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Mader Tools

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The deep theorem of Mader concerning the number of internally disjoint H-paths is a very powerfull tool. Nevertheless its use is very difficult, because one has to deal with a very reach family of separators. This paper shows several ways to strengthen Mader's theorem by certain additional restrictions of the appearing separators.

Keywords: graph, H-path, separator

1 Preliminaries and Results

For notations not defined here we refer to (1). Unless otherwise stated, k is an arbitrary integer, G is an arbitrary finite simple graph (loops and multiple edges are forbidden), U is an arbitrary subgraph of G, X and H are arbitrary disjoint subsets of $V(G)$ and Y is an arbitrary subset of $E(G - X - H)$. A path having exactly its endvertices in H is called an *H-path*. The maximum number of independent H-paths we denote by $p_G(H)$. $[Y]$ denotes the graph with edge set Y whose vertex set is the set of all vertices incident with at least one edge of Y. Let $\mathcal{C}(G)$ denote the set of components of G and $\partial_G(U)$ denote the set of vertices of U incident with at least one edge of $G - E(U)$. A pair (X, Y) is called *H-separator of G*, if each H-path of G contains a vertex of X or an edge of Y. Let $\mathcal{S}$ be the set of all H-separators of $G - E(G[H])$. A vertex x' of G is called *big brother* of a vertex x of G, if the neighborhood of x' in G contains the neighborhood of x in $G - x'$.

According to (1) we define the permeability of a pair (X, Y) by:

$$\mathrm{M}_G(X,Y) = |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{|\partial_{G-X}(C)|}{2} \right\rfloor$$

Mader's Theorem (cf. (2)) can be rewritten as follows (cf. (1).)

Theorem 1 (Mader, 1978)

$$p_G(H) = |E(G[H])| + \min\{M_G(X,Y) \mid (X,Y) \in \mathcal{S}\}$$

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Note, that here H is a set of vertices. To get this from the version of Mader's theorem in (1), you have to apply the version of (1) with the graph $G[H]$ instead of H.

Let a subset $\mathcal{S}'$ of $\mathcal{S}$ be a *Mader-Set*, whenever Theorem 1 remains valid if S is replaced by S'. In other words, a subset $\mathcal{S}'$ of $\mathcal{S}$ is a Mader-Set, iff for each element (X, Y) of $\mathcal{S}$ there is an element (X', Y') of $\mathcal{S}'$ with $\mathrm{M}_G(X', Y') \leq \mathrm{M}_G(X, Y)$. Note that a subset of $\mathcal{S}$ containing a Mader-Set is a Mader-Set, too.

The following conditions for elements (X, Y) of $\mathcal{S}$ will be discussed:

- *Odd Border Condition (OB)*

  For each component C of $[Y]$ the number $|\partial_{G-X} C|$ is odd.

- *Big Brother Vertex Condition (BV):* If $x \in X$ and x' is a big brother of x, then $x' \in X$.

- *Symmetric Edge Condition (SE):* If v and v' are two vertices of $G - H - X$ such that the neighborhood of v' in $G - v$ equals the neighborhood of v in $G - v'$, then the neighborhood of v' in $[Y] - v$ equals the neighborhood of v in $[Y] - v'$.

- *Edge Component Condition (EC):* For each edge e of $G - H - X - Y$ and each component C of $[Y] \cup (V(G - H - X), \emptyset)$ there is a path P in $G - X - Y - C$ containing an element of H and an endvertex of e.

- *Half Border Condition (HB):* For each $C \in [Y]$ and each $B \subseteq \partial_{G-X} C$ with $2|B| \geq |\partial_{G-X} C|$ there are two vertexdisjoint HB-paths in $G - X$.

For a subset Q of the set of conditions $\{$OB, BV, BE, EC$\}$ let $\mathcal{S}(Q)$ be the subset of $\mathcal{S}$ satisfying all conditions in Q. Our main results are as follows:

Theorem 2 $\mathcal{S}(\{$*OB,SE,HB,EC*$\})$ *is a Mader-Set.*

Theorem 3 $\mathcal{S}(\{$*BV,SE,HB,EC*$\})$ *is a Mader-Set.*

Theorem 4 *There is a graph G and a subset H of* $V(G)$ *such that* $\mathcal{S}(\{$*OB,BV*$\})$ *is not a Mader-Set.*

In other words, Theorem 2 and Theorem 3 state, that for each graph G and each subset H of $V(G)$ the set $\mathcal{S}^*(G, H)$ of H-separators of G with minimal permeability has (possibly equal) elements (X_1, Y_1) and (X_2, Y_2) such that (X_1, Y_1) satisfies the Odd Border Condition, the Symmetric Edge Condition, the Half Border Condition and the Edge Component Condition, and (X_2, Y_2) satisfies the Big Brother Vertex Condition, the Symmetric Edge Condition, the Half Border Condition and the Edge Component Condition.

Theorem 4 states, that there is a graph G and a subset H of $V(G)$, such that none of the elements of $\mathcal{S}^*(G, H)$ satisfies the Odd Border Condition and the Big Brother Vertex Condition.

2 Motivation

Why dealing with such mysterious conditions? The Odd Border Condition helps to simplify the formula for the permeability of an H separator:

Theorem 5 *Let G be a graph, H be a subset of G, and* (X, Y) *be an* H*-separator of G satisfying the Odd Border Condition. Then for the permeability of* (X, Y) *the following equation holds:*

$$M_G(X, Y) = |X| + \frac{|\partial_{G-X}[Y]| - |\mathcal{C}([Y])|}{2}$$

In order to motivate the remaining three conditions, we regard an application of Mader's Theorem: Suppose, a function f mapping H into the set of nonnegative integers is given. We are interested in a 'separator-like' condition for the existence of a set of p independent H-paths such that in the graph U being the union of all this paths $f(h) \geq d_U(h)$ holds for all $h \in H$. Such a problem appears for instance, if one wants to prove the f-factor theorem with help of Mader's Theorem.

Let the graph $R(G,f)$ be obtained from G by the following procedure: Let G' be the graph obtained from G by intersecting each edge e of $G[H]$ by a vertex h_e. In G' sequentially replace each vertex v of H by a complete bypartite graph R_v whose partition classes A_v and B_v satisfy $|A_v| = d_G(v)+1$ and $|B_v| = f(v)$. In each step each edge incident with v (say (u,v)) of G' has to be replaced by an edge (u,a) with $a \in A_v$ such that in the resulting graph $R(G,f)$ only one vertex a_v of A_v has all its neighbors in B_v. We call $R(G,f)$ the *f-replacement of G*.

The set $H_R(G,f) = \{a_v | v \in H\}$ we call *f-replacement of H in G*. With this definitions we find the following lemma:

Lemma 6 *G has a set of p independent H-paths such that each vertex v of H is contained in at most $f(v)$ of this paths if and only if $R(G,f)$ has a set of p independent $H_R(G,f)$-paths.*

Using Mader's Theorem for $R(G,f)$ instead of G and $H_R(G,f)$ instead of H we get

Lemma 7 *G has a set of p independent H-paths such that each vertex v of H is contained in at most $f(v)$ of this paths if and only if each $H_R(G,f)$-separator (X,Y) of $R(G,f)$ satisfies $p < p(X,Y)$.*

Now, we are nearly done. We have to retranslate this condition to a condition for the graph G, the set H, and the function f only. To reconstruct G from $R(G,f)$, for each $v \in H$ we have to contract the graph R_v to the vertex v, and after that for each $e \in E(G[H])$ we have to delete h_e and to add e. But, without any knowledge about a special structure of $H_R(G,f)$-separators in $R(G,f)$, we loose too much information by doing the contractions.

The situation changes rapidly, if we first apply Theorem 3 with $R(G,f)$ instead of G and $H_R(G,f)$ instead of H. Using this we prove the following Lemma:

Lemma 8 *G has a set of p independent H-paths such that each vertex v of H is contained in at most $f(v)$ of this paths if and only if each $H_R(G,f)$-separator (X,Y) of $R(G,f)$ that satisfies the following conditions also satisfies $p < p(X,Y)$.*

Here are the conditions:

For each element v of H one of the following statements holds:

1. *$V(R_v) \cap X = B_v$ and no edge of Y is incident with A_v,*
2. *$V(R_v) \cap X = \emptyset$ and Y contains each edge of $R(G,f)$ incident with $A_v \setminus \{a_v\}$.*
3. *$V(R_v) \cap X = \emptyset$ and no edge of Y is incident with A_v.*

For each edge e of $G[H]$ we have $h_e \in X$ if and only if for each edge v incident with e the third statement ($V(R_v) \cap X = \emptyset$ and no edge of Y is incident with A_v) holds.

By Lemma 8, it is possible to interpret the resulting structure in G, directly. For this, let a pair (X,Y) be (G,H)-*valid*, if $G - X - Y$ has no H-path and $\partial_{G-X}[Y]$ is disjoint to H.

We derive the following Theorem:

Theorem 9 *Given a graph G, a subset H of its vertex set, and a function f that maps H to the set of non-negative integers.*

The maximum number of independent H-paths, for which each vertex v of H is contained in at most $f(v)$ of this paths, equals the minimum of

$$|E(G[H \setminus (X \cup V([Y]))])| + |X \setminus H| + \sum_{x \in H \cap X} f(x) + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} \left(|\partial_{G-X} C| + \sum_{v \in H \cap V(C)} f(v) \right) \right\rfloor$$

taken over all (G, H)-valid pairs (X, Y).

References

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