

P_4 -Free Colorings and P_4 -Bipartite Graphs

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A vertex partition of a graph into disjoint subsets V_i s is said to be a P_4 -free coloring if each color class V_i induces a subgraph without a chordless path on four vertices (denoted by P_4). Examples of P_4 -free 2-colorable graphs (also called P_4 -bipartite graphs) include parity graphs and graphs with “few” P_4 s like P_4 -reducible and P_4 -sparse graphs. We prove that, given $k \geq 2$, P_4 -FREE k -COLORABILITY is NP-complete even for comparability graphs, and for P_5 -free graphs. We then discuss the recognition, perfection and the Strong Perfect Graph Conjecture (SPGC) for P_4 -bipartite graphs with special P_4 -structure. In particular, we show that the SPGC is true for P_4 -bipartite graphs with one P_3 -free color class meeting every P_4 at a midpoint.

Keywords: Perfect graph, the Strong Perfect Graph Conjecture, graph partition, cograph, NP-completeness

1 Introduction

A graph G is *perfect* if, for each induced subgraph H of G , the chromatic number of H is equal to the clique number of H . Claude Berge introduced perfect graphs and conjectured around 1960's that a graph is perfect if and only if it has no induced cycle of odd length at least five or the complement of such a cycle. Nowadays this conjecture is known as the Strong Perfect Graph Conjecture (SPGC) and is still open. We refer to [4] for more information on perfect graphs.

A measure of a graph's imperfection has been considered by Brown and Corneil [8] as follows. Given a graph G and a positive integer k , a map $\pi : V(G) \rightarrow \{1, \dots, k\}$ is a *perfect k -coloring* of G if the subgraphs induced by each color class $\pi^{-1}(i)$ is perfect. Thus, a graph is perfect if and only if it is perfect 1-colorable. Note also that, by the Perfect Graph Theorem [33], a graph G is perfect k -colorable if and only if its complement \overline{G} is perfect k -colorable. In this paper we consider a particular example of perfect colorings. Our discussion is motivated by the fact that the perfection of a graph depends only on the structure of its induced paths on four vertices (denoted by P_4); see [36]. In this sense, graphs with empty P_4 -structure (P_4 -free graphs) form a somewhat based graph class in discussing graph's perfection; they are indeed perfect by a result due to Seinsche [38] (see also Jung [31]).

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Now, we call a perfect k -coloring of a graph P_4 -free k -coloring if the subgraphs of that graph induced by the color classes are P_4 -free. Note that the P_4 is self-complementary, hence G is P_4 -free k -colorable if and only if \overline{G} is P_4 -free k -colorable. For general graphs, Brown [6] proved that P_4 -FREE k -COLORABILITY is NP-complete for $k \geq 3$, and in [1], Achlioptas proved a more general result implying the NP-completeness of P_4 -FREE k -COLORABILITY for $k \geq 2$. In the next section we shall prove that, for any integer $k \geq 2$, P_4 -FREE k -COLORABILITY is NP-complete even for (particular) perfect graphs, and for P_5 -free graphs. In Section 3 we shall give some examples of P_4 -free 2-colorable graphs, which we also call P_4 -bipartite graphs. Many well understood classes of perfect graphs consists of P_4 -bipartite graphs only. In Sections 4 and 5, perfect P_4 -bipartite graphs and the SPGC for P_4 -bipartite graphs with special P_4 -structure will be discussed.

The complement of a graph G is denoted by \overline{G} . Graphs having no induced subgraphs isomorphic to a given graph H are called H -free. If X is a set of vertices in G , $G[X]$ is the subgraph of G induced by X , and $N_G(X)$ is the *neighborhood* of X in G ; that is, the set of all vertices outside X adjacent to some vertex in X . If the context is clear, we simply write $N(X)$. The path on m vertices v_1, v_2, \dots, v_m with edges $v_i v_{i+1}$ ($1 \leq i < m$) is denoted by $P_m = v_1 v_2 \cdots v_m$. The vertices v_1 and v_m are the *endpoints* of that path, the other vertices are the *midpoints*. The cycle on m vertices v_1, v_2, \dots, v_m with edges $v_i v_{i+1}$ ($1 \leq i < m$) and $v_1 v_m$ is denoted by $C_m = v_1 v_2 \cdots v_m$. C_{2k+1} and $\overline{C_{2k+1}}$, $k \geq 2$, are also called *odd holes*, respectively, *odd antiholes*. Graphs without odd holes and odd antiholes are called *Berge graphs*.

2 NP-completeness results

We now consider the following problem for fixed positive integer k .

P_4 -FREE k -COLORABILITY *Is a given graph P_4 -free k -colorable?*

We show in this section that, for fixed $k \geq 2$, P_4 -FREE k -COLORABILITY is NP-complete for perfect graphs. Notice that P_4 -free 1-colorability (that is, recognizing P_4 -free graphs) is solvable in linear time [14]. We shall reduce the following NP-complete problem ([37], see also [16]) to P_4 -FREE k -COLORABILITY.

NOT-ALL-EQUAL 3SAT *Given a collection C of clauses over set V of Boolean variables such that each clause has exactly three literals. Is there a truth assignment for V such that each clause in C has at least one true literal and at least one false literal?*

A comparability graph G is one which admits a transitive orientation \vec{G} : If (x, y) and (y, z) are arcs of \vec{G} , then (x, z) is also an arc of \vec{G} . It is well known that comparability graphs are perfect. A typical example of comparability graphs are P_4 -free graphs, as proved by Jung [31].

Lemma 1 *Given a comparability graph G , it is NP-complete to decide whether G is P_4 -bipartite.*

Proof. The problem is clearly in NP. We shall reduce NOT-ALL-EQUAL 3SAT to our problem. Let $C = \{C_1, C_2, \dots, C_m\}$ be any set of clauses $C_i = (c_{i1}, c_{i2}, c_{i3})$ given as input for NOT-ALL-EQUAL 3SAT, where the literals c_{ik} ($1 \leq i \leq m, 1 \leq k \leq 3$) are taken from the set of variables V . We shall construct a comparability graph G which has a partition into two P_4 -free graphs if and only if C is satisfiable. For convenience, we call a vertex partition of a graph into two P_4 -free graphs a *good partition* of that graph. For each variable $v \in V$ let $G(v, \bar{v})$ be the graph shown in Figure 1 (left).

Observation 1 *$G(v, \bar{v})$ has a good partition. Every good partition of $G(v, \bar{v})$ must contain the labelled vertex v in one part and the labelled vertex \bar{v} in the other part. \diamond*

For each clause C_i , let $G(C_i)$ be the graph shown in Figure 1 (right).

Observation 2 $G(C_i)$ has a good partition. Every good partition of $G(C_i)$ must contain two of the labelled vertices c_{i1}, c_{i2}, c_{i3} in one part and the other labelled vertex in the other part. Moreover, every partition of $\{c_{i1}, c_{i2}, c_{i3}\}$ into two non-empty subsets can be extended to a good partition of $G(C_i)$. \diamond

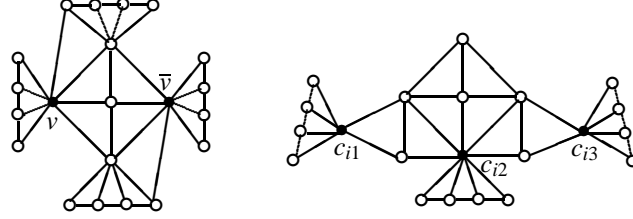


Fig. 1: The graphs $G(v, \bar{v})$ (left) and $G(C_i)$ (right)

The proofs of the observations will follow by inspection, hence are omitted. We now create the graph $G = G(C)$ from the graphs $G(v, \bar{v})$ ($v \in V$) and the graphs $G(C_i)$ ($1 \leq i \leq m$) as follows: For each $v \in V$ and each $1 \leq i \leq m$, connect the vertex $x \in \{v, \bar{v}\}$ in $G(v, \bar{v})$ with the vertex c_{ik} in $G(C_i)$ by an edge if, and only if, x is the literal c_{ik} in the clause C_i . Thus, in G , every c_{ik} ($1 \leq k \leq 3$) has exactly one neighbor outside $G(C_i)$ which is one of the labelled vertices v, \bar{v} in a graph $G(v, \bar{v})$ (with $c_{ik} \in \{v, \bar{v}\}$ in the given NOT-ALL-EQUAL 3SAT instance).

Suppose that G has a good partition into two P_4 -free graphs A and B . Then it is easy to see that, for all $v \in V$, if $x \in \{v, \bar{v}\}$ is adjacent to c_{ik} , then x and c_{ik} are in different parts A, B . We define a truth assignment for NOT-ALL-EQUAL 3SAT as follows:

v is true if and only if the labelled vertex v in $G(v, \bar{v})$ belongs to A .

By Observation 1, this assignment is well-defined. By Observation 2, it is clear that each clause C_i has at least one but not all true literals.

Conversely, suppose that there is a truth assignment satisfying NOT-ALL-EQUAL 3SAT. Then let $A(v, \bar{v}), B(v, \bar{v})$ be a good partition of $G(v, \bar{v})$ such that $A(v, \bar{v})$ contains the true vertex in $\{v, \bar{v}\}$ and $B(v, \bar{v})$ contains the false vertex of them. Such a good partition exists by Observation 1. Let A_i, B_i be a good partition of $G(C_i)$ such that A_i contains the false literals vertices in $\{c_{i1}, c_{i2}, c_{i3}\}$ and B_i contains the true vertices of them. Such a good partition exists by Observation 2, and the fact that every C_i has at least one but not all true literals. Set

$$A = \bigcup_{v \in V} A(v, \bar{v}) \cup \bigcup_{1 \leq i \leq m} A_i, \quad B = \bigcup_{v \in V} B(v, \bar{v}) \cup \bigcup_{1 \leq i \leq m} B_i.$$

Clearly, $V(G) = A \cup B$. Now, each $A(v, \bar{v})$ and each A_i is a P_4 -free graph, and no edge exists between two parts of the $A(v, \bar{v})$'s and A_i 's, hence A is a P_4 -free subgraph of G . similarly, B is P_4 -free. Thus, G is P_4 -bipartite.

To complete the proof, note that each $G(v, \bar{v})$ and each $G(C_i)$ admits a transitive orientation such that the labelled vertices v, \bar{v} are sinks and the labelled vertices c_{i1}, c_{i2}, c_{i3} are sources. To obtain a transitive orientation of G , direct the edges xy , $x \in \{v, \bar{v}\}$ and $y \in \{c_{i1}, c_{i2}, c_{i3}\}$ with $x = y$ in the given instance of NOT-ALL-EQUAL 3SAT, from y to x . \square

Theorem 1 *Given a comparability graph G and an integer $k \geq 2$, it is NP-complete to decide whether G is P_4 -free k -colorable.*

Proof. The case $k = 2$ is settled by Lemma 1. We shall make use of a construction for vertex-critical P_4 -free k -colorable graphs in [7] to reduce the case $k = 2$ to the case $k \geq 3$. Let H be a comparability graph, and let G be the graph obtained from an induced P_4 by substituting three (arbitrary) vertices by the graph H . Then G is clearly a comparability graph, and it can easily be seen that G is P_4 -free k -colorable if and only if H is P_4 -free $(k - 1)$ -colorable. \square

We shall remark that Brown [6] and Achlioptas [1] showed the NP-completeness of P_4 -FREE k -COLORABILITY for fixed $k \geq 3$ by reducing k -COLORABILITY to P_4 -FREE k -COLORABILITY. Since k -COLORABILITY can be decided in polynomial time when considering perfect graphs (see [17]), Brown's and Achlioptas's reduction cannot be used in proving NP-completeness of P_4 -FREE k -COLORABILITY for perfect graphs.

Since a graph is P_4 -free k -colorable if and only if its complement is, P_4 -FREE k -COLORABILITY is NP-complete for cocomparability graphs as well. Graphs which are both comparability graphs and cocomparability graphs are called *permutation graphs*. We do not know the complexity of P_4 -FREE COLORABILITY on permutation graphs.

Problem 1 *Find a polynomial time algorithm for solving P_4 -FREE k -COLORABILITY on permutation graphs, or prove that the problem is NP-complete for the class of permutation graphs.*

Notice that, using the construction mentioned in the proof of Theorem 1, one can show that for every fixed $k \geq 1$ there are P_4 -free k -colorable permutation graphs which are not P_4 -free $(k - 1)$ -colorable.

We now are going to show that P_4 -FREE k -COLORABILITY is NP-complete for (C_4, C_5) -free graphs. As a consequence, P_4 -FREE k -COLORABILITY is also NP-complete for P_5 -free graphs. This is best possible in the sense that the problem is trivial for P_4 -free graphs.

Lemma 2 *Given a (C_4, C_5) -free graph G , it is NP-complete to decide whether G is P_4 -bipartite.*

Proof. We shall reduce NOT-ALL-EQUAL 4SAT to our problem (the NP-completeness of NOT-ALL-EQUAL 4SAT follows easily from that of NOT-ALL-EQUAL 3SAT). Let $C = \{C_1, C_2, \dots, C_m\}$ be any set of clauses $C_i = (c_{i1}, c_{i2}, c_{i3}, c_{i4})$ given as input for NOT-ALL-EQUAL 4SAT, where the literals c_{ik} ($1 \leq i \leq m, 1 \leq k \leq 4$) are taken from the set of variables V . We may assume that,

$$\text{for every } v \in V, \text{ no clause } C_i \text{ contains both } v \text{ and } \bar{v}. \quad (1)$$

We now construct a (C_4, C_5) -free graph G which has a partition into two P_4 -free graphs if and only if C is satisfiable. For each variable $v \in V$ let $G(v, \bar{v})$ be the graph shown in Figure 2 (left). For each clause C_i , let $G(C_i)$ be the P_4 shown in Figure 2 (right). We create the graph $G = G(C)$ from the graphs $G(v, \bar{v})$ ($v \in V$) and the graphs $G(C_i)$ ($1 \leq i \leq m$) as follows: For each $v \in V$ and each $1 \leq i \leq m$, connect the vertex $x \in \{v, \bar{v}\}$ in $G(v, \bar{v})$ with the vertex c_{ik} in $G(C_i)$ by an edge if, and only if, x is the literal c_{ik} in the clause C_i . Clearly, the construction and assumption (1) guarantee that G cannot contain an induced C_4 or C_5 .

Now, we can show, similar to Lemma 1, that G is P_4 -bipartite if and only if C is satisfiable. \square

Theorem 2 *Given a (C_4, C_5) -free graph G and an integer $k \geq 2$, it is NP-complete to decide whether G is P_4 -free k -colorable.*

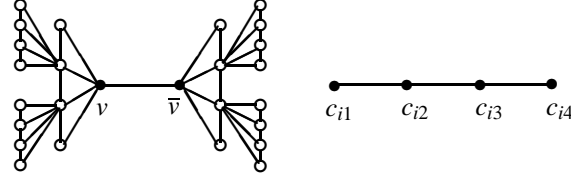


Fig. 2: The graphs $G(v, \bar{v})$ (left) and $G(C_i)$ (right)

Proof. The case $k = 2$ is settled by Lemma 2. Let $k \geq 3$. Let H be a (C_4, C_5) -free graph. Construct a graph G as follows: Take $k + 2$ disjoint copies G_1, \dots, G_{k+2} of H and $k + 2$ new vertices v_1, \dots, v_{k+2} , and connect every pair v_i, v_j ($1 \leq i \neq j \leq k + 2$) by an edge and connect every vertex in G_i with v_i ($1 \leq i \leq k + 2$) by an edge. Clearly, G is also (C_4, C_5) -free.

Suppose that H is P_4 -free k -colorable. Then G is P_4 -free $(k + 1)$ -colorable by coloring the vertices v_i 's with one new color.

Suppose, conversely, that G is P_4 -free $(k + 1)$ -colorable. Then H is P_4 -free k -colorable. If not, consider two distinct vertices $v_i, v_j \in \{v_1, \dots, v_{k+2}\}$ with the same color c in a P_4 -free $(k + 1)$ -coloring of G . Since H is not P_4 -free k -colorable, the color c must appear in every copy of H . Say, for some $i \neq j$, $x \in G_i$ and $y \in G_j$ are colored by c . But then xv_iv_jy is a monochromatic P_4 in G , a contradiction. Thus, H must be P_4 -free k -colorable, as claimed. \square

Since C_4 -free graphs are $\overline{P_5}$ -free, Theorem 2 implies that P_4 -FREE k -COLORABILITY is NP-complete for $\overline{P_5}$ -free graphs, and, by considering complementation, for P_5 -free graphs as well. This is best possible in the sense that P_4 -FREE k -COLORABILITY is trivial for P_4 -free graphs.

Also, Theorem 2 implies that P_4 -FREE k -COLORABILITY is NP-complete for $(C_5, \overline{C_4})$ -free graphs as well. Notice that graphs which are both (C_5, C_4) -free and $(C_5, \overline{C_4})$ -free, i.e., split graphs, are P_4 -free 2-colorable.

3 Examples of P_4 -bipartite graphs

P_4 -bipartite graphs generalize in a very natural way the well understood bipartite graphs, split graphs and cographs. Below we are going to list other well structured (perfect) graph classes that contain P_4 -bipartite graphs only. See [5] for a survey on these and related graph classes.

PROPER INTERVAL GRAPHS. Interval graphs without induced $K_{1,3}$ are called proper interval graphs. In [2], it was shown that every proper interval graph can be partitioned into two P_3 -free subgraphs. In particular, proper interval graphs are P_4 -bipartite. Notice that, for every k , there exists an interval graph that is P_4 -free k -colorable, but not P_4 -free $(k - 1)$ -colorable.

DISTANCE-HEREDITARY AND PARITY GRAPHS. Distance-hereditary graphs are those graphs in which for all vertices u, v , all induced paths connecting u and v have equal length [24]. In [9], Burlet and Uhry introduced the bigger class of parity graphs; these graphs are defined by the condition that all induced paths connecting u and v have equal parity. Let G be a parity graph, and let v be a vertex in G . In [9, Lemma 4] (see also [35]) it was shown that, for each i , the set $N^i(v)$ of vertices at distance exactly i from v induces a P_4 -free subgraph in G . Thus, $\bigcup N^{2i}(v)$ and $\bigcup N^{2i+1}(v)$ is a P_4 -free bipartition of G . We thank Stephan Olariu and Luitpold Babel for their hint to this fact on parity graphs.

In order to give other well known classes that consist of P_4 -bipartite graphs only we need the term of p -connectedness introduced by Jamison and Olariu [30]. A graph is called p -connected if, for every partition of its vertex set into two nonempty, disjoint subsets, there is an induced P_4 with vertices in both parts. A p -component of a graph is a maximal p -connected subgraph of that graph. Clearly, a graph is a P_4 -bipartite graph if and only if each of its p -components is a P_4 -bipartite graph.

P_4 -REDUCIBLE AND P_4 -SPARSE GRAPHS. P_4 -reducible graphs are those graphs in which each vertex belongs to at most one induced P_4 [26]. In [20], Hoàng introduced the bigger class of P_4 -sparse graphs; these are defined by the condition that each set of at most five vertices induces at most one P_4 . It was shown in [29] that every p -component of a P_4 -sparse graph is a split graph. Since split graphs are P_4 -bipartite, all P_4 -sparse graphs are P_4 -bipartite.

P_4 -EXTENDIBLE AND P_4 -LITE GRAPHS. P_4 -extendible graphs [28] are those graphs in which each p -component has at most five vertices. P_4 -lite graphs [27] are those graphs in which every induced subgraph with at most six vertices either has at most two P_4 s or is a (special) split graph. It was shown in [3] that every p -component of a P_4 -lite graph is a split graph or has at most six vertices. Notice that all graphs with at most six vertices are P_4 -bipartite, hence P_4 -lite and P_4 -extendible graphs are P_4 -bipartite.

COGRAPH CONTRACTIONS. In [25] Hujter and Tuza introduced the graphs called *cograph contractions*. These are graphs obtained from a cograph by contracting some pairwise disjoint stable sets and then making the ‘contracted vertices’ pairwise adjacent. It was shown in [32] that a graph is a cograph contraction if and only if it admits a clique meeting each P_4 in a midpoint and each \overline{P}_5 in both endpoints of the P_5 . In particular, cograph contractions are P_4 -bipartite graphs.

Notice that the complements of the graphs mentioned above are also P_4 -bipartite graphs.

4 Which P_4 -bipartite graphs are perfect?

Let G be a graph whose vertices are colored red and white (each vertex receives only one color). A P_4 $abcd$ of G is said to be of type

- 1 (or RRRR) if a, b, c, d are red,
- 2 (or WRRR) if a is white and b, c, d are red,
- 3 (or RWRR) if a, c, d are red and b is white,
- 4 (or RRWW) if a, b are red and c, d are white,
- 5 (or RWRW) if a, c are red and b, d are white,
- 6 (or RWWR) if a, d are red and b, c are white,
- 7 (or WRRW) if a, d are white and b, c are red,
- 8 (or RWWW) if a is red and b, c, d are white,
- 9 (or WRWW) if a, c, d are white and b is red,
- 10 (or WWWW) if a, b, c, d are white.

Clearly, G is P_4 -bipartite if and only if its vertices can be colored red and white in such a way that no P_4 is of type 1 or 10. We also write $G = (R, W, E)$ for P_4 -bipartite graph $G = (V, E)$ with partition $V = R \cup W$ such that $G[R]$ and $G[W]$ are P_4 -free subgraphs in G .

For non-empty subset $S \subseteq \{2, 3, \dots, 9\}$, we call a graph G a S -graph if the vertices of G can be colored red and white such that every P_4 of G is of type $t \in S$. Thus S -graphs are P_4 -bipartite. Bipartite graphs (respectively, complements of bipartite graphs) are, for instance, $\{5\}$ -graphs (respectively, $\{4\}$ -graphs).

Many classes of perfect P_4 -bipartite graphs have been described in terms of types of P_4 s. In [21], Hoàng proved that “odd P_4 -bipartite graphs” are perfect; here the P_4 -bipartite graph $G = (R, W, E)$ is *odd* if every P_4 of G has odd number of vertices in R (hence in W). Thus, odd P_4 -bipartite graphs are exactly the $\{2, 3, 8, 9\}$ -graphs. Chvátal, Lenhart and Sbihi [13, Theorem 2], and independently Gurvich [19] extended odd P_4 -bipartite graphs to a larger class of perfect P_4 -bipartite graphs; they proved that all $\{2, 3, 4, 5, 8, 9\}$ -graphs are perfect. These results and more related results in [12, 13] motivate the following question:

What are the maximal subsets $S \subset \{2, 3, \dots, 9\}$ with the property that all S -graphs are perfect?

We shall point out that the complete answer to this question already follows by the results in [12, 13].

Theorem 3 *Let S be a maximal subset of $\{2, 3, \dots, 9\}$ such that all S -graphs are perfect. Then S is exactly one of the following sets: $S_1 = \{4, 5, 6, 7\}$, $S_2 = \{2, 3, 4, 5, 8, 9\}$, $S_3 = \{3, 4, 5, 6, 8\}$, and $S_4 = \{2, 4, 5, 7, 9\}$.*

Proof. First, color the odd hole C_9 in the way RRWRRWRRW. Then every P_4 of this C_9 is of type 3 or 7, and every P_4 of the complement of this C_9 is of type 2 or 6. Second, color the odd hole C_9 in the way WWRWWRWR. Then every P_4 of this C_9 is of type 6 or 9, and every P_4 of the complement of this C_9 is of type 7 or 8. Therefore, as odd holes and odd antiholes are imperfect,

none of $\{3, 7\}$, $\{2, 6\}$, $\{6, 9\}$ and $\{7, 8\}$ is a subset of S .

Now, it is straightforward to show that S must be contained in one of the sets S_1, S_2, S_3 , or S_4 .

Finally, all S_1 -graphs are perfect [12], all S_2 -graphs are perfect [13, Theorem 2] (see also [19]), all S_3 -graphs and all S_4 -graphs are perfect [13, Theorem 6]. □

We now turn to the recognition problem for P_4 -bipartite graphs addressed in Theorem 3. Given a graph G , we consider the system of linear equations

$$w + x + y + z = 2 \quad (w, x, y, z \text{ induce a } P_4 \text{ in } G).$$

It is easy to see the G is a S_1 -graph if and only if this system of linear equations has a 0/1-solution. Thus, S_1 -graphs can be recognized in polynomial time. Also, S_3 -graphs can be recognized in polynomial time; the task reduces to the 2SAT problem as follows.

For each P_4 $wxyz$ in G , let $(x \vee y) \wedge (\bar{w} \vee \bar{z})$ be a Boolean formula.

The 2SAT formula for G is the product of such all formulas corresponding to the P_4 s in G . Now, if G is a S_3 -graph with a P_4 -free coloring $V(G) = R \cup W$, then the truth assignment $v := \text{true} \Leftrightarrow v \in W$ satisfies our 2SAT formula. If, conversely, our 2SAT formula is satisfied, then $W := \{v : v \text{ is true}\}$, $R := \{v : v \text{ is false}\}$ is a P_4 -free 2-coloring of G such that every P_4 of G is of type $t \in S_3$. Since a graph is a S_4 -graph if and only if its complement is a S_3 -graph, S_4 -graphs can be recognized in polynomial time, too.

The recognition problem of S_2 -graphs remains open; see also [10].

Problem 2 Given a graph G . Can you find in polynomial time a P_4 -free 2-coloring of G such that every P_4 of G is of type $t \in S_2$, or prove that such a coloring does not exist?

We remark that it can be shown that Problem 2 is NP-complete if S_2 is replaced by $S_2 \cup \{6\}$, or replaced by $S_2 \cup \{7\}$.

5 P_4 -bipartite graphs and the SPGC

The results in [21, 13, 19] mentioned in previous section will be implied by the truth of the following

Conjecture 1 *The SPGC is true for P_4 -bipartite graphs.*

Conjecture 1 has been proved for some particular cases. The following theorem is a consequence of previously known results (see also [23]). It proves Conjecture 1 for P_4 -free graphs with one color class being a stable set or a clique.

Theorem 4 *Let G have a stable set (or a clique) T such that T meets every P_4 of G . If G has no odd hole (respectively, no odd antihole), then G is perfect. \square*

Also, in [23], Conjecture 1 is proved for P_4 -bipartite graphs with one color class inducing a $(P_4, C_4, \overline{C_4})$ -free graph and meeting every P_4 in certain way as follows:

Theorem 5 *Let G have a subset $T \subseteq V(G)$ such that*

- (i) *T induces a threshold graph,*
- (ii) *T meets every P_4 in an endpoint, or meets every P_4 in a midpoint.*

If G is Berge, then G is perfect. \square

Theorem 4 suggests the following weaker conjecture for P_4 -bipartite graphs with one color class consisting of vertex-disjoint cliques.

Conjecture 2 *The SPGC is true for P_4 -bipartite graphs with one P_3 -free color class.*

The main result of this section is the following theorem which is related to Theorem 5 and proves Conjecture 2 for the case when the P_3 -free color class meets the P_4 s in a certain way.

Theorem 6 *Let G have a subset $T \subseteq V(G)$ such that*

- (i) *T induces a P_3 -free graph,*
- (ii) *T meets every P_4 in an endpoint, or meets every P_4 in a midpoint.*

If G is Berge, then G is perfect.

The proof of Theorem 6 relies on several known results on P_4 -free graphs and minimal imperfect graphs. First, Seinsche [38] proved that

$$\text{a } P_4\text{-free graph or its complement is disconnected.} \quad (2)$$

Two vertices x, y are *twins* if, for all other vertices z , z is adjacent to x if and only if z is adjacent to y . The next property of P_4 -free graphs is well known and can be derived from (2).

$$\text{Every } P_4\text{-free graph with at least two vertices has a pair of twins.} \quad (3)$$

A graph is *minimal imperfect* if it is not perfect but each of its proper induced subgraphs is. The well known Perfect Graph Theorem due to Lovász implies that

the complement of a minimal imperfect graph is also minimal imperfect. (4)

Two (nonadjacent) vertices x and y form an *even-pair* if every induced path connecting x to y has even length. Meyniel [34] showed that

no minimal imperfect graph has an even-pair. (5)

In particular, no minimal imperfect graph has a *two-pair* which is a pair of vertices x, y such that every induced path connecting x to y has exactly two edges.

A cutset S of G is called a *star-cutset*, respectively, a *stable-cutset*, respectively, a *complete multipartite-cutset* if $G[S]$ has a universal vertex, respectively, has no edge, respectively, is a complete multipartite graph (a complete multipartite graph is one whose vertex set can be partitioned into stable sets S_1, \dots, S_m such that, for $i \neq j$, every vertex in S_i is adjacent to every vertex in S_j). Chvátal [11] showed that

no minimal imperfect graph has a star-cutset. (6)

In particular,

no minimal imperfect graph has a clique-cutset, (7)

and

in a minimal imperfect graph, no vertex dominates another vertex. (8)

Here, the vertex x *dominates* the vertex y if $N(y) \subseteq N(x) \cup x$. The next property of minimal imperfect graphs was found by Tucker [39] saying that

no minimal imperfect graph has a stable-cutset, unless it is an odd hole. (9)

Finally, Cornuéjols and Reed [15] showed that

no minimal imperfect graph has a complete multipartite-cutset. (10)

Proof of Theorem 6. Suppose that T meets every P_4 in an endpoint. Color the vertices in T with color red and vertices outside T with color white. Then G has only P_4 s of types 3, 4, 5, 6, or 8. In particular, G is an S_3 -graph, hence perfect (see Theorem 3).

Let us consider the case when T meets every P_4 in a midpoint, and assume that G is a minimal imperfect Berge graph. Further, we may assume that

$G - T$ is disconnected.

Otherwise, by (2), $\overline{G} - T$ is disconnected and so T would be a stable-cutset or a complete multipartite-cutset of \overline{G} , contradicting (4) and (9) or (10). In particular, by (7),

T consists of $m \geq 2$ vertex-disjoint cliques.

For convenience, we call a P_4 *bad* if its both midpoints are outside T . By our hypothesis, no P_4 in G is bad.

CASE 1. $G - T$ has two adjacent twins x, y .

In this case, we claim that

x, y form an even-pair in \overline{G} .

To see this, consider an induced path $P = xx_1 \cdots x_k y$, $k \geq 2$, in \overline{G} connecting x and y . As x, y are twins in $\overline{G} - T$, x_1 must belong to T . Furthermore,

P has no edge in $\overline{G}[T]$.

For if P has an edge in $\overline{G}[T]$, then, since $\overline{G}[T]$ is a complete multipartite graph and $x_1 \in T$, this edge must be $x_1 x_2$, and P is the P_4 $xx_1 x_2 y$. But then $x_1 y x x_2$ is a bad P_4 in G , a contradiction.

P has no edge in $\overline{G} - T$.

Otherwise, let i be minimal such that $x_i x_{i+1}$ is an edge in $\overline{G} - T$. Note that $i > 1$. Set $x_0 := x$. Then $x_{i-1} \in T$ and $x_{i-2} \in \overline{G} - T$. But then $x_{i-1} x_{i+1} x_{i-2} x_i$ is a bad P_4 in G , a contradiction.

Thus, P has even number of edges, as claimed. This contradicts (4) and (5), and Case 1 is settled.

CASE 2. $G - T$ has no adjacent twins.

Write $G[T] = C_1 \cup C_2 \cup \cdots \cup C_m$ with vertex-disjoint cliques C_1, C_2, \dots, C_m . Recall that $m \geq 2$.

Observation 3 For all cliques $C = C_i$, $1 \leq i \leq m$, and all component H of $G - T$, if $N(C) \cap H \neq \emptyset$, then $H \subseteq N(C)$.

Proof of Observation 3 Assume the contrary, and let H be a component of $G - T$ and let C be a clique of T such that $N(C) \cap H \neq \emptyset$ and $H - N(C) \neq \emptyset$. Let $x \in N(C) \cap H$ having a neighbor y in $H - N(C)$, and let $v \in C$ be a neighbor of x .

By (8), there exists a vertex z adjacent to y but not to x . $z \in N(C) \cap H$, otherwise $zyxv$ would be a bad P_4 . The same argument shows that x and z have the same neighbors in C . Moreover,

$$\text{for all } u \in T - C, \text{ if } u \text{ is adjacent to } y, \text{ then } u \text{ is adjacent to both } x \text{ and } z. \quad (11)$$

(Else $uyxv$ or $uyzv$ would be a bad P_4), and

$$\text{for all } u \in T, u \text{ is adjacent to } x \text{ if and only if } u \text{ is adjacent to } z. \quad (12)$$

This is clear for $u \in C$. Suppose $u \in T - C$ is adjacent to x but not to z , then (11) implies that u is nonadjacent to y and so $uxyz$ is a bad P_4 , a contradiction. The case when u is adjacent to z but not x can be settled in a similar manner. Thus, (12) holds.

We now show that x, z form a two-pair. Let $P = xx_1 x_2 \cdots x_k z$ be a chordless path connecting x and z , and assume that $k \geq 2$. By (12), $x_1 \in H$, hence x_1 is adjacent to y (because H is P_4 -free). x_2 also belongs to H , otherwise, by (11), x_2 and y are nonadjacent and, by (12), x_2 and z are nonadjacent. But then $x_2 x_1 y z$ is a bad P_4 .

Thus, $x_1, x_2 \in H$. But then $x_3 x_2 x_1 x$ (or $zx_2 x_1 x$ if $k = 2$) is a bad P_4 . This contradiction proves Observation 3. \diamond

By (9) and $m \geq 2$, $G - T$ has a nontrivial component H . By (3), H has twins x, y which are nonadjacent by the hypothesis in this case. Write

$$N = N_H(x) = N_H(y), R = H - N - \{x, y\}.$$

Since H is connected, N is nonempty.

Observation 4 For all vertices $v \in T$, if v is adjacent to x or y but not both, then v is adjacent to all vertices in N .

Proof of Observation 4 Otherwise, there would be a bad P_4 . \diamond

By (8), there exists a vertex x' adjacent to x but nonadjacent to y , and a vertex y' adjacent to y but nonadjacent to x . As x, y are twins in $G - T$, x' and y' belong to T .

Observation 5 Such vertices x' and y' can be chosen in different cliques C_i, C_j .

Proof of Observation 5 Assume that there are vertices a, b in a clique C of T such that a is adjacent to x but not to y , and b is adjacent to y but not to x . As C is not a clique-cutset of G (see (7)), some vertex of H has a neighbor in another clique $C' \neq C$ of T . By Observation 3, x has a neighbor $c \in C'$. c cannot be adjacent to y , otherwise $cxabc$ would be a C_5 , contradicting the minimality of G . Now, Observation 5 follows by setting $C_i = C', C_j = C, x' = c$, and $y' = b$. \diamond

From now on, let $x' \in C_i, y' \in C_j$ with $i \neq j$. By Observation 4, x' and y' are adjacent to all vertices in N .

Observation 6 For all $C \in \{C_1, C_2, \dots, C_m\}$, $C \neq C_i$ or $C \neq C_j$, and for all $z \in N$, $N_C(x) \subseteq N_C(z)$ and $N_C(y) \subseteq N_C(z)$.

Proof of Observation 6 If there is a vertex $v \in N_C(x) - N_C(z)$, then, by Observation 4, v must be adjacent to y . But then $vyzx'$ (if $C \neq C_i$) or $vxyz'$ (if $C \neq C_j$) is a bad P_4 . Thus, $N_C(x) \subseteq N_C(z)$. By symmetry, $N_C(y) \subseteq N_C(z)$. \diamond

Observation 7 N cannot have a vertex z^* that is adjacent to all vertices in $N - z^*$.

Proof of Observation 7 Such a vertex z^* would dominate x (contradicting (8)): If v is a neighbor of x in T , and $v \in C$ for a clique $C \in \{C_1, C_2, \dots, C_m\}$, then, as C_i and C_j are different cliques, $C \neq C_i$ or $C \neq C_j$, hence, by Observation 6, v must be adjacent to z^* . \diamond

By Observation 7, there exist two nonadjacent vertices z_1, z_2 in N . We are going to show that z_1, z_2 form a two-pair. This contradiction to (5) settles Case 2.

Consider an induced path $P = z_1 t_1 t_2 \cdots t_k z_2$ in G , and assume that $k \geq 2$. Then

t_1 must belong to N .

For, if $t_1 \in R$, then $t_1 z_1 x z_2$ would be a bad P_4 ; if $t_1 \in C$, say $C \neq C_j$, then $t_1 z_1 x z_2$ (if t_1 is not adjacent to x) or $t_1 x z_2 y'$ (if t_1 is adjacent to x) is a bad P_4 , a contradiction. The case $t_1 \in C_j$ is similar. Now,

t_2 must belong to T ,

otherwise, $z_1 t_1 t_2 t_3$ would be a bad P_4 (set $t_{k+1} := z_2$). Thus, $t_2 \in C$ for a clique C of T , say $C \neq C_j$. Moreover,

t_2 is adjacent to x and y ,

otherwise, $t_2z_2xz_1$ or $t_2z_2yz_1$ (if t_2 and z_2 are adjacent), or $t_2t_1xz_2$ or $t_2t_1yz_2$ (if t_2 and z_2 are nonadjacent) would be a bad P_4 , a contradiction.

But then t_2xz_1y' is a bad P_4 . The case $t_2 \in C_j$ is similar. Thus, there is no induced path of length > 2 connecting x and y , and so x, y form a two-pair. The proof of Theorem 6 is complete. \square

The class of perfect graphs described in Theorem 6 contains all P_4 -free graphs, split graphs, cograph contractions, complements of cograph contractions, strongly P_4 -stable graphs, complements of strongly P_4 -stable graphs ([23]), bipartite graphs, and complements of bipartite graphs. In particular, this new class is not contained in BIP* ([11]), not in the class of strict-quasi parity graphs ([34]). We do not know whether there is a perfect graph described in Theorem 6 that is not quasi-parity ([34]). Also, we shall remark that these new perfect P_4 -bipartite graphs do not belong to any class of the classes of S_i -graphs, $i = 1, \dots, 4$, described in Theorem 3. This can be seen as follows. Let G be the graph obtained from the $\overline{C_6}$ by subdividing the three edges not belonging to a triangle (thus G has nine vertices). Then G satisfies the conditions of Theorem 6 with T consisting of the two disjoint triangles, but G is not an S_i -graph, for any $i = 1, \dots, 4$.

To conclude the paper, we remark that Fonlupt (see [22]) conjectures that no minimal imperfect Berge graph contains a cutset that induces a P_3 -free graph. Clearly, Conjecture 2 is implied by Fonlupt's conjecture together with (2) and (10).

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