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ON RECURRENT AND TRANSIENT SETS OF INHOMOGENEOUS SYMMETRIC RANDOM WALKS

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Abstract
We consider a continuous time random walk on the d-dimensional lattice $\mathbb{Z}^d$*: the jump rates are time dependent, but symmetric and strongly elliptic with ellipticity constants independent of time. We investigate the implications of heat kernel estimates on recurrence-transience properties of the walk and we give conditions for recurrence as well as for transience: we give applications of these conditions and discuss them in relation with the (optimal) Wiener test available in the time independent context. Our approach relies on estimates on the time spent by the walk in a set and on a* $0-1$ *law. We show also that, still via heat kernel estimates, one can avoid using a* $0-1$ *law, achieving this way quantitative estimates on more general hitting probabilities.*

1 Introduction and Results

Let $\mathbf{X} = \{X(t)\}_{t\geq s}$ be the random walk on $\mathbb{Z}^d$, $d \in \mathbb{N}$ which, starting from $x \in \mathbb{Z}^d$ at time $s \geq 0$ performs nearest–neighbor jumps with time and space dependent rates $c(x, y; t)$, but with the following restrictions:

Conditions 1.1.

1. Nearest–neighbor*:* $c(x, y; t) = 0$ *for any* $t \geq s$ *if* $\|x - y\| \neq 1$*;*
2. Symmetry*:* $c(x, y; t) = c(y, x; t)$ *for any* $x, y \in \mathbb{Z}^d$ *and* $t \geq s$*;*
3. Smoothness*:* $c(x, y; \cdot)$ *is a continuous function for any* $x, y \in \mathbb{Z}^d$*;*
4. Uniform ellipticity*: there are two constants* C_- *and* C_+ *such that* $0 < C_- \leq c(x, y; t) \leq C_+ < +\infty$ *for any* $x, y \in \mathbb{Z}^d$ *and* $t \geq s$*.*

Here and hereafter for $x \in \mathbb{Z}^d$ we denote with $\|x\|$ the Euclidean norm $\|x\| := \sqrt{\sum_{i=1}^d x_i^2}$ and with $|x|$ the norm $|x| := \max_{i=1,\ldots,d} |x_i|$. The existence of such a time inhomogeneous Markov process is standard (see e.g. [EK]), but we will give some details later on in the text.

We will investigate the problem of identifying recurrent and transient sets for $\mathbf{X}$. We want to stress that in absence of time dependence (in this case the jump rates will be denoted by $c(x,y)$) this problem has been intensively studied: one can exploit the strength of potential theory and an optimal criterion for recurrence (transience) of arbitrary sets is available (*Wiener test*: see e.g. [DY], [IMK]). This criterion is formulated in terms of capacities: the Dirichlet form of the semigroup of $\mathbf{X}$ is defined as

$$\mathcal{D}(f) = \frac{1}{2} \sum_{x,y} c(x,y) \left[f(x) - f(y) \right]^2 , \qquad f \in L^2(\mathbb{Z}^d), \tag{1.1}$$

and the capacity of the set $S \subset \mathbb{Z}^d$ is defined as

$$\mathrm{Cap}(S) = \inf \left\{ \mathcal{D}(f) : f \in L^2(\mathbb{Z}^d), f(x) \geq 1 \text{ if } x \in S \right\} . \tag{1.2}$$

The *Wiener test* then says that $S \subset \mathbb{Z}^d$ is recurrent if and only if

$$\sum_{k \in \mathbb{N}} \frac{\mathrm{Cap}\left(S \cap Q_k\right)}{2^{(d-2)k}} = +\infty, \tag{1.3}$$

where $Q_k = \{x : 2^k \leq \|x\| < 2^{k+1}\}$. Note that by Condition 1.1(4) the Dirichlet form (1.1) of the general walk is directly comparable with the Dirichlet form of the simple symmetric random walk ($c(\cdot,\cdot) \equiv 1$). So a set is recurrent (transient) for a strongly elliptic homogeneous symmetric walk if and only if it is recurrent (transient) for the simple symmetric walk.
The *ensemble of tools* available for the time independent case shrinks sensibly in the time dependent case: in particular the potential theory for elliptic operators seems to be of little help in the time inhomogeneous context (and the potential theory for parabolic operators is by far not as developed, besides addressing more general questions than the ones we are interested in, see [FU] for a detailed study of the simple random walk case and for further references). Note that also in the time inhomogeneous case the time dependent *Dirichlet form* is controlled, uniformly in time, by the Dirichlet form of the simple random walk, and therefore the validity of the Wiener test (1.3) is possibly a reasonable conjecture in the time inhomogeneous case too. However there does not seem to be an argument to corroborate this conjecture and, as a matter of fact, much of our intuition and essentially every basic estimate on time inhomogeneous symmetric walks arise from a very robust *non probabilistic* approach, initiated by the celebrated works of E. De Giorgi and J. Nash. This approach gives upper and lower bounds on the transition probabilities of the process. In the standard set–up ($\mathbb{R}^d$ and the walk is a diffusion generated by a strongly elliptic divergence form operator) these upper and lower bounds are of Gaussian type, see e.g. [FS]: in our discrete set–up the situation is, in practice, not much different (see subsection 1.2 below). We will investigate the consequences of these bounds on recurrence–transience issues.
We will give conditions for recurrence and transience and, though they are not optimal, they are sufficient to cover a variety of situations. In trying to understand the limitations of our approach, work by R. S. Bucy [Bu] turned out to be very relevant: reference [Bu] presents results obtained in the context of an active line of research in the sixties that concentrated on finding sufficient conditions for recurrence which can be handled more easily than the Wiener test. Bucy, for example, gave conditions which request to test *geometrical* properties similar to the ones that we propose in Theorem 1.2. As remarked in [Bu], in spite of looking like extreme simplification of (1.3), these *more user friendly* tests are, in a sense, *almost sharp* and certainly very useful *for practical purposes*.
Finally we remark that we have chosen to deal with the continuous time case for two reasons:

- These type of random walks appear in the Helffer–Sjöstrand representation for random interfaces [DGI], [GOS], and this has been the original motivation of our work. As a matter of fact some estimates in the spirit of the present work, Section 4, are already present in the analysis on non harmonic entropic repulsion [DG].
- To our knowledge, heat kernel estimates like the ones presented in Subsection 1.2 do not appear for the moment in the literature for the discrete time case. However we stress that if estimates like those in Subsection 1.2 hold in the discrete time contest, our arguments go through, with fewer technical difficulties.

1.1 Construction of the process I

The existence of the process $\mathbf{X}$ is classical, but it is simple and useful construct $\mathbf{X}$ explicitly in the following way (see Subsection 5.1 for a more formal construction).
Attach to each bond of $\mathbb{Z}^d$ a Poisson process with constant jump rate C_+: all these processes are chosen to be independent. Place a particle in $x = x_0$ at time $T_0 = s_0$ then:

1. the particle remains in x until one of the Poisson processes attached to one of the bonds exiting from x, say $b = (x, y)$, has a transition and call the transition time $\widetilde{T}$;
2. now flip a coin with probability to obtain a *head* equal to $c(x, y; \widetilde{T})/C_+$, then
   - if we obtain *head* the particle jumps to y and we restart the procedure from point 1 with $x = y$;
   - if we obtain *tail* the particle remains in x and we restart the procedure point 1.

We denote with $\mathbf{E}_{x,s}$ and $\mathbf{P}_{x,s}$ respectively the mean and the probability with respect to the process $\mathbf{X}$ started at time s in x, and with $\mathcal{L}_t$ its *pseudo–generator*:

$$(\mathcal{L}_t f)(x) = \sum_{y:\|x-y\|=1} c(x, y; t)\,[f(y) - f(x)] \qquad f : \mathbb{Z}^d \to \mathbb{R}, \tag{1.4}$$

in the sense that the transition kernel $p(x, s; y, t) = \mathbf{P}_{x,s}(X(t) = y)$ satisfies

$$\frac{d}{dt} p(x, s; y, t) = \mathcal{L}_t p(x, s; y, t), \tag{1.5}$$

where the action of the operator in the last term is on the y variable.

1.2 Heat Kernel estimates.

Symmetric walks fall in the realm of the De Giorgi–Nash–Moser Theory. In particular we will make use of the following:

Theorem 1.1 (Aronson estimates). *Let $p(x, s; y, t)$ be the heat kernel of a random walk on $\mathbb{Z}^d$ with pseudo-generator $\mathcal{L}_\cdot$ of the form* (1.4)*, with coefficients satisfying conditions 1.1. Then there exist $K_1, K_2, K_3 > 0$, depending only on d, C_- and C_+, such that*

$$p(x, s; y, t) \geq \frac{K_1}{1 \vee (t - s)^{d/2}} \tag{1.6}$$

for every $t > s \geq 0$ and every $x, y \in \mathbb{Z}^d$ such that $\|x - y\| \leq \sqrt{t - s}$, furthermore

$$p(x, s; y, t) \leq \frac{K_2}{1 \vee (t - s)^{d/2}} \exp\left(-K_3 \frac{\|x - y\|}{1 \vee \sqrt{t - s}}\right), \tag{1.7}$$

for every $x, y \in \mathbb{Z}^d$ and every $t \geq s \geq 0$.

Since in the literature we find the time independent case on $\mathbb{Z}^d$ treated in detail [CKS], [SZ], and since the time dependent case is very well understood for diffusions on $\mathbb{R}^d$ [FS], it is a matter of following the scheme of these proofs to get to the stated results (see Appendix B of [GOS]).

1.3 Recurrence and Transience.

We now give the definition of recurrence for a set $S \subset \mathbb{Z}^d$.

Definition 1.1 (Recurrence). *Let $S \subset \mathbb{Z}^d$ be a set. We say that S is* recurrent *for the process* $\mathbf{X}$ *if*

$$\mathbf{P}_{x,s}\left(\operatorname{diam}\{t \geq s : X(t) \in S\} = +\infty\right) = 1 \tag{1.8}$$

for any $(x, s) \in \mathbb{Z}^d \times \mathbb{R}_+$, otherwise we say that S is transient.

The diameter "diam" here and hereafter is taken with respect to the $|\cdot|$ norm.
For recurrence we have the following characterization:

Proposition 1.1. *Consider $S \subset \mathbb{Z}^d$ and define*

$$T_s^S = \text{“time spent by } \mathbf{X} \text{ in } S \text{ after } s\text{”} = \int_s^{+\infty} \mathbf{1}(X(u) \in S)\, du, \tag{1.9}$$

$$D_s^S = \operatorname{diam}\{t \geq s : X(t) \in S\}; \tag{1.10}$$

then $\mathbf{P}_{x,s}(D_s^S = +\infty) = \mathbf{P}_{x,s}(T_s^S = +\infty) \in \{0, 1\}$ and the value does not depend on $(x, s) \in \mathbb{Z}^d \times \mathbb{R}_+$. Furthermore $\mathbf{P}_{x,s}(D_s^S = +\infty) = 1 \Leftrightarrow \mathbf{P}_{x,s}(\mathbf{X}$ hits S after $s) = 1$.

1.4 Results and Applications

Set $\widetilde{S}_n = \{x \in S : |x| \leq n\}$ and $\theta_d(x) = 1/(1 + |x|^{d-2})$. Our main result is:

Theorem 1.2. *Any non empty set $S \subset \mathbb{Z}^d$ is recurrent for $d = 1, 2$. For $d \geq 3$ and $n \in \mathbb{N}$ define*

$$\sigma_1(n) \equiv \sum_{x \in \widetilde{S}_n} \theta_d(x), \tag{1.11}$$

$$\sigma_2^2(n) \equiv \sum_{x,y \in \widetilde{S}_n} \theta_d(x)\theta_d(y) \left[\frac{2 + |x|^{d-2} + |y|^{d-2}}{1 + |x - y|^{d-2}}\right] = 2 \sum_{x,y \in \widetilde{S}_n} \theta_d(x)\theta_d(y - x). \tag{1.12}$$

Then:

1. *if $\lim_{n\to+\infty} \sigma_1(n) < +\infty$ the set S is transient;*
2. *if $\lim_{n\to+\infty} \sigma_1(n) = +\infty$ and $\limsup_{n\to+\infty} \sigma_1(n)/\sigma_2(n) > 0$ then the set S is recurrent.*

We delay to Section 3 the application of this result to specific sets. Here we just list informally some of the outcomes and we make some considerations:

- In Corollary 3.1 we derive that any infinite connected cluster $S \subset \mathbb{Z}^3$ is recurrent, as a consequence of a stronger statement in $d = 3$.
- This implies that a line (or a half line) in $d = 3$ is recurrent: we will show that this fact is true even in higher dimension in the sense that a codimension 2 affine subspace is recurrent.
- Clearly the problem left open by Theorem 1.2 is: what happens if $\lim_{n\to+\infty} \sigma_1(n) = +\infty$ and $\lim_{n\to+\infty} \sigma_1(n)/\sigma_2(n) = 0$? In section 3, with the help of the simple random walk, we will consider how this may happen with both recurrent and transient sets.

Finally, we concentrate some attention on *explicit estimates*: as it will be clear from the sequel, our proof of the recurrence part of Theorem 1.2 relies on explicit bounds up to the last step, when a 0–1 law (proposition 1.1) is applied. Therefore the approach does not yield interesting quantitative estimates on what we may call *approximate recurrence*, that is finding an *explicit* lower bound of the type $1-\epsilon(n)$ ($\epsilon(n) \to 0$ as $n \to \infty$) on the probability that $\mathbf{X}$, starting (say) from the origin, hits a recurrent set before exiting $\widetilde{\mathbb{Z}^d}_n$.

2 Proof of Main Result

The strategy of the proof of theorem 1.2 is conceptually simple: consider the case $d \geq 3$ (the cases $d = 1, 2$ are simpler). Estimates (1.6) and (1.7) enable us to estimate the expectation of the time T spent by $\mathbf{X}$ in an arbitrary set $S \subset \mathbb{Z}^d$. It is easy to see that this expectation is finite if the set is finite and that it can be infinite, if the set is infinite. If it is finite necessarily $\mathbf{X}$ spends an almost surely finite amount of time in S, so that S is transient. If the expected value of T is infinite the time T spent by the process $\mathbf{X}$ in S can be infinite but also almost surely finite. To investigate if the latter is the case a crude approach is to perform a second moment calculation on T and compare it with the first moment. To compare the two first moments of T we need to truncate in some way the variable T, because its moments are not finite. We make this by considering the random variables $\widetilde{T}_n$ defined as the "time spent by $\mathbf{X}$ in $\widetilde{S}_n$. Theorem 1.1 enables us to estimate the expectation of $\widetilde{T}_n$ with $\sigma_1(n)$ and to estimate from below the second moment of $\widetilde{T}_n$ with $\sigma_2^2(n)$, then we can perform our moment comparison, namely Lemma 2.1. Proposition 1.1 completes the proof.

Lemma 2.1. *Let $0 < Y_1 \leq Y_2 \leq \cdots$ be an increasing sequence of positive random variables, such that $\mathbf{E}(Y_n) < +\infty$ and $\mathbf{E}(Y_n^2) < +\infty$ for any $n = 1, 2, \ldots$. Define $Y = \lim_{n\to+\infty} Y_n$,*

1. *if $\lim_{n\to+\infty} \mathbf{E}(Y_n) < +\infty$ then $\mathbf{E}(Y) < +\infty$ and $\mathbf{P}(Y = +\infty) = 0$;*
2. *if $\lim_{n\to+\infty} \mathbf{E}(Y_n) = +\infty$ then $\mathbf{E}(Y) = +\infty$, furthermore if*

$$\limsup_{n\to\infty} \frac{\mathbf{E}(Y_n)}{\sqrt{\mathbf{E}(Y_n^2)}} > 0$$

then $\mathbf{P}(Y = +\infty) > 0$.

Proof. The proof of (1) is immediate. (2) is essentially the Paley–Zygmund inequality: first of all observe that, by the monotone convergence theorem, $\mathbf{E}(Y) = \lim_{n\to+\infty} \mathbf{E}(Y_n) = +\infty$. By passing to a subsequence, we can assume that $\mathbf{E}(Y_n)/\sqrt{\mathbf{E}(Y_n^2)} \geq \epsilon > 0$ for any $n \in \mathbb{N}$. Now fix $c \in [0, 1)$, then:

$$\begin{aligned} \mathbf{E}(Y_n) &= \mathbf{E}(Y_n; Y_n \leq c\,\mathbf{E}(Y_n)) + \mathbf{E}(Y_n; Y_n > c\,\mathbf{E}(Y_n)) \\ &\leq c\,\mathbf{E}(Y_n) + \sqrt{\mathbf{E}(Y_n^2)\,\mathbf{P}(Y_n > c\,\mathbf{E}(Y_n))}, \end{aligned} \tag{2.1}$$

which implies that:

$$\mathbf{P}(Y_n > c\,\mathbf{E}(Y_n)) \geq (1-c)^2 \frac{\mathbf{E}(Y_n)^2}{\mathbf{E}(Y_n^2)} \geq (1-c)^2\epsilon^2, \tag{2.2}$$

and finally

$$\mathbf{P}(Y > c\,\mathbf{E}(Y_n)) = \mathbf{P}((Y - Y_n) + Y_n > c\,\mathbf{E}(Y_n)) \geq \mathbf{P}(Y_n > c\,\mathbf{E}(Y_n)) > \epsilon, \tag{2.3}$$

for any $n \in \mathbb{N}$, from which, because $\lim_{n\to+\infty} \mathbf{E}(Y_n) = +\infty$, clearly follows $\mathbf{P}(Y = +\infty) > 0$.

□

Proof of Theorem 1.2. We start by proving that any $S \subset \mathbb{Z}^d$ is recurrent for $d = 1, 2$. Clearly this is equivalent to prove that any point $x \in \mathbb{Z}^d$ is recurrent, and because of Proposition 1.1 it is sufficient to show that for any $x \in \mathbb{Z}^d$ the process $\mathbf{X}$ starting from 0 at time 0 spends an infinite amount of time in x with positive probability.
Fix $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, and define the random variables

$$\overline{T} = \text{"time spent by } \mathbf{X} \text{ in } x\text{"} = \int_0^{+\infty} \mathbf{1}(X(u) = x) du, \tag{2.4}$$

$$\overline{T}_n = \text{"time spent by } \mathbf{X} \text{ in } x \text{ before } n\text{"} = \int_0^{n} \mathbf{1}(X(u) = x) du. \tag{2.5}$$

Clearly $\overline{T}_n \uparrow \overline{T}$ as $n \to +\infty$. We want to apply Lemma 2.1 to the variables $\overline{T}_n$; we start by estimating from below the first moment of $\overline{T}_n$. Assume that $n \geq |x|^2$, then by (1.6) we have

$$\mathbf{E}_{0,0}(\overline{T}_n) = \int_0^n p(0,0;x,s) ds \geq \int_{|x|^2}^n p(0,0;x,s) ds \geq C_1 \left[\mathbf{1}(x = 0) + \int_{1 \vee |x|^2}^n s^{-d/2} ds \right], \tag{2.6}$$

and therefore $\mathbf{E}_{0,0}(\overline{T}_n) \uparrow +\infty$ for $d = 1, 2$.
To estimate from above the second moment of $\overline{T}_n$ first observe that

$$\begin{aligned} \mathbf{E}_{0,0}(\overline{T}_n^2) &= \int_0^n \int_0^n \mathbf{P}_{0,0}(X(s) = x, X(t) = x) ds\, dt \\ &= \int_0^n ds \int_s^n dt\ \mathbf{P}_{0,0}(X(s) = x, X(t) = x) + \int_0^n dt \int_t^n ds\ \mathbf{P}_{0,0}(X(s) = x, X(t) = x) \\ &= \int_0^n ds \int_s^n dt\, p(x,s;x,t) p(0,0;x,s) + \int_0^n dt \int_t^n ds\, p(x,t;x,s) p(0,0;x,t). \end{aligned} \tag{2.7}$$

Then by using (1.7) and (2.6):

$$\begin{aligned} \int_0^n ds \int_s^n dt\, p(x,s;x,t) p(0,0;x,s) &\leq C_2^2 \int_0^n \frac{\mathrm{e}^{-C_3 \frac{|x|}{1 \vee \sqrt{s}}}}{1 \vee s^{d/2}} ds \int_s^n \frac{1}{1 \vee (t-s)^{d/2}} dt \\ &\leq C_2^2 \int_0^n \frac{ds}{1 \vee s^{d/2}} \int_s^n \frac{dt}{1 \vee (t-s)^{d/2}} \leq C_2^2 \int_0^n \frac{ds}{1 \vee s^{d/2}} \int_0^n \frac{dt}{1 \vee t^{d/2}} \leq C_1\, \mathbf{E}_{0,0}(\overline{T}_n)^2 \end{aligned} \tag{2.8}$$

for n large enough. Thus we can apply the second part of Lemma 2.1 to the variables $\overline{T}_n$ and claim that $\overline{T} = \lim_{n \to +\infty} \overline{T}_n$ is infinite with positive probability and Proposition 1.1 implies that any $x \in \mathbb{Z}^d$ is recurrent for $d = 1, 2$.
We can now consider the case $d > 2$. Fix $S \subset \mathbb{Z}^d$, because of Proposition 1.1 , in order to show that S is recurrent it is sufficient to show that the process $\mathbf{X}$ starting from 0 at time 0 spends an infinite amount of time in S with positive probability. Define (and recall) the sets

$$S_n = \{x \in S : |x| = n\} \qquad \text{and} \qquad \widetilde{S}_n = \{x \in S : |x| \leq n\} = \bigcup_{r=0}^n S_r$$

for any $n \in \mathbb{Z}_+$ and the random variables

$$\begin{aligned} T_n &= \text{"time spent by } \mathbf{X} \text{ in } S_n\text{"} = \int_0^{+\infty} \mathbf{1}(X(u) \in S_n) du \\ \widetilde{T}_n &= \text{"time spent by } \mathbf{X} \text{ in } \widetilde{S}_n\text{"} = \int_0^{+\infty} \mathbf{1}(X(u) \in \widetilde{S}_n) du \\ T &= \text{"time spent by } \mathbf{X} \text{ in } S\text{"} = \int_0^{+\infty} \mathbf{1}(X(u) \in S) du. \end{aligned} \tag{2.9}$$

We start by estimating the first moment of $\widetilde{T}_n$. By (1.7), we obtain

$$\mathbf{E}_{0,0}(\widetilde{T}_n) = \sum_{x\in\widetilde{S}_n} \int_0^{+\infty} p(0,0;x,s)ds \leq \sum_{x\in\widetilde{S}_n} \int_0^{+\infty} \frac{C_2}{1\vee s^{d/2}} \mathrm{e}^{-C_3\frac{|x|}{1\vee\sqrt{s}}} ds. \tag{2.10}$$

Note that $\frac{1}{2} \leq \frac{1\vee s}{1+s} \leq 1$, thus

$$\int_0^{+\infty} \frac{C_2}{1\vee s^{d/2}} \mathrm{e}^{-C_3\frac{|x|}{1\vee\sqrt{s}}} ds \leq \int_0^{+\infty} \frac{C_{2,1}}{(1+s)^{d/2}} \mathrm{e}^{-C_{3,1}\frac{|x|}{\sqrt{1+s}}} ds = C_{2,1} \int_1^{+\infty} \frac{\mathrm{e}^{-C_{3,1}\frac{|x|}{\sqrt{s}}}}{s^{d/2}} ds, \tag{2.11}$$

and one can find C_4 such that

$$\int_0^{+\infty} \frac{C_{2,1}}{1\vee s^{d/2}} \mathrm{e}^{-C_{3,1}\frac{|x|}{1\vee\sqrt{s}}} ds \leq \frac{C_4}{1+|x|^{d-2}}. \tag{2.12}$$

Recalling (2.10) we have

$$\mathbf{E}_{0,0}(\widetilde{T}_n) \leq \sum_{x\in\widetilde{S}_n} \frac{C_4}{1+|x|^{d-2}} = C_4\sigma_1(n), \tag{2.13}$$

so if $\lim_{n\to+\infty}\sigma_1(n) < +\infty$ (see (1.11)) then, by lemma 2.1, $\mathbf{P}_{0,0}(T=+\infty) = 0$.
By (1.6) instead we obtain

$$\mathbf{E}_{0,0}(\widetilde{T}_n) = \sum_{x\in\widetilde{S}_n} \int_0^{+\infty} p(0,0;x,s)ds \geq \sum_{x\in\widetilde{S}_n} \int_{|x|^2}^{+\infty} \frac{C_1}{1\vee s^{d/2}} ds \geq \sum_{x\in\widetilde{S}_n} \frac{C_5}{1+|x|^{d-2}} = C_5\sigma_1(n), \tag{2.14}$$

so if $\lim_{n\to+\infty}\sigma_1(n) = +\infty$ then by lemma 2.1 $\mathbf{E}_{0,0}(T) = +\infty$.
In order to apply lemma 2.1 and understand if T is almost surely finite or not, we need to estimate the second moment of $\widetilde{T}_n$; first notice that

$$\begin{aligned}\mathbf{E}_{0,0}(\widetilde{T}_n^2) &= \sum_{x,y\in\widetilde{S}_n} \int_0^{+\infty}\int_0^{+\infty} \mathbf{P}_{0,0}(X(t)=x, X(s)=y)dt\, ds \\ &= \sum_{x,y\in\widetilde{S}_n} \int_0^{+\infty} p(0,0;s,y)ds \int_s^{+\infty} p(s,x;t,y)dt \\ &\quad + \sum_{x,y\in\widetilde{S}_n} \int_0^{+\infty} p(0,0;t,x)dt \int_t^{+\infty} p(t,y;s,x)ds. \end{aligned} \tag{2.15}$$

Then we use (1.7) to bound the integrals in the last lines, obtaining:

$$\begin{aligned}\int_0^{+\infty} p(0,0;s,y)ds \int_s^{+\infty} p(s,x;t,y)dt &\leq C_2^2 \int_0^{+\infty} \frac{\mathrm{e}^{-C_3\frac{|y|}{1\vee\sqrt{s}}}}{1\vee s^{d/2}} ds \int_s^{+\infty} \frac{\mathrm{e}^{-C_3\frac{|x-y|}{1\vee\sqrt{t-s}}}}{1\vee(t-s)^{d/2}} dt \\ = C_2^2 \int_0^{+\infty} \frac{\mathrm{e}^{-C_3\frac{|y|}{1\vee\sqrt{s}}}}{1\vee s^{d/2}} ds \int_0^{+\infty} \frac{\mathrm{e}^{-C_3\frac{|x-y|}{1\vee\sqrt{t}}}}{1\vee t^{d/2}} dt &\leq \frac{C_6}{(1+|x|^{d-2})(1+|x-y|^{d-2})}, \end{aligned} \tag{2.16}$$

where we used (2.12) in the last step. Since we have a similar estimate (exchange x and y) for the last term in (2.15), we obtain

$$\mathbf{E}_{0,0}(\widetilde{T}_n^2) \leq 2C_6 \sum_{x,y\in\widetilde{S}_n} \frac{1}{1+|x-y|^{d-2}} \left(\frac{1}{1+|x|^{d-2}} + \frac{1}{1+|y|^{d-2}}\right) = 2C_6\sigma_2(n) \tag{2.17}$$

Equations (2.14) and (2.17) together imply

$$\frac{\mathbf{E}_{0,0}(\widetilde{T}_n)}{\sqrt{\mathbf{E}_{0,0}(\widetilde{T}_n^2)}} \geq C_7 \frac{\sigma_1(n)}{\sigma_2(n)}, \tag{2.18}$$

so that $\limsup_{n\to+\infty} \sigma_1(n)/\sigma_2(n) > 0$ and, by lemma 2.1, $\mathbf{P}_{0,0}(T = +\infty) > 0$ and, recalling Proposition 1.1, S is recurrent. □

3 Applications and Counterexamples

3.1 Examples

We give now some applications of Theorem 1.2: they can be extended in several natural ways.

Proposition 3.1. *Let $S \subset \mathbb{Z}^3$ and assume that there exists $r_0 \geq 0$ such that $|S_r| \geq 1$ for any $r > r_0$, then S is recurrent.*

Proof. It is clear that if we prove that a set $S \subset \mathbb{Z}^3$ such that $|S_r| = 1$ for any $r \geq r_0 \geq 0$ is recurrent then we have done. So let $S \subset \mathbb{Z}^3$ be such that $|S_r| = 1$ for any r large enough then:

$$\sigma_1(n) = \sum_{j=0}^{n} \sum_{x\in S_j} \frac{1}{1+|x|} \geq \sum_{j=r_0}^{n} \frac{1}{1+j} \xrightarrow[n\to+\infty]{} +\infty. \tag{3.1}$$

Furthermore

$$\begin{aligned}
\sigma_2^2(n) &= \sum_{i,j=1}^{n} \frac{2+i+j}{(i+1)(1+j)} \sum_{\substack{x\in S_i\\ y\in S_j}} \frac{1}{1+|x-y|} \\
&\leq \sum_{i,j=1}^{n} \frac{2+i+j}{(i+1)(1+j)(1+|i-j|)} \leq 2 \sum_{1\leq i\leq j\leq n} \frac{2+i+j}{(i+1)(1+j)(1+j-i)} \\
&\leq 2 \sum_{1\leq i\leq j\leq n} \frac{2+2j}{(i+1)(1+j)(1+j-i)} = 4\sum_{i=1}^{n} \frac{1}{i+1} \sum_{j=i}^{n} \frac{1}{1+j-i} \\
&\leq 4\sum_{i=1}^{n} \frac{1}{i+1} \sum_{j=1}^{n} \frac{1}{1+j} = O(\sigma_1^2(n)),
\end{aligned} \tag{3.2}$$

this implies that condition 2 of Theorem 1.2 is satisfied and S is recurrent. □
As an immediate consequence we have:

Corollary 3.1. *Any infinite connected cluster $S \subset \mathbb{Z}^3$ is recurrent.*

This corollary implies that the straight line $\pi^1 := \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_2 = x_3 = 0\}$ is recurrent. This is a general property, in fact we have:

Proposition 3.2. *The $(d-2)$-dimensional "affine variety"*

$$\pi^{d-2} := \left\{(x_1, \ldots, x_d) \in \mathbb{Z}^d : x_{d-1} = x_d = 0\right\} \tag{3.3}$$

is recurrent for any integer $d \geq 3$.

Proof. Express $\sigma_2(n)$ as discrete convolution restricted to $\widetilde{S}_n$:

$$(\sigma_2(n))^2 = 2 \sum_{x \in \widetilde{S}_n} \frac{1}{1+|x|} \sum_{y \in \widetilde{S}_n} \frac{1}{1+|y-x|}, \tag{3.4}$$

and specializing to $S = \pi^{d-2}$ we see that

$$(\sigma_2(n))^2 \leq 2 \sum_{x \in \widetilde{S}_n} \frac{1}{1+|x|} \sum_{y \in \widetilde{S}_{2n}} \frac{1}{1+|y|} = 2\sigma_1(n)\sigma_1(2n). \tag{3.5}$$

By direct computation we have that $\sigma_1(n) = O(\log n)$ and therefore $\liminf_{n\to\infty} \sigma_1(n)/\sigma_2(n) \geq 1/2$. In other words, recurrence is established. □

3.2 On the limitations of the approach

With some effort, one can build sets S for which $\sigma_1(n) \to +\infty$ and $\sigma_1(n)/\sigma_2(n) \to 0$. Theorem 1.2 is of little help in this case: a quick look at the proof however will be sufficient to convince the reader that $\sigma_1(n) \to +\infty$ implies that the expectation of the time spent by $\mathbf{X}$ in S is infinite.
The example we present is due to R. S. Bucy [Bu]. Let $\mathbf{X}$ be the standard random walk on $\mathbb{Z}^3$ with jump rates $c(x, y; t) = \mathbf{1}(|x - y| = 1)$ and define the set

$$S = \left\{(x, 0, 0) \in \mathbb{Z}^3 : \exists\, r \in \mathbb{N} \text{ such that } 2^r \leq x < 2^r(1 + r^{-1})\right\}. \tag{3.6}$$

Then it is easy to prove that for this set $\lim_{n\to+\infty} \sigma_1(n) = +\infty$, but the standard *Wiener test* states that S is *transient* for the standard discrete time random walk Y on $\mathbb{Z}^3$ (see [Bu, page 543]). Therefore Theorem 1.2 implies that $\sigma_1(n)/\sigma_2(n) \to 0$, fact which can be verified directly without much difficulty.
We believe that there exist *recurrent* sets S such that $\sigma_1(n) \to +\infty$ and $\sigma_1(n)/\sigma_2(n) \to 0$. A promising candidate is the set $S = \{([n \log n], 0, 0) : n = 1, 2, \ldots\}$: in [Bu] it is shown that S is recurrent for the simple random walk and numerical computations suggest $\sigma_1(n)/\sigma_2(n) \to 0$ (we have not been able to prove it), while it is easy to verify that $\sigma_1(n) \to \infty$.

4 Explicit estimates

The approach via heat kernel bounds and first–second moment estimates is very direct and constructive, but, in order to get recurrence one has to complete it, because by itself it would yield only recurrence with a positive probability. The completion of the argument via proving a 0–1 law is very natural. Consider however the following problems:

- If $d = 2$, what is the probability that $\mathbf{X}$, $X(0) = 0$, hits the finite set S before hitting the boundary of $\widetilde{\mathbb{Z}^d}_n$, n very large?
- If $d \geq 3$, what is the probability that the walk, starting from the origin, hits a finite but very large set close to the origin, before setting off to infinity?

While the second moment argument is a good start to answer these questions, it will not suffice in most of the cases if we want interesting quantitative estimates. For the sake of brevity, we will address the first of the two questions, as a prototype of several questions (among which the second one). It will be clear from what we will explain below that this technique could also substitute the 0–1 law in the proof of Theorem 1.2.

Let us set $\widetilde{Z}_n = \widetilde{\mathbb{Z}^d}_n$ and let us restrict to $d = 2$. Choose $e = (1, 0)$ and set $\tau_n = \inf\{t : X(t) \notin \widetilde{S}_n\}$, $n = 1, 2, \ldots$, as well as $\tau_0 = \inf\{t : X(t) = 0\}$.

Proposition 4.1. *There exists a positive constant $c = c(C_-, C_+)$ and a natural number $\overline{n}$ such that*

$$\mathbf{P}_{e,0}\left(\tau_0 < \tau_n\right) \geq 1 - \frac{1}{(\log n)^c}, \tag{4.1}$$

for every $n \geq \overline{n}$.

The computation in the simple random walk case yields $c = 1$ (with a suitable constant that multiplies $1/\log n$).
The proof is based on recursive estimates and it will be preceded by preparatory lemmas. All the estimates can be done explicitly, but, to make the argument more fluent, we will not always carry all the constants explicitly (already from the first lemma...).

Lemma 4.1. *For every $\alpha > 1$ there exists $\overline{k}$ such that*

$$\inf_{k>\overline{k}} \inf_{|y|=k} \mathbf{P}_{y,0}\left(\tau_0 < k^{2\alpha}\right) \geq K_1^2/4K_2^2 \left(1 - \frac{1}{\alpha}\right)^2 \equiv 2\delta. \tag{4.2}$$

Proof. Recall notations and basic idea from the proof of the first part of Theorem 1.2. We estimate from below the expectation of the time spent in zero up to time $k^{2\alpha}$ starting from y, $|y| = k$:

$$\mathbf{E}_{y,0}\left[\overline{T}_{k^{2\alpha}}\right] = \int_0^{k^{2\alpha}} p(y,0;0,s)ds \geq \int_{k^2}^{k^{2\alpha}} \frac{K_1}{s} ds = 2K_1(\alpha - 1)\log k, \tag{4.3}$$

and we estimate the expectation of the square of the same random variable from above

$$\mathbf{E}_{y,0}\left[\overline{T}^2_{k^{2\alpha}}\right] \leq K_2^2 \left(1 + \int_1^{k^{2\alpha}} \frac{ds}{s}\right) = K_2^2 \left(1 + 2\alpha \log k\right)^2. \tag{4.4}$$

Therefore for k sufficiently large

$$\frac{\mathbf{E}_{y,0}\left[\overline{T}_{k^{2\alpha}}\right]}{\sqrt{\mathbf{E}_{y,0}\left[\overline{T}^2_{k^{2\alpha}}\right]}} \geq \frac{K_1}{2K_2}\left(1 - \frac{1}{\alpha}\right), \tag{4.5}$$

and by using (2.2) we have that

$$\mathbf{P}_{y,0}\left(\overline{T}_{k^{2\alpha}} > 0\right) \geq \frac{K_1^2}{4K_2^2}\left(1 - \frac{1}{\alpha}\right)^2. \tag{4.6}$$

Since $\mathbf{P}_{y,0}(\overline{T}_{k^{2\alpha}} > 0) = \mathbf{P}_{y,0}(\tau_0 < k^{2\alpha})$ the lemma is proven. □

In the very same way of the previous proof one can prove the following:

Lemma 4.2. *There exists $\widetilde{k} \in \mathbb{N}$ such that*

$$\mathbf{P}_{e,0}\left(\tau_0 < \widetilde{k}\right) \geq K_1^2/4K_2^2 (> 2\delta). \tag{4.7}$$

Another important ingredient is the following lemma, that follows by repeating step by step the proof of Proposition 6.5 (see also Prop. 8.1) of chapter VII in [Ba]: note that the proof relies only on the upper bound (1.7) on the heat kernel. We give it for arbitrary d.

Lemma 4.3. *There exists a constant K_4, depending only on the ellipticity bounds and on d, such that for every $x \in \mathbb{Z}^d$, every $T \geq 0$ and every $\lambda > 0$*

$$\mathbf{P}_{x,0}\left(\sup_{s\in[0,T]} |X_s - x| \geq \lambda\right) \leq K_4 \exp\left(-\frac{\lambda}{K_4\sqrt{T}}\right). \tag{4.8}$$

Proof of Proposition 4.1. Let $\{r_j\}_{j\in\mathbb{Z}_+}$ be a strictly increasing sequence of natural numbers. For $j > 1$ and $x \in \partial^+\widetilde{Z}_{r_{j-1}}$, with $r_j < n$ (therefore $j \leq N$, for some N that we assume larger than 2), we have the following

$$\begin{aligned} \mathbf{P}_{x,s}\left(\tau_0 < \tau_n\right) &= \mathbf{P}_{x,s}\left(\tau_0 < \tau_n, \tau_{r_j} < \tau_0\right) + \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right) \\ &= \mathbf{E}_{x,s}\left[\mathbf{P}_{x,s}\left(\tau_0 < \tau_n | \mathcal{F}_{\tau_{\widetilde{Z}_{r_j}}}\right); \tau_{r_j} < \tau_0\right] + \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right) \\ &\geq \inf_{y\in\partial^+\widetilde{Z}_{r_j}} \inf_{s'\geq s} \mathbf{P}_{y,s'}\left(\tau_0 < \tau_n\right)\left(1 - \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right)\right) + \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right), \end{aligned} \tag{4.9}$$

and in the last step we have used the strong Markov property. Take now the infimum over $x \in \partial^+\widetilde{Z}_{r_{j-1}}$ and over $s \geq 0$ to get, with the notation $Q_j = \inf_{y\in\partial^+\widetilde{Z}_{r_j}} \inf_{s\geq 0} \mathbf{P}_{y,s}\left(\tau_0 < \tau_n\right)$, that

$$Q_{j-1} \geq Q_j\left(1 - \inf_{x\in\partial^+\widetilde{Z}_{r_{j-1}}} \inf_s \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right)\right) + \inf_{x\in\partial^+\widetilde{Z}_{r_{j-1}}} \inf_s \mathbf{P}_{x,s}\left(\tau_{r_j} > \tau_0\right). \tag{4.10}$$

We now look for a lower bound on $\inf_{x\in\partial^+\widetilde{Z}_{r_{j-1}}} \inf_s \mathbf{P}_{x,s}(\tau_{r_j} > \tau_0)$ which is uniform in j. In order to do this we observe that for $x \in \partial^+\widetilde{Z}_{r_{j-1}}$ and every $T > 0$

$$\mathbf{P}_{x,s}\left(\tau_0 < \tau_{r_j}\right) \geq \mathbf{P}_{x,s}\left(\tau_0 < T\right) - \mathbf{P}_{x,s}\left(\tau_{r_j} < T\right), \tag{4.11}$$

that follows by inserting in the left–hand side the characteristic function of the event $\{\tau_0 < T\}$. Choose $T = (r_{j-1} + 1)^{2\alpha}$, $j \geq 2$. By applying Lemma 4.1 we obtain that

$$\inf_j \inf_s \inf_{x\in\partial^+\widetilde{Z}_{r_{j-1}}} \mathbf{P}_{x,s}\left(\tau_0 < T\right) \geq 2\delta, \tag{4.12}$$

and by Lemma 4.3, there exist two (sufficiently large) constant $q(> 1)$ and $k' \in Z^+$ such that if we set $r_j \geq q(r_{j-1})^\alpha$ and $r_1 > k'$

$$\sup_{j\geq 2} \sup_{x\in\widetilde{Z}_{r_{j-1}}} \mathbf{P}_{x,s}\left(\tau_{r_j} < T\right) \leq \delta. \tag{4.13}$$

Fix now $r_1 = \max(\overline{k}, \widetilde{k}, k')$, $\overline{k}$ and $\widetilde{k}$ were introduced respectively in Lemma 4.1 and Lemma 4.2. Formulas (4.10), (4.11), (4.12) and (4.13) imply the recursion relation

$$Q_{j-1} \geq Q_j(1 - \delta) + \delta, \tag{4.14}$$

for $j \geq 2$. Set then $Q_0 = \mathbf{P}_{e,0}(\tau_0 < \tau_n)$: of course $Q_0 \geq \inf_s \inf_{x\in\partial^+\{0\}} \mathbf{P}_{x,s}(\tau_0 < \tau_n)$ and, recalling the steps in (4.9) and the choice of $r_1 \geq \widetilde{k}$, we can extend the validity of the recursion (4.14) down to $j = 1$. We can then solve this recursive chain of inequalities and, recalling that j ranges from 1 to N, we obtain

$$Q_0 \geq 1 - (1 - \delta)^N. \tag{4.15}$$

Therefore all that is left to determine is how many iterations N are allowed. Since the condition is that $r_N < n$ and since we can choose $r_N = [\beta^{\alpha^{N-1}}]$, $\beta = r_1(2q)^{1/(\alpha-1)}$, one obtains that

$$Q_0 \geq 1 - \frac{c_1}{(\log n)^{c_2}}, \tag{4.16}$$

$c_1 = 1/(1 - \delta)^{\beta/\log\alpha}$ and $c_2 = |\log(1 - \delta)|/\log\alpha$: the proof is therefore complete. □

5 The 0–1 law

We start by giving a more detailed construction of the process.

5.1 Construction of the process II

Denote with $\mathcal{B}$ the set of bonds of $\mathbb{Z}^d$:

$$\mathcal{B} = \left\{(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d : \|x-y\| = 1\right\}, \tag{5.1}$$

and for any $b = (x,y) \in \mathcal{B}$ consider a Poisson process $\mathbf{N}(b) = \{N(t;b)\}_{t\geq 0}$ with intensity C_+ and a sequence $U_1(b), U_2(b), \dots$ of *i.i.d* random variables uniformly distributed in $[0,1]$. Let $0 = \widetilde{T}_0(b), \widetilde{T}_1(b), \dots$ be the jump times of the process $\mathbf{N}(b)$. Then it is clearly possible to construct for any $b \in \mathcal{B}$ a probability space $(\Omega_b, \mathcal{F}_b, \mu_b)$ where these objects live, and define the probability space $(\Omega, \mathcal{F}, \mathbf{P}) = (\bigotimes_{b\in\mathcal{B}} \Omega_b, \bigotimes_{b\in\mathcal{B}} \mathcal{F}_b, \bigotimes_{b\in\mathcal{B}} \mu_b)$ where are defined $\mathbf{N}(b)$ and $U_1(b), U_2(b), \dots$ for any $b \in \mathcal{B}$.
On this probability space we define our random walk $\mathbf{X}$ starting from $x_0 \in \mathbb{Z}^d$ at time $T_0 = s_0 \geq 0$ in the following way:
Step 1: $X(t) = x_0$ for any $t \in [T_0, T_1)$, where:

$$T_1 = \inf\left\{\widetilde{T}_n(b) > T_0 : b = (x_0, y) \in \mathcal{B}, U_n(b) \leq \frac{c(x_0, y; \widetilde{T}_n(b))}{C_+}\right\} = \widetilde{T}_{n_1}(b_1) \tag{5.2}$$

and $b_1 = (x_0, x_1)$;
Step 2: $X(t) = x_1$ for any $t \in [T_1, T_2)$, where:

$$T_2 = \inf\left\{\widetilde{T}_n(b) > T_1 : b = (x_1, y) \in \mathcal{B}, U_n(b) \leq \frac{c(x_1, y; \widetilde{T}_n(b))}{C_+}\right\} = \widetilde{T}_{n_2}(b_2) \tag{5.3}$$

and $b_2 = (x_1, x_2)$; ...
Step k: in general $X(t) = x_k$ for any $t \in [T_k, T_{k+1})$, where:

$$T_{k+1} = \inf\left\{\widetilde{T}_n(b) > T_n : b = (x_k, y) \in \mathcal{B}, U_n(b) \leq \frac{c(x_k, y; \widetilde{T}_n(b))}{C_+}\right\} = \widetilde{T}_{n_{k+1}}(b_{k+1}). \tag{5.4}$$

and $b_{k+1} = (x_k, x_{k+1})$.

Remark. Notice that by the construction of the process we have $\mathbf{P}_{x_0,s_0} = 1$ that

$$\begin{aligned} T_1 \geq \overline{T}_1 &= \inf\left\{\widetilde{T}_n(b) > T_0 : b = (x_0, y) \in \mathcal{B}\right\} \\ T_2 \geq \overline{T}_2 &= \inf\left\{\widetilde{T}_n(b) > T_1 : b = (x_1, y) \in \mathcal{B}\right\} \\ &\vdots \end{aligned}$$

and $\overline{T}_1, \overline{T}_2, \dots$ are *i.i.d* exponentially distributed random variables, with mean $1/2dC_+$. We will use this remark in the sequel.

5.2 Proof of Proposition 1.1

We start by showing that $\mathbf{P}_{x,s}(D_s^S = +\infty)$ can assume only the two values 0 or 1, and that this is independent of the choice of $(x,s) \in \mathbb{Z}^d \times \mathbb{R}_+$. Notice that $f(x,s) = \mathbf{P}_{x,s}(D_s^S = +\infty)$ satisfies the equation

$$f(x,s) = \sum_{y \in \mathbb{Z}^d} p(x,s;y,t) f(y,t) \tag{5.5}$$

for any $x \in \mathbb{Z}^d$ and any $t > s$. Call any function f satisfying (5.5) *harmonic*. We are going to show that *any bounded harmonic function is constant*. By (1.6) we have that if $\|x-y\| \leq 1$ and $0 < \Delta s < 1$ then there exists $C_1 > 0$ which depends only on d, C_-, C_+ such that

$$\sum_{y:\|x-y\|\leq 1} p(x,s;y,s+\Delta s) = 1 - \delta \geq C_1. \tag{5.6}$$

Let ϕ be a bounded harmonic function and $M = \sup_{x,s} \phi(x,s)$, then for any $\epsilon > 0$ there exists (x_0,s_0) such that $\phi(x_0,s_0) > M - \epsilon$ and by using (5.5) and (5.6), it is easy to show that for every x such that $\|x - x_0\| \leq 1$ we have $\phi(x, s_0 + \Delta s) > M - \epsilon/C_1$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the canonical base of $\mathbb{R}^d$, by iterating the above procedure we can construct a sequence (x_0,s_0), $(x_1,s_1) = (x_0 + \mathbf{e}_1, s_0 + \Delta s)$, $\ldots$, $(x_n,s_n) = (x_{n-1} + \mathbf{e}_1, s_{n-1} + \Delta s)$, $\ldots$, such that $\phi(x_n,s_n) > M - \epsilon/C_1^n$. So if $M > 0$ then for any $N > 0$ it is possible to chose $n > 0$, $\epsilon > 0$ and consequently (x_0,s_0) such that: $\sum_{k=0}^n \phi(x_k,s_k) \geq N$. By repeating the same reasoning for $m = \inf_{x,s} \phi(x,s)$ we obtain that if $m < 0$ for any $N > 0$ it is possible to choose $n > 0$, $\epsilon > 0$ and (x_0,s_0) such that $\sum_{k=0}^n \phi(x_k,s_k) \leq -N$.
Let now f be an harmonic bounded function, $\phi(x,s) = f(x + \mathbf{e}_1, s + \Delta s) - f(x,s)$ is harmonic and bounded; if $\sup \phi > 0$ then for any $N > 0$ there exists (x_0,s_0) such that

$$f(x_0 + n(\Delta s)\mathbf{e}_1, s_0 + n(\Delta s)) - f(x_0,s_0) = \sum_{k=0}^n \phi(x_k,s_k) \geq N,$$

which contradicts the fact that f is bounded. This implies that $\sup \phi \leq 0$. Similarly we get $\inf \phi \geq 0$, in conclusion $\phi \equiv 0$.
We can repeat this argument substituting $\mathbf{e}_1$ with $\mathbf{e}_2$ and iterate. We obtain $f(x + \mathbf{e}_1, s + \Delta s) - f(x,s) = \cdots = f(x + \mathbf{e}_d, s + \Delta s) - f(x,s) = 0$ for any $\Delta s \in (0,1)$ *i.e.* f is constant.
We know that $\mathbf{P}_{x,s}(D_s^S = +\infty)$ is bounded, so because it is harmonic it is constant. Now we prove that it can only assume the values 0 or 1.
Let $\tau_s = \inf\{t \geq s : X(t) \in S\}$ the time of first hitting of S after $s \geq 0$. Then by strong Markov property:

$$\begin{aligned}
&\mathbf{P}_{x,s}(D_s^S = +\infty) = \\
&\quad = \sum_{y \in S} \int_s^{+\infty} \mathbf{P}_{x,s}(D_s^S = +\infty | \tau_s = t, X(\tau_s) = y)\, \mathbf{P}_{x,s}(\tau_s \in dt, X(\tau_s) = y) \\
&\quad = \sum_{y \in S} \int_s^{+\infty} \mathbf{P}_{y,t}(D_s^S = +\infty)\, \mathbf{P}_{x,s}(\tau_s \in dt, X(\tau_s) = y) \\
&\quad = \mathbf{P}_{x,s}(D_s^S = +\infty) \sum_{y \in S} \int_s^{+\infty} \mathbf{P}_{x,s}(\tau_s \in dt, X(\tau_s) = y) = \\
&\quad = \mathbf{P}_{x,s}(D_s^S = +\infty)\, \mathbf{P}_{x,s}(\mathbf{X} \text{ hits } S \text{ after } s),
\end{aligned} \tag{5.7}$$

so $\mathbf{P}_{x_0,s_0}(\mathbf{X} \text{ hits } S \text{ after } s_0) < 1$ for some $(x_0,s_0) \in \mathbb{Z}^d \times \mathbb{R}_+$ implies $\mathbf{P}_{x,s}(D_s^S = +\infty) = 0$ for any $(x,s) \in \mathbb{Z}^d \times \mathbb{R}_+$. Assume that $\mathbf{P}_{x,s}(\mathbf{X} \text{ hits } S \text{ after } s) \equiv 1$ and define $C_n = \{\mathbf{X} \text{ does not hit } S \text{ for any } t > n\}$. Clearly $(D_s^S < +\infty) \subset \bigcup_{k \geq s} C_k$ and by the assumption $\mathbf{P}_{x,s}(C_n) = 0$ for any (x,s) and $n \geq s$. This implies $\mathbf{P}_{x,s}(D_s^S < +\infty) \equiv 0$.

We proved that $\mathbf{P}_{x,s}(D_s^S = +\infty) \in \{0,1\}$ and that $\mathbf{P}_{x,s}(D_s^S = +\infty) \equiv 1$ if and only if

$$\mathbf{P}_{x,s}(\mathbf{X} \text{ hits } S \text{ after } s) \equiv 1 \,.$$

If $\mathbf{P}_{x,s}(D_s^S = +\infty) \equiv 0$ then because $\mathbf{P}_{x,s}(T_s^S = +\infty) \leq \mathbf{P}_{x,s}(D_s^S = +\infty)$ we have that $\mathbf{P}_{x,s}(T_s^S = +\infty) \equiv 0$. On the contrary assume $\mathbf{P}_{x,s}(D_s^S = +\infty) \equiv 1$ and define the sequence of entry and exit times of $\mathbf{X}$ in S:

$$\begin{aligned}
\tau_{1,s}^S &= \inf\{t \geq s : X(t) \in S\} = \text{“instant of first entry of } \mathbf{X} \text{ in } S \text{ after } s\text{”};\\
\sigma_{1,s}^S &= \inf\{t \geq \tau_{1,s}^S : X(t) \notin S\} = \text{“instant of first exit of } \mathbf{X} \text{ from } S \text{ after } s\text{”};\\
\tau_{2,s}^S &= \inf\{t \geq \sigma_{1,s}^S : X(t) \in S\} = \text{“instant of second entry of } \mathbf{X} \text{ in } S \text{ after } s\text{”};\\
\sigma_{2,s}^S &= \inf\{t \geq \tau_{2,s}^S : X(t) \notin S\} = \text{“instant of second exit of } \mathbf{X} \text{ from } S \text{ after } s\text{”};\\
&\ \vdots
\end{aligned}$$

with the usual convention that $\inf \emptyset = +\infty$. Now define the random variable

$$N_s^S = \text{“number of times that } \mathbf{X} \text{ hits } S\text{”} = \sup\{n \geq 1 : \tau_{n,s}^S < +\infty\} \,,$$

and notice that $\mathbf{P}_{x,s}(T_s^S < +\infty) = \mathbf{P}_{x,s}(T_s^S < +\infty, D_s^S = +\infty) \leq \mathbf{P}_{x,s}(T_s^S < +\infty, N_s^S = +\infty)$. Assume for a moment to know that $\mathbf{P}_{x,s}(N_s^S = +\infty) \in \{0,1\}$. If $\mathbf{P}_{x,s}(N_s^S = +\infty) = 0$ then $\mathbf{P}_{x,s}(D_s^S < +\infty) = 0$, *i.e.* $\mathbf{P}_{x,s}(T_s^S = +\infty) = 1$. On the contrary if $\mathbf{P}_{x,s}(N_s^S = +\infty) = 1$, the time spent from $\mathbf{X}$ into S during the k^{th} visit is $\delta_{k,s}^S = \sigma_{k,s}^S - \tau_{k,s}^S$ and $T_s^S = \sum_{k=1}^{N_s^S} \delta_{k,s}^S = \sum_{k=1}^{+\infty} \delta_{k,s}^S$. Now define $\widetilde{\sigma}_{k,s}^S = \inf\{t > \tau_{k,s}^S : X(t) \neq X(\tau_{k,s}^S)\}$, $k = 1, \ldots, N_s^S$ then clearly $\widetilde{\delta}_{k,s}^S = \widetilde{\sigma}_{k,s}^S - \tau_{k,s}^S \leq \delta_{k,s}^S$, furthermore $\widetilde{\delta}_{k,s}^S \geq \overline{\delta}_{k,s}^S$ where $\overline{\delta}_{k,s}^S$, $k = 1, \ldots, N_s^S$ are independent random variables exponentially distributed with mean $1/2dC_+$ (see remark at the end of Section 5.1). Thus we have

$$\begin{aligned}
\mathbf{P}_{x,s}(T_s^S < +\infty) &= \mathbf{P}_{x,s}(T_s^S < +\infty, N_s^S = +\infty) =\\
&= \mathbf{P}_{x,s}\left(\sum_{k=1}^{N_s^S} \delta_{k,s}^S < +\infty, N_s^S = +\infty\right) \leq \mathbf{P}_{x,s}\left(\sum_{k=1}^{N_s^S} \overline{\delta}_{k,s}^S < +\infty, N_s^S = +\infty\right)\\
&\qquad = \mathbf{P}_{x,s}\left(\sum_{k=1}^{+\infty} \overline{\delta}_{k,s}^S < +\infty, N_s^S = +\infty\right) = 0. \qquad (5.8)
\end{aligned}$$

It remains to prove that $\mathbf{P}_{x,s}(N_s^S = +\infty) \in \{0,1\}$. It is easy to show that $\mathbf{P}_{x,s}(N_s^S = +\infty)$ is a harmonic, trivially bounded, function, so it is constant. Let now $n \in \mathbb{N}$ be a fixed number, then again by strong Markov property

$$\begin{aligned}
&\mathbf{P}_{x,s}(N_s^S = +\infty) =\\
&\quad = \sum_{y_1,\ldots,y_n \in S} \int \cdots \int_{s<s_1<\cdots<s_n} \mathbf{P}_{x,s}\left(N_s^S = +\infty \,\middle|\, \begin{array}{c} \tau_{1,s}^S = s_1, \ldots, \tau_{n,s}^S = s_n \\ X(\tau_{1,s}^S) = y_1, \ldots, X(\tau_{n,s}^S) = y_n \end{array}\right)\\
&\qquad\qquad \times \mathbf{P}_{x,s}\left(\begin{array}{c} \tau_{1,s}^S \in ds_1, \ldots, \tau_{n,s}^S \in ds_n \\ X(\tau_{1,s}^S) = y_1, \ldots, X(\tau_{n,s}^S) = y_n \end{array}\right) =\\
&= \sum_{y_1,\ldots,y_n \in S} \int \cdots \int_{s<s_1<\cdots<s_n} \mathbf{P}_{y_n,s_n}(N_{s_n}^S = +\infty) \times \mathbf{P}_{x,s}\left(\begin{array}{c} \tau_{1,s}^S \in ds_1, \ldots, \tau_{n,s}^S \in ds_n \\ X(\tau_{1,s}^S) = y_1, \ldots, X(\tau_{n,s}^S) = y_n \end{array}\right)\\
&= \mathbf{P}_{x,s}(N_s^S = +\infty) \sum_{y_1,\ldots,y_n \in S} \int \cdots \int_{s<s_1<\cdots<s_n} \mathbf{P}_{x,s}\left(\begin{array}{c} \tau_{1,s}^S \in ds_1, \ldots, \tau_{n,s}^S \in ds_n \\ X(\tau_{1,s}^S) = y_1, \ldots, X(\tau_{n,s}^S) = y_n \end{array}\right)\\
&\qquad\qquad = \mathbf{P}_{x,s}(N_s^S = +\infty)\, \mathbf{P}_{x,s}(N_s^S \geq n).
\end{aligned}$$

By taking the limit for $n \to +\infty$ in the above equation we get $\mathbf{P}_{x,s}(N_s^S = +\infty) = \left[\mathbf{P}_{x,s}(N_s^S = +\infty)\right]^2$ which implies $\mathbf{P}_{x,s}(N_s^S = +\infty) \in \{0,1\}$. $\square$

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