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A LARGE WIENER SAUSAGE FROM CRUMBS

OMER ANGEL
Department of Mathematics, The Weizmann Institute of Science,
Rehovot, Israel 76100
email: `omer@wisdom.weizmann.ac.il`

ITAI BENJAMINI
Department of Mathematics, The Weizmann Institute of Science,
Rehovot, Israel 76100
email: `itai@wisdom.weizmann.ac.il`
`http://www.wisdom.weizmann.ac.il/~itai/`

YUVAL PERES
Institute of Mathematics, The Hebrew University, Jerusalem
and Department of Statistics, University of California, Berkeley, USA
email: `peres@math.huji.ac.il`
`http://www.math.huji.ac.il/~peres/`

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Abstract
Let $B(t)$ denote Brownian motion in $\mathbb{R}^d$. It is a classical fact that for any Borel set A in $\mathbb{R}^d$, the volume $V_1(A)$ of the Wiener sausage $B[0,1]+A$ has nonzero expectation iff A is nonpolar. We show that for any nonpolar A, the random variable $V_1(A)$ is unbounded.

1. **Introduction**

The impetus for this note was the following message, that was sent to one of us (Y. P.) by Harry Kesten:

> "... *First a question, though. It is not of major importance but has bugged me for a while in connection with some large deviation result for the Wiener sausage (with Yuji Hamana). Let $V_1(A)$ be the volume of the Wiener sausage at time* 1*, that is, $V_1(A) = \mathrm{Vol}_d\Big(\cup_{s\le 1} B_s + A\Big)$, where B_s is d-dimensional Brownian motion, and A is a d-dimensional*

set of positive capacity. Is it true that the support of $V_1(A)$ is unbounded, i.e., *is $\mathbf{P}[V_1(A) > x] > 0$ for all x? This is easy if A has a section of positive $(d-1)$-dimensional Lebesgue measure, but I cannot prove it in general. Do you have any idea? "*

We were intrigued by this question, because it led us to ponder the source of the volume of the Wiener sausage when A is a "small" set (*e.g.*, a nonpolar set of zero Hausdorff dimension, in the plane). Is it due to the macroscopic movement of B (in which case $V_1(A)$ would not be bounded) or to the microscopic fluctuations (in which case $V_1(A)$ might be bounded, like the quadratic variation)?

Our proof of the following theorem indicates that while the microscopic fluctuations of B are necessary for the positivity of $V_1(A)$, the macroscopic behaviour of B certainly affects the magnitude of $V_1(A)$.

Theorem 1. *If the capacity $\mathcal{C}(A)$ of $A \subset \mathbb{R}^d$ is positive, then $V_1(A)$ is not bounded.*

The relevant capacity can be defined for $A \subset \mathbb{R}^d$ with $d \geq 3$, by

$$\begin{aligned} \mathcal{C}(A) &= \sup_\mu \frac{\mu(A)^2}{\mathcal{E}(\mu)} \\ \text{where} \quad \mathcal{E}(\mu) &= \iint \frac{c_d \, d\mu(x) \, d\mu(y)}{|x-y|^{d-2}}, \end{aligned}$$

and the supremum is over measures supported on A. (the constant c_d is unimportant for our purpose). A similar formula holds for $d = 2$ with a logarithmic kernel; in that case $\mathcal{C}(A)$ is often called Robin's constant, and it will be convenient to restrict attention to sets A of diameter less than 1.

Denote by τ_A the hitting time of A by Brownian motion. By Fubini's theorem

$$\mathbf{E}[V_1(A)] = \int_{\mathbb{R}^d} \mathbf{P}_x[\tau_A \leq 1] \, dx .$$

It follows from the relation between potential theory and Brownian motion, that $\mathbf{E}[V_1(A)]$ is nonzero if and only if A has positive capacity; see, *e.g.*, [3], [2], or [4].

2. The recipe

For any kernel $K(x, y)$, the corresponding capacity is defined by $\mathcal{C}_K(A) = \sup_\mu \frac{\mu(A)^2}{\mathcal{E}_K(\mu)}$ where $\mathcal{E}_K(\mu) = \iint K(x, y) \, d\mu(x) \, d\mu(y)$ and the supremum is over measures on A. We assume that $K(x, x) = \infty$ for all x, and that for $0 < |x - y| < R_K$, the kernel K is continuous and $K(x, y) > 0$.

The following lemma holds for all such kernels.

Lemma 1. *If a set $A \subset \mathbb{R}^d$ has $\mathcal{C}_K(A) > 0$, then for any $L < \infty$ there exists $\epsilon > 0$ and subsets $A_1, A_2, \ldots, A_m$ of A such that $\sum_{i=1}^m \mathcal{C}_K(A_i) \geq L$, and the distance between A_i and A_j is at least ϵ for all $i \neq j$. (m and ϵ depend on A and L).*

Proof: We can assume that $\mathrm{diam}(A) < R_K$, for otherwise we can replace A by a subset of positive capacity and diameter less than R_K.

Let μ be a measure supported on A such that $\mu(A) = 1$ and $\mathcal{E}_K(\mu) < \infty$.

By dominated convergence,

$$\lim_{\delta\to 0}\iint_{|x-y|\le\delta} K(x,y)\,d\mu(x)\,d\mu(y) = 0.$$

Choose δ so that this integral is less then $2^{-2d}L^{-1}$. Let $\epsilon = \delta d^{-1/2}$ and let $\mathcal{F}$ be a grid of cubes of side ϵ, *i.e.*,

$$\mathcal{F} = \Big\{ \prod_{i=1}^{d} [\epsilon\ell_i, \epsilon\ell_i + \epsilon) \,:\, (\ell_1, \dots, \ell_d) \in \mathbb{Z}^d \Big\}.$$

We can partition $\mathcal{F}$ into 2^d subcollections $\Big\{\mathcal{F}_v \,:\, v \in \{0,1\}^d\Big\}$ according to the vector of parities of $(\ell_1, \dots, \ell_d)$. Then the distance between any two cubes in the same $\mathcal{F}_v$ is at least ϵ. Since μ is a probability measure, there exists $v \in \{0,1\}^d$ such that

$$\sum_{Q\in\mathcal{F}_v} \mu(Q) \ge 2^{-d}\,. \tag{1}$$

Let $A_1, A_2, \dots, A_m$ be all the nonempty sets among $\{A \cap Q \,:\, Q \in \mathcal{F}_v\}$. Since μ is supported on A, we can rewrite (1) as $\sum_{i=1}^m \mu(A_i) \ge 2^{-d}$.
Denote by $e_i = \iint_{A_i\times A_i} K(x,y)\,d\mu\,d\mu$ the energy in A_i. Then

$$\sum_{i=1}^{m} e_i \le \iint_{|x-y|\le\delta} K(x,y)\,d\mu\,d\mu < 2^{-2d}L^{-1}\,. \tag{2}$$

By Cauchy-Schwarz,

$$\Big(\sum_{i=1}^{m} e_i\Big)\Big(\sum_{i=1}^{m}\frac{\mu(A_i)^2}{e_i}\Big) \ge \Big(\sum_{i=1}^{m}\mu(A_i)\Big)^2 \ge 2^{-2d}\,. \tag{3}$$

We have $\mathcal{C}_K(A_i) \ge \mu(A_i)^2/e_i$, whence

$$\sum_{i=1}^{m}\mathcal{C}_K(A_i) \ge \sum_{i=1}^{m}\frac{\mu(A_i)^2}{e_i} \ge L\,,$$

by (2) and (3).

□

Proof of Theorem 1: Suppose that

$$\operatorname{esssup} V_1(A) = M < \infty\,. \tag{4}$$

Let $V_t(A)$ denote the volume of the Wiener sausage $B[0,t] + A$. From Spitzer [3] (see also [2] or [1]) it follows that $\mathbf{E}[V_1(A)] > 2\alpha_d\mathcal{C}(A)$ for some absolute constant α_d. (If $d = 2$ we assume that $\operatorname{diam}A < 1$). We infer that $\mathbf{E}[V_t(A)] > \alpha_d t\mathcal{C}(A)$ for $0 < t < 1$, by subadditivity of Lebesgue measure and monotonicity of $V_t(A)$,
Fix $L > 6M/\alpha_d$, and let $A_1, \dots, A_m$ be the subsets of A given by the lemma. A Wiener sausage on A contains the union of Wiener sausages on the A_i, and the sum of their volumes is expected to be large. If we can arrange for the intersections to be small, then $V_1(A)$ will be large as well.

Consider the event

$$H_n = \Big\{ \max_{0 \le s \le \frac{1}{2n}} |B_s| < \frac{\epsilon}{2} \Big\} .$$

By Brownian scaling and standard estimates for the maximum of Brownian motion,

$$\mathbf{P}[H_n^c] \le 4d \exp(-\frac{n\epsilon^2}{4d}) .$$

Choose n large enough so that the right-hand side is less than $\frac{1}{nm}$. For each i, we have $\mathbf{E}[V_{\frac{1}{2n}}(A_i) \mid H_n^c] \le M$ by (4), so

$$\mathbf{E}[V_{\frac{1}{2n}}(A_i) \mid H_n] \ge \mathbf{E}[V_{\frac{1}{2n}}(A_i)] - M\mathbf{P}[H_n^c] \ge \frac{\alpha_d}{2n}\mathcal{C}(A_i) - \frac{M}{mn} . \qquad (5)$$

For $0 \le j < n$, denote by G_j the event that

$$\max_{\frac{2j}{2n} \le s \le \frac{2j+1}{2n}} |B_s - B_{\frac{2j}{2n}}| < \epsilon/2$$

and the first coordinate of the increment $B_{\frac{2j+2}{2n}} - B_{\frac{2j+1}{2n}}$ is greater than the diam$(A)+2\epsilon$. Define $G = \cap_{j=0}^{n-1} G_j$. We will see that the expectation of $V_1(A)$ given G is large.
On the event G, for each fixed j, the m sausages $\{B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A_i\}_{i=1}^m$ are pairwise disjoint due to the separation of the A_i and the localization of B in the time interval $[\frac{2j}{2n}, \frac{2j+1}{2n}]$. Therefore,

$$\mathbf{E}\Big[\mathrm{Vol}_d\Big(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A\Big) \;\Big|\; G\Big] \ge \sum_{i=1}^m \Big(\frac{\alpha_d}{2n}\mathcal{C}(A_i) - \frac{M}{mn}\Big) \ge \frac{\alpha_d L}{2n} - \frac{M}{n} > \frac{2M}{n} .$$

Also, on G, the sausages on the odd intervals, $B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A$ for $0 \le j < n$, are pairwise disjoint due to the large increments of B (in the first coordinate) on the even intervals. We conclude that

$$\mathbf{E}[V_1(A) \mid G] \ge \sum_{j=0}^{n-1} \mathbf{E}\Big[\mathrm{Vol}_d\Big(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A\Big) \;\Big|\; G\Big] > 2M . \qquad (6)$$

This contradicts the assumption (4) and completes the proof. ∎

Questions:

- Can the event G that we conditioned on at the end of the preceding proof, be replaced by a simpler event involving just the endpoint of the Brownian path? In particular, does every nonpolar $A \subset \mathbb{R}^d$ satisfy

$$\lim_{R\to\infty} \mathbf{E}\Big[V_1(A) \;\Big|\; |B(1)| > R\Big] = \infty \,?$$

- Can one estimate precisely the tail probabilities $\mathbf{P}[V_1(A) > v]$ for specific nonpolar fractal sets A and large v, *e.g.*, when $d = 2$ and A is the middle-third Cantor set on the x-axis ?

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