrix}$$

solves

$$\left[-i\alpha_r\mathcal{D}_r + i\alpha_r\left(\frac{S}{r} + \varphi'\right) + Q\right]\xi = g := -\mu^{-1/2}(i\alpha_r)\chi' e^\varphi \zeta,$$

where

$$Q = \mu I_N + (V-q)\begin{pmatrix} F^{-2}I_{N/2} & 0 \\ 0 & F^2 I_{N/2} \end{pmatrix} - \frac{m_0 q'}{2q(q+2m_0)}(i\alpha_r\beta).$$

So our virial relation (A.8) now reads

$$\begin{aligned}&\int \{\langle[\partial_r(rQ)]\xi,\xi\rangle + \langle(\alpha\cdot Bx)\xi,\xi\rangle\}\\ &= \int 2r\varphi'\mathrm{Re}\langle -i\alpha_r\mathcal{D}_r\xi,\xi\rangle - \int 2r\mathrm{Re}\langle g,\mathcal{D}_r\xi\rangle. \end{aligned}\tag{6.2}$$

Before beginning our estimates we observe that the assumption (i) implies

$$r\mu \geq \text{const.}\, r^{\delta_0} \quad \text{or} \quad q^{-1/2} \leq \text{const.}\, r^{1-\delta_0}. \tag{6.3}$$

In view of (6.1) and our assumptions (ii), (H.1) we therefore have

$$\begin{aligned}\frac{1}{\mu}(rF^{-2})'(V-q) &= \frac{1}{\mu F^2}\left[1-\frac{m_0 rq'}{q(q+2m_0)}\right](V-q)\\ &= o(1)\frac{V-q}{q} = o(1),\end{aligned}$$

$$\begin{aligned}\frac{1}{\mu}(rF^{2})'(V-q) &= \frac{F^2}{\mu}\left[1+\frac{m_0 rq'}{q(q+2m_0)}\right](V-q)\\ &= O(1)\frac{V-q}{q} = o(1),\end{aligned}$$

and from

$$\left[\frac{rq'}{q(q+2m_0)}\right]' = \frac{1}{q+2m_0}\left[\frac{q'}{q}+\frac{rq''}{q}-r\left(\frac{q'}{q}\right)^2-\frac{r(q')^2}{q(q+2m_0)}\right]$$

we find

$$\frac{1}{\mu}\left[\frac{rq'}{q(q+2m_0)}\right]' = o(1),$$

taking advantage of (6.3) again. Since

$$\int\langle(\alpha\cdot Bx)\xi,\xi\rangle \geq -\int\frac{a}{\mu}\,\mu\,\|\xi\|^2 = -\int o(1)\,\mu\,\|\xi\|^2,$$

the left-hand side of (6.2) can be estimated from below by

$$\int[\delta_0+o(1)]\mu\|\xi\|^2.$$

Lemma 6.1. *Let* $\varphi' \geq 0$ *and*

$$\begin{aligned}k_\varphi &:= -r\varphi'\varphi'' - (\varphi')^2 + \frac{1}{2}(r\varphi'')',\\ c &:= \frac{a}{\mu} + \frac{r(V-q)^2}{\mu q} + \frac{1}{2\mu}\left[\left(r\frac{\mu'}{\mu}\right)' + m_0\left|\left(r\frac{q'}{\mu^2}\right)'\right|\right].\end{aligned}$$

*Then*

$$\begin{aligned} I \quad &:= \quad \int 2r\varphi' \mathrm{Re}\langle -i\alpha_r \mathcal{D}_r \xi, \xi\rangle \\ &\le \quad \int \left(\frac{k_\varphi}{\mu^2} + \frac{\varphi'}{\mu}c + \text{const.}\,\frac{|\varphi''|}{\mu^2}\right)\mu\|\xi\|^2 + J, \end{aligned}$$

*where*

$$\begin{aligned} J \quad &:= \quad \int \frac{r}{\mu^2}[\varphi'(\chi')^2 + |\varphi''||\chi''|]\,\|e^\varphi \zeta\|^2 \\ &\le \quad \text{const.}\left\{\int_{s-1}^{s} r^3(\varphi' + |\varphi''|)\,\|e^\varphi\zeta\|^2 \right. \\ &\qquad \left. + \int_{t_k}^{t_{k+1}} r^{2-2\delta_0}\left(\frac{\varphi'}{r} + |\varphi''|\right)\|e^\varphi\zeta\|^2\right\} \end{aligned}$$

*with a constant which is independent of $\varphi$ and $t_k := 2^k$. (Note that the assumptions of Theorem 2.10 imply $c = o(1)$.)*

Proof. Since $w := \mu^{-1/2}\xi$ satisfies

$$\left[-i\alpha_r\mathcal{D}_r + i\alpha_r\left(\frac{S}{r} + \varphi'\right) + Q - i\frac{\mu'}{2\mu}\alpha_r\right]w = f := \mu^{-1/2}g,$$

we have

$$\begin{aligned} &\int j\left\langle \left(\frac{S}{r} + \varphi'\right)w, w\right\rangle + \frac{1}{2}\int\left(j' - j\frac{\mu'}{\mu}\right)\|w\|^2 \\ &\quad + \int j\,\mathrm{Im}\langle Qw, \alpha_r w\rangle + \int \frac{j}{\mu^2}\chi\chi'\|e^\varphi\zeta\|^2 = 0 \end{aligned}$$

as a substitute for identity (A.12).
Let

$$\begin{aligned} I_1 \quad &:= \quad \int 2r\varphi'\,\mathrm{Re}\left\langle -i\alpha_r\mathcal{D}_r w, \frac{m_0 q'}{2q(q+2m_0)}(i\alpha_r\beta)w\right\rangle, \\ I_2 \quad &:= \quad \int 2r\varphi'\,\mathrm{Re}\left\langle -i\alpha_r\mathcal{D}_r w, (V-q)\begin{pmatrix} F^{-2}w_1 \\ F^2 w_2\end{pmatrix}\right\rangle. \end{aligned}$$

Replacing $q$ by $\mu$ in (A.10), we can write

$$\begin{aligned} I &= \int 2r\varphi'\,\mathrm{Re}\,\langle -i\alpha_r\mathcal{D}_r w, \mu w\rangle = \int 2r\varphi'\,\mathrm{Re}\,\langle -i\alpha_r\mathcal{D}_r w, Qw\rangle + I_1 + I_2 \\ &= \int r\varphi'(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r\mathcal{D}_r w\|^2) - \int \varphi'\langle Aw, w\rangle \\ &\quad + \int\left[k_\varphi + \frac{1}{2}\left(r\varphi'\,\frac{\mu'}{\mu}\right)' - \frac{1}{2}r\varphi''\,\frac{\mu'}{\mu}\right]\|w\|^2 + \int \frac{r\varphi''}{\mu^2}\chi\chi'\|e^\varphi\zeta\|^2 \\ &\quad + \int r\varphi''\,\mathrm{Im}\langle Qw, \alpha_r w\rangle + I_1 + I_2. \end{aligned}$$

Integrating by parts in $I_1$, we obtain

$$\begin{aligned} I_1 &= -\int 2r\varphi' \operatorname{Re}\left\langle \partial_r w, \frac{q'}{2q(q+2m_0)}(m_0\beta)w \right\rangle \qquad (6.4)\\ &= \frac{1}{2}\int \left\langle w, \left\{ r\varphi'' \frac{q'}{q(q+2m_0)}(m_0\beta) + \varphi' \left[\frac{rq'}{q(q+2m_0)}\right]'(m_0\beta)\right\} w\right\rangle \end{aligned}$$

and note that the first term in (6.4) cancels

$$\int r\varphi'' \operatorname{Im}\langle Qw, \alpha_r w\rangle = -\frac{1}{2}\int r\varphi'' \left\langle \frac{q'}{q(q+2m_0)}(m_0\beta)w, w\right\rangle.$$

Furthermore, since

$$F^{-4}|w_1|^2 + F^4|w_2|^2 = \frac{q}{q+2m_0}|w_1|^2 + \frac{q+2m_0}{q}|w_2|^2,$$

we have

$$I_2 \le \int r\varphi' \|\mathcal{D}_r w\|^2 + \text{const.} \int r\varphi' \frac{(V-q)^2}{q}\|w\|^2.$$

Collecting terms, the assertion follows. □

LEMMA 6.2. *Let $\varphi' \ge 0$. Then*

$$\begin{aligned} T &:= -\int 2r \operatorname{Re}\langle g, \mathcal{D}_r\xi\rangle \\ &\le \left\{ \text{const.} \int_{s-1}^{s} r^2 \left[(1+\varphi')\|e^{\varphi}\zeta\|^2 + e^{2\varphi}\|\mathcal{D}_r\zeta\|^2\right] \right.\\ &\qquad \left. + \int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1+|\varphi''|)\|e^{\varphi}\zeta\|^2 \right\} \end{aligned}$$

*with a constant which is independent of $\varphi$ and $t_k$.*

Proof. Abbreviating $\phi := e^{\varphi}\zeta$, we have

$$T = -\int 2r \operatorname{Re}\langle g, \mathcal{D}_r\xi\rangle = \int 2r\frac{\chi\chi'}{\mu}\operatorname{Re}\langle i\alpha_r\phi, \mathcal{D}_r\phi\rangle.$$

On the interval $[s-1, s]$ the integral can be estimated by

$$\int_{s-1}^{s} 2r\frac{\chi'}{\mu}\|\phi\|\,\|\mathcal{D}_r\phi\| \le \text{const.} \int_{s-1}^{s} r^{2-\delta_0} e^{2\varphi}\|\zeta\|\,\|\mathcal{D}_r\zeta + \varphi'\zeta\|,$$

using (6.3). On $[t_k, t_{k+1}]$ we majorise the integral by

$$\int_{t_k}^{t_{k+1}} \frac{r}{\mu}(-\chi')(\|\phi\|^2 + \|\mathcal{D}_r\phi\|^2) \qquad (6.5)$$

and note that it is permitted to use (A.9), (A.11) with $\chi = 1$ (i.e., $g = 0$) and $\xi = \phi$, since

$$h = \frac{r}{\mu}(-\chi'), \quad j = r\left(\frac{h}{r}\right)' = h' - \frac{h}{r}$$

have compact support. Hence on $[t_k, t_{k+1}]$

$$\begin{aligned}&\int \frac{r}{\mu}(-\chi')\left[\|\mathcal{D}_r\phi\|^2 + \left\|\left(\frac{S}{r} + \varphi'\right)\phi\right\|^2\right] \\ &= \int \frac{r}{\mu}(-\chi')\|Q\phi\|^2 + \int \frac{\chi'}{\mu}\langle A\phi, \phi\rangle + \int\left[\frac{j'}{2} - \left(\frac{\varphi'}{r} + \varphi''\right)h\right]\|\phi\|^2 \\ &\quad + \int j\,\mathrm{Im}\langle Q\phi, \alpha_r\phi\rangle.\end{aligned}$$

Now, $-\varphi'' h \leq \text{const.}|\varphi''|r^{1-\delta_0}$ by (6.3), while $-(\varphi'/r)h \leq 0$ on $[t_k, t_{k+1}]$. From $h' = o(1) = h''$ we conclude $j = o(1) = j'$. The integral (6.5) can therefore be estimated by

$$\text{const.}\int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + |\varphi''|)\|\phi\|^2,$$

which concludes the proof. □

Summing up, we have

$$\begin{aligned}\int [\delta_0 + o(1)]\|\chi e^{\varphi}\zeta\|^2 &\leq \int\left(\frac{k_\varphi}{\mu^2} + \frac{\varphi'}{\mu}c + \text{const.}\frac{|\varphi''|}{\mu^2}\right)\|\chi e^{\varphi}\zeta\|^2 \\ &+ \text{const.}\left\{\int_{s-1}^{s} r^3 e^{2\varphi}[(1 + \varphi' + |\varphi''|)\|\zeta\|^2 + \|\mathcal{D}_r\zeta\|^2]\right. \\ &\left. + \int_{t_k}^{t_{k+1}} r^{1-\delta_0}(1 + \varphi' + r|\varphi''|)e^{2\varphi}\|\zeta\|^2\right\},\end{aligned}$$

which does not differ from (5.1) in any essential way. The previous bootstrap argument can therefore be repeated almost verbatim, proving $\zeta = 0$ and so $u = 0$ a.e. on $E_{R_1}$ for some $R_1 > R$. □

# Appendix

## A Identities in Connection with the Virial Theorem

### A.1 Algebraic Relations

The principal part in

$$(\alpha \cdot D + Q)u = 0, \quad D := p - b, \quad p = -i\nabla \tag{A.1}$$

can be decomposed with the operators

$$\mathcal{D}_r := \partial_r - i\frac{x}{r}\cdot b, \quad \alpha_r := \alpha \cdot \frac{x}{r},$$
$$S := \frac{n-1}{2} - \sum_{1\le j<k\le n} i\alpha_j\alpha_k(x_jD_k - x_kD_j) \tag{A.2}$$

as follows:

$$\alpha \cdot D = \alpha_r\left(-ir^{(1-n)/2}\,\mathcal{D}_r r^{(n-1)/2} + \frac{i}{r}S\right). \tag{A.3}$$

$S$ is a symmetric operator in $L^2(S^{n-1})^N$ which commutes with every operator which solely depends on the radial variable $r$; it anticommutes with $\alpha_r$. In two dimensions we have

$$S = \frac{1}{2} - i\sigma_1\sigma_2(x_1D_2 - x_2D_1) = \frac{1}{2} + \sigma_3(x_1D_2 - x_2D_1),$$

where $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices. For $n = 3$ it is convenient to define $\sigma := (-i\alpha_2\alpha_3, -i\alpha_3\alpha_1, -i\alpha_1\alpha_2)$. Then

$$S = 1 + \sigma\cdot L \quad\text{with}\quad L := x \times D,$$

but the operator $K := \beta S$ is also used instead of $S$.
We notice that $\alpha \cdot Bx$ anticommutes with $\alpha_r$, since $B$ is skew-symmetric. A longer but completely elementary calculation shows

$$A := [\mathcal{D}_r, S] = -i\alpha_r(\alpha\cdot Bx).$$

Since $\alpha_r^2 = 1$, this implies $A^2 = |Bx|^2$ and

$$[\mathcal{D}_r, i\alpha_r S] = \alpha\cdot Bx. \tag{A.4}$$

Furthermore,

$$T := (\beta + \eta\,\alpha\cdot Bx)(1 + \zeta\, i\alpha_r)$$

is an Hermitian $N\times N$ matrix with square

$$T^2 = (1 + \eta^2|Bx|^2)(1+\zeta^2),$$

if $\eta$, $\zeta \in \boldsymbol{R}$.
Finally we note that the $(n+1)$ Dirac matrices can be given the following block structure :

$$\alpha_j = \begin{pmatrix} 0 & a_j \\ a_j^* & 0\end{pmatrix} \quad (j = 1, 2, \cdots, n), \quad \beta = \begin{pmatrix} I_{N/2} & 0 \\ 0 & I_{N/2}\end{pmatrix}. \tag{A.5}$$

The $a_j$ are $(N/2)\times(N/2)$ matrices (Hermitian if $n$ is odd) which satisfy the following commutation relations :

$$a_ja_k^* + a_ka_j^* = 2\delta_{jk}I_{N/2}, \quad a_j^*a_k + a_k^*a_j = 2\delta_{jk}I_{N/2}.$$

For $n = 2$, 3 this is of course well-known; the Pauli matrices take the role of the $\alpha_j$ if $n = 2$ and of the $a_j$ if $n = 3$. For general $n$ see [KY].

## A.2 Analytic Tools

Let $\varphi$, $\chi$, $q$, $h$ and $j$ be smooth real-valued functions which depend only on $r$; we assume that $\chi$ has compact support and $q$ is positive. When $u$ is a solution of (A.1), then (A.3) implies that

$$\xi := \chi e^{\varphi} U \quad \text{with} \quad U := r^{(n-1)/2} u$$

and

$$w := q^{-(1/2)} \xi$$

are solutions of

$$\left[-i\alpha_r \mathcal{D}_r + i\alpha_r \left(\frac{S}{r} + \varphi'\right) + Q\right] \xi = g := -i\alpha_r \chi' e^{\varphi} U, \tag{A.6}$$

$$\left[-i\alpha_r \mathcal{D}_r + i\alpha_r \left(\frac{S}{r} + \varphi'\right) + Q - i\alpha_r \frac{q'}{2q}\right] w = f := q^{-(1/2)} g. \tag{A.7}$$

Splitting $Q$ into an Hermitian part $Q_1$ and $Q_2 := Q - Q_1$, the following virial relation holds :

$$\begin{aligned} \int \langle [\partial_r (r Q_1] \xi, \xi \rangle &= -\int \langle (\alpha \cdot Bx)\xi, \xi \rangle \qquad \text{(A.8)} \\ &+ \int 2\,\mathrm{Re}\langle r\, Q_2 \xi, \mathcal{D}_r \xi \rangle + \int 2r\varphi' \,\mathrm{Re}\langle -i\alpha_r \mathcal{D}_r \xi, \xi \rangle - \int 2r\,\mathrm{Re}\langle g, \mathcal{D}_r \xi \rangle. \end{aligned}$$

Norm and scalar product in $L^2(S^{n-1})^N$ are denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. We write $\|\xi\|$ rather than $\|\xi(r\cdot)\|$ and similarly for the scalar product. Integration is over $(0, \infty)$.
Starting from

$$\int 2r\,\mathrm{Re}\langle g, \mathcal{D}_r \xi \rangle$$

(A.8) can immediately be verified, using (A.6) and (A.4) as well as an integration by parts in the term containing $Q_1$ (see [KOY], Proposition 3.1 and the remark on p. 40). In case $\chi$ can be replaced by 1, the virial theorem in its familiar form follows from (A.8) by setting $\varphi = 0$, $Q_2 = 0$ and observing that $g$ is zero.
As a consequence of (A.6) – (A.7) we note the following energy relations:

$$\begin{aligned} \int h \left[ \|\mathcal{D}_r \xi\|^2 + \left\| \left(\frac{S}{r} + \varphi'\right) \xi \right\|^2 \right] &= \int h \|g - Q\xi\|^2 \qquad \text{(A.9)} \\ - \int \frac{h}{r} \langle A\xi, \xi \rangle - \int r \left(\frac{h}{r}\right)' &\left\langle \frac{S}{r} \xi, \xi \right\rangle - \int (h\varphi')' \|\xi\|^2, \end{aligned}$$

$$\int 2h \,\mathrm{Re}\langle -i\alpha_r \mathcal{D}_r w, Qw\rangle \tag{A.10}$$
$$= \int h(\|f\|^2 - \|\mathcal{D}_r w\|^2 - \|f + i\alpha_r \mathcal{D}_r w\|^2) - \int \frac{h}{r}\langle Aw, w\rangle$$
$$- \int r\left(\frac{h}{r}\right)'\left\langle \frac{S}{r}w, w\right\rangle + \int \left[\left(h\frac{q'}{2q}\right)' - (h\varphi')'\right]\|w\|^2.$$

(A.9) follows from

$$\int h\|i\alpha_r(g - Q\xi)\|^2 = \int h\left\|\mathcal{D}_r\xi - \left(\frac{S}{r} + \varphi'\right)\xi\right\|^2$$

by undoing the square on the right-hand side and integrating by parts (cf. [KOY], Proposition 3.3 and the remark on p.40). Similarly, (A.10) can be proved by inserting into the left-hand side the expression for $Qw$ which arises from (A.7) (cf. [KOY], Proposition 3.2 and the remark on p.40).
Since $S$ is unbounded from above and from below, the corresponding term on the right-hand side of (A.9)–(A.10) has to be eliminated. This can be done with the help of the auxiliary identities

$$\int j\left\langle \left(\frac{S}{r} + \varphi'\right)\xi, \xi\right\rangle + \frac{1}{2}\int j'\|\xi\|^2 + \int j\,\mathrm{Im}\langle Q\xi, \alpha_r\xi\rangle$$
$$+ \int j\chi\chi'\|e^{\varphi}U\|^2 = 0, \tag{A.11}$$

$$\int j\left\langle \left(\frac{S}{r} + \varphi'\right)w, w\right\rangle + \frac{1}{2}\int \left(j' - j\frac{q'}{q}\right)\|w\|^2 + \int j\,\mathrm{Im}\langle Qw, \alpha_r w\rangle$$
$$+ \int \frac{j}{q}\chi\chi'\|e^{\varphi}U\|^2 = 0. \tag{A.12}$$

(A.11), for example, results at once from

$$\int \mathrm{Im}\left\langle i\alpha_r\left(\frac{S}{r} + \varphi'\right)\xi, j\alpha_r\xi\right\rangle,$$

inserting (A.6) and integrating by parts (cf. [KOY], p.23).
Let $F = F(r) > 0$ be a smooth function, $m_0 > 0$ and $\lambda \in \boldsymbol{R}$. When $V$ is a scalar function and $Q = V + m_0\beta - \lambda$, it may be advantageous to split a solution $u$ of (A.1) into two vectors $u_1$, $u_2$ with $N/2$ components and use the block structure (A.5) of the Dirac matrices jointly with the transformation

$$\zeta := \begin{pmatrix} F\,U_1 \\ (1/F)\,U_2 \end{pmatrix}, \quad U_j = r^{(n-1)/2}u_j \tag{A.13}$$

(cf. [KOY], p.37). Then $\alpha_r = \begin{pmatrix} 0 & a_r \\ a_r^* & 0 \end{pmatrix}$ where $a_r := \sum_{j=1}^n (x_j/r)a_j$. With a smooth function $q = q(r)$ and

$$P := \frac{F'}{F} i\alpha_r\beta + \begin{pmatrix} (1/F^2)(V - q + q + m_0 - \lambda)I_{N/2} & 0 \\ 0 & F^2(V - q + q - m_0 - \lambda)I_{N/2} \end{pmatrix}, \tag{A.14}$$

we then have

$$\left(-\alpha_r\mathcal{D}_r + \frac{i}{r}\alpha_r S + P\right)\zeta = 0. \tag{A.15}$$

If, for example, $q - m_0 - \lambda > 0$, requiring

$$\frac{1}{F^2}(q + m_0 - \lambda) = \mu = F^2(q - m_0 - \lambda), \tag{A.16}$$

leads to

$$\mu = [(q - \lambda)^2 - m_0^2]^{1/2}, \quad F = \left(\frac{q + m_0 - \lambda}{q - m_0 - \lambda}\right)^{1/4} \tag{A.17}$$

Since

$$\frac{F'}{F} = -\frac{m_0}{2}\frac{q'}{\mu^2},$$

the potential (A.14) becomes

$$P = \mu I_N + (V - q)\begin{pmatrix} F^{-2}I_{N/2} & 0 \\ 0 & F^2 I_{N/2} \end{pmatrix} - \frac{m_0}{2}\frac{q'}{\mu^2} i\alpha_r\beta. \tag{A.18}$$

## B Asymptotic Behaviour of Solutions

In case $Q$ in

$$(-i\alpha \cdot \nabla + Q)u = 0 \tag{B.1}$$

is $Q = m_0\beta + \lambda - q$ and $q$ is a rotationally symmetric scalar function, it suffices to discuss the ordinary differential equation

$$u' = \begin{pmatrix} -(k/r) & -q + m_0 + \lambda \\ q + m_0 - \lambda & (k/r) \end{pmatrix} u, \tag{B.2}$$

where $k$ is an eigenvalue of the angular momentum operator $S$ in (A.2) with $b = 0$. $k$ is an integer or half-integer such that $|k| \geq (n-1)/2$. (For general $n$, the reduction of (B.1) to (B.2) can be found in [KY].)

### B.1 CASE $m_0 = 0 = \lambda$

In the new variables

$$t = \log r, \quad v(t) = u(e^t) \tag{B.3}$$

(B.2) reads

$$v' = \left\{ \begin{pmatrix} -k & 0 \\ 0 & k \end{pmatrix} + \begin{pmatrix} 0 & -e^t\, q(e^t) \\ e^t\, q(e^t) & 0 \end{pmatrix} \right\} v. \tag{B.4}$$

If $e^t\, q(e^t) \to 0$ as $t \to \infty$, then (B.4) has a fundamental system of solutions $v_\pm$ with the property

$$\lim_{t\to\infty} \frac{1}{t} \log |v_\pm(t)| = \pm k.$$

(We refer to the references in [AKS] for this theorem which goes back to Perron, Lettenmeyer and Hartman-Wintner.) Hence (B.2) has a solution which is in $L^2$ at infinity if $rq(r) \to 0$ as $r \to \infty$ and $2|k| > 1$. Note that (B.4) has a fundamental system of solutions

$$v_\pm(t) = e^{\pm\sqrt{k^2-c^2}\,t} v^0_\pm$$

if $q = c/r$. Hence (B.2) has an $L^2$-solution at infinity if $|k| > [(1/2) + c^2]^{1/2}$.

### B.2 CASE $\lambda = -m_0 < 0$

In this case (B.2) reads

$$u' = \left\{ \begin{pmatrix} 0 & 0 \\ 2m_0 & 0 \end{pmatrix} + \begin{pmatrix} -(k/r) & -q \\ q & (k/r) \end{pmatrix} \right\} u =: (J+R)u. \tag{B.5}$$

The Jordan matrix $J$ can be removed by observing that

$$\phi := \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix} \quad \text{with} \quad \sigma := 2m_0\, r$$

has the properties $\phi' = J = J\phi$. Hence $z := \phi^{-1}u$ satisfies

$$\phi' z + \phi z' = (J+R)\,\phi z \quad \text{or} \quad z' = \phi^{-1} R\, \phi z$$

(see [Ea], p.43 for this trick). Since the asymptotically leading term in $\phi^{-1}R\phi$ still has the double eigenvalue zero, a second transformation is required. With

$$D := \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}, \quad w := D^{-1} z$$

we obtain

$$\begin{aligned} w' &= [(D^{-1})' D + (\phi D)^{-1} R \phi D] w \qquad\qquad \text{(B.6)} \\ &= \left\{ \begin{pmatrix} -k & 0 \\ 2k & k-1 \end{pmatrix} + \begin{pmatrix} -\sigma r q & -\sigma r q \\ \sigma r q + rq/\sigma & -\sigma r q \end{pmatrix} \right\} \frac{1}{r}\, w. \end{aligned}$$

The constant matrix in (B.6) has the eigenvalues

$$\mu_{\pm} := -\frac{1}{2} \pm \sqrt{\frac{1}{4} + k(k-1)}.$$

So, if $k \neq 0$, 1, and if $r^2\, q \to 0$ $r \to \infty$, (B.6) has a fundamental system of solutions $w_{\pm}$ with

$$|w_{\pm}(r)| = r^{\,\mu_{\pm}+o(1)} \quad \text{as} \quad r \to \infty.$$

Since $u = \phi D w$ and $|\phi D| \leq \text{const}.r$, (B.5) has a solution which is of integrable square at infinity if $2\mu_{-} + 3 < 0$, i.e., $|k - (1/2)| > 1$.
For $q = c/r^2$ system (B.6) reads

$$w' = \left\{ A + \begin{pmatrix} 0 & 0 \\ \frac{c}{2m_0} & 0 \end{pmatrix} \frac{1}{r^2} \right\} \frac{1}{r}\, w, \tag{B.7}$$

where

$$A := \begin{pmatrix} -(k+\sigma_0) & -\sigma_0 \\ 2k + \sigma_0 & k - 1 - \sigma_0 \end{pmatrix}$$

and $\sigma_0 := 2mc$. Introducing new variables as in (B.3), (B.7) becomes

$$\tilde{w}'(t) = \left\{ A + e^{-2t} \begin{pmatrix} 0 & 0 \\ \frac{c}{2m_0} & 0 \end{pmatrix} \right\} \tilde{w}(t). \tag{B.8}$$

Since the eigenvalues of $A$ are

$$\mu_{\pm} := -\frac{1}{2}(1 + 2\sigma_0) \pm \left[ \frac{1}{4} + k(k - 1 - 2\sigma_0) - \sigma_0^2 \right]^{1/2},$$

it follows from the Levinson theorem ([Ea], Theorem 1.8.1) that (B.8) has a solution which is in $L^2$ at infinity if $|k|$ is sufficiently large. The same is therefore true of (B.5)

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# The Variation of the Monodromy Group in Families of Stratified Bundles in Positive Characteristic

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Abstract. In this article we study smooth families of stratified bundles in positive characteristic and the variation of their monodromy group. Our aim is, in particular, to strengthen the weak form of the positive equicharacteristic $p$-curvature conjecture stated and proved by Esnault and Langer in [9]. The main result is that if the ground field is uncountable then the strong form holds. In the case where the ground field is countable we provide positive and negative answers to possible generalizations.

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## 1. Introduction

Let $(E, \nabla)$ be a vector bundle endowed with a flat connection on a smooth complex variety $X$. Then, there exists a smooth scheme $S$ over (some open subscheme of) $\operatorname{Spec} \mathbb{Z}$ such that $(E, \nabla) = (E_S, \nabla_S) \otimes_S \mathbb{C}$ and $X = X_S \otimes_S \mathbb{C}$ with $X_S$ smooth over $S$ and $(E_S, \nabla_S)$ flat connection on $X_S$ relative to $S$. The $p$-curvature conjecture of Grothendieck and Katz predicts (see [2, Conj. 3.3.3]) that if for all closed points $s$ of a dense open subscheme $\tilde{S} \subset S$ we have that $E_S \times_S s$ is spanned by its horizontal sections, then $(E, \nabla)$ must be trivialized by an étale finite cover of $X$.

An analogue problem can be studied in equicharacteristic zero, and in fact it reduces the $p$-curvature conjecture to the number field case. Y. André in [2, Prop. 7.1.1] and E. Hrushovski in [14, 116] stated and proved the following equicharacteristic zero version of the $p$-curvature conjecture: let $X \to S$ be a smooth morphism of varieties over a field $K$ of characteristic zero; let $(E, \nabla)$ be a flat connection on $X$ relative to $S$ such that, for every closed point $s$ in a dense open $\tilde{S} \subset S$, the flat connection $(E, \nabla) \times_S s$ is trivialized by a finite étale cover. Then, there exists a finite étale cover of the generic geometric fiber over $\bar{\eta}$ that trivializes $(E, \nabla) \times_S \bar{\eta}$, where $\bar{\eta}$ is a geometric generic point of $S$.

The theorem of André and Hrushovsky translates naturally in positive characteristic, providing a positive equicharacteristic analogue to the $p$-curvature conjecture. Here, the role of relative flat connections is played by relative stratified bundles. A *stratified bundle on $X$ relative to $S$* is a vector bundle of finite rank with an action of the ring of differential operators $\mathscr{D}_{X/S}$ on $X$ relative to $S$.
In [9, Cor. 4.3, Rmk. 5.4.1] H. Esnault and A. Langer proved, using an example of Y. Laszlo (see [18]), that there exists $X \to S$ a projective smooth morphism of varieties over $\bar{\mathbb{F}}_2$ and a stratified bundle over $X$ relative to $S$ which is trivialized by a finite étale cover on every closed fiber but not on the geometric generic one. In particular, this provides a counterexample to the positive equicharacteristic version of André and Hrushovsky's theorem.
Nevertheless, they were able to prove what they call a weak form of the theorem (see [9, Thm. 7.2]): let $X \to S$ be a *projective* smooth morphism and let $\mathbb{E} = (E, \nabla)$ be a stratified bundle on $X$ relative to $S$ such that, for all closed points of a dense open $\tilde{S} \subset S$, the stratified bundle $\mathbb{E} \times_S s$ is trivialized by a finite étale cover *of order prime to $p$*. Then, if $K \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover *of order prime to $p$* of the generic geometric fiber that trivializes $\mathbb{E} \times_S \bar{\eta}$. In case $K = \bar{\mathbb{F}}_p$, there exists a finite étale cover *of order prime to $p$* such that the pullback of $\mathbb{E} \times_S \bar{\eta}$ is a direct sum of stratified line bundles.

In this article we answer two natural questions aiming at generalizing this last theorem: the first one is whether we can relax the assumption of coprimality to $p$ of the order of the trivializing covers of the $\mathbb{E} \times_S s$, while keeping the assumption that $X$ is proper over $S$. The second one is if we can drop this last assumption as well. Our main result, in particular, is that if $K$ is uncountable then the positive equicharacteristic version of André and Hrushovsky's theorem holds.
Let assume from now on that $K$ has positive characteristic $p$. Bearing in mind the counterexample of Esnault and Langer ([9, Cor. 4.3]) we cannot hope in general to completely eliminate the assumption of coprimality to $p$ of the order of the trivializing covers of the $\mathbb{E} \times_S s$. Still, we can answer positively the first question proving that it suffices to impose to the power of $p$ dividing the order of such trivializing covers to be bounded:

Theorem 1 (See Theorem 4.3). Let $K$ be an algebraically closed field, $X \to S$ a smooth proper morphism of $K$-varieties and $\mathbb{E} = (E, \nabla)$ a stratified bundle on $X$ relative to $S$. Assume that for every closed point $s$ in a dense open $\tilde{S} \subset S$ the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover *whose order is not divisible by $p^N$ for some fixed $N \geq 0$*. Then, if $K \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$. In case $K = \bar{\mathbb{F}}_p$, there exists a finite étale cover such that the pullback of $\mathbb{E}_{\bar{\eta}}$ is the direct sum of stratified line bundles.

The second question has a more involved answer. The assumption on $X$ being proper over $S$ is more delicate to eliminate; the order of the trivializing covers does not play any role while the cardinality of $K$ becomes the main obstruction:

Counterexample (See Proposition 5.1). If $K$ is a countable field, then there exists a stratified bundle on $\mathbb{A}^2_K$ relative to $\mathbb{A}^1_K$ which is trivial on every closed fiber but is not trivialized by any finite étale cover on the generic geometric fiber.

On the other hand the main result of this article is that in case $K$ is uncountable the strong version of the theorem holds, namely:

Theorem 2 (See Theorem 6.1). Let $K$ be an *uncountable* algebraically closed field, $X \to S$ a smooth morphism of $K$-varieties and $\mathbb{E} = (E, \nabla)$ a stratified bundle on $X$ relative to $S$ such that, for every closed point $s$ in a dense open $\tilde{S} \subset S$, the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover. Then, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$.

In the case where $K$ is countable and $X$ is not proper over $S$ there is still room for improvement, using the theory of regular singular stratified bundles (introduced in [11]). Roughly speaking, a stratified bundle is regular singular if it has only mild (that is logarithmic) singularities along the divisor at infinity. In characteristic zero there is a parallel notion of regular singular flat connections, and one of the first steps in the proof of André's theorem is to show that if a relative flat connection $(E, \nabla)$ on $X$ over $S$ is regular singular on the fiber over all closed points of a dense subset of $S$ then it is regular singular on the generic fiber (see [2, Lemma 8.1.1]). In positive characteristic this is no longer true, as our counterexample shows. The converse still holds (see the proof of Lemma 7.4): if $X$ admits a good compactification over $S$ and $\mathbb{E} = (E, \nabla)$ is a stratified bundle on $X$ relative to $S$ such that $\mathbb{E}_{\bar{\eta}}$ is regular singular then for every closed point $s$ of some dense open $\tilde{S} \subset S$ the stratified bundle $\mathbb{E}_s$ is regular singular as well. Moreover, assuming $\mathbb{E}_{\bar{\eta}}$ to be regular singular we obtain the same results than in the proper case:

Theorem 3 (See Theorem 7.7). Let $K$ be an algebraically closed field of any cardinality and $X \to S$ a smooth morphism of $K$-varieties. Let $\mathbb{E} = (E, \nabla)$ be a stratified bundle on $X$ relative to $S$ such that, for every closed point $s$ in a dense open $\tilde{S} \subset S$, the stratified bundle $\mathbb{E}_s = \mathbb{E} \times_S s$ is trivialized by a finite étale cover whose order is not divisible by $p^N$ for some fixed $N \geq 0$. Assume moreover that $\mathbb{E}_{\bar{\eta}} = \mathbb{E} \times_S \bar{\eta}$ is *regular singular*. Then, if $K \neq \bar{\mathbb{F}}_p$, there exists a finite étale cover of the generic geometric fiber that trivializes $\mathbb{E}_{\bar{\eta}}$. In case $K = \bar{\mathbb{F}}_p$, there exists a finite étale cover such that the pullback of $\mathbb{E}_{\bar{\eta}}$ is the direct sum of stratified line bundles.

The proofs of these generalizations are of two different kinds. The ones of Theorem 1 and Theorem 3 rely on a reduction to Esnault and Langer's result ([9, Thm. 7.2]). Theorem 3 is reduced to Theorem 1 using the theory of exponents

for regular singular stratified bundles ([11],[16]) and an adaptation of Kawamata coverings ([15, Thm. 17]) to positive characteristic (see Theorem 7.6). Theorem 1 is then proved by reduction to [9, Thm. 7.2] constructing a suitable finite étale cover of $X$ that kills the $p$-powers in the orders of the trivializing covers on the closed fibers.
The proof of Theorem 2 is of another flavour: it relies on the invariance of the Tannakian monodromy group under algebraically closed extension of fields for finite stratified bundles (Lemma 3.4), and on the easy but fundamental fact that a (relative) stratified bundle is defined by countably many data.

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*Notation.* If $S$ is an integral scheme $k(S)$ will denote its field of fractions, $\eta$ its generic point and $\bar{\eta}$ a geometric generic point given by the choice of an algebraically closure $\overline{k(S)}$ of $k(S)$. If $K$ is a field a *variety* over $K$ is a separated integral scheme of finite type over $K$.

## 2. The category of stratified bundles

Throughout the whole article $K$ will denote an algebraically closed field of positive characteristic $p$ and $u : X \to S$ a smooth morphism of varieties over $K$, of relative dimension $d$. Let $\mathscr{D}_{X/S}$ be the quasi-coherent $\mathscr{O}_X$-module of relative differential operators as defined in [12, §16]; recall that if $\mathcal{U}$ is an open subscheme of $X$ admitting global coordinates $x_1, \dots, x_d$ relative to $S$, then for every $k \in \mathbb{N}$ there are $\mathscr{O}_S$-linear maps $\partial_{x_i}^{(k)} : \mathscr{O}_{\mathcal{U}} \to \mathscr{O}_{\mathcal{U}}$ given by

$$\partial_{x_i}^{(k)}(x_j^h) = \delta_{ij}\binom{h}{k}(x_j^{h-k})$$

where $\delta_{ij}$ is the Kronecker delta. These maps are differential operators of order $k$ and generate locally the ring of differential operators:

$$\mathscr{D}_{X/S|\mathcal{U}} = \mathscr{O}_{\mathcal{U}}\big[\partial_{x_i}^{(k)} \mid i \in \{1, \dots, d\}, k \in \mathbb{N}_{>0}\big].$$

For higher differential operators we have an extension of the Leibniz rule, namely if $f, g \in \mathscr{O}_{\mathcal{U}}$ then

$$\partial_{x_i}^{(k)}(fg) = \sum_{\substack{a+b=k \\ a,b\geq 0}} \partial_{x_i}^{(a)}(f)\partial_{x_i}^{(b)}(g). \tag{1}$$

Definition 2.1. A *stratified bundle* $\mathbb{E}$ *(relative to $S$)* is a $\mathscr{O}_X$-locally free module $E$ of finite rank $r$ endowed with a $\mathscr{D}_{X/S}$-action extending the $\mathscr{O}_X$-module structure via the inclusion $\mathscr{O}_X \subset \mathscr{D}_{X/S}$. A *morphism* of stratified bundles is a morphism of $\mathscr{D}_{X/S}$-modules. We denote by $\mathrm{Strat}(X/S)$ the category of stratified bundles on $X$ relative to $S$; if $S = \operatorname{Spec} K$ we use the notation $\mathrm{Strat}(X/K)$

for $\mathrm{Strat}(X/\operatorname{Spec} K)$. The structure sheaf $\mathscr{O}_X$ together with the natural $\mathscr{D}_{X/S}$-action is denoted by $\mathbb{I}_{X/S}$; if $S = \operatorname{Spec} K$ we use the notation $\mathbb{I}_{X/K}$ for $\mathbb{I}_{X/\operatorname{Spec} K}$. A stratified bundle is *trivial* if it is isomorphic to $\mathbb{I}_{X/S}^{\oplus r}$ for some $r \in \mathbb{N}$.

If $h : Y \to X$ is a morphism of smooth $S$-varieties then the pullback along $h$ induces a functor $h^* : \mathrm{Strat}(X/S) \to \mathrm{Strat}(Y/S)$ and if $h$ is finite and étale then the pushforward along $h$ induces a functor $h_* : \mathrm{Strat}(Y/S) \to \mathrm{Strat}(X/S)$. For $\mathbb{E}, \mathbb{F} \in \mathrm{Strat}(X/S)$ we can construct the dual $\mathbb{E}^\vee$, the tensor product $\mathbb{E} \otimes \mathbb{F}$ and the direct sum $\mathbb{E} \oplus \mathbb{F}$, all of which are objects of $\mathrm{Strat}(X/S)$.

## 3. The monodromy group

If $X$ is a smooth connected $K$-variety, $\mathrm{Strat}(X/K)$ is an abelian tensor rigid $K$-linear category and the choice of a rational point $x \in X(K)$ defines a fiber functor to the category of finite dimensional $K$-vector spaces by:

$$\begin{aligned} \omega_x : \mathrm{Strat}(X/S) &\to \mathrm{Vecf}_K \\ \mathbb{E} &\mapsto E_x \end{aligned}$$

where $E$ is the vector bundle underlying $\mathbb{E}$ ([21, §VI.1]). Hence $(\mathrm{Strat}(X/S), \omega_x)$ is a neutral Tannakian category and by Tannakian duality ([7, Thm. 2.11]) there exists an affine group scheme $\pi_1^{\mathrm{Strat}}(X, x) \doteq \pi(\mathrm{Strat}(X/S), \omega_x) = \underline{\mathrm{Aut}}_K^{\otimes}(\omega_x)$ over $K$ such that $\mathrm{Strat}(X/K)$ is equivalent via $\omega_x$ to the category of finite dimensional representations of $\pi_1^{\mathrm{Strat}}(X, x)$ over $K$. For every $\mathbb{E} \in \mathrm{Strat}(X/K)$ we denote by $\langle \mathbb{E} \rangle_\otimes \subset \mathrm{Strat}(X/K)$ the full Tannakian subcategory spanned by $\mathbb{E}$ with fiber functor $\omega_x$ defined as above. The affine group scheme $\pi(\mathbb{E}, x) \doteq \pi(\langle \mathbb{E} \rangle_\otimes, \omega_x)$ is called the *monodromy group* of $\mathbb{E}$. If $\mathcal{U} \subset X$ is an open dense subscheme of $X$ then by [16, Lemma 2.5(a)] the restriction functor $\rho_{\mathcal{U}} : \langle \mathbb{E} \rangle_\otimes \to \langle \mathbb{E}_{|\mathcal{U}} \rangle_\otimes$ is an equivalence; hence, in particular, the monodromy group of $\mathbb{E}$ is invariant under restriction to dense open subschemes. Moreover, as $K$ is algebraically closed, the monodromy group does not depend on the choice of $x$, up to non-unique isomorphism (this can be deduced from [7, Thm. 3.2]). We will hence sometimes use the notation $\pi(\mathbb{E})$ instead of $\pi(\mathbb{E}, x)$.

Definition 3.1. We say that $\mathbb{E} \in \mathrm{Strat}(X/K)$ is *finite* if its monodromy group is finite. By what we have just remarked, this property is independent of the choice of $x$. We say that $\mathbb{E}$ is *isotrivial* if it is étale trivializable; that is, there exists $h : Y \to X$ finite étale cover such that $h^*\mathbb{E}$ is trivial in $\mathrm{Strat}(Y/K)$.

These two properties are equivalent:

Lemma 3.2. *For a stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/K)$ the following are equivalent:*

i) $\mathbb{E}$ *is isotrivial;*
ii) $\mathbb{E}$ *is finite.*

*Moreover, if $\mathbb{E}$ is finite, then there exists an étale $\pi(\mathbb{E}, x)$-torsor $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$, called the* Picard–Vessiot torsor *of $\mathbb{E}$ such that, for any $\mathbb{E}' \in \mathrm{Strat}(X/K)$, the pullback $h_{\mathbb{E},x}^*\mathbb{E}'$ is trivial if and only if $\mathbb{E}' \in \langle \mathbb{E} \rangle_\otimes$.*

*Finally, for a finite étale cover $h : Y \to X$ such that $h^*\mathbb{E}$ is trivial, the following conditions are equivalent:*

i) *$h : Y \to X$ is the Picard–Vessiot torsor for $\mathbb{E}$;*
ii) *every finite étale cover trivializing $\mathbb{E}$ factors (non-uniquely) through $h : Y \to X$;*
iii) *$h : Y \to X$ is Galois and $\langle \mathbb{E} \rangle_\otimes = \langle h_* \mathbb{I}_{Y/K} \rangle_\otimes$;*
iv) *$h : Y \to X$ is Galois of Galois group $\pi(\mathbb{E}, x)(K)$.*

*Proof.* The first part of the lemma is [9, Lemma 1.1]. As for the second part, first notice that point (b) and (f) of [16, Prop. 2.15], together with [16, Cor. 2.16] imply that if $h : Y \to X$ is a finite étale cover trivializing $\mathbb{E}$ then $\langle \mathbb{E} \rangle_\otimes \subset \langle h_* \mathbb{I}_{Y/K} \rangle_\otimes$ and that $\langle \mathbb{E} \rangle_\otimes = \langle h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/K} \rangle_\otimes$. Moreover, if $h : Y \to X$ is Galois of Galois group $G$, then $\pi(h_* \mathbb{I}_{Y/K}, x)$ is the finite constant group $G$ and if $\tilde{h} : \tilde{Y} \to X$ is an étale cover factoring through $h$ then $\langle h_* \mathbb{I}_{Y/K} \rangle_\otimes \subset \langle \tilde{h}_* \mathbb{I}_{\tilde{Y}/K} \rangle_\otimes$. We are now ready to prove the rest of the lemma.

(i)⇒(ii) Because $\langle \mathbb{E} \rangle_\otimes = \langle h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/K} \rangle_\otimes$, a cover $\tilde{h} : \tilde{Y} \to X$ trivializes $\mathbb{E}$ if and only if it trivializes $h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/K}$. Let $Z = \tilde{Y} \times_X Y_{\mathbb{E},x}$, and let $p_1$ and $p_2$ be the projections on the first and second factor. Then by flat base change (notice that the flat base change morphism is compatible with the $\mathscr{D}_{\tilde{Y}/K}$-action) there is an isomorphism of $\mathscr{D}_{\tilde{Y}/K}$-modules $\tilde{h}^* h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/K} \simeq p_{1*} \mathbb{I}_{Z/K}$; hence, the latter is also a trivial stratified bundle. This, together with [16, Cor. 2.17], implies that $p_1 : Z \to \tilde{Y}$ is a trivial covering. In particular, it admits a section $s$; hence, $\tilde{h} = s \circ p_2 \circ h_{\mathbb{E},x}$ and $\tilde{h}$ factors through $h$.

(ii)⇒(iii) Because $h$ trivializes $\mathbb{E}$, we have the inclusion $\langle \mathbb{E} \rangle_\otimes \subset \langle h_* \mathbb{I}_{Y/K} \rangle_\otimes$. On the other side, by assumption, $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ factors through $h : Y \to X$; hence, $\langle h_* \mathbb{I}_{Y/K} \rangle_\otimes \subset \langle h_{\mathbb{E},x*} \mathbb{I}_{Y_{\mathbb{E},x}/K} \rangle_\otimes = \langle \mathbb{E} \rangle_\otimes$.

(iii)⇒(iv) As $\langle \mathbb{E} \rangle_\otimes = \langle h_* \mathbb{I}_{Y/K} \rangle_\otimes$, then we have the equality $\pi(\mathbb{E}, x) = \pi(h_* \mathbb{I}_{Y/K}, x)$ and as $h : Y \to X$ is Galois, then its Galois group is $\pi(h_* \mathbb{I}_{Y/K})(K) = \pi(\mathbb{E}, x)(K)$.

(iv)⇒ (i) By what we already proved there must be a factorization $f : Y \to Y_{\mathbb{E},x}$ such that $h = h_{\mathbb{E},x} \circ f$. Hence, if $G$ is the Galois group of $h : Y \to X$ then $h_\mathbb{E} : Y_\mathbb{E} \to X$ corresponds to a normal subgroup $H$ of $G$. But by assumption $G = \pi(\mathbb{E}, x)(K) = H$; hence, $h = h_\mathbb{E}$. □

COROLLARY 3.3. *If $\mathbb{E} \in \mathrm{Strat}(X/K)$ is finite then the set of finite étale covers of $X$ trivializing $\mathbb{E}$ has a minimal element which is Galois of Galois group $\pi(\mathbb{E}, x)(K)$.*

By [8, Cor. 12] for every $\mathbb{E} \in \mathrm{Strat}(X/K)$ the group scheme $\pi(\mathbb{E}, x)$ is smooth (which is equivalent to being reduced). In particular, as $K$ is algebraically closed, if $\mathbb{E}$ is finite we are allowed to identify the abstract group $\pi(\mathbb{E}, x)(K)$ and the algebraic group $\pi(\mathbb{E}, x)$. Given a finite stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/K)$ it is straightforward to see that for every $L \supset K$ algebraically closed field extension $\mathbb{E}_L = \mathbb{E} \otimes_K L \in \mathrm{Strat}(X_L/L)$ is finite as well. In fact, the following stronger statement holds:

Lemma 3.4. *Let $\mathbb{E} \in \mathrm{Strat}(X/K)$ and let $L \supset K$ be an algebraically closed field extension such that $\mathbb{E}_L$ is finite. Then for every $L' \supset K$ algebraically closed such that $\mathrm{trdeg}_K L' \geq \mathrm{trdeg}_K L$ (or, if both are infinite, such that there exists an immersion $L \hookrightarrow L'$ which is the identity on $K$) we have that $\mathbb{E}_{L'}$ is finite. Moreover for any $x \in X(K)$*

$$\pi(\mathbb{E}_L, x)(L) \simeq \pi(\mathbb{E}_{L'}, x)(L'),$$

*where we consider $x \in X_L(L)$ via $K \subset L$ and similarly for $L'$.*

*Proof.* Let $L$ and $L'$ as in the hypothesis, then we can construct an immersion $L \hookrightarrow L'$ which is the identity on $K$, just by sending any transcendence basis of $L$ to a algebraically independent set in $L'$ over $K$ and using the fact that $L'$ is algebraically closed to see that this extends to an immersion $L \hookrightarrow L'$. Hence, we have reduced the problem to proving that if $\mathbb{E}$ is finite and $L \supset K$ is an algebraically closed field extension then $\mathbb{E}_L$ is finite and has the same monodromy group of $\mathbb{E}$ as abstract groups. In order to do so we need first to establish a result on Galois covers:

*Claim.* Let $h : Y \to X$ be a Galois cover of Galois group $G$ and let $h_L : Y_L \to X_L$ the extension of scalars of $h : Y \to X$ to $L$, then $h_L$ is a Galois cover of Galois group $G$.

*Proof.* Certainly $h_L : Y_L \to X_L$ is a finite étale morphism as these properties are stable under base change. We are left to check that (i) $Y_L$ is connected, (ii) $\mathrm{Aut}(Y_L/X_L)$ acts transitively on the fiber over some geometric point of $X_L$ and finally (iii) $\mathrm{Aut}(Y_L/X_L) \simeq \mathrm{Aut}(Y/X)$.

i) As $K$ is algebraically closed (hence, in particular, separably closed) $Y$ is connected if and only if $Y_L$ is connected for any field extension $L \supset K$. In particular, $Y_L$ is connected.

ii) Let $x \in X_L(L)$ be any closed (in particular, geometric) point of $X_L$, then the composition $\bar{x} : \mathrm{Spec}(L) \to X_L \to X$ is a geometric point for $X$. As $h : Y \to X$ is Galois, $\mathrm{Aut}(Y/X)$ acts transitively on $Y_{\bar{x}} = Y \times_X \bar{x} = Y_L \times_{X_L} x = Y_{L,x}$. Now, the action of $\mathrm{Aut}(Y/X)$ on $Y_{L,x}$ factors through $\mathrm{Aut}(Y_L/X_L)$ via the inclusion $\mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y_L/X_L)$ defined by $\phi \mapsto \phi_L$. Hence, the action of $\mathrm{Aut}(Y_L/X_L)$ on $Y_{L,x}$ is transitive as well; therefore, $h_L : Y_L \to X_L$ is Galois.

iii) As $Y_{\bar{x}} = Y_{L,x}$ and both $h$ and $h_L$ are Galois, it follows that the order of their Galois group is the same, as it is the cardinality of the respective geometric fibers over $\bar{x}$ and over $x$. Moreover we have a natural inclusion $\mathrm{Aut}(Y/X) \subset \mathrm{Aut}(Y_L/X_L)$ so as they have the same cardinality they must be equal; hence, $\mathrm{Aut}(Y_L/X_L) = G$. □

Until the end of the proof let us denote by $h_{\mathbb{E},x} : Y \to X$ the Picard–Vessiot torsor of $\mathbb{E}$ (see Lemma 3.2), then $h_{\mathbb{E},x} \otimes_K L : Y_L \to X_L$ is a Galois cover trivializing

$\mathbb{E}_L$ which is then finite, by Lemma 3.2. Recall that $\langle (h_{\mathbb{E},x})_* \mathbb{I}_{Y/K} \rangle_\otimes = \langle \mathbb{E} \rangle_\otimes$. But then in particular, $\langle (h_{\mathbb{E},x})_* \mathbb{I}_{Y/K} \otimes_K L \rangle_\otimes = \langle \mathbb{E}_L \rangle_\otimes$ and as $(h_{\mathbb{E},x})_* \mathbb{I}_{Y/K} \otimes_K L = (h_{\mathbb{E},x} \otimes_K L)_* \mathbb{I}_{Y_L/L}$, it follows that $\langle (h_{\mathbb{E},x} \otimes_K L)_* Y_L \rangle_\otimes = \langle \mathbb{E}_L \rangle_\otimes$. Hence, by Lemma 3.2, we have that $h_{\mathbb{E},x} \otimes_K L : Y_L \to X_L$ is the minimal trivializing cover for $\mathbb{E}_L$. Now, the Galois group of $h_{\mathbb{E},x}$ is the same as that of $h_{\mathbb{E},x} \otimes_K L$ by the previous claim; hence, again by Lemma 3.2, we have that $\pi(\mathbb{E}_L, x)(L) = \pi(\mathbb{E}, x)(K)$. □

Finite stratified bundle have an additional propriety that will turn out to be very useful to prove that some stratified bundle cannot be isotrivial:

Lemma 3.5. *Let $\mathbb{E} \in \mathrm{Strat}(X/K)$ be a finite stratified bundle. Then there exists a subfield $K' \subset K$ of finite type over $\mathbb{F}_p$ over which $X$ and $\mathbb{E}$ are defined; that is, there exists $X'$ smooth variety over $K'$ and $\mathbb{E}' \in \mathrm{Strat}(X'/K')$ such that $X = X' \times_{\mathrm{Spec}\, K'} \mathrm{Spec}\, K$ and $\mathbb{E} = \mathbb{E}' \otimes_{K'} K$.*

*Proof.* Let $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ be the Picard–Vessiot torsor of $\mathbb{E}$ (see Lemma 3.2), and let $\mathbb{H} = (h_{\mathbb{E},x})_* \mathbb{I}_{Y_{\mathbb{E},x}/K}$, then (see Lemma 3.2) $\mathbb{E} \in \langle \mathbb{H} \rangle_\otimes$. Certainly there exists $K''$ of finite type over $\mathbb{F}_p$ on which $h_{\mathbb{E},x} : Y_{\mathbb{E},x} \to X$ is defined; hence, $\mathbb{H}$ is also defined over $K''$. Notice that $\mathbb{E}$ is a subquotient of $\mathbb{P}$ where $\mathbb{P} \in \mathbb{Z}[\mathbb{H}, \mathbb{H}^\vee]$ (see e.g. [16, def. 2.4]); that is, $\mathbb{E} \simeq \tilde{\mathbb{P}}/\bar{\mathbb{P}}$ with $\bar{\mathbb{P}} \subset \tilde{\mathbb{P}} \subset \mathbb{P}$. It is clear that $\tilde{\mathbb{P}}$ and $\bar{\mathbb{P}}$ are defined over some extension $K'$ of finite type of $K''$ (thus over $\mathbb{F}_p$). Therefore, so does $\mathbb{E} \simeq \tilde{\mathbb{P}}/\bar{\mathbb{P}}$. □

In particular, we have the following:

Corollary 3.6. *Let $\mathbb{E} \in \mathrm{Strat}(X/K)$ with $K$ algebraically closed, such that for some algebraically closed field extension $L \supset K$ the stratified bundle $\mathbb{E}_L$ is finite. Assume that $K$ has infinite transcendence degree over $\mathbb{F}_p$, then $\mathbb{E}$ is finite as well.*

*Proof.* As $\mathbb{E}_L$ is finite then by Lemma 3.5 there exists $K' \subset L$ algebraically closed of finite transcendence degree over $\mathbb{F}_p$ over which $\mathbb{E}_L$ and is defined. But then $\mathbb{E}$ is defined over $K'$ as well, hence we can assume that $K' \subset K$. Let $h_{\mathbb{E}_L} : Y \to X_L$ be the Picard–Vessiot torsor of $\mathbb{E}_L$, then it is defined over some algebraically closed field $K''$ of finite transcendence degree over $K'$. In particular $\mathbb{E}_{K''}$ is finite and as $\infty = \mathrm{trdeg}_{K'} K \geq \mathrm{trdeg}_{K'} K''$ by Lemma 3.2 $\mathbb{E} = \mathbb{E}_{K'} \otimes_{K'} K$ is finite. □

## 4. Families of finite stratified bundles

We can view a relative stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/S)$ as a family of stratified bundles parametrized by the points of $S$. In particular, for every $s \in S(K)$, let $\mathbb{E}_s \in \mathrm{Strat}(X_s/k(s))$ denote the restriction of $\mathbb{E}$ on $X_s$ and $\mathbb{E}_{\bar{\eta}} \in \mathrm{Strat}(X_{\bar{\eta}}/k(\bar{\eta}))$ its restriction on the geometric generic fiber given by a choice of an algebraic closure $\overline{k(S)}$ of $k(S)$. It is natural then to ask how the property of being isotrivial behaves in families: the main question we want to study is whether it is true that if $\mathbb{E}_s$ is finite for every $s \in S(K)$ then so is $\mathbb{E}_\eta$. André proved in [2, Prop. 7.1.1] that the analogous result in characteristic zero holds. In positive characteristic, following an idea of Laszlo, in [9, Cor. 4.3, Rmk. 5.4.1] the authors proved that there exists $X \to S$ a projective smooth morphism of varieties over $\bar{\mathbb{F}}_2$ and a stratified bundle on $X$ relative to $S$ which is finite on every closed fiber but not on the geometric generic one. Nevertheless, assuming $X$ to be projective over $S$ and imposing a coprimality to $p$ condition on the order of the monodromy group on the closed fibers, they proved the following:

Theorem 4.1. [9, Thm. 7.2] *Let $X \to S$ be a smooth* projective *morphism of $K$-varieties with geometrically connected fibers and let $\mathbb{E} \in \mathrm{Strat}(X/S)$. Assume that there exists a dense subset $\tilde{S} \subset S(K)$ such that, for every $s \in \tilde{S}$, the stratified bundle $\mathbb{E}_s$ has finite monodromy* of order prime to $p$. *Then*

i) *there exists $f_{\bar\eta} : Y_{\bar\eta} \to X_{\bar\eta}$ a finite étale cover* of order prime to $p$ *such that $f^*\mathbb{E}_{\bar\eta}$ decomposes as direct sum of stratified line bundles;*
ii) *if $K \neq \bar{\mathbb{F}}_p$ then $\mathbb{E}_{\bar\eta}$ is trivialized by a finite étale cover* of order prime to $p$.

Note that the cover of order prime to $p$ in the second point of the theorem factors through the Picard–Vessiot torsor of $\mathbb{E}_{\bar\eta}$ by its minimality (see Lemma 3.2). In particular, this implies that the order of the monodromy group of $\mathbb{E}_{\bar\eta}$ is prime to $p$.
This article is devoted to determine how the assumptions of $X$ being projective over $S$ and of the order of the monodromy groups to be prime to $p$ can or cannot be relaxed in order to get similar results. A first strengthening of the theorem comes rather directly from the ideas in the proof of Theorem 4.1. In order to prove it we need first to establish the following

Lemma 4.2. *Let $h : X \to S$ be a proper flat separable morphism of connected varieties with geometrically connected fibers over an algebraically closed field $K$ and suppose it has a section $\sigma : S \to X$. Let $\tilde{S} \subset S(K)$ be any subset of the closed points of $S$, let $N \in \mathbb{N}$ and let us fix for every $s \in \tilde{S}$ a finite étale cover $g_s : Z_s \to X_s$ of degree less than $N$. Then there exists an open subscheme $\mathcal{U} \subset S$ and a finite étale cover $f : W \to X \times_S \mathcal{U}$ dominating all the $g_s$ for $s \in \tilde{S} \cap \mathcal{U}$; that is, for every $s \in \tilde{S} \cap \mathcal{U}$ the finite étale cover $f_s : W_s \to X_s$ factors through $g_s : Z_s \to X_s$.*

*Proof.* The proof of this lemma is a generalization of the construction that one can find in the beginning of the proof of [9, Thm. 5.1].
First notice that if the order of the $g_s : Z_s \to X_s$ is bounded by $N$ then the order of their Galois closures is bounded by $N!$, hence we can assume all the $g_s : Z_s \to X_s$ to be Galois. Moreover if $S'$ is connected and $S' \to S$ is étale and generically finite then there is a non-trivial open $\mathcal{U}$ over which $S' \times_S \mathcal{U} \to \mathcal{U}$ is finite and étale. As $X \to S$ is smooth its image is open; hence, by shrinking $S$, we can assume that $X \to S$ is surjective.
Let $S' \to S$ be finite étale, then so is $X' = X \times_S S' \to X$. Let $s \in \tilde{S}$ and $s' \in S'(K)$ lying over $s$. Assume that we have found $f' : W \to X'$ such that $f'_s : W_s \to X'_s$ factors through $g_{s'} : Z_{s'} = Z_s \times_{k(s)} k(s') \to X'_{s'}$ then the

composition $f : W \to X' \to X$ is a finite étale cover of $X$ and $f_s : W_s \to X_s$ factors through $g_s$.
As $h : X \to S$ has geometrically connected fibers so does $h' : X' \to S'$ as if $s' \in S$ lies over $s \in S$ then $X'_{s'} = X_s \otimes_{k(s)} k(s')$. Therefore, $h' : X' \to S'$ is proper, flat, separable and has geometrically connected fibers. So if $S'$ is connected the morphism $h' : X' \to S'$ together with the section $\sigma' : S' \to X'$ induced by $\sigma : S \to X$ satisfy the assumptions of the theorem.

For any $s \in \tilde{S}$ let $G_s \subset \pi_1^{\text{ét}}(X_s, \sigma(s))$ be the open normal subgroup corresponding via Galois duality to the cover $g_s : Z_s \to X_s$. Let $\bar{\eta}$ be a generic geometric point of $S$ given by the choice of an algebraic closure $\overline{k(S)}$ of $k(S)$. The fibers of $X \to S$ are geometrically connected and the morphism is proper, flat and separable; hence, the specialization map $\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s))$ is surjective. Composing it with the quotient of $\pi_1^{\text{ét}}(X_s, \sigma(s))$ by $G_s$ we get

$$\rho_s : \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s))/G_s.$$

Notice that the index of $\ker(\rho_s)$ in $\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))$ is bounded by $N$. Let $\tau : S' \to S$ be any finite étale cover and let $s' \in S$ lying over $s$. As $K$ is algebraically closed, then $k(s') \simeq k(s)$ and hence $X'_{s'} \simeq X_s$. In particular, the natural morphism $\pi_1^{\text{ét}}(X'_{s'}, \sigma'(s')) \to \pi_1^{\text{ét}}(X_s, \sigma(s))$ is an isomorphism. Let $G_{s'} \subset \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))$ be the open subgroup corresponding to $g_{s'} : Z_{s'} \to X'_{s'}$, that is, the preimage of $G_s$ under this isomorphism and let us denote by

$$\rho_{s'} : \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \twoheadrightarrow \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s')) \twoheadrightarrow \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))/G_{s'},$$

then $\ker(\rho_{s'}) = \ker(\rho_s) \subset \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))$.
As $X_{\bar{\eta}}$ is a projective $\overline{k(S)}$-variety, then $\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))$ is topologically finitely generated and hence has finitely many subgroups of index less than $N$, which are all opens by Nikolov–Segal theorem ([19, Thm 1.1]), the intersection of which we denote by $G$: It is a normal open subgroup and it has finite index. Moreover

$$G \subset \bigcap_{s \in \tilde{S}} \ker(\rho_s) = \bigcap_{s' \in \tau^{-1}(\tilde{S})} \ker(\rho_{s'}).$$

At this point, we need an additional step before concluding similarly than in the proof of [9, Thm. 5.1]. By Galois duality $G$ corresponds to a finite étale cover $Z_{\bar{\eta}} \to X_{\bar{\eta}}$. Let $k(S)^{\text{sep}}$ be the separable closure of $k(S)$ in $\overline{k(S)}$. The base change functor from the category of finite étale covers over $X \otimes_S k(S)^{\text{sep}}$ to the one of finite étale covers over $X_{\bar{\eta}}$ is an equivalence. Hence, $Z_{\bar{\eta}}$ is defined over some separable extension of $k(S)$. In particular, there exists an étale generically finite cover $S' \to S$ such that $Z_{\bar{\eta}}$ descends to a finite étale cover of $X' = X \times_S S'$.

Let $\bar{\eta}'$ be the geometric generic point of $S'$ given by $k(S) \subset k(S') \subset \overline{k(S)}$. Then $X'_{\bar{\eta}'} = X_{\bar{\eta}}$ and $\sigma(\bar{\eta}) = \sigma'(\bar{\eta}')$. Hence, the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{\text{ét}}(X'_{\bar{\eta}}, \sigma'(\bar{\eta}')) \arrow[r] \arrow[d, equal] & \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}')) \arrow[d] \\
\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \arrow[r] & \pi_1^{\text{ét}}(X, \sigma(\bar{\eta})).
\end{tikzcd}$$

Let $K'$ be the kernel of $\pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \to \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}'))$. As $X \to S$ is projective we have the following exact sequence:

$$\begin{tikzcd}
& \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \arrow[d, "q"] \arrow[dr, "\alpha"] & & & \\
\{1\} \arrow[r] & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/K' \arrow[r, "i"] & \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}')) \arrow[r, shift right] & \pi_1^{\text{ét}}(S', \bar{\eta}') \arrow[l, shift right, "\sigma'_*"'] \arrow[r] & \{1\}.
\end{tikzcd}$$

By [1, V Cor 6.7] we have the inclusion $G \supset K'$. Moreover if we denote by $\Pi_{K'} = \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/K'$, then the section $\sigma'_*$ induces a splitting

$$\pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}')) \simeq \Pi_{K'} \rtimes \sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta}'))$$

as abstract groups. It is also a splitting of topological groups (see for example [4, §2.10 Prop. 28] and following discussion). In particular, the topology on $\pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}'))$ is the product topology. Note that $\overline{G} = q(G)$ is invariant by the action of $\sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta}'))$; hence, we can define

$$H = \overline{G} \rtimes \sigma'_*(\pi_1^{\text{ét}}(S', \bar{\eta}')).$$

By definition $\alpha^{-1}(H) = G$, and $H$ has finite index in $\pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}'))$. It is also open: $\pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}'))$ is endowed with the product topology, and $\overline{G}$ is open because $G = q^{-1}(\overline{G})$ is open as well. Hence, $H$ corresponds to a finite étale cover $W \to X'$ and as the composition $H \subset \pi_1^{\text{ét}}(X', \sigma'(\bar{\eta}')) \twoheadrightarrow \pi_1^{\text{ét}}(S', \bar{\eta}')$ is surjective, then $W$ has geometrically connected fibers over $S'$. In particular, the specialization map is again surjective. Let $z \in W$ be a closed point lying over $s'$ and let $\bar{\zeta}$ be a geometric generic point lying over $\sigma(\bar{\eta})$, then we have the following commutative diagram

$$\begin{tikzcd}
\pi_1^{\text{ét}}(W_{\bar{\eta}}, \bar{\zeta}) \arrow[r] \arrow[d] \arrow[rr, bend left, "0"] & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta})) \arrow[r] \arrow[d, two heads] & \pi_1^{\text{ét}}(X_{\bar{\eta}}, \sigma(\bar{\eta}))/G \arrow[d] \\
\pi_1^{\text{ét}}(W_s, z) \arrow[r] & \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s')) \arrow[r] & \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))/G_{s'}.
\end{tikzcd}$$

Using the surjectivity of the specialization map on $W$ it follows that the composition of the morphisms on the second line is zero as well; hence, if $\tilde{G}_{s'} \subset \pi_1^{\text{ét}}(X'_{s'}, \sigma'(s'))$ is the open normal subgroup corresponding to $f_{s'} : W_{s'} \to X'_{s'}$ then $\tilde{G}_s \subset G_{s'}$. Therefore, $f_{s'}$ factors through $g_{s'}$. $\square$

We can summarize the previous lemma by saying that with the assumptions of the theorem, up to shrinking $S$ every family of finite étale covers of the closed fibers with bounded order can be dominated by a finite étale cover of $X$ (notice that the existence of the section $\sigma : X \to S$ is not essential for the proof).

Theorem 4.3. *Let $X \to S$ be a smooth* proper *morphism of $K$-varieties with geometrically connected fibers and $\mathbb{E} \in \mathrm{Strat}(X/S)$ of rank $r$. Assume that there exists a dense subset $\tilde{S} \subset S(K)$ such that, for every $s \in \tilde{S}$, the stratified bundle $\mathbb{E}_s$ has finite monodromy and that the highest power of $p$ dividing $|\pi(\mathbb{E}_s)|$ is bounded over $\tilde{S}$. Then*

i) *there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover such that $f^*\mathbb{E}_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;*
ii) *if $K \neq \bar{\mathbb{F}}_p$ then $\mathbb{E}_{\bar{\eta}}$ is finite.*

*Proof.* We will reduce this theorem to Theorem 4.1. By the invariance of the monodromy group it suffices to prove the theorem for $f^*\mathbb{E}$ where $f : Y \to X$ is a morphism of smooth $S$-varieties which is generically finite étale. By Chow's lemma and using de Jong alterations ([6]) there exists $f : Y \to X$ projective and generically finite étale, hence we can assume $X$ to be projective. Up to taking an étale open of $S$ we can assume that there exists a section $\sigma : S \to X$. For any $s \in \tilde{S}$ let $\Gamma_s = \pi(\mathbb{E}, \sigma(s))$ and $h_s : Y_s \to X_s$ the Picard–Vessiot torsor of $\mathbb{E}_s$ (see Lemma 3.2). Let $G_s \subset \pi_1^{\text{ét}}(X_s, \sigma(s))$ be the normal open subgroup corresponding via Galois duality to the cover $h_s$. By Tannakian duality $\mathbb{E}_s$ corresponds to the image of an $r$-dimensional representation of $\pi_1^{\mathrm{Strat}}(X_s, \sigma(s))$ ([7, Prop. 2.21]) and as $\mathbb{E}_s$ is finite by [8, Prop. 13] this representation factors through the étale fundamental group, considered as a constant group scheme

$$\pi_1^{\mathrm{Strat}}(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s))\big/G_s = \Gamma_s \subset GL_r(K)$$

where $r$ is the rank of $\mathbb{E}$. By Brauer–Feit generalization of Jordan's theorem [5, Theorem], as the orders of the Sylow-p-subgroups of every $G_s$ are bounded by $p^N$, there exists an integer $M = f(r, N)$ and, for every $s \in \tilde{S}$, a normal abelian subgroup $A_s$ such that $|\Gamma_s : A_s| < M$. This gives for every $s \in S(K)$ a Galois cover $g_s : Z_s \to X_s$ of order bounded by $M$ and a factorization

$$Y_s \to Z_s \to X_s$$

where $Y_s \to Z_s$ is Galois of Galois group $A_s$. Therefore, by Lemma 4.2, up to shrinking $S$ there exists a cover $g' : Z' \to X$ such that $g'_s : Z'_s \to X_s$ factors through $g_s : Z_s \to X_s$. In particular, if $\mathbb{E}'$ is the pullback of $\mathbb{E}$ via $g'$ then $\pi(\mathbb{E}'_s)$ is abelian for every $s$. Up to taking an étale open of $S$ the section $\sigma : S \to X$ extends to a section $\sigma' : S \to Z'$. Let $\Gamma'_s = \pi(\mathbb{E}', \sigma'(s))$, as we just noticed for every $s \in \tilde{S}$ we have that $\Gamma'_s$ is abelian; hence, we can write it as the direct product of its $p$ part with its prime to $p$ part:

$$\Gamma'_s = \Gamma_s^p \times \Gamma_s^{p'}$$

and $\Gamma_s^{p'}$ corresponds to a Galois cover over $Z'_s$ whose index is by assumption bounded by $p^N$ for some $N \in \mathbb{N}$. Applying Lemma 4.2 and up to shrinking $S$

we get a Galois cover $g'': Z'' \to Z'$ dominating all such covers. Let $\mathbb{E}''$ be the pullback of $\mathbb{E}'$ along $g''$, then $\pi(\mathbb{E}''_s)$ is (abelian) of order prime to $p$ for every $s \in \tilde{S}$. Therefore, we have reduced the problem to Theorem 4.1. $\square$

## 5. A counterexample over countable fields

Our next aim is to drop the assumption of $X$ being projective over $S$. However, before getting to the positive results, let us present a counterexample to understand what we can reasonably expect to hold without this assumption. Assume for the rest of this section $K$ to be an algebraically closed countable field. Let $X = \mathbb{A}^2_K$, $S = \mathbb{A}^1_K$ and let $X \to S$ be given by $K[y] \to K[x,y]$. The main result of this section is the following:

Proposition 5.1. *There exists* $\mathbb{E} \in \mathrm{Strat}(X/S)$ *such that* $\mathbb{E}_s$ *is trivial for every point* $s \in S(K)$ *but* $\mathbb{E}_{\bar{\eta}}$ *is not isotrivial.*

The rest of the section will be spent constructing such a stratified bundle and proving it satisfies the proposition.
As $x$ is a global coordinate of $X$ relative to $S$, it is clear that

$$\mathscr{D}_{X/S} = \mathscr{O}_X[\partial_x^{(k)} \mid k \in \mathbb{N}_{>0}].$$

Moreover any vector bundle is free over $X$. If $E$ is a vector bundle on $X$, then a $\mathscr{D}_{X/S}$-module structure on $E$ is a $\mathscr{O}_S$-linear morphism

$$\phi : \mathscr{D}_{X/S} \to \mathrm{End}_{\mathscr{O}_S}(E)$$

extending the $\mathscr{O}_X$-module structure on $E$; hence, the image of $\mathscr{O}_X \subset \mathscr{D}_{X/S}$ under $\phi$ is always fixed. Therefore, to determine the action of the whole $\mathscr{D}_{X/S}$ it is enough to consider the image of the generators $\partial_x^{(k)}$ under $\phi$.
Let $e_1, \dots, e_r$ be a basis for $E$, and let $A_k = (a_{ij}^k)$ be given by $\partial_x^{(k)}(e_i) = \sum a_{ij}^k e_j$. Then the $A_k$, for $k \in \mathbb{N}_{>0}$, determine the $\mathscr{D}_{X/S}$-action: If $s = \sum_{i=1}^r f_i \cdot e_i$ is a section of $E$, with $f_i \in \mathscr{O}_X$, then using (1) we have

$$\partial_{x_i}^{(k)}(s) = \sum_{i=1}^{r} \sum_{\substack{a+b=k \\ a,b \geq 0}} \partial_x^{(a)}(f_i) A_b \cdot e_i. \tag{2}$$

Note that if $e'_1, \dots, e'_r$ is an other basis of $E$ and $U = (u_{ij}) \in \mathrm{H}^0(X, \mathrm{GL}_r)$ is given by $e'_i = \sum u_{ij} e_j$ then by (1) it follows that in this new basis the matrices $A'_k = (a'^k_{ij})$ describing the action of $\partial_x^{(k)}$ are given by

$$A'_k = \Big[ \sum_{\substack{a+b=k \\ a,b \geq 0}} \partial_x^{(a)}(U) A_b \Big] U^{-1}. \tag{3}$$

In order to construct our example, let us fix a bijection $n \mapsto a_n$ between the natural numbers and $K = S(K)$. Let $\mathbb{E} \in \mathrm{Strat}(X/S)$ be the rank-two relative stratified bundle $E = \mathscr{O}_X \cdot e_1 \oplus \mathscr{O}_x \cdot e_2$ with $\mathscr{D}_{X/S}$-action given by $\partial_x^{(k)}(e_1) = 0$ and

$$\partial_x^{(k)}(e_2) = \begin{cases} \prod_{i=0}^{h}(y-a_i)\cdot e_1 & \text{if } k = p^h, \\ 0 & \text{else.} \end{cases} \tag{4}$$

In order to prove Proposition 5.1 we need to show that this actually defines an action of $\mathscr{D}_{X/S}$ over $E$ and that $\mathbb{E}$ satisfies the two properties of the proposition, namely that it is trivial on every closed fiber and not isotrivial on the geometric generic fiber.

Lemma 5.2. *The formulae in* (4) *define a $\mathscr{D}_{X/S}$-module structure on $E$.*

*Proof.* As we fixed the action of the generators of $\mathscr{D}_{X/S}$, for it to extend to a $\mathscr{D}_{X/S}$-action we only need to check that the relations of the generators in the ring of differential operators are satisfied by their images in $\mathrm{End}_{\mathscr{O}_X}(E)$. By [3, Cor. 2.5] the only relations are

$$[\partial_x^{(l)}, \partial_x^{(k)}] = 0$$
$$\partial_x^{(k)} \circ \partial_x^{(l)} = \binom{k+l}{k}\partial_x^{(k+l)}$$
$$[\partial_x^{(k)}, x] = \partial_x^{(k-1)}.$$

Let us begin with the second relation: for $k, l > 0$

$$\partial_x^{(k)} \circ \partial_x^{(l)}(e_1) = 0$$
$$\partial_x^{(k)} \circ \partial_x^{(l)}(e_2) = \begin{cases} \partial_x^{(k)}(\prod_{i=0}^{h}(y-a_i)\cdot e_1) = 0 & \text{if } l = p^h \\ 0 & \text{else} \end{cases}$$

Hence, we just need to verify that if $k+l = p^h$ then $\binom{k+l}{k} = 0$ but this holds by Lucas's theorem and the first relation follows immediately. Moreover by (2) we have

$$\partial_x^{(k)} \cdot x(e_i) = \partial_x^{(k)}(x \cdot e_i) = \sum_{\substack{a+b=k \\ a,b\geq 0}} \partial_x^{(a)}(x)\partial_x^{(b)}(e_i) = x\partial_x^{(k)}(e_i) + \partial_x^{(k-1)}(e_i);$$

hence, the third relation trivially holds. □

In order to prove that $\mathbb{E}_s$ is trivial for every closed fiber, let us fix $n \in \mathbb{N}$ and let $s = a_n \in S(K)$, that is, $X_s = \{y = a_n\} \subset X$. Let us consider the basis change on $\mathscr{O}_{X_s} \cdot e_1 \oplus \mathscr{O}_{X_s} \cdot e_2 = E_s$ given by $e_1' = e_1$ and

$$e_2' = e_2 - \Big[(y-a_0)x + (y-a_0)(y-a_1)x^p + \cdots + \big[\prod_{i=0}^{n-1}(y-a_i)\big]x^{p^{n-1}}\Big]\cdot e_1$$

then by (3) in this new basis the action of $\mathscr{D}_{X_s/k(s)}$ is given by $\partial_x^{(k)}(e_1') = \partial_x^{(k)}(e_2') = 0$ hence is the trivial action.

We are now left to prove that $\mathbb{E}_{\bar\eta}$ is not isotrivial:

Lemma 5.3. *Let $\mathbb{E} \in \mathrm{Strat}(X/S)$ be the stratified bundle defined by* (4)*, then $\mathbb{E}_{\bar\eta}$ is not isotrivial.*

*Proof.* In order to prove that $\mathbb{E}_{\bar{\eta}}$ is not isotrivial it suffices by Lemma 3.5 to show that it cannot be defined over any $K'$ of finite type over $\mathbb{F}_p$. Remark that $\mathbb{E}_{\bar{\eta}}$ is defined over $\mathbb{A}^1_{\bar{\eta}}$ which descends to $\mathbb{A}^1_{K'}$ for any $K' \subset K$. By way of contradiction assume there exists such a $K'$ and let $\mathbb{E}'$ be the descent of $\mathbb{E}_{\bar{\eta}}$ over $\mathbb{A}^1_{K'}$. This means that there is a basis $e'_1, e'_2$ of $\mathbb{E}_{\bar{\eta}}$ such that the matrices $A'_k$ in this new basis take values in $K'[x]$.
Let $U$ be the basis change matrix between $e_i$ and $e'_i$, then $U$ is defined over some $K''$ of finite type over $K'$, hence over $\mathbb{F}_p$, so by (3) we have that $\prod_{i=0}^h (y-a_i) \in K''[x]$. In particular, if we denote by $\mathcal{A} = \mathbb{F}_p[\prod_{i=0}^h (y-a_i) \mid h \in \mathbb{N}]$, our assumption implies that $\mathcal{A} \subset K''[x]$.
To see that this leads to a contradiction it suffices to show that $\mathcal{K} \nsubseteq K''(x)$ where $\mathcal{K}$ is the quotient field of $\mathcal{A}$. Note that $K \subset \mathcal{K}$; therefore, it is enough to prove that for every $K'$ of finite type over $\mathbb{F}_p$ we have that $K \nsubseteq K'(x)$ and as $\bar{\mathbb{F}}_p \subset K$ it is sufficient to show $\bar{\mathbb{F}}_p \nsubseteq K'(x)$, which follows from the following:

*Claim.* Let $\mathbb{F}_q$ be a finite field with $q = p^n$ for some $n \in \mathbb{N}$ and let $K \supset \mathbb{F}_q$ an algebraic extension such that $[K : \mathbb{F}_q] = +\infty$. Then for every $m \in \mathbb{N}$ and every $\varepsilon_1, \dots, \varepsilon_m$ non-algebraic over $\mathbb{F}_q$ we have that

$$K \nsubseteq \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m).$$

*Proof.* By induction on $m$, the case $m = 0$ being evident. Let $m = 1$, and $\gamma \in K - \mathbb{F}_q$ and let $\mu_\gamma(t)$ its minimal polynomial over $\mathbb{F}_q$. By way of contradiction assume $\gamma \in \mathbb{F}_q(\varepsilon_1)$; then $\gamma = f(\varepsilon_1)/g(\varepsilon_1)$ and

$$g(\varepsilon_1)^{\deg \mu_\gamma} \cdot \mu_\gamma\left(\frac{f(\varepsilon_1)}{g(\varepsilon_1)}\right) = 0$$

gives an algebraic dependence of $\varepsilon_1$ over $\mathbb{F}_q$ which is a contradiction with our assumption that $\varepsilon_1$ is not algebraic over $\mathbb{F}_q$. Let now $m \geq 1$, by induction step we know that for every $n \in \mathbb{N}$, $q = p^n$, then no infinite algebraic extension of $\mathbb{F}_q$ is contained in $\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1})$; hence, there exists an $r$ such that $\mathbb{F}_{q^r} = \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1}) \cap K$. Then

$$\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_{m-1})(\varepsilon_m) \cap K \subset \mathbb{F}_{q^r}(\varepsilon_m) \neq K$$

by the $m = 1$ step applied to $q = p^{rn}$. In particular, $K \nsubseteq \mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m)$. □

Note that if $K'$ is of finite type over $\mathbb{F}_p$ then $K'$ can be always be written as $\mathbb{F}_q(\varepsilon_1, \dots, \varepsilon_m)$ for some $q = p^n$ and $\varepsilon_i$ non-algebraic over $\mathbb{F}_q$; hence, $\bar{\mathbb{F}}_p \nsubseteq K'(x)$. Therefore, $\mathbb{E}$ cannot be defined over any $K'$ of finite type over $\mathbb{F}_p$ and by Lemma 3.5 it cannot be finite. □

*Remark* 5.4. Let us observe that if $K$ is uncountable the same construction provides an example of a relative stratified bundle $\mathbb{E} \in \mathrm{Strat}(\mathbb{A}^2_K/\mathbb{A}^1_K)$ and a dense subset $\tilde{S} \subset \mathbb{A}^1_K(K)$ such that $\mathbb{E}_s$ is trivial for every $s \in \tilde{S}$ but $\mathbb{E}_{\bar{\eta}}$ is not isotrivial. Therefore, the density condition on $\tilde{S}$ of Theorem 4.1 will not be sufficient for our purposes, in parallel with the similar problem that one encounters in the equicharacteristic zero case (see [2, Rmk. 7.2.3]).

## 6. The main theorem

From the example in previous section it appears that in the case where $X$ is not projective over $S$ the situation is significantly different from the one in Theorem 4.1. In the latter one big obstruction for the theorem to hold is related to $p$ dividing the order of the monodromy group on the closed fibers. In the counterexample of Section 5 these are trivial and the obstruction seems more related to the cardinality of $K$. As noticed in Section 3, the monodromy group does not depend (up to a non-unique isomorphism) on the choice of $x \in X$. Therefore, in this section we will denote the monodromy group of a stratified bundle $\mathbb{E}$ simply by $\pi(\mathbb{E})$.

In order to phrase the statement of the main theorem let us introduce the following notation: we will denote by $(X, S; \mathbb{E})$ (and call it *a triple over* $K$) any triple consisting of $X \to S$ smooth morphism of $K$-varieties with geometrically connected fibers and $\mathbb{E} \in \mathrm{Strat}(X/S)$. We denote furthermore by $K' = K'(X, S; \mathbb{E})$ a (minimal) algebraically closed subfield of $K$ such that $(X, S; \mathbb{E})$ is defined over $K'$, and by $(X', S'; \mathbb{E}')$ the descent of the triple $(X, S; \mathbb{E})$ to $K'$. Then the following result holds

Theorem 6.1. *Let $(X, S; \mathbb{E})$ be a triple over $K$, let $K' = K'(X, S; \mathbb{E}) \subset K$ and $(X', S'; \mathbb{E}')$ the descended triple to $K'$. Let $k(S')$ be the function field of $S'$. Let us assume:*

$$\exists \quad i : k(S') \hookrightarrow K \text{ extending } K' \subset K. \qquad (*)$$

*Assume that there exists a dense open $\tilde{S} \subset S$ such that $\mathbb{E}_s$ is finite for every $s \in \tilde{S}(K)$, then so is $\mathbb{E}_{\bar{\eta}}$. Moreover there exists $s \in \tilde{S}(K)$ such that $\pi(\mathbb{E}_s)(K) \simeq \pi(\mathbb{E}_{\bar{\eta}})(\overline{k(S)})$, in particular*

- i) *$|\pi(\mathbb{E}_s)|$ is bounded over $\tilde{S}(K)$;*
- ii) *if $p \nmid |\pi(\mathbb{E}_s)|$ for every $s \in \tilde{S}(K)$ then $p \nmid |\pi(\mathbb{E}_{\bar{\eta}})|$;*
- iii) *any group property holding for $\pi(\mathbb{E}_s)$ for every $s \in \tilde{S}(K)$ holds for $\pi(\mathbb{E}_{\bar{\eta}})$.*

*Proof.* Up to shrinking $S$, we can assume $\tilde{S} = S$. Let $\Delta : S' \to S' \times_{\mathrm{Spec}\, K'} S'$ be the diagonal morphism and $i : k(S') \hookrightarrow K$ be an immersion as in $(*)$. Then the base change along

$$\mathrm{Spec}\, K \xrightarrow{i} \mathrm{Spec}\, k(S') \longrightarrow \mathrm{Spec}\, S'$$

induces $(s : \mathrm{Spec}\, K \to S) \in S(K)$ such that $X' \otimes_{k(S')} K \simeq X_s$ and

$$i^*\mathbb{E}' = \mathbb{E}_s,$$

where we are considering $i : k(S') \hookrightarrow K$ as a geometric generic point of $S'$. In particular, $\pi(i^*\mathbb{E}') = \pi(\mathbb{E}_s)$. Therefore, $i^*\mathbb{E}'$ is finite; hence, by Lemma 3.4, $\mathbb{E}_{\bar{\eta}}$ is finite as well, moreover their monodromy group are isomorphic as constant groups. $\square$

Corollary 6.2. *If $K$ is uncountable then the assumption $(*)$, hence the theorem, always holds.*

*Proof.* It suffices to show that if $K$ is uncountable, then for any triple $(X, S; \mathbb{E})$ over $K$ there exists $K' = K'(X, S; \mathbb{E})$ and an inclusion $k(S') \subset K$ extending $K' \subset K$. But it is easy to check that a triple $(X, S; \mathbb{E})$ is defined by countably many data; hence, we can choose $K'$ such that it has countable transcendence degree over $\mathbb{F}_p$. As $K$ is uncountable, it has infinite transcendence degree over $K'$; hence, there always exists $k(S') \subset K$ as in $(*)$. $\square$

*Remark* 6.3. Notice that if the smooth morphism $X \to S$ does not have geometrically connected fibers then we lose the notion of monodromy group on the closed and geometric generic fibers: if $X$ is a $K$-variety which is not connected and $\mathbb{I}_{X/K}$ is the trivial stratified bundle on $X$ then $\mathrm{End}(\mathbb{I}_{X/K}) \neq K$, hence $\mathrm{Strat}(X/K)$ is not a Tannakian category. Nevertheless, if we do not assume $X \to S$ to have geometrically connected fibers, the same proof shows that if $\mathbb{E}_s$ is finite when restricted to every connected component of $X_s$, then the same holds for $\mathbb{E}_{\bar\eta}$ on every connected component of $X_{\bar\eta}$.

## 7. Regular singularity and a refinement of the theorem

Regardless of the example in Section 5, there is a way to broaden Theorem 4.1 in the case where $K$ is countable, making the additional assumption that the stratified bundle is regular singular on the geometric generic fiber.

Let $X$ be a smooth variety over $K$ and let $(X, \overline{X})$ be a *good partial compactification* of $X$; that is: $\overline{X}$ is a smooth variety over $K$ such that $X \subset \overline{X}$ is an open subscheme and $D = \overline{X} \backslash X$ is a strict normal crossing divisor. Let $\mathscr{D}_{\overline{X}/K}(\log D) \subset \mathscr{D}_{\overline{X}/K}$ the subalgebra generated by the differential operators that locally fix all powers of the ideal of definition of $D$. If $\mathcal{U} \subset \overline{X}$ admits global coordinates $x_1, \dots, x_d$ and $D$ is smooth and given by $\{x_1 = 0\}$ then

$$\mathscr{D}_{\overline{X}/K}(\log D)_{|\mathcal{U}} = \mathscr{O}_{\mathcal{U}}\big[x_1^k \partial_{x_1}^{(k)}, \partial_{x_i}^{(k)} \mid i \in \{2, \dots, d\}, k \in \mathbb{N}_{>0}\big].$$

Definition 7.1. A stratified bundle $\mathbb{E} \in \mathrm{Strat}(X/K)$ is called $(X, \overline{X})$-*regular singular* if it extends to a locally free $\mathscr{O}_{\overline{X}}$-coherent $\mathscr{D}_{\overline{X}/k}(\log D)$-module $\overline{\mathbb{E}}$ on $\overline{X}$. It is *regular singular* if it is $(X, \overline{X})$-regular singular for every partial good compactification $(X, \overline{X})$.

*Remark* 7.2. There is a parallel notion of regular singularities in characteristic zero. Despite the fact that isotrivial implies regular singular over the complex numbers, this is not longer true in positive characteristic, due to the existence of wild coverings (for a more precise statement, see [16, Thm. 1.1]).

For a $(X, \overline{X})$-regular singular stratified bundle $\mathbb{E}$ we have a theory of exponents (see [11, §3]) of $\mathbb{E}$ along $D$: it is a finite subset $\mathrm{Exp}_D(\mathbb{E}) \subset \mathbb{Z}_p/\mathbb{Z}$ given by the following:

Proposition 7.3. [11, Lemma 3.8],[16, Prop. 4.12] *Let* $\overline{X} = \mathrm{Spec}\, A$ *be a smooth variety over* $K = \bar{K}$ *with global coordinates* $x_1, \dots, x_d$ *and let* $D$ *be the smooth divisor defined by* $\{x_1 = 0\}$*. Let* $\mathbb{E} \in \mathrm{Strat}(X/K)$ *a* $(X, \overline{X})$*-regular singular stratified bundle and* $\overline{\mathbb{E}}$ *a locally free* $\mathscr{D}_{\overline{X}/K}(\log D)$*-module extending* $\mathbb{E}$ *. Then* *there exists a decomposition of* $\overline{\mathbb{E}}_{|D} = \bigoplus K_\alpha$ *with* $\alpha \in \mathbb{Z}_p$ *such that* $x_1^k \partial_{x_1}^{(k)}$ *acts on* $K_\alpha$ *by multiplication by* $\binom{\alpha}{k}$*. The image in* $\mathbb{Z}_p/\mathbb{Z}$ *of the* $\alpha \in \mathbb{Z}_p$ *such that* $K_\alpha \neq 0$ *are called the* exponents *of* $\mathbb{E}$ *along* $D$ *and do not depend on the choice of* $\overline{\mathbb{E}}$*.*

If $D$ is not smooth $\mathrm{Exp}_D(\mathbb{E})$ is defined to be the union of the exponents along all the irreducible components of $D$. By [16, Cor. 5.4] $\mathbb{E}$ extends to a stratified bundle $\overline{\mathbb{E}}$ on $\overline{X}$ if and only if its exponents are zero. In particular, [16, Prop. 4.11] implies that if $\mathbb{E}$ is finite then its exponents are torsion. Moreover:

Lemma 7.4. *Let* $\mathbb{E}$ *be a* $\mathscr{D}_{X/S}$*-module such that* $\mathbb{E}_s$ *is finite for every* $s \in \tilde{S}$ *a dense subset of* $S(K)$*. If* $\mathbb{E}_{\bar{\eta}}$ *is regular singular then the exponents of* $\mathbb{E}_{\bar{\eta}}$ *with respect to any partial good compactification of* $X_{\bar{\eta}}$ *are torsion.*

*Proof.* Let us fix $(X_{\bar{\eta}}, \overline{X}_{\bar{\eta}})$ a partial good compactification and let $D_{\bar{\eta}} = \overline{X}_{\bar{\eta}} \backslash X_{\bar{\eta}}$. As the exponents can be checked locally, we can shrink $\overline{X}_{\bar{\eta}}$ around the generic point of one of the irreducible components of $D_{\bar{\eta}}$ at a time. Moreover in order to prove the lemma we are allowed to take a generically finite étale open $S'$ of $S$ and substitute $X$ by $X \times_S S'$ (and $\tilde{S}$ by its preimage) as the geometric generic fiber is the same. Finally for every $s \in S(K)$ we have that $X'_s$ is either empty or a finite union of copies of $X_s$; hence, we will still denote by $s$ any point $s' \in S'$ lying over it.
Hence, without loss of generality, we can assume that we are in the following situation: the partial good compactification $(X_{\bar{\eta}}, \overline{X}_{\bar{\eta}})$ is the restriction of a relative good partial compactification $(X, \overline{X})$ defined on the whole $S$, $\overline{X}$ is the spectrum of a ring $A$, with global relative coordinates $x_1, \dots, x_d$ over $S$ and finally $D = \overline{X} \backslash X$ is defined by $\{x_1 = 0\}$. Moreover we can assume that $\mathbb{E}$ is globally free and that on the geometric generic fiber $\mathbb{E}_{\bar{\eta}}$ extends to a globally free $\mathscr{D}_{\bar{X}_{\bar{\eta}}/\overline{k(S)}}(\log D_{\bar{\eta}})$-module $\overline{\mathbb{E}}_{\bar{\eta}}$.
Let $s \in \tilde{S}$ be any point such that $X_s \cap D \neq \emptyset$, and let us consider the globally free $\mathscr{O}_X$-module $\bar{E} = \mathscr{O}_{\overline{X}} \bar{e}_1 \oplus \cdots \oplus \mathscr{O}_{\overline{X}} \bar{e}_r$. Then the $\bar{e}_i$ induce a basis on the restriction of $\overline{E}$ to the closed fiber over $s$ (as well as to the geometric generic one) and to the boundary divisor (as well as to its complement) as in the following commutative diagram:

$$\begin{tikzcd}
E_s = \bigoplus_{i=1}^r \mathscr{O}_{X_s} e_i^s & E = \bigoplus_{i=1}^r \mathscr{O}_X e_i \arrow[l, "\otimes k(s)"'] \arrow[r, "\otimes \overline{k(S)}"] & E_{\bar{\eta}} = \bigoplus_{i=1}^r \mathscr{O}_{X_{\bar{\eta}}} \varepsilon_i \\
\overline{E}_s = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}_s} \bar{e}_i^s \arrow[u, "|X_s"] \arrow[d, "|_{D_s}"'] & \overline{E} = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}} \bar{e}_i \arrow[l, "\otimes k(s)"'] \arrow[r, "\otimes \overline{k(S)}"] \arrow[u, "|_X"] \arrow[d, "|_D"'] & \overline{E}_{\bar{\eta}} = \bigoplus_{i=1}^r \mathscr{O}_{\overline{X}_{\bar{\eta}}} \bar{\varepsilon}_i \arrow[u, "|_{X_{\bar{\eta}}}"] \arrow[d, "|_{D_{\bar{\eta}}}"'] \\
\overline{E}_{|D_s} = \bigoplus_{i=1}^r \mathscr{O}_{D_s} \tilde{e}_i^s & \overline{E}_{|D} = \bigoplus_{i=1}^r \mathscr{O}_D \tilde{e}_i \arrow[l, "\otimes k(s)"'] \arrow[r, "\otimes \overline{k(S)}"] & \overline{E}_{|D_{\bar{\eta}}} = \bigoplus_{i=1}^r \mathscr{O}_{D_{\bar{\eta}}} \tilde{\varepsilon}_i.
\end{tikzcd}$$

Consider the first line of the diagram: on the first (respectively second and third) column there is an action of $\mathscr{D}_{X_s/k(s)}$ (respectively $\mathscr{D}_{X/S}$ and $\mathscr{D}_{X_{\bar{\eta}}/\overline{k(S)}}$),

compatible with each other. On the last column this action extends to a logarithmic action on $\overline{E}_{\bar\eta}$ that we want to extend compatibly to $\overline{E}$.
Similarly as in Section 5, let $A_{i,k}$ be the matrices describing the action of $\partial_{x_i}^{(k)} \in \mathscr{D}_{X/S}$ in the basis $e_i$, then the same ones describe the action of $\partial_{x_i}^{(k)} \in \mathscr{D}_{X_{\bar\eta}/\overline{k(S)}}$ in the basis $\varepsilon_i$. By regular singularity of $\mathbb{E}_{\bar\eta}$ this action extends to a $\mathscr{D}_{\overline{X}_{\bar\eta}/\overline{k(S)}}(\log D_{\bar\eta})$-action. Therefore, there is a second basis $\varepsilon'_1, \dots, \varepsilon'_d$ on the geometric generic fiber such that in the new basis the matrices $A'_{i,k}$ have no poles in $x_1$ for $i \neq 1$ and logarithmic poles for $i = 1$. Let $U \in \mathrm{H}^0(X_{\bar\eta}, \mathrm{GL}_r)$ the basis change matrix from $\varepsilon_i$ to $\varepsilon'_i$. Taking a generically finite étale open of $S$ we can assume that $U$ is defined on the whole $S$; hence, the $A'_{i,k}$ are defined over the whole $S$ as well and this defines an action of

$$\mathscr{D}_{X/S}(\log D) \doteq \mathscr{O}_X\left[x_1^k \partial_{x_1}^{(k)}, \partial_{x_i}^{(k)} \mid i \in \{2, \dots, d\}, k \in \mathbb{N}_{>0}\right]$$

on $\overline{E}$, compatible with the logarithmic action on the fibers over $\bar\eta$. In particular, this induces a $\mathscr{D}_{\overline{X}_s}(\log D_s)$-action on $\overline{E}_s$; hence, $\mathbb{E}_s$ is $(X_s, \overline{X}_s)$-regular singular (notice that if $S'$ is an étale open of $S$ then for $s \in S(K)$ the fiber $X'_s$ of $X' = X \times_S S'$ is either empty or the disjoint union of finitely many copies of $X_s$).
We want now to compare $\mathrm{Exp}_{D_{\bar\eta}}(\mathbb{E}_{\bar\eta})$ and $\mathrm{Exp}_{D_s}(\mathbb{E}_s)$. By Proposition 7.3 we have that $\overline{\mathbb{E}}_{\bar\eta|_{D_{\bar\eta}}} = \oplus F_\alpha$; hence, there exists $\tilde\varepsilon_i$ a basis of $\overline{E}_{|D_{\bar\eta}}$ such that the matrices $\tilde B_k$ defining the action of $x_1^k \partial_{(k),x_1}$ are diagonal with values $\binom{\alpha}{k} \in \mathbb{F}_p$. Let $\bar\varepsilon_i$ be a lift of $\tilde\varepsilon_i$, then up to taking an étale generically finite open of $S$ we can assume that $\bar\varepsilon_i$ is a restriction of a basis $\bar e_i$ of $\overline{E}$ over $\overline{X}$. In particular, the decomposition extends as well and $\overline{\mathbb{E}}_{|D} = \oplus F_\alpha$ induces a decomposition on $\overline{\mathbb{E}}_{s|_{D_s}}$. This decomposition must coincide with the one given by Proposition 7.3; hence, the exponents must be the same of the ones of $\mathbb{E}_{\bar\eta}$. As $\mathbb{E}_s$ is isotrivial, its exponents are torsion; hence, so must be the ones of $\mathbb{E}_{\bar\eta}$. □

*Remark* 7.5. While the previous proof shows that if $\mathbb{E}_{\bar\eta}$ is regular singular so are the $\mathbb{E}_s$ for every $s \in S(K)$, the example in Section 5, together with Theorem 7.7, shows that the converse does not hold in general (however, one can prove it is the case when $K$ is uncountable). On the contrary, in characteristic zero it is always true that if a relative flat connection is regular with respect to some smooth good compactification on the fibers over a dense set of points of $S$, then it is regular on the geometric generic fiber, as proven in [2, Lemma 8.1.1].

Before stating and proving the main theorem of this section we need to prove the existence of Kawamata coverings in positive characteristic. Analogously to the original construction in characteristic zero ([15, Thm. 17]) we have the following

Theorem 7.6. *Let $X$ be a projective smooth variety of dimension $d$ over an algebraically closed field $K$ of characteristic $p$ and let $D$ be a simple normal crossing divisor on $X$. Let $m \in \mathbb{N}$ prime to $p$, then there exist a projective smooth variety $Y$ and a finite surjective mapping $f : Y \to X$ such that $(f^*D)_{red}$*

*is a simple normal crossing divisor on* $Y$ *and if* $f^*D = \sum m_i \tilde{D}_i$ *is the decomposition in irreducible components with* $\tilde{D}_i \neq \tilde{D}_j$ *for* $i \neq j$ *then* $m \mid m_i$ *for all* $i$ *and* $m_i$ *are all prime to* $p$.

*Proof.* The proof follows the one of the original theorem ([15, Thm. 17], see also [10, Lemma 3.17]). One does the construction one irreducible component $D'$ of $D$ at a time, choosing an ample line bundle $\mathcal{M}$ on $X$ and $N \gg 0$ such that $N\mathcal{M} - D'$ is very ample. The only additional care that needs to be taken, is to choose $N$ so that $m \mid N$ and $(N, p) = 1$, which is possible as $m$ is prime to $p$ (this will be enough to prove that $Y$ is smooth in the very same way by [13, Lemma 1.8.6]). One needs moreover to use [17, Cor. 12] instead of the classical smoothness theorem for general members of a very ample linear system. $\square$

We can now state and prove the following

Theorem 7.7. *Let* $X \to S$ *be a smooth morphism of* $K$*-varieties with geometrical connected fibers and let* $\mathbb{E} \in \mathrm{Strat}(X/S)$. *Assume that there exists a dense subset* $\tilde{S} \subset S(K)$ *such that, for every* $s \in \tilde{S}$, *the stratified bundle* $\mathbb{E}_s$ *has finite monodromy and that the highest power of* $p$ *dividing* $|\pi(\mathbb{E}_s)|$ *is bounded over* $\tilde{S}$. *Assume moreover that* $\mathbb{E}_{\bar{\eta}}$ *is regular singular, then*

i) *there exists* $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ *a finite étale cover such that* $f^*\mathbb{E}_{\bar{\eta}}$ *decomposes as direct sum of stratified line bundles;*
ii) *if* $K \neq \bar{\mathbb{F}}_p$ *then* $\mathbb{E}_{\bar{\eta}}$ *is finite.*

*Proof.* Let $\mathcal{U} \subset X$ be a dense open, then by invariance of the monodromy group it is enough to show the theorem for $\mathbb{E}_{|\mathcal{U}}$ moreover it is enough to prove finiteness for its pullback along any finite étale cover. Therefore, we can always work up to generically finite étale covers. Using [6] we can find an alteration generically finite étale $f : X' \to X$ such that $X'$ admits a good projective compactification relative to $S$. By [16, Prop. 4.4] the pullback of a regular singular stratified bundle is again regular singular. Hence, without loss of generality, we can assume that $X$ admits a good projective compactification $\overline{X}$ relative to $S$. We will denote by $D = \overline{X} \backslash X$ the divisor at infinity.
Let $\mathrm{Exp}_D(\mathbb{E}_{\bar{\eta}}) \subset \mathbb{Z}_p/\mathbb{Z}$ be the finite set of exponents of $E_{\bar{\eta}}$ along $D_{\bar{\eta}}$ (as defined in Lemma 7.3). As $\mathbb{E}_{\bar{\eta}}$ is regular singular then by Lemma 7.4 the exponents of $\mathbb{E}_{\bar{\eta}}$ are torsion; let $m \in \mathbb{N}$ an integer prime to $p$ killing the torsion of $\mathrm{Exp}_D(\mathbb{E}_{\bar{\eta}})$ and let $f : \overline{Y}_{\bar{\eta}} \to \overline{X}_{\bar{\eta}}$ be the Kawamata covering constructed in Theorem 7.6: it ramifies on a simple normal crossing divisor $\tilde{D}_{\bar{\eta}}$ containing the divisor at infinity $D_{\bar{\eta}} = \overline{X}_{\bar{\eta}} - X_{\bar{\eta}}$ and it is Kummer on $\overline{X}_{\bar{\eta}} - \tilde{D}_{\bar{\eta}}$. As $m$ divides the ramification order along $D_{\bar{\eta}}$ by [16, Prop. 4.11] the exponents of the pullback of $\mathbb{E}_{\bar{\eta}}$ along $(f^*D_{\bar{\eta}})_{red}$ are zero; hence, it extends to the whole $Y_{\bar{\eta}}$. Up to taking an étale open of $S$ and using a similar argument as in the proof of Lemma 7.4 we can assume that this extension is defined on the whole $S$. Therefore, we have reduced the problem to Theorem 4.3. $\square$

## 8. Finite vector bundles

The notion of isotriviality has as well relevance in the category of vector bundles over a proper smooth $K$-variety, even though it is not equivalent to the notion of finiteness (see [20, Lemma 3.1] and following definition) for vector bundles, at least in positive characteristic. In the same spirit of Theorem 4.1, Esnault and Langer proved in the same paper the following:

Theorem 8.1. [9, Thm. 5.1] *Let $X \to S$ be a smooth projective morphism of $K$-varieties with geometrically connected fibers and let $E$ be a locally free sheaf over $X$. Assume that there exists a dense subset $\tilde{S} \subset S(K)$ such that, for every $s \in \tilde{S}$, there is a finite étale Galois cover $h_s : Y_s \to X_s$* of order prime to $p$ *such that $h_s^*(E_s)$ is trivial. Then*

i) *there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover* of order prime to $p$ *such that $f^* E_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;*
ii) *if $K \neq \bar{\mathbb{F}}_p$ then $E_{\bar{\eta}}$ is trivialized by a finite étale cover* of order prime to $p$.

Then, a reasoning similar to the proof of Theorem 4.3 proves the following:

Theorem 8.2. *Let $X \to S$ be a smooth* projective *morphism of $K$-varieties with geometrically connected fibers and let $E$ be a locally free sheaf over $X$. Assume that there exists a dense subset $\tilde{S} \subset S(K)$ such that, for every $s \in \tilde{S}$, there is a finite étale Galois cover $h_s : Y_s \to X_s$ such that $h_s^*(E_s)$ is trivial and that the highest power of $p$ dividing the order of such covers is bounded over $\tilde{S}$. Then*

i) *there exists $f_{\bar{\eta}} : Y_{\bar{\eta}} \to X_{\bar{\eta}}$ a finite étale cover such that $f^* E_{\bar{\eta}}$ decomposes as direct sum of stratified line bundles;*
ii) *if $K \neq \bar{\mathbb{F}}_p$ then $E_{\bar{\eta}}$ is trivialized by a finite étale cover.*

*Proof.* We will reduce this theorem to Theorem 8.1. By taking an étale open of $S$ we can assume there exists a section $\sigma : S \to X$. Let $r$ be the rank of $E$ and fix $s$ a closed point in $S$. As $X_s$ is a smooth $k(s)$-variety, then (see [9, Definition 3.2] and following discussion) every étale trivializable vector bundle is Nori semistable. In particular, the Galois cover $h_s : Y_s \to X_s$ corresponds to a representation of rank $r$ of the Nori fundamental group scheme $\pi_1^N(X_s, \sigma_s)$ (for the definition of the Nori group scheme see [20]) that factors through the étale fundamental group:

$$\pi_1^N(X_s, \sigma(s)) \twoheadrightarrow \pi_1^{\text{ét}}(X_s, \sigma(s)) \twoheadrightarrow \Gamma_s \subset \mathrm{GL}_r(K),$$

where $\Gamma_s$ is the Galois group of $h_s : Y_s \to X_s$. The rest of the proof follows exactly as in Theorem 4.3. □

If the morphism $X \to S$ is not projective but only smooth we get a similar result to Corollary 6.2:

Theorem 8.3. *Let $K$ be an algebraically closed field of positive characteristic with infinite transcendental degree over $\mathbb{F}_p$. Let $X \to S$ be a smooth morphism* *of varieties over $K$ and $E$ a vector bundle over $X$. Assume that there exists a dense open $\tilde{S} \subset S$ such that $E_s$ is isotrivial for every $s \in \tilde{S}(K)$, then so is $E_{\bar{\eta}}$.*

*Proof.* There exists $K'$ a subfield of $K$ of finite type over $\mathbb{F}_p$ such that $X \to S$ and $E$ descend to $X' \to S'$ and $E'$. Moreover as $K$ has infinite transcendence degree over $\mathbb{F}_p$ there exists an immersion $k(S') \hookrightarrow K$ over $K'$ and a point $s \in S(K)$, like in the proof of Theorem 6.1, such that the morphism $i : \operatorname{Spec} K \to \operatorname{Spec} k(S')$ given by $k(S') \subset K$ is a geometric generic point of $S'$, and on $X' \otimes_{k(S')} K \simeq X_s$

$$i^* E' = E_s.$$

Note that there exists an immersion $\iota : K \hookrightarrow \overline{k(S)}$ (which is not the natural one given by the fact that $S$ is a $K$-variety) that is the identity on $k(S')$, hence via $\iota$ we have that $X_{\bar{\eta}} \simeq X' \otimes_{k(S')} \overline{k(S)} \simeq X_s \otimes_K \overline{k(S)}$. In particular, if we continue to consider $K$ as a subfield of $\overline{k(S)}$ via the immersion $\iota$, then $h_s \otimes_K \overline{k(S)} : Y_s \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{k(S)} \to X_{\bar{\eta}}$ trivializes $E_{\bar{\eta}}$. $\square$

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