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On the number of series parallel and outerplanar graphs†

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We show that the number g_n of labelled series-parallel graphs on n vertices is asymptotically $g_n \sim g \cdot n^{-5/2}\gamma^n n!$, where γ and g are explicit computable constants. We show that the number of edges in random series-parallel graphs is asymptotically normal with linear mean and variance, and that the number of edges is sharply concentrated around its expected value. Similar results are proved for labelled outerplanar graphs.

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A graph is series-parallel (SP for short) if it does not contain the complete graph K_4 as a minor; equivalently, if it does not contain a subdivision of K_4. Since both K_5 and $K_{3,3}$ contain a subdivision of K_4, by Kuratowski's theorem a SP graph is planar. Another characterization, justifying the name, is the following. A connected graph is SP if it can be obtained from a single edge by means of the the following two operations: subdividing an edge (series); and duplicating an edge (parallel). In addition, a 2-connected graph is SP if it can be obtained from a double edge by means of series and parallel operations; in particular, this implies that a simple 2-connected SP graph has always a vertex of degree two. Although SP operations may give rise to multiple edges, in this paper all graphs considered are simple.

Yet another characterization is that SP graphs are precisely the graphs with treewidth at most two. Equivalently they are subgraphs of 2-trees, where a 2-tree is a graph formed by, starting from a triangle, adding repeatedly a new vertex and joining it to an existing edge.

An outerplanar graph is a planar graph that can be embedded in the plane so that all vertices are in the outer face. They are characterized as those graphs not containing a minor isomorphic to (or a subdivision of) either K_4 or $K_{2,3}$. They constitute an important subclass of the class of SP graphs.

Series-parallel graphs have been widely studied in graph theory and computer science. They are simple in structure but yet rich enough so that several theoretical and computational problems are still unsolved on SP graphs. In fact, they are often used as a benchmark for analyzing the complexity of graph problems. The same thing can be said, maybe even more, about outerplanar graphs.

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In this paper we study the enumeration of labelled series-parallel and outerplanar graphs. From now on, unless stated otherwise, all graphs are labelled. Next we summarize what is known about this problem. An SP graph on n vertices has at most $2n-3$ edges. Those having this number of edges are precisely the 2-trees; it is known that the number of labelled 2-trees on n vertices is equal to $\binom{n}{2}(2n-3)^{n-4}$.

On the other hand, an outerplanar graph is 2-connected if and only it has a unique Hamilton cycle. It follows that a 2-connected outerplanar graph is in fact equivalent to a dissection of a convex polygon, the boundary of the polygon being the unique Hamilton cycle. Hence counting 2-connected outerplanar graphs amounts essentially to counting dissections of a convex polygon, a classical and well-known problem. It is also worth mentioning that an outerplanar *map* (a map is a planar graph together with a particular embedding in the plane) on n vertices can be encoded with $3n$ bits (2); hence the number of outerplanar graphs is at most $2^{3n} = 8^n$.

The main goal of this paper is to give precise asymptotic estimates for the number of SP and outerplanar graphs.

Theorem 1 *Let b_n, c_n and g_n be, respectively, the number of 2-connected, connected and arbitrary labelled SP graphs on n vertices. Then we have the following asymptotic estimates:*

$$\begin{aligned} b_n &\sim b \cdot n^{-5/2} R^{-n} n!, \\ c_n &\sim c \cdot n^{-5/2} \rho^{-n} n!, \\ g_n &\sim g \cdot n^{-5/2} \rho^{-n} n!, \end{aligned}$$

where b, c, g, R and ρ are computable constants. In particular, $R \approx 0.128003$ and $\rho \approx 0.110213$.

Our second result has to do with the number of edges in random series-parallel graphs. Recall that a sequence of discrete random variables X_n with mean μ_n and variance σ_n^2 is asymptotically normal if the normalized variables $X_n^* = (X_n - \mu_n)/\sigma_n$ converge in law to the standard normal distribution $\mathcal{N}(0,1)$; convergence in law means, as usual, point-wise convergence of the corresponding distribution functions.

Theorem 2 *Let X_n denote the number of edges in random series-parallel graphs with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 of X_n satisfy*

$$\mu_n \sim \kappa n, \qquad \sigma_n^2 \sim \lambda n,$$

where $\kappa \approx 1.616734$ and $\lambda \approx 0.553479$. As a consequence, the number of edges is sharply concentrated around its expected value.

For the class of outerplanar graphs we obtain similar results, that we summarize in the next theorem.

Theorem 3 *The number h_n of labelled outerplanar graphs on n vertices satisfies the estimate*

$$h_n \sim h \cdot n^{-5/2} \sigma^{-n} n!,$$

where $\sigma \approx 0.136593$. Moreover, the distribution of the number of edges in a random outerplanar graph with n vertices is asymptotically normal with mean and variance

$$\mu_n \sim \zeta n, \qquad \sigma_n^2 \sim \eta n,$$

where $\zeta \approx 1.56251$ and $\eta \approx 0.223992$.

We remark that the best result known so far with respect to the previous theorem was $\zeta \geq 7/5$, proved in (6).

Our last result has to do with the number of connected components. A sequence X_n of discrete random variables converges to a discrete random variable X if, for every integer k,

$$\mathrm{Prob}\{X_n = k\} \to \mathrm{Prob}\{X = k\}, \quad \text{as } n \to \infty.$$

In the next statement, convergence is to a *shifted* Poisson law because the number of components is always strictly positive.

Theorem 4 *The distribution of the number of connected components in random series-parallel graphs is asymptotically a shifted Poisson law* $1 + P(\nu)$ *with parameter equal to* $\nu \approx 0.117614$*. The same result holds for outerplanar graphs, in this case the parameter of the Poisson law being equal to* $\xi \approx 0.148404$*. As a consequence the probability that a random SP graph is connected tends to* $e^{-\nu} \approx 0.889038$*, and to* $e^{-\xi} \approx 0.862082$ *for outerplanar graphs.*

The proofs of the previous results are based on singularity analysis of generating functions (see (4; 5)), and on several ideas developed in (1) and (7) for solving similar problems for the class of planar graphs. Because of space limitations we just outline the main ingredients of our analysis.

The first thing is to analyze the exponential generating function

$$B(x, y) = \sum b_{n,q} y^q \frac{x^n}{n!},$$

where $b_{n,q}$ is the number of 2-connected SP graphs with n vertices and q edges.

Following Walsh (8), define a *network* as a graph with two distinguished vertices, called poles, such that adding the edge between the poles the resulting multigraph is 2-connected. If $D(x, y)$ is the EGF for networks, where again x marks vertices and y marks edge then, as shown in (1), we have

$$\frac{\partial B(x, y)}{\partial y} = \frac{x^2}{2} \left(\frac{1 + D(x, y)}{1 + y} \right).$$

Since a 2-connected SP graph has always a vertex of degree two, it follows that there are no 3-connected SP graphs; in the terminology of (8) there are only s-networks and p-networks and there are no h-networks. Hence equation (12) in (1) simplifies to

$$\log \left(\frac{1 + D}{1 + y} \right) = \frac{xD^2}{1 + xD}.$$

From the two previous equations it is possible to perform a full singularity analysis of $B(x, y)$. For a fixed value of y in a suitable (complex) neighborhood of 1, we determine the dominant singularity $R(y)$ of $B(x, y)$ and we show that the following singular expansion holds

$$B(x, y) = B_0(y) + B_2(y)X^2 + B_3(y)X^3 + \mathcal{O}(X^4),$$

where $X = \sqrt{1 - x/R(y)}$ and $B_0(y), B_2(y), B_3(y)$ are explicit analytic functions of y (they are too involved to be reproduced in this abstract).

Then we set $y = 1$, so that $B(x) = B(x, 1) = \sum b_n x^n/n!$. Applying singularity analysis, we obtain the first part of Theorem 1. The constant R appearing there is precisely $R(1)$.

Next we consider the generating functions $C(x, y)$ and $G(x, y)$, defined analogously for connected and arbitrary SP graphs, respectively. The series B, C and G are related through the following two equations

$$G(x, y) = \exp(C(x, y)), \qquad xC'(x, y) = x \exp\left(B'(xC'(x, y), y)\right),$$

where derivatives are always with respect to the first variable. This is due to the fact that a graph is SP if and only its connected and 2-connected components are themselves SP.

The second equation can be reinterpreted by saying that

$$\psi(x, y) = xe^{-B'(x,y)}$$

is the functional inverse of $F(x, y) = xC'(x, y)$. We show that for real y close to 1, $\psi'(x, y)$ has a positive root $\tau(y)$. By the general principles of singularity analysis, it follows that the radius of convergence of $F(x, y)$ is $\rho(y) = \psi(\tau(y), y)$. We next find the singular expansion of $F(x, y)$ at $\rho(y)$, and from this the singular expansions of $C(x, y)$ and $G(x, y)$, whose dominant singularity is also $\rho(y)$. Again by singularity analysis, the estimates for c_n and g_n in Theorem 1 follow.

The singular expansion of $G(x, y)$ is of the form

$$G(x, y) = G_0(y) + G_2(y)X^2 + G_3(y)X^3 + \mathcal{O}(X^4),$$

where now $X = \sqrt{1 - x/\rho(y)}$ and the G_i are (again explicit) analytic functions of y. Using the extensions of the central limit theorem based on perturbation of singularities (5), we are able to prove Theorem 2; the constants κ and λ are computed as

$$\kappa = -\frac{\rho'(1)}{\rho(1)}, \qquad \lambda = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2.$$

The analysis for outerplanar graphs is similar but simpler, since the analogous generating function $B(x, y)$ is obtained directly from the (ordinary) generating for dissections of a convex polygon (3). In fact, $B'(x, y)$ is given by

$$B'(x, y) = \frac{1 + xy(3 + 2y) - \sqrt{1 - xy(2 + 4y) + x^2y^2}}{4(1 + y).}$$

Finally, for the proof of Theorem 3, the key observation is that, for fixed k, the generating function of SP graphs with exactly k connected components is $C(x)^k/k!$. Since we have a full singular expansion of $C(x)$, we can estimate precisely the coefficient of x^n in $C(x)^k$, and this is all that is needed in order to derive the Poisson limit law. The parameter ν stated in Theorem 4 is equal to $C(\rho)$, where as before ρ is the dominant singularity of $C(x)$. The situation for outerplanar graphs is analogous.

References

[1] E. A. Bender, Z. Gao, N. C. Wormald, *The number of 2-connected labelled planar graphs*, Elec. J. Combinatorics 9 (2002), #43.

[2] N. Bonichon, C. Gavoille, and N. Hanusse, *Canonical Decomposition of Outerplanar Maps and Application to Enumeration, Coding and Generation*, Springer Lecture Notes in Computer Science, vol.. 2280, pages 81–92, 2003.

[3] P. Flajolet, M. Noy, *Analytic Combinatorics of Non-crossing Configurations*, Discrete Math. (1999).

[4] P. Flajolet, A. Odlyzko, *Singularity analysis of generating functions*, SIAM J. Discrete Math. 3 (1990), 216–240.

[5] P. Flajolet, R. Sedgewick, *Analytic Combinatorics* (book in preparation), preliminary version available at `http://algo.inria.fr/flajolet/Publications`

[6] S. Gerke, C. McDiarmid, *On the Number of Edges in Random Planar Graphs*, Comb. Prob. and Computing 13 (2004), 165–183.

[7] O. Giménez, M. Noy, *Asymptotic enumeration and limit laws of planar graphs*, math.CO/0501269, 14 pages.

[8] T. R. S. Walsh, *Counting labelled three-connected and homeomorphically irreducible two-connected graphs*, J. Combin. Theory Ser. B 32 (1982), 1–11.