A RANDOM WALK ON THE ROOK PLACEMENTS ON A FERRERS BOARD

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There are many positive ways in which Dominique Foata has impacted our Combinatorics community. Foremost are his fundamental scholarly contributions, many of them breaking new ground or setting directions for the future. Just as important has been his support of students and colleagues. I've had the pleasure of visiting Dominique twice in Strasbourg and each time have been struck by his enthusiasm for, and interest in the work of others. I'm pleased to have the opportunity to contribute to this special issue in his honor.

Abstract: Let B be a Ferrers board, i.e., the board obtained by removing the Ferrers diagram of a partition from the top right corner of an $n \times n$ chessboard. We consider a Markov chain on the set R of rook placements on B in which you can move from one placement to any other legal placement obtained by switching the columns in which two rooks sit. We give sharp estimates for the rate of convergence of this Markov chain using spectral methods. As part of this analysis we give a complete combinatorial description of the eigenvalues of the transition matrix for this chain. We show that two extremes cases of this Markov chain correspond to random walks on groups which are analyzed in the literature. Our estimates for rates of convergence interpolate between those two results.

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Section 1: Background

In a 1970 paper, Foata and Schützenberger [FS] consider rook polynomials of Ferrers boards. A Ferrers board is an $n \times n$ board with the Ferrers diagram of a partition removed from the Northeast corner. In that paper they show that a minimal set of representatives of Ferrers boards up to rook equivalence is given by those where the partition removed has distinct parts.

In a series of papers [GJW 1-4], Goldman, Joichi and White establish elegant enumerative results concerning rook polynomials of Ferrers boards. We recall some of these results in Section 2.

In this paper we study a Markov chain whose states correspond to rook placements on a fixed Ferrers board B. A step in this Markov chain is taken as follows. Suppose we are at a state α corresponding to a placement of rooks on B. Let S_α be the multiset consisting of all placements of rooks on B that are obtained from α by switching the columns in which two rooks are contained. We allow a column to switch with itself so α occurs in S_α with multiplicity n. In our Markov chain we move from α to each state in S_α with probability $|S_\alpha|^{-1}$.

We give a combinatorial description of the eigenvalues (and their multiplicities) of the transition matrix of this Markov chain. A special case of this result agrees with a special case of an interesting recent result of Hozo [H] on the spectrum of the Laplacian of the Koszul complex for certain nilpotent Lie algebras. We use this information to give sharp estimates for the rate of convergence of this Markov chain to its stationary distribution in some cases.

The analysis of this Markov chain is related to recent results concerning random walks on groups (see [D] for an overview). Perhaps the earliest sharp rates of convergence for such random walks were given by Diaconis and Shahshahani [DS1] in 1981 who studied the walk by random transpositions on the symmetric group. They show that this walk converges to its stationary distribution after $\frac{1}{2} n \log n$ steps (a precise statement of their result appears in Section 5).

Diaconis and Shahshahani have applied a similar analysis to the natural random walk on the hypercube $\mathbb{Z}_2^{n-1}$. In a 1987 paper [DS2], they show that $\frac{1}{4} n \log n$ steps are necessary and sufficient for convergence to the uniform distribution. The exact asymptotics of the total variation distance was worked out in a 1991 paper of Diaconis, Graham, and Morisson [DGM].

The main result in this paper interpolates between the two cases S_n and $\mathbb{Z}_2^{n-1}$ described above. But unlike these two extreme cases, the author knows of no group structure (or association scheme structure,) that can be used to analyze the spectrum of the transition matrix. Nevertheless, the final results give a characterization of the spectrum of the transition matrix which neatly interpolates between the representation-theoretic characterizations in the two extreme cases.

Some comments on notation. All tensor products are over the complex numbers unless otherwise specified. At various times, when considering the permutation representation of a group G on a set S, we will sometimes abuse notation by letting S also denote the vector space with basis consisting of the elements of S. Lastly, there will be cases in this work when we consider left-modules, other times when we consider right-modules and still other times when we consider two-sided modules. The situation will be clear in every case from context. We will leave it to the reader to determine in each instance which sort of module is being discussed.

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Section 2: $\underline{b}$-Regular Permutations

Let $\underline{b}=(b_1,\ldots,b_n)$ be a sequence of positive integers each taken from $\{1,2,\ldots,n\}$. A permutation $\alpha=a_1a_2\ldots a_n$ is *$\underline{b}$-regular* if $a_i \geq b_i$ for all i. We let $R_n(\underline{b})$ denote the set of $\underline{b}$-regular permutations in S_n. Usually we will think of n as having a fixed value and we will write $R(\underline{b})$ instead of $R_n(\underline{b})$.

We say that α and γ in $R(\underline{b})$ are *neighbors* if they differ by a transposition. Of course this is a reflexive relation on $R(\underline{b})$ and so determines a graph $G(\underline{b})$ whose vertices are the elements of $R(\underline{b})$ with two vertices adjacent if they are neighbors. We are going to consider the random walk on this graph (in the usual sense).

Note that $(b_1,\ldots,b_n)$ simply restricts the positions in which a given number can be placed. So if $\underline{b}=(b_1,\ldots,b_n)$ and $\underline{b}'=(b_{\alpha 1},\ldots,b_{\alpha n})$ differ by a permutation then $G(\underline{b})$ and $G(\underline{b}')$ are isomorphic as graphs. From the point of view of the random walk on $G(\underline{b})$ the order of the numbers $b_1,\ldots,b_n$ is unimportant.

We will henceforth assume that the numbers $b_1,\ldots,b_n$ are written in weakly increasing order.

We now catalogue some combinatorial properties of the sets $R(\underline{b})$ and the graphs $G(\underline{b})$. It is straightforward to see that $R(\underline{b})$ is empty if $b_i > i$ for any i. So in the next theorem we will assume that $1 \leq b_1 \leq b_2 \leq \ldots \leq b_n \leq n$ and that $b_i \leq i$ for all i.

Proposition 2 .1: Let $\underline{b}=(b_1,\ldots,b_n)$ be as above. For π a permutation, let $c(\pi)$ and $i(\pi)$ denote the number of cycles and inversions of π respectively. Then

(a) $|R(\underline{b})|=\prod_{i=1}^{n}(1+(i-b_i))$

(b) $\sum_{\pi \in R(\underline{b})} q^{c(\pi)} = \prod_{i=1}^{n} (q + (i - b_i))$

(c) $\sum_{\pi \in R(\underline{b})} q^{i(\pi)} = \prod_{i=1}^{n} \frac{(1-q^{i+1-b_i})}{(1-q)}$

(d) The graph $G(\underline{b})$ is regular of degree $\sum_{i=1}^{n} (i - b_i) = \binom{n}{2} - \sum_{i=1}^{n} (b_i - 1)$.

Proof:

Part (a) follows immediately from either (b) or (c). Both (b) and (c) are proved in various places in the literature (for example [GJW1-4]). We will prove (d).

Fix $\alpha = a_1 \dots a_n$ in $R(\underline{b})$. We will count the neighbors of α in $G(\underline{b})$. Let N_i denote the set of neighbors obtained from α by interchanging a_i with some a_j where $j < i$. There are $(n+1) - b_i$ numbers x in the set $\{a_1, \dots, a_n\}$ which satisfy $x \geq b_i$. Since $b_i \leq b_{i+1} \leq \dots \leq b_n$, each of $a_i, a_{i+1}, \dots, a_n$ are greater than or equal to b_i. So there are $(n+1) - b_i - (n - i + 1) = i - b_i$ numbers a_j with $j < i$ and $a_j \geq b_i$. This shows that $|N_i| = i - b_i$ which completes the proof. ■

Section 3: Examples

In this section we will discuss some special cases that arise by varying the sequence $\underline{b}$.

Example 1: Suppose first that $b_i = i$ for some $i \geq 2$. Since the sequence $\underline{b}$ weakly increases this implies that $b_j \geq i$ for all $j \geq i$. So, if $\alpha = a_1 a_2 \dots a_n$ is in $R_n(\underline{b})$ then

$$\{a_1, \dots, a_{i-1}\} = \{1, 2, \dots, i-1\} \text{ and } \{a_i, \dots, a_n\} = \{i, i+1, \dots, n\}.$$

There is a natural bijection φ between $R_n(\underline{b})$ and $R_{i-1}(\underline{c}) \times R_{n-i+1}(\underline{d})$ where $\underline{c} = (b_1, \dots, b_{i-1})$ and $\underline{d} = (b_i - (i-1), \dots, b_n - (i-1))$.

Our goal in this paper is to study the Markov chain on $G_n(\underline{b})$ described earlier. In the case where $b_i = i$, the transition matrix for this Markov chain is easily seen to be the tensor product of the transition matrices for the corresponding chains on $G_{i-1}(\underline{c}) \times G_{n-i+1}(\underline{d})$. So the random walk on $R_n(\underline{b})$ is put together in a simple way from two smaller random walks.

Amongst sequences $\underline{b}$ with $b_i < i$ for all $i \geq 2$ there are two extreme cases - one where the numbers in the sequence $\underline{b}$ are as large as possible and one where the numbers in $\underline{b}$ are as small as possible.

Example 2: Suppose $(b_1, \ldots, b_n) = (1, 1, \ldots, 1)$.

This case is very familiar. The set $R_n(\underline{b})$ is all of S_n and the Markov chain on $G_n(\underline{b})$ is the well-studied random transposition walk on the symmetric group (see [DS1]).

Example 3: Suppose $(b_1, \ldots, b_n) = (1, 1, 2, \ldots, n-1)$.

Again this case is very familiar, although it takes some work to recognize it given its present formulation. For $\alpha = a_1 a_2 \ldots a_n$ in $R_n(\underline{b})$ let $E(\alpha)$ denote the set of indices i such that $a_i = i - 1$. So $E(\alpha) \subseteq \{2, 3, \ldots, n\}$.

Claim 1: Given a subset $S \subseteq \{2, 3, \ldots, n\}$, there is exactly one $\alpha = a_1 a_2 \ldots a_n$ in $R_n(\underline{b})$ satisfying $E(\alpha) = S$.

Proof: We prove this by induction on n, the result being easy to check in the base case $n = 2$.

Assume the result holds for $n - 1$ and consider a set $S \subseteq \{2, 3, \ldots, n\}$.

Case 1: $n \notin S$.

By our induction hypothesis, there is a unique $\alpha' = a_1 a_2 \ldots a_{n-1}$ in $R_{n-1}(b_1, \ldots, b_{n-1})$ satisfying $E(\alpha') = S$. Let $\alpha = a_1 a_2 \ldots a_{n-1} n$. Clearly $E(\alpha) = S$. Moreover, if $\gamma = c_1 c_2 \ldots c_n$ satisfies $E(\gamma) = S$ then $c_n = n$ (since $n \notin E(\gamma)$), and $E(c_1 c_2 \ldots c_{n-1}) = S$. The uniqueness of α' implies that $\gamma = \alpha$ and this completes the induction step in this case.

Case 2: $n \in S$.

Let $S' = S \backslash n$ so $S' \subseteq \{2, 3, \ldots, n-1\}$. By our induction hypothesis there exists exactly one α' in $R_{n-1}(b_1, \ldots, b_{n-1})$ satisfying $E(\alpha') = S'$. Let $\overline{\alpha'}$ be identical to α' except that $n - 1$ is replaced by n. Note that $E(\overline{\alpha'}) = E(\alpha') = S'$. Now define α to be $\alpha = \overline{\alpha'}(n-1)$ (in other words append $n - 1$ to the end of $\overline{\alpha'}$). Clearly $\alpha \in R_n(\underline{b})$ and $E(\alpha) = S$. This shows existence.

For uniqueness, suppose $\gamma = c_1 \ldots c_n$ is in $R_n(\underline{b})$ with $E(\gamma) = S$. Then $c_n = n - 1$ and $\overline{\gamma'} = c_1 \ldots c_{n-1}$ satisfies $E(\overline{\gamma'}) = S \backslash \{n\}$. Let γ' be $\overline{\gamma'}$ but with n replaced by $n - 1$. Then $E(\gamma') = E(\overline{\gamma'}) = S'$ and by the uniqueness of α' we have $\gamma' = \alpha'$. So $\gamma = \alpha$ which completes the induction step. ∎

This shows that the elements of $R_n(\underline{b})$ are naturally identified with the subsets of $\{2, 3, \ldots, n\}$. We will further identify the subsets of $\{2, 3, \ldots, n\}$ with the vertices of the $(n-1)$ dimensional hypercube $C_{n-1} = \{0, 1\}^{n-1}$ in the obvious way. $S \subseteq \{2, 3, \ldots, n\}$ is identified with the vector $(\varepsilon_1, \ldots, \varepsilon_{n-1})$ given by $\varepsilon_i = \begin{cases} 1 & \text{if } i+1 \in S \\ 0 & \text{if } i+1 \notin S \end{cases}$

Claim 2: Let α and γ be in $R_n(\underline{b})$ and let ε and ρ be the corresponding vertices in C_{n-1}. Then α and γ are adjacent in $G_n(\underline{b})$ if and only if ε and ρ are adjacent in C_{n-1}.

Proof: The proof is by induction on n. The result is easy to check in the base case $n = 2$. Assume the result is true for $n-1$ and let α and γ be elements of $R_n(\underline{b})$ with $\alpha = a_1 a_2 ... a_n$ and $\gamma = c_1 c_2 ... c_n$.

Case 1: $a_n = c_n = n$.

Let $\alpha' = a_1 ... a_{n-1}$ and $\gamma' = c_1 ... c_{n-1}$. Note that $\varepsilon = (\varepsilon', 0)$ and $\rho = (\rho', 0)$ where ε' and ρ' are the elements in H_{n-2} corresponding to α' and γ' respectively.

Note that α and γ are adjacent in $G_n(\underline{b})$ if and only if α' and γ' are adjacent in $G_{n-1}(\underline{b})$. By our induction hypothesis, α' and γ' are adjacent if and only if ε' and ρ' are adjacent in H_{n-2} which happens if and only if ε and ρ are adjacent in H_{n-1}.

Case 2: $a_n = c_n = n - 1$.

This is a similar argument to Case 1.

Case 3: $a_n = n - 1$ and $c_n = n$.

Let $\alpha' = a_1 a_2 \ldots a_{n-1}$ and let $\overline{\alpha'}$ be α' but with n replaced by $n - 1$. Let $\gamma' = c_1 \ldots c_{n-1}$. Note that α and γ are adjacent in $G_n(\underline{b})$ if and only if γ is obtained from α by interchanging the numbers n and $n - 1$ if and only if $\overline{\alpha'} = \gamma'$.

Also note that ε ends in 1 and ρ ends in 0. So ε and ρ are adjacent in H_{n-1} if and only if their first $n - 2$ coordinates are identical, if and only if $\overline{\alpha'} = \gamma'$.

Case 4: $a_n = n$ and $c_n = n - 1$.

This is identical to Case 3. This completes the induction step. ■

This shows that the graph $G_n(\underline{b})$ has the structure of the $(n - 1)$-cube.

Section 4: The Random Walk on $G_n(\underline{b})$

In this section we will begin the study of the random walks on $G_n(\underline{b})$. This random walk is described by a Markov chain whose states are $R_n(\underline{b})$. Roughly speaking, to take a step in this Markov chain you do the following. You are at a state $\alpha = a_1 ... a_n$ which you think of as being the cards $a_1 ... a_n$ laid out in front of you from left to right. Put your left hand down on a card a_i chosen at random. Put your right hand down on a card a_j (perhaps the same card a_i) chosen at random. If you can switch the two cards a_i and a_j and remain in the set $R_n(\underline{b})$ then do so. That is your step in the Markov chain. However, if you leave the set $R_n(\underline{b})$ by switching a_i and a_j then go back and pick two new cards. Continue this until you finally pick a pair that you can switch.

Let $\triangle = \sum_i (i - b_i)$. We know that $G_n(\underline{b})$ is regular of degree $\triangle$. Suppose you are at a vertex α in the Markov chain described above. If γ is a neighbor of α in $G_n(\underline{b})$ then there are two ways to pick the pair whose interchange takes you from α to γ. There are n ways to remain at α. So the transition matrix for our Markov chain is

$$< 4.1 > \qquad \frac{1}{n+2\triangle}(n\mathcal{I} + 2\mathcal{U}) = \mathcal{T}$$

where $\mathcal{U}$ is the adjacency matrix of $G_n(\underline{b})$.

In this section we will determine the spectrum of $\mathcal{U}$ hence (by $< 4.1 >$) the eigenvalues of $\mathcal{T}$ and their multiplicities.

It is worth pointing out that our Markov chain is primitive because there is a non-trivial probability of remaining at each state. Without a non-zero staying probability, the Markov chain would have period two. You would be in states which are indexed by permutations with the same sign as your starting state after an even number of steps and opposite sign after an odd number of steps.

Definition 4.2: Let $\underline{b} = (b_1, b_2, \ldots b_n)$ be a sequence with $1 \leq b_1 \leq b_2 \leq \ldots \leq b_n \leq n$ and $b_i \leq i$ for all i, and let $\underline{n}$ denote the set $\{1, 2, \ldots, n\}$. We say that $u, v \in \underline{n}$ are *left-equivalent* if $b_u = b_v$. Let $L_1, \ldots, L_s$ denote the left-equivalence classes of $\underline{n}$ ordered so that the numbers in L_i are less than the numbers in L_j whenever i is less than j.

We say that $u, v \in \underline{n}$ are *right-equivalent* if for some i, $b_i \leq u \leq v < b_{i+1}$(where b_{n+1} is defined to be $n+1$). Let $R_1, \ldots, R_s$ be the right-equivalence classes ordered so that the numbers in R_i are less than the numbers in R_j whenever i is less than j.

It is straightforward to check that the number of left-equivalence classes is the same as the number of right-equivalence classes. Also the right-equivalence classes are described easily in terms of the sequence of $\underline{b}$ by: $R_i = \{b_i, b_i + 1, \ldots, b_{i+1} - 1\}$.

For A a subset of $\underline{n}$ we let $Sym(A)$ denote the group of all permutations of $\underline{n}$ which fix every number outside A. So, $Sym(A) \cong S_{|A|}$. In terms of this notation, define the groups L and R by:

$$L = Sym(L_1) \times Sym(L_2) \times \ldots \times Sym(L_s) \leq S_n$$
$$R = Sym(R_1) \times Sym(R_2) \times \ldots \times Sym(R_s) \leq S_n.$$

We will think of L and R as acting on $\mathbb{C}S_n$ by the left-regular representation and the right-regular representation respectively.

From the definitions of L and R, it is straightforward to verify that they each preserve the set $R_n(\underline{b})$ and that the linear transformation $\mathcal{U}$ commutes with the actions of L and R. Since the actions of L and R commute, we have an action of the group G

$$G = L \times R$$

on the set $R_n(\underline{b})$ which commutes with the action of $\mathcal{U}$. The main theorem in this section will describe the eigenvalues of $\mathcal{U}$ and the G-module structure of the corresponding eigenspaces.

Definition 4.3: A *$\underline{b}$-partition* $\beta = (\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_s)$ is a sequence of partitions such that:
1) $\lambda_i \supseteq \mu_i \subseteq \lambda_{i+1}$ for all $i = 1, 2, \ldots, s-1$
2) $|\lambda_i/\mu_i| = |R_i|$ for all $i = 1, 2, \ldots, s$ (where $\mu_s = \emptyset$)
3) $|\lambda_{i+1}/\mu_i| = |L_{i+1}|$ for all $i = 0, 1, \ldots, s-1$ (where $\mu_0 = \emptyset$).

Given a $\underline{b}$-partition β, a an *up-down tableau of type $\underline{b}$ and shape β* (or *$\underline{b}$-tableau* of *shape* β, for short) is a sequence of partitions (Ferrers diagrams) $\tau = (\emptyset = \tau_0, \tau_1, \tau_2, \tau_3, \ldots, \tau_{2n} = \emptyset)$ such that each τ_i is obtained from τ_{i-1} by either adding or subtracting a box and such that the τ_i's initially increase in size until they reach λ_1, then decrease in size until they reach μ_1, then increase in size until they reach λ_2, etc. Let $P(\underline{b})$ denote the collection of $\underline{b}$-partitions.

We now pause to do some examples.

Example 4.4: Let $\underline{b} = (1, 1, 1, 2, 4)$.
The sets R_i and L_i are given below:

$L_1 = \{1, 2, 3\}$ $\quad$ $R_1 = \{1\}$
$L_2 = \{4\}$ $\quad$ $R_2 = \{2, 3\}$
$L_3 = \{5\}$ $\quad$ $R_3 = \{4, 5\}$

A typical $\underline{b}$-partition β is: $\beta = (21, 1^2, 1^3, 1, 2)$

A $\underline{b}$-tableau of shape β is given by $\tau = (\emptyset, 1, 1^2, 21, 1^2, 1^3, 1^2, 1, 2, 1, \emptyset)$

Example 4.5: Let $\underline{b} = (1, 1, 2, 2)$. In this case
$L_1 = \{1, 2\}$ $\quad$ $R_1 = \{1\}$
$L_2 = \{3, 4\}$ $\quad$ $R_2 = \{2, 3, 4\}$

There are 6 $\underline{b}$-partitions which are listed below. To the right of each $\beta \in P(\underline{b})$ we see a listing of the $\underline{b}$-tableaux of shape β.

$\beta_1 = (2, 1, 3)$ $\quad$ $\tau^{(1)} = (\emptyset, 1, 2, 1, 2, 3, 2, 1, \emptyset)$

$\beta_2 = (2, 1, 21)$ $\quad$ $\tau^{(1)} = (\emptyset, 1, 2, 1, 2, 21, 2, 1, \emptyset)$

$$\tau^{(2)} = (\emptyset, 1, 2, 1, 2, 21, 1^2, 1, \emptyset)$$
$$\tau^{(3)} = (\emptyset, 1, 2, 1, 1^2, 21, 2, 1, \emptyset)$$
$$\tau^{(4)} = (\emptyset, 1, 2, 1, 1^2, 21, 1^2, 1, \emptyset)$$

$\beta_3 = (2, 1, 1^3)$ $\qquad$ $\tau^{(1)} = (\emptyset, 1, 2, 1, 1^2, 1^3, 1^2, 1, \emptyset)$

$\beta_4 = (1^2, 1, 3)$ $\qquad$ $\tau^{(1)} = (\emptyset, 1, 1^2, 1, 2, 3, 2, 1, \emptyset)$

$\beta_5 = (1^2, 1, 21)$ $\qquad$ $\tau^{(1)} = (\emptyset, 1, 1^2, 1, 2, 21, 2, 1, \emptyset)$
$$\tau^{(2)} = (\emptyset, 1, 1^2, 1, 2, 21, 1^2, 1, \emptyset)$$
$$\tau^{(3)} = (\emptyset, 1, 1^2, 1, 1^2, 21, 2, 1, \emptyset)$$
$$\tau^{(4)} = (\emptyset, 1, 1^2, 1, 1^2, 21, 1^2, 1, \emptyset)$$

$\beta_6 = (1^2, 1, 1^3)$ $\qquad$ $\tau^{(1)} = (\emptyset, 1, 1^2, 1, 1^3, 1^2, 1, \emptyset)$

Note that the total number of $\underline{b}$-tableaux is 12 which is the size of $R_4(1, 1, 2, 2)$.

Example 4.6: Suppose $\underline{b} = (1, 1, \dots, 1)$.

Then $L_1 = R_1 = \{1, 2, \dots, n\}$ so $G = S_n \times S_n$. Each $\beta \in P(\underline{b})$ consists just of a partition (λ) of n. There are f_λ^2 $\underline{b}$-tableaux of shape λ (where f_λ is the number of standard Young tableaux of shape λ) and so $n!$ $\underline{b}$-tableaux in total.

Example 4.7: Suppose $\underline{b} = (1, 1, 2, \dots, n-1)$. In this case

$L_1 = \{1, 2\}$ $\qquad$ $R_1 = \{1\}$
$L_2 = \{3\}$ $\qquad$ $R_2 = \{2\}$
$L_3 = \{4\}$ $\qquad$ $R_3 = \{3\}$
$\vdots$ $\qquad$ $\vdots$
$L_{n-1} = \{n\}$ $\qquad$ $R_{n-1} = \{n-1, n\}$

A $\underline{b}$-partition β is a sequence $\beta = (\lambda_1, \square, \lambda_2, \square, \dots, \square, \lambda_{n-1})$ where each λ_i is either $\square\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. Thus the number of $\underline{b}$-partitions is 2^{n-1}. Given a $\underline{b}$-partition β, there is a unique $\underline{b}$-tableau τ of shape β which is obtained from β by adding $\emptyset, \square$ at the beginning and $\square, \emptyset$ at the end. So the total number of $\underline{b}$-tableaux is 2^{n-1} which once again agrees with the size of $R_n(\underline{b})$.

We will need a fact from the representation theory of the symmetric groups which is a slight generalization of a well-known result. For the moment let $W = \mathbb{C}S_n$. Then W

can be considered as a right S_n-module via the right-regular representation and as a left S_n-module via the left-regular representation. Moreover these two actions of S_n commute so we get an action of the direct product $S_n \times S_n$. A well-known result (see [F]) gives a decomposition of W into $S_n \times S_n$ irreducibles:

$$< 4.8 > \qquad \mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} V[\lambda] \otimes V[\lambda]$$

where $V[\lambda]$ is the irreducible for S_n indexed by λ.

We consider a more general situation. Let $0 \leq m \leq n$ and let μ be a partition of m. Consider the vector space $\hat{W} = V[\mu] \otimes_{\mathbb{C}} \mathbb{C}S_n$. Let H denote the symmetric group on the letters $[m+1, ..., n]$. If $v \in V[\mu], \pi \in S_m, \alpha \in S_n$ and $\sigma \in H$ then

$$v \otimes \sigma(\pi\alpha) = v \otimes \pi(\sigma\alpha).$$

So there is a well-defined left-action of H on the induction

$$W = V[\mu] \otimes_{\mathbb{C}S_m} \mathbb{C}S_n.$$

Of course W is also a right S_n-module and the actions of H and S_n commute. We need the following statement about the decomposition of W as an $(H \times S_n)$-module.

Proposition 4.9: As an $(H \times S_n)$-module, W is isomophic to

$$W \cong \bigoplus_{\substack{\lambda \vdash n \\ \mu \subseteq \lambda}} V[\lambda/\mu] \otimes V[\lambda].$$

Proof: Let λ be a partition of n. If we restrict $V[\lambda]$ to $S_m \times H$ and apply Frobenius Reciprocity we have

$$V[\lambda] \downarrow^{S_n}_{S_m \times H} = \bigoplus_{\alpha,\beta} g_{\alpha\beta\lambda} V[\alpha] \otimes V[\beta]$$

So as a left H-module,

$$V[\mu] \otimes_{\mathbb{C}S_m} V[\lambda] = \bigoplus_{\beta} g_{\mu\beta\lambda} V[\beta] = V[\lambda/\mu].$$

The result now follows from equation 4.8. ■

We are ready now to state our first main theorem which gives a decomposition of $R_n(\underline{b})$ as a G-module.

Theorem 4.10: The vector space $R_n(\underline{b})$ is isomorphic to a direct sum of G-invariant subspaces $R_n(\beta)$ for $\beta \in P(\underline{b})$. Given $\beta = (\lambda_1, \mu_1, \lambda_2, \mu_2, ..., \mu_{s-1}, \lambda_s)$:

(1) As a module for $G = Sym(L_1) \times ... \times Sym(L_s) \times Sym(R_1) \times ... \times Sym(R_s)$

$$R_n(\beta) \cong V_1 \otimes \cdots \otimes V_s \otimes W_1 \otimes \cdots \otimes W_s$$

where V_i is the skew Specht module $V[\lambda_i/\mu_{i-1}]$ for $Sym(L_i)$ and W_i is the skew Specht module $V[\lambda_i/\mu_i]$ for $Sym(R_i)$. As usual we are using the convention $\mu_o = \mu_s = \emptyset$.

(2) The dimension of $R_n(\beta)$ is equal to the number of $\underline{b}$-tableaux of shape β.

Proof: The proof will be by induction on s. We consider a sequence $\underline{b} = (b_1, \ldots, b_n)$ with s L_i and R_i-sets.

Case 1: $s = 1$

In this case, $b_i = 1$ for all $i \leq n$, so $R_n(\underline{b}) = \mathbb{C}S_n$ and $G = S_n \times S_n$. The decomposition of $\mathbb{C}S_n$ as a G-module is given by $< 4.8 >$

$$R_n(\underline{b}) = \bigoplus_{\lambda \vdash n} V[\lambda] \otimes V[\lambda].$$

In this case the $\underline{b}$-partitions consist just of a partition of n, and so we take $R_n(\beta)$ for $\beta = (\lambda)$ to be the summand $V[\lambda] \otimes V[\lambda]$. This proves (1).

The dimension of $V[\lambda]$ is the number of standard Young tableaux of shape λ. There is a simple bijection between standard Young tableaux of shape λ and sequences of partitions which begin with the empty partition and increase by one square at each step until λ is reached. So f_λ^2, the dimension of $R_n(\beta)$, is the number of $\underline{b}$-tableaux of shape $\beta = (\lambda)$. This proves (2).

Case 2: $s > 1$.

In this case $b_n > 1$ and we can write L_s and R_s as $L_s = \{a, a+1, \ldots, n\}$ and $R_s = \{b_n, b_{n+1}, \ldots, n\}$ for some $a > 1$. By the definitions of the L_i and R_i we see that

$$b_{a-1} < b_a = b_{a+1} = \ldots = b_n.$$

Since $b_n = b_a \leq a$ this implies that $L_s \subseteq R_s$.

Let $\underline{b}' = (b_1, \ldots, b_{a-1})$, let L'_i and R'_i be the corresponding L and R sets for $\underline{b}'$ and let s' be the number of L'_i. Note that

$$
\begin{aligned}
s' &= s - 1 \\
L'_i &= L_i \text{ for } 1 \leq i \leq s-1 \\
R'_i &= R_i \text{ for } 1 \leq i \leq s-2 \\
R'_{s-1} &= R_{s-1} \cup (R_s \setminus L_s).
\end{aligned}
\tag{$< 4.11 >$}
$$

In keeping with this notation we will let G' be the automorphism group associated with $\underline{b}'$, i. e.,

$$G' = Sym(L'_1) \times \ldots \times Sym(L'_{s-1}) \times Sym(R'_1) \times \ldots \times Sym(R'_{s-1}).$$

We will also need to consider a subgroup $\hat{G}$ of G' namely

$$\hat{G} = Sym(L'_1) \times \ldots \times Sym(L'_{s-1}) \times Sym(R'_1) \times \ldots \times (Sym(R_{s-1}) \times Sym(R_s \setminus L_s)).$$

(recall from $< 4.11 >$ that $R'_{s-1} = R_{s-1} \cup (R_s \setminus L_s)$).

Let $\mathcal{Z}$ denote the collection of injective mappings from $\{1, 2, \ldots, n-a+1\}$ into $\{b_n, b_n+1, \ldots, n\}$ (which is nonempty sice $b_n \leq a$), written as sequences. For each $\mathrm{Z} = (\mathrm{Z}_1, \ldots, \mathrm{Z}_{n-a+1}) \in \mathcal{Z}$ let $R_n(\underline{b}; \mathrm{Z})$ be the subspace of $R_n(\underline{b})$ spanned by all permutations that have the sequence Z in their last $n-a+1$ positions. Since $b_a = b_{a+1} = \ldots = b_n$, every permutation must end with some element of $\mathcal{Z}$, i. e.,

$$R_n(\underline{b}) = \bigoplus_{\mathrm{Z} \in \mathcal{Z}} R_n(\underline{b}; \mathrm{Z})$$

Fix a sequence $\mathrm{Z} \in \mathcal{Z}$ and let $\overline{\mathrm{Z}}$ denote the set of numbers that appear in Z. Then $|\overline{\mathrm{Z}}| = |L_s| = n - a + 1$ so there is a permutation $\pi_{\mathrm{Z}} \in S_n$ which sends the set $R_s \setminus L_s$ to the set $R_s \setminus \overline{\mathrm{Z}}$ and keeps the relative order of the numbers intact.

Define f_{Z} to be the linear map from $R_{a-1}(\underline{b}')$ to $R_n(\underline{b}; \mathrm{Z})$ which sends a permutation σ to the permutation obtained by first right-multiplying σ by π_{Z} then adding the sequence Z on the right. Notationally we will represent this as:

$$\mathrm{f}_{\mathrm{Z}}(\sigma) = (\sigma \pi_{\mathrm{Z}})\mathrm{Z}.$$

As an example, suppose that $\underline{b} = (1, 1, 3, 3, 3, 4, 4)$. Then $s = 3$, $L_3 = \{6, 7\}$ and $R_3 = \{4, 5, 6, 7\}$. Thus $a = 6$, $b_n = 4$, and $\underline{b}' = (1, 1, 3, 3, 3)$. Let $\mathrm{Z} = (6, 4)$. The permutation π_{Z} must send $R_3 \setminus L_3$ to $R_3 \setminus \overline{\mathrm{Z}}$ in an order-preserving way so we can choose $\pi_{\mathrm{Z}} = (4, 5, 7)(6)$.

There are four permutations in $R_s(b')$. We see these below along with their images under $\mathrm{f_Z}$:

$\sigma \in R_{a-1}(\underline{b}')$	$\sigma\pi_{\mathrm{Z}}$	$\mathrm{f_Z}(\sigma) = (\sigma\pi_{\mathrm{Z}})\mathrm{Z}$
12345	12357	1235764
21345	21357	2135764
12354	12375	1237564
21354	21375	2137564

<u>Claim 1</u>: The linear map $\mathrm{f_Z}$ is an isomorphism between $R_{a-1}(\underline{b}')$ and $R_n(\underline{b};\mathrm{Z})$.

<u>Proof</u>: The map $\sigma \to (\sigma\pi_{\mathrm{Z}})\mathrm{Z}$ is one-to one as a function from $\mathbb{C}S_{a-1}$ to $\mathbb{C}S_n$. So it is enough to show that $\mathrm{f_Z}(R_{a-1}(\underline{b}')) \subseteq R_n(\underline{b};\mathrm{Z})$ and that $\mathrm{f_Z}$ maps $R_{a-1}(\underline{b}')$ onto $R_n(\underline{b};\mathrm{Z})$.

Let $\sigma = c_1c_2\cdots c_{a-1}$ be in $R_{a-1}(\underline{b}')$. Write $(\sigma\pi_{\mathrm{Z}})Z = d_1d_2\cdots d_n$. For $1 \le i \le a-1$ we have

$$b_i = b_i' \le c_i \le c_i\pi_{\mathrm{Z}} = d_i.$$

For $a \le i \le n$ we have $d_i \in \{b_n, \ldots, n\}$ so $b_i = b_n \le d_i$. Thus $\mathrm{f_Z}(\sigma) \in R_n(b')$.

Now let $\tau = d_1 \ldots d_{a-1}Z$ be in $R_n(\underline{b};\mathrm{Z})$. Then $\tau = \mathrm{f_Z}(\sigma)$ where $\sigma = (d_1 \ldots d_{a-1})\pi_{\mathrm{Z}}^{-1}$. Either $d_i\pi_{\mathrm{Z}}^{-1} = d_i$ in which case $b_i' = b_i \le d_i$ or $d_i\pi_{\mathrm{Z}}^{-1} \in R_s \setminus L_s$ and in this case $b_i' = b_i \le b_a \le d_i\pi_{\mathrm{Z}}^{-1}$. So $\sigma \in R_{a-1}(\underline{b}')$ which shows that $\mathrm{f_Z}$ is onto. This proves Claim 1. ■

<u>Claim 2</u>: As a module for $Sym(R_s)$, $R_n(\underline{b})$ is isomorphic to the induction of $R_{a-1}(\underline{b}')$ from $Sym(R_s \setminus L_s)$ to $Sym(R_s)$.

<u>Proof</u>: Let A denote $Sym(R_s \setminus L_s)$ and B denote $Sym(R_s)$. Note that the collection $\{\pi_{\mathrm{Z}} : \mathrm{Z} \in \mathcal{Z}\}$ is a set of coset representatives for A in B. So there is a vector space isomorphism between $R_{a-1}(\underline{b}') \bigotimes_A B$ and $R_n(\underline{b})$ given by:

$$\psi(\sigma \otimes \pi_{\mathrm{Z}}) = (\sigma\pi_{\mathrm{Z}})\mathrm{Z}.$$

Let $g \in B$ and let $\mathrm{Z}g = Y$. Then g acts on $(\sigma\pi_{\mathrm{Z}})\mathrm{Z}$ by

$$(\sigma\pi_{\mathrm{Z}})\mathrm{Z} \cdot g = ((\sigma\pi_{\mathrm{Z}})g)Y = (\sigma \cdot \pi_{\mathrm{Z}}g\pi_{\mathrm{Z}}^{-1})\pi_Y Y. \qquad < 4.12 >$$

From $< 4.12 >$ and the usual construction of induced modules (see [F]) one sees that ψ is a B-equivariant isomorphism which proves Claim 2. ■

We are now in a position to finish part (1) of Theorem 4.10. By our induction hypothesis, $R_{a-1}(\underline{b}')$ is isomorphic, as a G'-module, to the direct sum of modules $R_{a-1}(\beta')$ for $\beta' \in P(\underline{b}')$. Each β' has the form

$$\beta' = (\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \mu_{s-2}, \lambda_{s-1})$$

and the corresponding module $R_{a-1}(\beta')$ is isomorphic to the tensor product

$$V_1 \otimes \ldots \otimes V_{s-1} \otimes W_1 \otimes \ldots \otimes W_{s-1}$$

where $V_i \cong V[\lambda_i/\mu_{i-1}]$ and $W_i \cong V[\lambda_i/\mu_i]$. When we restrict $R_{a-1}(\beta')$ to $\hat{G}$ we obtain a similar direct sum but this time indexed by sequences

$$\hat{\beta} = (\lambda_1, \mu_1, \ldots, \mu_{s-2}, \lambda_{s-1}, \mu_{s-1})$$

where $\mu_s \subseteq \lambda_{s-1}$ and $|\lambda_{s-1}/\mu_{s-1}| = |R_{s-1}|$ (or equivalently $|\mu_{s-1}| = |R_s \setminus L_s|$). The representation of $\hat{G}$ on this $R_n(\hat{\beta})$ is

$$V_1 \otimes \ldots \otimes V_{s-1} \otimes W_1 \otimes \ldots \otimes W_{s-2} \otimes (V[\lambda_s/\mu_{s-1}] \otimes V[\mu_{s-1}]).$$

By Claim 2, $R_n(\hat{\beta})$ indexes a G-invariant subspace of $R_n(\underline{b})$ which as a G-module is isomorphic to

$$< 4.13 > \qquad V_1 \otimes \ldots \otimes V_{s-1} \otimes W_1 \otimes \ldots \otimes W_{s-2} \otimes W_{s-1} \otimes \text{ ind}\,{}^{Sym(R_s)}_{Sym(R_s \setminus L_s)}(V[\mu_{s-1}]).$$

But in order to regard $< 4.13 >$ as a G-module it is important to view the right-most tensor factor as a module over $Sym(L_s) \times Sym(R_s)$ where $Sym(L_s)$ acts from the left and $Sym(R_s)$ acts from the right.

Now applying Proposition 4.9 gives us part (1) of Theorem 4.10.

Part (2) of Theorem 4.10 follows easily from part (1) and the fact that the dimension of $V[\lambda/\mu]$ is the number of standard Young tableaux of shape λ/μ. ∎

We now return to the transition matrix $\mathcal{U}$ encountered in our random walk on $R_n(\underline{b})$.

Definition 4.14: Let $\beta = (\lambda_1, \mu_1, \ldots, \mu_{s-1}, \lambda_s)$ be in $P(\underline{b})$. Define the *indicator tableau* $T(\beta)$ to be the tableau whose entry in a square x is the number of skew shapes λ_i/μ_{i-1} that contain x. We denote this entry by $T_x(\beta)$.

Theorem 4.15: The matrix $\mathcal{U}$ preserves the space $R_n(\beta)$. Moreover, $\mathcal{U}$ restricted to $R_n(\beta)$ is the scalar

$$\Lambda(\beta) = \sum_x T_x(\beta) c_x$$

where c_x is the content of x.

Proof: The proof will proceed by induction on n. The cases $n = 1, 2$ are straightforward to verify. Assume the result is true for $n-1$ and consider a sequence $\underline{b} = (b_1, \ldots, b_n)$.

From this point on, our proof will proceed in a series of steps. This argument depends on appropriately ordering our basis for $R_n(\underline{b})$. So we begin with a discussion of this ordering. Whenever we have an N and a sequence $\gamma = (\gamma_1, \ldots, \gamma_N)$ we will assume that the set $R_N(\gamma)$ is ordered in reverse lexicographic order reading permutations from left to right. In other words, to decide whether $\alpha = a_1 a_2 \ldots a_N$ comes before $\delta = d_1 d_2 \ldots d_N$ first compare a_N to d_N. If a_N is bigger then α comes first. If a_N is smaller then δ comes first. If $a_N = d_N$ then move on and compare a_{N-1} to d_{N-1}, etc... For example, if $\underline{b} = (1, 1, 2)$ then the appropriate ordering of $R_3(\underline{b})$ is

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{array}$$

STEP 1: Split the ordered basis $R_n(\underline{b})$ as the disjoint union of $R_n^{(n)}(\underline{b}) \cup R_n^{(n-1)}(\underline{b}) \cup \ldots \cup R_n^{(b_n)}(\underline{b})$ where $R_n^{(i)}(\underline{b})$ consists of all permutations in $R_n(\underline{b})$ that have i in their final position. Note that each set $R_n^{(i)}(\underline{b})$ is identical (as an *ordered* set) to a copy of $R_{n-1}(\underline{b}')$ where $\underline{b}' = (b_1, \ldots, b_{n-1})$.

STEP 2: Write the linear transformation $\mathcal{U}$ as $\mathcal{L} + \mathcal{M}$ where $\mathcal{L}_{\alpha,\gamma} = 0$ unless the last positions of α and γ are identical and where $\mathcal{M}_{\alpha,\gamma} = 0$ if the last positions of α and γ are identical. To put this another way, $\mathcal{L}$ represents all switches in which neither occurs in the last position and $\mathcal{M}$ represents all switches in which one of the two elements occurs in the last position.

It is easy to see that $\mathcal{L}$, in terms of the decomposition of the ordered basis $R_n(\underline{b})$ given in STEP 1, is block diagonal with all blocks equal to the matrix $\mathcal{U}'$, the matrix corresponding to $\mathcal{U}$ in the case $\underline{b}' = (b_1, \ldots, b_{n-1})$.

STEP 3: $\mathcal{L}$ and $\mathcal{M}$ commute.

We will compare $\mathcal{M} \cdot \mathcal{L}(\alpha)$ to $\mathcal{L} \cdot \mathcal{M}(\alpha)$. $\mathcal{M} \cdot \mathcal{L}(\alpha)$ is a sum of permutations obtained by first switching a_i and a_j where $i < j < n$ and then switching a_ℓ and a_n where $\ell < n$. If ℓ is distinct from i and j then the same term occurs in $\mathcal{L} \cdot \mathcal{M}(\alpha)$. Suppose now that $\ell = j$. Then the resulting permutation looks like

$$\begin{array}{ccccccc} & & i & & j & & n \\ & & \downarrow & & \downarrow & & \downarrow \\ a_1 & \ldots & a_n & \ldots & a_i & \ldots & a_j \end{array}$$

The same term occurs in $\mathcal{L} \cdot \mathcal{M}(\alpha)$ when we first switch a_j with a_n then we switch a_i with a_n (this second switch is included in $\mathcal{L}$ because a_n is in position j after the first switch).

The only subtlety is when $\ell = i$. The permutation in $\mathcal{M} \cdot \mathcal{L}(\alpha)$ which results from switching a_i with a_j then a_i with a_n is

$$\begin{array}{ccccccc} & & i & & j & & n \\ & & \downarrow & & \downarrow & & \downarrow \\ a_1 & \ldots & a_j & \ldots & a_n & \ldots & a_i \end{array}$$

The corresponding term in $\mathcal{L} \cdot \mathcal{M}(\alpha)$ is where we first switch a_i with a_n then switch a_n with a_j. But we need to be sure that the first switch can be made, i. e. , that $a_n \geq b_i$. However this is clear since a_n started in position n so $a_n \geq b_n \geq b_i$.

This shows that every term which occurs in $\mathcal{M} \cdot \mathcal{L}(\alpha)$ also occurs in $\mathcal{L} \cdot \mathcal{M}(\alpha)$. A similar argument shows that every term occuring in $\mathcal{L} \cdot \mathcal{M}(\alpha)$ also occurs in $\mathcal{M} \cdot \mathcal{L}(\alpha)$ which completes STEP 3.

STEP 4: We now apply our induction hypothesis to the sequence $\underline{b}' = (b_1, \ldots, b_{n-1})$. Let β' be a $\underline{b}'$-partition which ends in some partition λ' of $n - b_{n-1}$. Let $\tau' = (\emptyset, \tau_1, \tau_2, \ldots, \lambda')$ be an up-down tableau of type $\underline{b}'$ and shape β', but truncated at the (last) position where λ' occurs. Let W be the corresponding subspace of $\mathbb{C}R_{n-1}(\underline{b}')$ which is isomorphic to $S^{\lambda'}$ as an $S_{n-b_{n-1}}$-module and on which $\mathcal{U}'$ acts as the scalar $\Lambda(\beta')$. Recall that $S_{n-b_{n-1}}$ acts on $R_{n-1}(\underline{b}')$ by permuting the numbers $b_{n-1}, b_{n-1}+1, \ldots, n-1$. Let S_{n-b_n} denote the subgroup which permutes $b_n, b_n+1, \ldots, n-1$.

Restrict W to S_{n-b_n} and let μ be a partition of $n - b_n$ contained in λ'. The number of occurrences of S^μ in W is exactly the number of chains of partitions which begin at λ' and end at μ in which each partition is obtained from the previous one by removing a square. So we can think of the S^μ-isotypic component of W as being the direct sum of copies of S^μ, each copy indexed by such a chain from λ' down to μ. Let V be the copy of S^μ in W indexed by one such chain.

$$\begin{array}{c} (\overbrace{\emptyset, \tau_1, \tau_2, \ldots, \lambda'}^{\tau'}, \mu_1, \ldots, \mu = \mu_{b_n - b_{n-1}}) \\ \qquad\qquad \updownarrow \qquad\quad \updownarrow \\ \mathbb{C}R_{n-1}(\underline{b}') \geq W \quad \geq \quad V \end{array}$$

STEP 5: Recall that V is a subspace of $\mathbb{C}R_{n-1}(\underline{b}')$. For each $i = b_n, b_n+1, \ldots, n$, let $V^{(i)}$ denote the copy of V obtained from the identification of $R_n^{(i)}(\underline{b})$ with $R_{n-1}(\underline{b}')$.

From our choice of V in STEP 4 we have that $\mathcal{U}'$ acts like the scalar $\Lambda(\beta')$ on V. Hence $\mathcal{L}$ acts like the scalar $\Lambda(\beta')$ on each $V^{(i)}$.

STEP 6: Let $X = V^{(b_n)} \oplus V^{(b_n+1)} \oplus \ldots \oplus V^{(n)}$. One important observation about X is that it is isomorphic as an S_{n+1-b_n}-module (where S_{n+1-b_n} permutes $b_n, b_n+1, \ldots, n$) to the induction from S_{n-b_n} to S_{n+1-b_n} of V. This follows easily by the construction of the sets $R_n^{(i)}(\underline{b})$. It follows that X is isomorphic as an S_{n+1-b_n}-module to the direct sum of irreducibles S^λ where λ contains μ, each S^λ occurring with multiplicity 1.

Next we examine the transformation $\mathcal{M}$. Recall that $\mathcal{M}$ accounts for all possible switches which involve the entry in the n^{th} position. So $\mathcal{M}$ is left multiplication by the Murphy element $\underline{m}$ in S_{n+1-b_n},

$$\underline{m} = (b_n, n) + (b_n + 1, n) + \ldots + (n-1, n).$$

It is well-known that the Murphy element acts like a scalar on each irreducible S^λ occurring in $ind_{S_{n-b_n}}^{S_{n+1-b_n}}(S^\mu)$ and that scalar is just the content of the square λ/μ. So $\mathcal{M}$ acts like the scalar on $j-i$ on S^λ where λ/μ is the square in row i and column j. Also $\mathcal{L}$ acts like the scalar $\Lambda(\beta')$ on each $V^{(i)}$ hence on X, so $\mathcal{L}+\mathcal{M} = \mathcal{U}$ acts like the scalar $\Lambda(\beta) = \Lambda(\beta')+j-i$ on S^λ where this action is being parametrized by the truncated up-down tableau τ of type $\underline{b}$ and shape β given by

$$\tau = (\overbrace{\emptyset, \tau_1, \tau_2, \ldots, \lambda'}^{\tau'}, \mu_1, \ldots, \overbrace{\mu_{b_n - b_{n-1}}}^{\mu}, \lambda).$$

This completes the proof of the theorem. ■

Before listing corollaries of this theorem we do an example. Let $\underline{b} = (1, 1, 2, 2)$. The permutations in $R_n(\underline{b})$ listed according to our canonical ordering are:

1	2	3	4
2	1	3	4
1	3	2	4
3	1	2	4
1	2	4	3
2	1	4	3
1	4	2	3
4	1	2	3
1	3	4	2
3	1	4	2
1	4	3	2
4	1	3	2

The matrix $\mathcal{U}$ written with respect to that ordered basis is:

$$\begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}$$

The symmetric group S_3 (permuting the numbers 2, 3, 4) acts on $R_4(\underline{b})$ and the matrix $\mathcal{U}$ commutes with this action. Theorem 4.13 gives the eigenvalues of $\mathcal{U}$ and the decomposition of the corresponding eigenspaces as S_3-modules.

The $\underline{b}$-partitions, the truncated up-down tableaux of type $\underline{b}$, the indicator tableaux and the eigenvalues in this case are:

β	τ	$T(\beta)$	$\Lambda(\beta)$
$(2,1,3)$	$(\emptyset,1,2,1,2,3)$	121	4
$(1^2,1,3)$	$(\emptyset,1,1^2,1,2,3)$	111 1	2
$(2,1,21)$	$(\emptyset,1,2,1,2,21)$	12 1	1
$(1^2,1,21)$	$(\emptyset,1,1^2,1,2,21)$	11 2	-1
$(2,1,21)$	$(\emptyset,1,2,1,1^2,21)$	12 1	1
$(1^2,1,21)$	$(\emptyset,1,1^2,1,1^2,21)$	11 2	-1
$(2,1,1^3)$	$(\emptyset,1,2,1,1^2,1^3)$	11 1 1	-2
$(1^2,1,1^3)$	$(\emptyset,1,1^2,1,1^2,1^3)$	1 2 1	-4

Reading from this chart we obtain the following list of eigenvalues and S_3-decompositions of the corresponding eigenspaces:

eigenvalueΛ	multiplicity	$S_3 -$ decomposition of the eigenspace
4	1	3
2	1	3
1	4	$21 \oplus 21$
-1	4	$21 \oplus 21$
-2	1	1^3
-4	1	1^3

Theorem 4.15 has some interesting corollaries:

Corollary 4.16: 1) The eigenvalues of $\mathcal{U}$ are integers.
2) The multiplicity of Λ as an eigenvalue is the same as the multiplicity of $-\Lambda$, but the S_{n+1-b_n} structure of their corresponding eigenspaces differ by tensoring with the sign representation.

Corollary 4.17: The cardinality of $R_n(\underline{b})$ is equal to the number of up-down tableaux of type $\underline{b}$. The multiplicity of Λ as an eigenvalue of $\mathcal{U}$ is equal to the number of up-down tableaux τ of type $\underline{b}$ with $\Lambda(\tau) = \Lambda$.

Section 5: Applications

We can use the spectral information from Section 4 to obtain sharp estimates for the rate of convergence of our Markov chain to the uniform distribution. We will consider two cases and in each one sketch how the results from the previous section apply.

We model our analysis after that given in [D] for the case $\underline{b} = (1, 1, \ldots, 1)$. However several of the bounds given in [D] do not apply directly and must be modified to handle our general case. Details will appear in a subsequent paper containing work which is joint with Persi Diaconis.

The general method goes as follows. First establish that the $G_n(\underline{b})$ has a transitive automorphism group (this makes the starting point of your Markov chain irrelevant). Given that, you have the following upper bound for the total variation distance (see [DS] for a derivation of this bound).

Lemma 5.1 For $\alpha \in R_n(\underline{b})$ let $\rho_\alpha^k(\sigma)$ be probability that you are in state σ after k steps in the Markov chain on the graph $G_n(\underline{b})$ from starting state α. Let U be the uniform distribution on $R_n(\underline{b})$. Then $||\rho_\alpha^k - U||$ does not depend on α and an upper bound is given by

$$||\rho_\alpha^k - U|| \leq \frac{1}{2} \left\{ \sum_{\Lambda \neq 1} \Lambda^{2k} \right\}^{\frac{1}{2}}$$

where the sum over $\Lambda \neq 1$ is over eigenvalues of the transition matrix $\mathcal{U}$.

The results of Section 4 tell us all the eigenvalues that appear in the sum

$$\sum_{\Lambda \neq 1} \Lambda^{2k} \tag{$<5.2>$}$$

The idea is to carefully analyze the contribution made by a few Λ that dominate the sum and to bound the contribution made by the others. Certainly the second largest eigenvalue will be amongst those that dominate the sum.

The last step is to find a lower bound to $||\rho_\alpha^k - U||$ that agrees with our upper bound. This is accomplished by using an alternative definition of total variation distance, namely

$$||\rho_\alpha^k - U|| = \max_{A \subseteq R_n(\underline{b})} \sum_{x \in A} |\rho_\alpha^k(x) - U(x)|. \tag{$<5.3>$}$$

where the sum is over all subsets A in $R_n(\underline{b})$. If we choose a subset A so that its indicator function has a significant projection onto the eigenspace of some Λ which dominates the sum $< 5.2 >$ then $\sum_{x \in A} |\rho_\alpha^k(x) - U(x)|$ will be of the same order of magnitude as the upper bound $< 5.2 >$. By $< 5.3 >$ this gives a lower bound to total variation distance.

As mentioned above, most of this work will appear in an upcoming paper. Here we will first indicate how to find the second largest eigenvalue and show how that leads to a cut-off value of k in two interesting special cases.

We begin by restating the main result from Section 4 in a friendlier way. Visualize the Ferrers board B from Sec.1 as the $n \times n$ grid whose southwest corner is at $(0,0)$. Hence the northwest corner is at $(0,n)$ and the southeast corner is at $(n,0)$. In column j of this grid, shade in the top $b_j - 1$ squares. Let P be the path from the northwest corner to the southeast corner which runs along the top of B until the shaded portion is reached, then runs along the southern boundary of the shaded portion and finally along the eastern boundary of B. We view P as a path consisting of $2n$ steps, each one going one unit to the right or one unit down. The normalization factor $n + 2\triangle$ is equal to n^2 minus twice the shaded area.

Theorem 4 states that the eigenvalues of $\mathcal{U}$ are indexed by sequences of tableaux, one labelling each point of P, where the tableaux τ_1 and τ_2 indexing consecutive points differ by adding a square or subtracting a square depending on whether the corresponding step in P goes to the right or down. The eigenvalues associated to such a sequence is the sum of the contents of the squares added at the right-hand steps.

The eigenvalues associated to a sequence of tableaux depends only on the particular tableaux which occur at the corners of the path P. Let $2s-1$ denote the number of corners of P, let $\gamma_1, \ldots, \gamma_{2s-1}$ be the corners of P and let (u_i, v_i) denote the coordinates of the corner γ_i. It is easy to see that the odd corners γ_{2i-1} point to the northeast whereas the even corners γ_{2i} point to the southwest.

If $\tau = (\emptyset, \tau_1, \tau_2, \ldots, \tau_{2n-1}, \emptyset)$ is a sequence of tableaux labelling P then the size of the tableau labelling the corner γ_i is $(u_i + v_i - n)$. The eigenvalue of $\mathcal{U}$ associated to τ depends only on the subsequence $(\lambda_1, \mu_1, \lambda_2, \ldots, \mu_{s-1}, \lambda_s)$ of τ (following the notation of Sec. 4). Here λ_i (μ_i resp.) is the element of τ corresponding to the corner γ_{2i-1} (γ_{2i} resp.). According to Theorem 4.15 this eigenvalue is the sum over i of the sum of the contents in λ_i/μ_{i-1}. Here i ranges from 1 to s, where μ_0 is defined to be $\emptyset$. The multiplicity of this eigenvalue is a product over i of $\deg(\chi_{\lambda_i/\mu_{i-1}})\deg(\chi_{\lambda_i/\mu_i})$. The maximum eigenvalue of $\mathcal{U}$ is $\triangle$ and this corresponds to π_i being a single row for all i. The second largest eigenvalue occurs when exactly one of the λ_i is $(u_{2i+1} + v_{2i+1} - n - 1, 1)$. This eigenvalue is $\triangle - (u_{2i+1} + v_{2i+1} - n)$ and the corresponding multiplicity is $(v_{2i+1} - v_{2i} - 1)(u_{2i+1} - u_{2i+2} - 1)$.

We now see how this works in two special cases.

Case 1: Let $\underline{b} = (1, 1, \ldots, 1, 2, \ldots, 2)$ where there are n^α 1's and $(n - n^\alpha)$ 2's. Here α is a fixed element of $(0, 1]$. In terms of permutations, $\sigma = a_1 \cdots a_n$ is in $R_n(\underline{b})$ if and only if 1 occurs in one of the first n^α positions. It is straightforward to check that $G_n(\underline{b})$ has a transitive automorphism group and so Lemma 5.1 applies.

There are three corners in the path P, one at (n^α, n) the next at $(n^\alpha, n-1)$ and the last at $(n, n-1)$. So P starts at $(0, n)$, goes to the right n^α steps, goes down 1, goes to right another $n - n^\alpha$ steps and then down $n-1$. In this case we have $n + 2\triangle = n^2 - 2(n - n^\alpha)$. There are two candidates for second largest eigenvalue.

Eigenvalue Λ_1: This eigenvalue corresponds to $(\pi_1, \pi_2, \pi_3) = ((n^\alpha - 1, 1), (n^\alpha - 1), (n-1))$. The eigenvalue Λ_1 of $\mathcal{T}$ is

$$1 - \frac{2n^\alpha}{n^2 - 2n + 2n^\alpha}$$

and the multiplicity is $n^\alpha - 1$.

Eigenvalue Λ_2: This eigenvalue corresponds to $(\pi_1, \pi_2, \pi_3) = ((n^\alpha), (n^\alpha - 1), (n - 2, 1))$. This eigenvalue Λ_2 of $\mathcal{T}$ is

$$1 - \frac{2(n-1)}{n^2 - 2n + 2n^\alpha}$$

and the multiplicity is $(n-n^\alpha)(n-2)$.

In order to get $\sum\limits_{\Lambda\neq 1}\Lambda^{2k}$ close to 0 we must choose k so that both

$$< 5.4 > \qquad (n^\alpha-1)\left(1-\frac{2n^\alpha}{n^2-2n+2n^\alpha}\right)^{2k} \sim e^{\log(n^\alpha)+2k\log\left(1-\frac{2n^\alpha}{n^2}\right)}$$

$$< 5.5 > \qquad (n-n^\alpha)(n-2)\left(1-\frac{2(n-1)}{n^2-2n+2n^\alpha}\right)^{2k} \sim e^{\log(n^2)+2k\log\left(1-\frac{2}{n}\right)}$$

both approach 0. To bound $< 5.4 >$ we must have $\alpha\log(n) = 2k(n^{\alpha-2})$, i. e. , $k = \left(\frac{\alpha}{2}\right)n^{2-\alpha}\log n$. To bound the second sum we require $2\log n = 2k\left(\frac{2}{n}\right)$, i. e. , $k=\frac{1}{2}n\log n$. For $\alpha<1$, the first condition is more restrictive. In our upcoming paper, we will prove the following result:

Theorem 5.6: Let $\underline{b} = (1^{n^\alpha}, 2^{n-n^\alpha})$ where α is fixed. Let $k = \left(\frac{\alpha}{2}\right)n^{2-\alpha}\log n + cn^2$. For $c>0$ we have

$$||P^k-U|| \leq \frac{1}{2}e^{-c}$$

For $c<0$, $||P^k-U||$ is bounded away from 0 by a function of c which is independent of n.

Notice that this matches the result obtained by Diaconis and Shahshahani in the case $\alpha=1$. The case $\alpha=0$ merits some discussion. As this is set up, the case $\alpha=0$ would be $\underline{b}=(1,2,2,\ldots,2)$ which requires 1 to always remain in position 1. So this is the usual random transposition walk on S_{n-1}. The eigenvalue $< 5.4 >$ does not occur and $< 5.5 >$ gives the cut-off $k=\frac{1}{2}n\log n$ as follows from the work in [DS]. A more interesting interpolation of the $\alpha=0$ case is to consider $(1^\ell, 2^{n-\ell})$ where ℓ is bounded. In this case the eigenvalue $< 5.4 >$ does occur with multiplicity $(\ell-1)$ and we obtain cut-off at $k=0(n^2)$ which agrees with the statement in Theorem 5.6 up to the constant.

Case 2: Let $\underline{b}=(1,1,\ldots,1,2,3,4,\ldots,n-t+1)$ where this sequence begins with t ones. The path up P takes t steps to the right then alternately steps down and to the right until the line $x=n$ is reached and finishes with t steps down. Equivalently, the sequences of tableaux labelling P are obtained by adding t squares, then alternately subtracting and adding a square $n-t$ times, and then subtracting t squares. The second largest eigenvalue of $\mathcal{T}$ can be determined using the ideas described earlier in this section. It is $2nt-n-t-t^2$ with multiplicity $(n+t-3)$. The dominant term in $\sum\limits_{\Lambda\neq 1}\Lambda^{2k}$ is

$$< 5.7 > \qquad (n+t-3)\left(1-\frac{2t}{n(2t-1)-t^2+t}\right)^{2k}$$

If t is fixed (relative to n) this gives a cut-off at $k = \left(\frac{2t-1}{4t}\right) n \log n$. A special case of this is when $t = 2$. We have seen that $G_n(\underline{b})$ is the hypercube of dimension $n-1$. In this case, the cut-off occurs at $\frac{3}{8} n \log n$. This differs from the $\frac{1}{4} n \log n$ derived in [DS]. The difference results from a difference in "staying probabilities". In our case there is a $\frac{1}{3}$ probability of staying in each state as opposed to a probability of $1/(n+1)$ in the case analyzed by Diaconis and Shashahani.

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