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A CONDITION FOR WEAK DISORDER FOR DIRECTED POLYMERS IN RANDOM ENVIRONMENT

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Abstract
We give a sufficient criterion for the weak disorder regime of directed polymers in random environment, which extends a well-known second moment criterion. We use a stochastic representation of the size-biased law of the partition function.

We consider the so-called directed polymer in random environment, being defined as follows: Let $p(x,y) = p(y-x)$, $x, y \in \mathbb{Z}^d$ be a shift-invariant, irreducible transition kernel, $(S_n)_{n\in\mathbb{N}_0}$ the corresponding random walk. Let $\xi(x,n), x \in \mathbb{Z}^d, n \in \mathbb{N}$ be i.i.d. random variables satisfying

$$\mathbb{E}[\exp(\beta\xi(x,n))] < \infty \quad \text{for all } \beta \in \mathbb{R}, \tag{1}$$

we denote their cumulant generating function by

$$\lambda(\beta) := \log \mathbb{E}[\exp(\beta\xi(x,n))]. \tag{2}$$

We think of the graph of S_n as the (directed) polymer, which is influenced by the random environment generated by the $\xi(x,n)$ through a reweighting of paths with

$$e_n := e_n(\xi, S) := \exp\Big(\sum_{j=1}^{n} \big\{\beta\xi(S_j, j) - \lambda(\beta)\big\}\Big),$$

that is, we are interested in the random probability measures on path space given by

$$\mu_n(ds) = \frac{1}{Z_n}\mathbb{E}[e_n \mathbf{1}(S \in ds) \,|\, \xi(\cdot,\cdot)],$$

where the normalising constant (or partition function) is given by

$$Z_n = \mathbb{E}[e_n|\xi] = \sum_{s_1,\dots,s_n\in\mathbb{Z}^d} \prod_{j=1}^{n} p(s_{j-1}, s_j) \exp\Big(\sum_{k=1}^{n} \big\{\beta\xi(s_k,k) - \lambda(\beta)\big\}\Big).$$

Note that (Z_n) is a martingale, and hence converges almost surely. This model has been studied by many authors, see e.g. [2] and the references given there. It is known that the behaviour of μ_n as $n \to \infty$ depends on whether $\lim_n Z_n > 0$ or $\lim_n Z_n = 0$. One speaks of *weak disorder* in the first, and of *strong disorder* in the second case. Our aim here is to give a condition for the weak disorder regime.

Let (S_n) and (S'_n) be two independent p-random walks starting from $S_0 = S'_0 = 0$, and let $V := \sum_{n=1}^{\infty} \mathbf{1}(S_n = S'_n)$ be the number of times the two paths meet. Define

$$\alpha_* := \sup\big\{\alpha \geq 1 : \mathbb{E}[\alpha^V | S'] < \infty \text{ almost surely}\big\}. \tag{3}$$

Proposition 1 *If $\lambda(2\beta) - 2\lambda(\beta) < \log \alpha_*$, then*

$$\lim\nolimits_{n\to\infty} Z_n > 0 \quad \textit{almost surely},$$

that is, the directed polymer is in the weak disorder regime.

Note that Proposition 1 implicitly requires that the difference random walk $S - S'$ be transient, for otherwise we would have $\log \alpha_* = 0$, but we also have $\lambda(2\beta) - 2\lambda(\beta) \geq 0$ by convexity. For symmetric simple random walk in dimension $d = 1, 2$ we have $Z_n \to 0$ almost surely for any $\beta \neq 0$, see [2], Thm. 2.3 (b).

Observe that

$$\alpha_* \geq \alpha_2 := \sup\big\{\alpha \geq 1 : \mathbb{E}[\alpha^V] < \infty\big\} = \frac{1}{1 - \mathbb{P}_{(0,0)}(S_n \neq S'_n \text{ for } n \geq 1)}.$$

An easy calculation shows that (Z_n) is an L^2-bounded martingale iff $\lambda(2\beta) - 2\lambda(\beta) < \log \alpha_2$, cf. e.g. [2], equation (1.8) and the paragraph below it on p. 707 and the references given there (note that for symmetric simple random walk, $\mathbb{P}_{(0,0)}(S_n \neq S'_n \text{ for } n \geq 1) = \mathbb{P}_0(S_n \neq 0 \text{ for } n \geq 1) =: q$).

If $S - S'$ is transient and p satisfies

$$\sup_{n,x} \frac{p_n(x)}{\sum_y p_n(y) p_n(-y)} < \infty \tag{4}$$

then we have

$$\alpha_* = 1 + \left(\sum_{n=1}^{\infty} \exp\big(-H(p_n)\big)\right)^{-1} > \alpha_2, \tag{5}$$

where $p_n(x) := \mathbb{P}_0(S_n = x)$ is the n-step transition probability of a p-random walk, and $H(p_n) = -\sum_x p_n(x) \log(p_n(x))$ is its entropy, see [1], Thm. 5. Note that (4) is automatically satisfied if a local central limit theorem holds for p, in particular, it holds for symmetric simple random walk. Thus, Proposition 1 is an extension of the second moment condition (1.8) in [2].

Let $\hat{Z}_n$ have the size-biased law of Z_n, i.e.

$$\mathbb{E}[f(\hat{Z}_n)] = \mathbb{E}[Z_n f(Z_n)]$$

for any bounded, measurable f. The proof of Proposition 1 hinges on the representation of the size-biased law given in the following lemma.

Lemma 1 *Let (S'_n) be a p-random walk starting from $S'_0 = 0$, and let $\big\{(e(n,x), \hat{e}(n,x))\big\}_{n,x}$ be i.i.d., independent of S', with values in $\mathbb{R}^2$ such that*

$$\begin{aligned} \mathbb{P}(e(n,x) \in dr) &= \mathbb{P}(\exp(\beta\xi(n,x) - \lambda(\beta)) \in dr) \\ \mathbb{P}(\hat{e}(n,x) \in dr) &= \mathbb{E}[e(n,x); e(n,x) \in dr], \end{aligned}$$

i.e. $\hat{e}(n,x)$ has the size-biased law of $e(n,x)$. Put

$$\begin{aligned} \tilde{e}_x(n,y) &:= \delta_{xy}\hat{e}(n,x) + (1-\delta_{xy})e_x(n,y) \quad \textit{and} \\ \tilde{Z}_n &:= \mathbb{E}\Big[\prod_{1\le j\le n} \tilde{e}_{S'_j}(j,S_j)\,\Big|\, e,\hat{e},S'\Big]. \end{aligned}$$

Then $\hat{Z}_n$ and $\tilde{Z}_n$ have the same distribution.

Proof. Note that $\tilde{Z}_n$ is a function of S', e and $\hat{e}$, namely

$$\tilde{Z}_n = \sum_{s_1,\dots,s_n\in\mathbb{Z}^d} \prod_{j=1}^n p(s_{j-1},s_j) \times \prod_{k=1}^n \tilde{e}_{S'_k}(k,s_k).$$

We have by definition for a bounded $f : \mathbb{R}_+ \to \mathbb{R}$

$$\begin{aligned} \mathbb{E}[f(\hat{Z}_n)] &= \mathbb{E}[Z_n f(Z_n)] \\ &= \sum_{s_1,\dots,s_n\in\mathbb{Z}^d} \prod_{j=1}^n p(s_{j-1},s_j)\mathbb{E}\Big[f(Z_n)\prod_{1\le k\le n} e(k,s_k)\Big] \\ &= \mathbb{E}\Big[f(Z_n)\prod_{1\le k\le n} e(k,S'_k)\Big] \\ &= \mathbb{E}\Big[f\Big(\sum_{y_1,\dots,y_n}\prod_1^n p(y_{j-1},y_j)\times\prod_{1\le k\le n} e(k,y_k)\Big)\prod_{1\le \ell\le n} e(\ell,S'_\ell)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[\dots\,\big|\,S'\big]\Big] = \mathbb{E}\Bigg[f\Big(\sum_{y_1,\dots,y_n}\prod_1^n p(y_{j-1},y_j)\times\prod_{1\le k\le n}\tilde{e}_{S'_k}(k,y_k)\Big)\Bigg] \\ &= \mathbb{E}[f(\tilde{Z}_n)]. \end{aligned}$$

□

Proof of Proposition 1. As $\mathbb{P}(Z_\infty > 0) \in \{0,1\}$ by Kolmogorov's $0-1$ law (see e.g. (1.7) in [2]), the proposition will be proved if we can show that under the given condition, the sequence Z_n, $n \in \mathbb{N}$ is uniformly integrable. This, in turn, is equivalent to tightness of the sequence $\hat{Z}_n$, see e.g. Lemma 9 in [1]. We see from Lemma 1 that this is equivalent to whether the family $\mathcal{L}(\tilde{Z}_n)$, $n \in \mathbb{N}$, is tight. Let us denote by $\alpha := \mathbb{E}\exp(\beta\hat{\xi} - \lambda(\beta)) = \exp(\lambda(2\beta) - 2\lambda(\beta))$, then

$$\mathbb{E}[\tilde{Z}_n|S'] = \mathbb{E}\Big[\alpha^{\#\{1\le i\le n: S_i = S'_i\}}\Big|S'\Big],$$

hence $\alpha < \alpha_*$ implies $\sup_n \mathbb{E}[\tilde{Z}_n|S'] < \infty$ almost surely, which in particular shows that the family of laws $\mathcal{L}(\tilde{Z}_n)$ is tight. □

Remark. Note that we obtain a sufficient condition for weak disorder by averaging out $\xi(\cdot,\cdot)$ and $\tilde{\xi}(\cdot,\cdot)$ in the construction of $\tilde{Z}_n$ given in Lemma 1. In order to obtain a sharp criterion one would have to analyse the distribution of $\tilde{Z}_n$ itself. Unfortunately, this seems a rather hard problem.

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