Rationally Isotropic Exceptional Projective Homogeneous Varieties Are Locally Isotropic

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Abstract. Assume that R is a regular local ring that contains an infinite field and whose field of fractions K has charactertistic $\neq 2$. Let X be an exceptional projective homogeneous scheme over R. We prove that in most cases the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.

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1. Introduction

The main result of the present article extends the main results of [Pa3] and [PP] to the case of exceptional groups. In the latter paper one can find historical remarks which might help the general reader. All the rings in the present paper are *commutative* and *Noetherian*. We prove the following theorem.

Theorem 1. *Let R be a regular local ring that contains an infinite field and whose field of fractions K has characteristic $\neq 2$. Let G be a split simple group of exceptional type (that is, E_6, E_7, E_8, F_4, or G_2), P be a parabolic subgroup of G, $[\xi]$ be a class from $\mathrm{H}^1(R,G)$, and $X = (G/P)_\xi$ be the corresponding homogeneous space over R. Assume that $P \neq P_7, P_8, P_{7,8}$ in case $G = E_8$, $P \neq P_7$ in case $G = E_7$, and $P \neq P_1$ in case $G = E_7^{ad}$. Then the condition $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

The results of the present paper depend on the following yet unpublished results: [FP, Corollary of Theorem 1] and [Pa, Theorem 10.0.30].

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2. Purity of some H^1 functors

Let R be a commutative noetherian domain of finite Krull dimension with a fraction field F. We say that a functor $\mathcal{F}$ from the category of commutative R-algebras to the category of sets *satisfies purity* for R if we have

$$\mathrm{Im}\,[\mathcal{F}(R) \to \mathcal{F}(F)] = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_\mathfrak{p}) \to \mathcal{F}(F)].$$

An element $a \in \mathcal{F}(F)$ *is called R-unramified* if it belongs to $\bigcap_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{Im}\,[\mathcal{F}(R_\mathfrak{p}) \to \mathcal{F}(F)]$. If $\mathfrak{p}$ is a height one prime ideal in R, the element *a is called $\mathfrak{p}$-unramified*, if it belongs to $\mathrm{Im}\,[\mathcal{F}(R_\mathfrak{p}) \to \mathcal{F}(F)]$.
If $\mathcal{H}$ is an étale group sheaf we write $\mathrm{H}^i(-,\mathcal{H})$ for $\mathrm{H}^i_{\text{ét}}(-,\mathcal{H})$ below through the text.
The following theorem is proven in the characteristic zero case [Pa2, Theorem 4.0.3]. We extend it here to reductive group schemes. Let R be a commutative noetherian ring. Recall that an R-group scheme G is called *reductive*, if it is affine and smooth as an R-scheme and if, moreover, for each algebraically closed field Ω and for each ring homomorphism $R \to \Omega$ the scalar extension G_Ω is a connected reductive algebraic group over Ω. This definition of a reductive R-group scheme coincides with [SGA, Exp. XIX, Definition 2.7].

Theorem 2. *Let R be the local ring of a closed point on a smooth scheme over an infinite field. Let G be a reductive R-group scheme. Let $i\colon Z \hookrightarrow G$ be a closed subgroup scheme of the center* $\mathrm{Cent}(G)$. *It is known that Z is of multiplicative type. Let $G' = G/Z$ be the factor group, $\pi\colon G \to G'$ be the projection.*
If the functor $\mathrm{H}^1(-,G')$ *satisfies purity for R, then the functor* $\mathrm{H}^1(-,G)$ *satisfies purity for R as well.*

It is known that π is surjective and strictly flat. Thus the exact sequence of R-group schemes

$$(*) \qquad \{1\} \to Z \xrightarrow{i} G \xrightarrow{\pi} G' \to \{1\}$$

induces an exact sequence of group sheaves in the fppf-topology.

Lemma 1. *Consider the category of R-algebras. The functor*

$$R' \mapsto \mathcal{F}(R') = \mathrm{H}^1_{\mathrm{fppf}}(R', Z)/\,\mathrm{Im}\,(\delta_{R'}),$$

where δ is the connecting homomorphism associated to sequence (*), satisfies purity for R.

Proof. The lemma coincides with [Pa, Theorem 10.0.30]. □

Lemma 2. *The map*

$$\mathrm{H}^2_{\mathrm{fppf}}(R, Z) \to \mathrm{H}^2_{\mathrm{fppf}}(K, Z)$$

is injective.

Proof. See [C-TS, Theorem 4.3]. □

Proof of Theorem 2. Reproduce the diagram chase from the proof of [Pa2, Theorem 4.0.3]. For this purpose consider the commutative diagram

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathcal{F}(K) & \xrightarrow{\delta_K} & \mathrm{H}^1(K,G) & \xrightarrow{\pi_K} & \mathrm{H}^1(K,G') & \xrightarrow{\Delta_K} & \mathrm{H}^2_{\mathrm{fppf}}(K,Z) \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \alpha} \\
\{1\} & \longrightarrow & \mathcal{F}(R) & \xrightarrow{\delta} & \mathrm{H}^1(R,G) & \xrightarrow{\pi} & \mathrm{H}^1(R,G') & \xrightarrow{\Delta} & \mathrm{H}^2_{\mathrm{fppf}}(R,Z)
\end{array}$$

Let $[\xi] \in \mathrm{H}^1(K,G)$ be an R-unramified class and let $[\bar{\xi}] = \pi_K([\xi])$. Clearly, $[\bar{\xi}] \in \mathrm{H}^1(K,G')$ is R-unramified. Thus there exists an element $[\bar{\xi}'] \in \mathrm{H}^1(R,G')$ such that $[\bar{\xi}']_K = [\bar{\xi}]$. The map α is injective by Lemma 2. One has $\Delta([\bar{\xi}']) = 0$, since $\Delta_K([\bar{\xi}]) = 0$. Thus there exists $[\xi'] \in \mathrm{H}^1(R,G)$ such that $\pi([\xi']) = [\bar{\xi}']$. Twisting G by ξ' we may assume that $[\bar{\xi}] = *$, so that $[\xi]$ comes from some $a \in \mathcal{F}(K)$.

Lemma 3. *The above constructed element $a \in \mathcal{F}(K)$ is R-unramified.*

Assume Lemma 3; we use it to complete the proof of Theorem 2. By Lemma 1 the functor $\mathcal{F}$ satisfies the purity for regular local rings containing the field k. Thus there exists an element $a' \in \mathcal{F}(R)$ with $a'_K = a$. It is clear that $[\delta(a')]_K = [\xi]$. It remains to prove Lemma 3. First we need a small variation of Nisnevich's theorem.

Lemma 4. *Let H be a reductive group scheme over a discrete valuation ring A. Let K be the fraction field of A. Then the map*

$$\mathrm{H}^1(A,H) \to \mathrm{H}^1(K,H)$$

is injective.

Proof. Let $[\xi_0], [\xi_1]$ be classes from $\mathrm{H}^1(A,H)$. Let $\mathcal{H}_0$ be a principal homogeneous H-bundle representing the class ξ_0. Let H_0 be the inner form of the group scheme H, corresponding to $\mathcal{H}_0$. Let $X = Spec(A)$. For each X-scheme S there is a well-known bijection $\phi_S\colon \mathrm{H}^1(S,H) \to \mathrm{H}^1(S,H_0)$ of non-pointed sets. That bijection takes the principal homogeneous H-bundle $\mathcal{H}_0 \times_X S$ to the trivial principal homogeneous H_0-bundle $H_0 \times_X S$. That bijection is functorial with respect to morphisms of X-schemes.
Assume that $[\xi_0]_K = [\xi_1]_K$. Then one has $* = \phi_K([\xi_0]_K) = \phi_K([\xi_1]_K) \in \mathrm{H}^1(K,H_0)$. The kernel of the map $\mathrm{H}^1(A,H_0) \to \mathrm{H}^1(K,H_0)$ is trivial by Nisnevich's theorem [Ni]. Thus $\phi_A([\xi]_1) = * = \phi_A([\xi]_0) \in \mathrm{H}^1(A,H_0)$. Whence $[\xi]_1 = [\xi]_0 \in \mathrm{H}^1(A,H)$. □

Now we go back to the proof of Lemma 3. Consider a height 1 prime ideal $\mathfrak{p}$ in R. Since $[\xi]$ is R-unramified there exists its lift up to an element $[\tilde{\xi}]$ in $\mathrm{H}^1(R_{\mathfrak{p}},G)$.
The map

$$\mathrm{H}^1(R_{\mathfrak{p}},G') \to \mathrm{H}^1(K,G')$$

is injective by Lemma 4. But

$$(\pi_{\mathfrak{p}}([\tilde{\xi}]))_K = \pi_K([\xi]) = *,$$

so $\pi_{\mathfrak{p}}[\tilde{\xi}] = *$. Therefore there exists a unique class $a_{\mathfrak{p}} \in \mathcal{F}(R_{\mathfrak{p}})$ such that $\delta(a_{\mathfrak{p}}) = [\tilde{\xi}] \in \mathrm{H}^1(R_{\mathfrak{p}}, G)$. So, $\delta_K(a_{\mathfrak{p},K}) = [\xi] \in \mathrm{H}^1(K, G)$ and finally $a = a_{\mathfrak{p},K}$. Lemma 3 is proven and Theorem 2 is proven as well. □

3. Purity of some H^1 functors, continued

Theorem 3. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{PGL}_n)$ satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{PGL}_n)$ be an R-unramified element. Let $\delta\colon \mathrm{H}^1(-, \mathrm{PGL}_n) \to \mathrm{H}^2(-, \mathbb{G}_m)$ be the boundary map corresponding to the short exact sequence of étale group sheaves

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

Let D_ξ be a central simple K-algebra of degree n corresponding ξ. If $D_\xi \cong M_l(D')$ for a skew-field D', then there exists $[\xi'] \in \mathrm{H}^1(K, \mathrm{PGL}_{n'})$ such that $D' = D_{\xi'}$. Then $\delta([\xi']) = [D'] = [D] = \delta(\xi)$. Replacing ξ by ξ', we may assume that $D := D_\xi$ is a central skew-field over K of degree n and the class $[D]$ is R-unramified. Since the functor $\mathrm{H}^2(-, \mathbb{G}_m)$ satisfies purity for R, there exists an Azumaya R-algebra A and an integer d such that $A_K = M_d(D)$.

There exists a projective left A-module P of finite rank such that each projective left A-module Q of finite rank is isomorphic to the left A-module P^m for an appropriative integer m (see [DeM, Cor.2]). In particular, two projective left A-modules of finite rank are isomorphic if they have the same rank as R-modules. One has an isomorphism $A \cong P^s$ of left A-modules for an integer s. Thus one has R-algebra isomorphisms $A \cong \mathrm{End}_A(P^s) \cong \mathrm{M}_s(\mathrm{End}_A(P))$. Set $B = \mathrm{End}_A(P)$. Observe, that $B_K = \mathrm{End}_{A_K}(P_K)$, since P is a finitely generated projective left A-module.

The class $[P_K]$ is a free generator of the group $K_0(A_K) = K_0(M_d(D)) \cong \mathbb{Z}$, since $[P]$ is a free generator of the group $K_0(A)$ and $K_0(A) = K_0(A_K)$. The P_K is a simple A_K-module, since $[P_K]$ is a free generator of $K_0(A_K)$. Thus $\mathrm{End}_{A_K}(P_K) = B_K$ is a skew-field.

We claim that the K-algebras B_K and D are isomorphic. In fact, $A_K = M_r(B_K)$ for an integer r, since P_K is a simple A_K-module. From the other side $A_K = M_d(D)$. As D, so B_K are skew-fields. Thus $r = d$ and D is isomorphic to B_K as K-algebras.

We claim further that B is an Azumaya R-algebra. That claim is local with respect to the étale topology on $\mathrm{Spec}(R)$. Thus it suffices to check the claim assuming that $\mathrm{Spec}(R)$ is strictly henselian local ring. In that case $A = M_l(R)$ and $P = (R^l)^m$ as an $M_l(R)$-module. Thus $B = \mathrm{End}_A(P) = M_m(R)$, which proves the claim.

Since B_K is isomorphic to D, one has $m = n$. So, B is an Azumaya R-algebra, and the K-algebra B_K is isomorphic to D. Let $[\zeta] \in \mathrm{H}^1(R, \mathrm{PGL}_n)$ be class corresponding to B. Then $[\zeta]_K = [\xi]$, since $\delta([\zeta])_K = [B_K] = [D] = \delta([\xi]) \in \mathrm{H}^2(K, \mathbb{G}_m)$. □

We denote by Sim_n the group of similitudes of a *split* quadratic form of rank n and by Sim_n^+ its connected component. Recall that $\mathrm{H}^1(-, \mathrm{Sim}_n)$ classifies similarity classes of nondegenerate quadratic forms of rank n (see [KMRT, (29.15)]).

THEOREM 4. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{Sim}_n)$ satisfies purity for R.*

Proof. Let $[\xi] \in \mathrm{H}^1(K, \mathrm{Sim}_n)$ be an R-unramified element. Let φ be a quadratic form over K whose similarity class represents $[\xi]$. Diagonalizing φ we may assume that $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ for certain non-zero elements $f_1, f_2, \dots, f_n \in K$. For each i write f_i in the form $f_i = \frac{g_i}{h_i}$ with $g_i, h_i \in R$ and $h_i \neq 0$.
There are only finitely many height one prime ideals $\mathfrak{q}$ in R such that there exists $0 \leq i \leq n$ with f_i not in $R_\mathfrak{q}$. Let $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$ be all height one prime ideals in R with that property and let $\mathfrak{q}_i \neq \mathfrak{q}_j$ for $i \neq j$.
For all other height one prime ideals $\mathfrak{p}$ in R each f_i belongs to the group of units $R_\mathfrak{p}^\times$ of the ring $R_\mathfrak{p}$.
If $\mathfrak{p}$ is a height one prime ideal of R which is not from the list $\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_s$, then $\varphi = \sum_{i=1}^n f_i \cdot t_i^2$ may be regarded as a quadratic space over $R_\mathfrak{p}$. We will write ${}_\mathfrak{p}\varphi$ for that quadratic space over $R_\mathfrak{p}$. Clearly, one has $({}_\mathfrak{p}\varphi) \otimes_{R_\mathfrak{p}} K = \varphi$ as quadratic spaces over K.
For each $j \in \{1, 2, \dots, s\}$ choose and fix a quadratic space ${}_j\varphi$ over $R_{\mathfrak{q}_j}$ and a non-zero element $\lambda_j \in K$ such that the quadratic spaces $({}_j\varphi) \otimes_{R_{\mathfrak{q}_j}} K$ and $\lambda_j \cdot \varphi$ are isomorphic over K. The ring R is factorial since it is regular and local. Thus for each $j \in \{1, 2, \dots, s\}$ we may choose an element $\pi_j \in R$ such that firstly π_j generates the only maximal ideal in $R_{\mathfrak{q}_j}$ and secondly π_j is an invertible element in $R_\mathfrak{n}$ for each height one prime ideal $\mathfrak{n}$ different from the ideal $\mathfrak{q}_j$.
Let $v_j \colon K^\times \to \mathbb{Z}$ be the discrete valuation of K corresponding to the prime ideal $\mathfrak{q}_j$. Set $\lambda = \prod_{i=1}^s \pi_j^{v_j(\lambda_j)}$ and

$$\varphi_{new} = \lambda \cdot \varphi.$$

Claim. The quadratic space φ_{new} is R-unramified. In fact, if a height one prime ideal $\mathfrak{p}$ is different from each of $\mathfrak{q}_j$'s, then $v_\mathfrak{p}(\lambda) = 0$. Thus, $\lambda \in R_\mathfrak{p}^\times$. In that case $\lambda \cdot ({}_\mathfrak{p}\varphi)$ is a quadratic space over $R_\mathfrak{p}$ and moreover one have isomorphisms of quadratic spaces $(\lambda \cdot ({}_\mathfrak{p}\varphi)) \otimes_{R_\mathfrak{p}} K = \lambda \cdot \varphi = \varphi_{new}$. If we take one of $\mathfrak{q}_j$'s, then $\frac{\lambda}{\lambda_j} \in R_{\mathfrak{q}_j}^\times$. Thus, $\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi)$ is a quadratic space over $R_{\mathfrak{q}_j}$. Moreover, one has

$$\frac{\lambda}{\lambda_j} \cdot ({}_j\varphi) \otimes_{R_\mathfrak{q}} K = \frac{\lambda}{\lambda_j} \cdot \lambda_j \cdot \varphi = \varphi_{new}.$$

The Claim is proven.
By [PP, Corollary 3.1] there exists a quadratic space $\tilde{\varphi}$ over R such that the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ_{new} are isomorphic over K. This shows that the

similarity classes of the quadratic spaces $\tilde{\varphi} \otimes_R K$ and φ coincide. The theorem is proven. □

Theorem 5. *Let R be such as in Theorem 1. The functor $\mathrm{H}^1(-, \mathrm{Sim}_n^+)$ satisfies purity for R.*

Proof. Consider an element $[\xi] \in \mathrm{H}^1(K, \mathrm{Sim}_n^+)$ such that for any $\mathfrak{p}$ of height 1 $[\xi]$ comes from $[\xi_\mathfrak{p}] \in \mathrm{H}^1(R_\mathfrak{p}, \mathrm{Sim}_n^+)$. Then the image of $[\xi]$ in $\mathrm{H}^1(K, \mathrm{Sim}_n)$ by Theorem 4 comes from some $[\zeta] \in \mathrm{H}^1(R, \mathrm{Sim}_n)$. We have a short exact sequence

$$1 \to \mathrm{Sim}_n^+ \to \mathrm{Sim}_n \to \mu_2 \to 1,$$

and $R^\times/(R^\times)^2$ injects into $K^\times/(K^\times)^2$. Thus the element $[\zeta]$ comes actually from some $[\zeta'] \in \mathrm{H}^1(R, \mathrm{Sim}_n^+)$. It remains to show that the map

$$\mathrm{H}^1(K, \mathrm{Sim}_n^+) \to \mathrm{H}^1(K, \mathrm{Sim}_n)$$

is injective, or, by twisting, that the map

$$\mathrm{H}^1(K, \mathrm{Sim}^+(q)) \to \mathrm{H}^1(K, \mathrm{Sim}(q))$$

has trivial kernel. The latter follows from the fact that the map

$$\mathrm{Sim}(q)(K) \to \mu_2(K)$$

is surjective (indeed, any reflection goes to $-1 \in \mu_2(K)$). □

4. Proof Theorem 1

Till the end of the proof of Lemma 9 we suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. Let $[\xi]$ be a class from $\mathrm{H}^1(R, G)$, and $X = (G/P)_\xi$ be the corresponding homogeneous space. Denote by L a Levi subgroup of P.

Lemma 5. *Consider a parabolic subgroup P_1 in PGO_n^+, which is the stabilizer of an isotropic line. A Levi subgroup of P_1 is isomorphic to Sim_{n-2}^+.*

Proof. Is is clear from the matrix representation that a Levi subgroup of a parabolic subgroup P_1 in O_n^+ is isomorphic to $\mathrm{O}_{n-2}^+ \times \mathbb{G}_m$. Now the homomorphism

$$\mathrm{O}_{n-2}^+ \times \mathbb{G}_m \to \mathrm{Sim}_{n-2}^+$$

induced by the natural inclusions is surjective in the sense of groups schemes, and its kernel is μ_2. The claim follows. □

Recall that a subset Ψ of a root system Φ is called *closed* if for any $\alpha, \beta \in \Psi$ such that $\alpha + \beta \in \Phi$ we have $\alpha + \beta \in \Psi$.

Lemma 6. *Let L modulo its center be isomorphic to PGO_{2m}^+ (resp., PGO_{2m+1}^+ or $\mathrm{PGO}_{2m}^+ \times \mathrm{PGL}_2$). Denote by Φ the root system of G with respect to T, and by Ψ the root system of L with respect to T, where T is a maximal split torus in L. Assume that there is a root $\lambda \in \Phi$ such that the smallest closed set of roots Ψ' containing Ψ and $\pm\lambda$ is a root subsystem of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$), and Ψ is the standard subsystem of type D_m (resp.*

B_m or $D_m + A_1$) therein. Then there is a surjective map $L \to \mathrm{Sim}^+_{2m}$ (resp., $L \to \mathrm{Sim}^+_{2m+1}$ or $L \to \mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) whose kernel is a central closed subgroup scheme in L. In particular, the functor $\mathrm{H}^1(-, L)$ satisfies purity for R.

Proof. Consider the subgroup $H_{\Psi'}$ of G corresponding to Ψ' in the sense of [SGA, Exp. XXII, Definition 5.4.2]. Then $H_{\Psi'}$ is split reductive of type D_{m+1} (resp. B_{m+1} or $D_{m+1} + A_1$) by [SGA, Exp. XXII, Proposition 5.10.1], so it maps onto the split adjoint group of the same type. Under this map L maps onto a Levi subgroup of a parabolic subgroup P_1, which is isomorphic to Sim^+_{2m} (resp. Sim^+_{2m+1} or $\mathrm{Sim}^+_{2m} \times \mathrm{PGL}_2$) by Lemma 5. The purity claim follows from Theorem 5, Theorem 3 and Theorem 2. □

Lemma 7. *For any semi-local R-algebra S the map*

$$\mathrm{H}^1(S, L) \to \mathrm{H}^1(S, G)$$

is injective. Moreover, $X(S) \neq \emptyset$ if and only if $[\xi]_S$ comes from $\mathrm{H}^1(S, L)$.

Proof. See [SGA, Exp. XXVI, Cor. 5.10]. □

Lemma 8. *Assume that the functor $\mathrm{H}^1(-, L)$ satisfies purity for R. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. By Lemma 7 $[\xi]_K$ comes from some $[\zeta] \in \mathrm{H}^1(K, L)$, which is uniquely determined. Since X is smooth projective, for any prime ideal $\mathfrak{p}$ of height 1 we have $X(R_\mathfrak{p}) \neq \emptyset$. By Lemma 7 $\xi_{R_\mathfrak{p}}$ comes from some $[\zeta_\mathfrak{p}] \in \mathrm{H}^1(R_\mathfrak{p}, L)$. Now $[\zeta_\mathfrak{p}]_K = [\zeta]$, and so by the purity assumption there is $[\zeta'] \in \mathrm{H}^1(R, L)$ such that $[\zeta']_K = [\zeta]$.
Set $[\xi']$ to be the image of ζ' in $\mathrm{H}^1(R, G)$. We claim that $[\xi'] = [\xi]$. Indeed, by the construction $[\xi']_K = [\xi]_K$. It remains to recall that the map $\mathrm{H}^1(R, G) \to \mathrm{H}^1(K, G_K)$ is injective by [FP, Corollary of Theorem 1]. □

Lemma 9. *Let $Q \leq P$ be another parabolic subgroup, $Y = (G/Q)_\xi$. Assume that $X(K) \neq \emptyset$ implies $Y(K) \neq \emptyset$, and $Y(K) \neq \emptyset$ implies $Y(R) \neq \emptyset$. Then $X(K) \neq \emptyset$ implies $X(R) \neq \emptyset$.*

Proof. Indeed, there is a map $Y \to X$, so $Y(R) \neq \emptyset$ implies $X(R) \neq \emptyset$. □

Proof of Theorem 1. We first suppose that R is the local ring of a closed point on a smooth scheme over an infinite field. By Lemma 9 we may assume that P_K is a minimal parabolic subgroup of $(G_\xi)_K$. All possible types of such P_K are listed in [T, Table II]: the Dynkin diagram with circled vertices erased corresponds to the type of L. We show case by case that $\mathrm{H}^1(-, L)$ satisfies purity for R, hence we are in the situation of Lemma 8.
If $P = B$ is the Borel subgroup, obviously $\mathrm{H}^1(S, L) = \{*\}$ for any semi-local R-algebra S. In the case of index $E^9_{7,4}$ (resp. ${}^1E^{16}_{6,2}$) L modulo its center is isomorphic to $\mathrm{PGL}_2 \times \mathrm{PGL}_2 \times \mathrm{PGL}_2$ (resp. $\mathrm{PGL}_3 \times \mathrm{PGL}_3$), and we may apply Theorem 2 and Theorem 3. In the all other cases we provide an element $\lambda \in \mathrm{X}^*(T)$ such that the assumption of Lemma 6 holds ($\tilde{\alpha}$ stands for the maximal root, enumeration follows [B]). The indices $E^{78}_{7,1}$, $E^{133}_{8,1}$ and $E^{78}_{8,2}$ are

not in the list below since in those cases the L does not belong to one of the type D_m, B_m, $D_m \times A_1$. The index $E^{66}_{7,1}$ is not in the list below since in that case we need a weight λ which is not in the root lattice. So, the indices $E^{78}_{7,1}$, $E^{133}_{8,1}$, $E^{78}_{8,2}$ and $E^{66}_{7,1}$ are the exceptions in the statement of the Theorem.

Index	${}^1E^{28}_{6,2}$	$E^{48}_{7,1}$	$E^{31}_{7,2}$	$E^{28}_{7,3}$	$E^{91}_{8,1}$	$E^{66}_{8,2}$	$E^{28}_{8,4}$	$F^{21}_{4,1}$
λ	α_1	$-\tilde{\alpha}$	α_1	α_1	$-\tilde{\alpha}$	α_8	α_1	$-\tilde{\alpha}$

It remains to settle the case $P = P_1$ for $G = E^{sc}_7$. Denote by $\tilde{E}_7$ a Levi subgroup of a parabolic subgroup P_8 in E_8. Comparing the exact sequences

$$\mathrm{H}^1(R, E^{sc}_7) \to \mathrm{H}^1(R, E^{ad}_7) \to \mathrm{H}^2(R, \mu_2)$$

and

$$\mathrm{H}^1(R, \tilde{E}^{sc}_7) \to \mathrm{H}^1(R, E^{ad}_7) \to \mathrm{H}^2(R, \mathbb{G}_m)$$

and one sees that the image of $[\xi]$ in $\mathrm{H}^1(R, E^{ad}_7)$ comes from some $[\zeta] \in \mathrm{H}^1(R, \tilde{E}_7)$. Let $\tilde{P}_1$ denote the corresponding parabolic subgroup in $\tilde{E}_7$; then we have $(E^{sc}_7/P_1)_\xi \simeq (\tilde{E}_7/\tilde{P}_1)_\zeta$.

We claim that $\mathrm{H}^1(-, \tilde{L})$ satisfies purity for R, where $\tilde{L}$ is a Levi subgroup of $\tilde{P}_1$. Indeed, consider a Levi subgroup G' of a parabolic subgroup P_1 inside E_8; then G' has type D_7 and $\tilde{L}$ is a Levi subgroup of a parabolic subgroup P_1 in G'. The rest of the proof goes exactly the same way as in Lemma 6.

Now suppose that R is a regular local ring containing an infinite field k. We first prove a general lemma. Let k' be an infinite field, X be a k'-smooth irreducible affine variety, Denote by $k'[X]$ the ring of regular functions on X and by $k'(X)$ the field of rational functions on X. Let $\mathfrak{p}$ be prime ideal in $k'[X]$, and let $\mathcal{O}_\mathfrak{p}$ be the corresponding local ring.

Lemma 10. *Theorem 1 holds for the local ring $\mathcal{O}_\mathfrak{p}$.*

Proof. Choose a maximal ideal $\mathfrak{m} \subset k'[X]$ containing $\mathfrak{p}$. One has inclusions of k'-algebras $\mathcal{O}_\mathfrak{m} \subset \mathcal{O}_\mathfrak{p} \subset k'(X)$. We already proved Theorem 1 for the ring $\mathcal{O}_\mathfrak{m}$. Thus Theorem 1 holds for the ring $\mathcal{O}_\mathfrak{p}$. □

The rest of the proof of Theorem 1 follows the arguments from [FP, page 5], which we reproduce here. Namely, let $\mathfrak{m}$ be the maximal ideal of R. Let k' be the algebraic closure of the prime field of R in k. Note that k' is perfect. It follows from Popescu's theorem ([P, Sw]) that R is a filtered inductive limit of smooth k'-algebras R_α. Modifying the inductive system R_α if necessary, we can assume that each R_α is integral. There are an index α, a 1-cocycle $\xi_\alpha \in Z^1(R_\alpha, G)$, and an element $f_\alpha \in R_\alpha$ such that $\xi = \varphi_\alpha(\xi_\alpha)$, f is the image of f_α under the homomorphism $\phi_\alpha : R_\alpha \to R$, the homogeneous space $X_\alpha := (G/H)_{\xi_\alpha}$ over R_α has a section over $(R_\alpha)_{f_\alpha}$.

If the field k' is infinite, then set $\mathfrak{p} = \phi^{-1}_\alpha(\mathfrak{m})$. The homomorphism ϕ_α induces a homomorphism of local rings $(R_\alpha)_\mathfrak{p} \to R$. By Lemma 10 one has $X_\alpha(R_\alpha) \neq \emptyset$, whence $X(R) \neq \emptyset$.

If the field k' is finite, then k contains an element t transcendental over k'. Thus R contains the subfield $k'(t)$ of rational functions in the variable t. So, if $R'_\alpha := R_\alpha \otimes_{k'} k'(t)$, then ϕ_α can be decomposed as follows $R_\alpha \xrightarrow{i_\alpha} R_\alpha \otimes_{k'} k'(t) = R'_\alpha \xrightarrow{\psi_\alpha} R$. Let $\xi' = i_\alpha(\xi_\alpha)$, $f'_\alpha = f_\alpha \otimes 1 \in R'_\alpha$, then the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$.
Let $\mathfrak{q} = \psi_\alpha^{-1}(\mathfrak{m})$. The ring R'_α is a $k'(t)$-smooth algebra over the infinite field $k'(t)$, and the homogeneous space $X'_\alpha := (G/H)_{\xi'_\alpha}$ over R'_α has a section over $(R'_\alpha)_{f'_\alpha}$. By Lemma 10 one has $X'_\alpha((R'_\alpha)_\mathfrak{q}) \neq \emptyset$. The homomorphism ψ_α can be factored as $R'_\alpha \to (R'_\alpha)_\mathfrak{q} \to R$. Thus $X(R) \neq \emptyset$. □

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