

SHARP BOUNDS FOR GREEN FUNCTIONS AND JUMPING FUNCTIONS OF SUBORDINATE KILLED BROWNIAN MOTIONS IN BOUNDED $C^{1,1}$ DOMAINS

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Abstract

In this paper we obtain sharp bounds for the Green function and jumping function of a subordinate killed Brownian motion in a bounded $C^{1,1}$ domain, where the subordinating process is a subordinator whose Laplace exponent has certain asymptotic behavior at infinity.

1 Introduction

Let $X = (X_t, \mathbb{P}_x)$ be a Brownian motion in \mathbb{R}^d with generator Δ . Let $D \subset \mathbb{R}^d$ be a bounded domain, and we use $X^D = (X_t^D, \mathbb{P}_x)$ to denote the killed Brownian motion in D . Let $T = (T_t : t \geq 0)$ be a subordinator independent of X , and define $Y_t^D := X^D(T_t)$, $t \geq 0$. The process $Y^D = (Y_t^D : t \geq 0)$ is called a subordinate killed Brownian motion.

The study of the process Y^D was initiated in [7], where T is assumed to be an $\alpha/2$ -stable subordinator, $\alpha \in (0, 2)$. Recently a lot of progress has been made in the study of the potential theory of Y^D . In [12], under the assumption that T is an $\alpha/2$ -stable subordinator, upper and lower bounds on the Green function and jumping function of Y^D were established when D is a bounded $C^{1,1}$ domain. However, the upper and lower bounds provided in [12] were different near the boundary. In [11], new lower bounds for the Green function and jumping function of Y^D , that agree with the upper bounds of [12] up to multiplicative constants, were established. In this sense, sharp bounds of the Green function and jumping function of Y^D , in the case when T is an $\alpha/2$ -stable subordinator, were obtained.

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The purpose of this paper is to obtain sharp bounds for the Green function and jumping function of Y^D for a much larger class of subordinating processes T . The sharp bounds obtained in this paper are very useful. For instance, they can be used to give concrete criteria for functions or measures to belong to the classes $\mathbf{S}_1(X)$ and $\mathbf{A}_\infty(X)$, defined in [2], [3] and [4], when the process X is the subordinate killed Brownian motion dealt with in this paper. The content of this paper is organized as follows. In Section 2 we recall some basic facts about special subordinators and subordinate killed Brownian motion and in Section 3 we establish our main results.

In this paper we write $f \sim g$ as $x \rightarrow \infty$ (respectively, $x \rightarrow 0$), if $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ (respectively, $\lim_{x \rightarrow 0} f(x)/g(x) = 1$).

2 Preliminaries

Let $T = (T_t : t \geq 0)$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty]$ with $T_0 = 0$. The Laplace exponent of the subordinator T is a function $\phi : (0, \infty) \rightarrow [0, \infty)$ such that

$$\mathbb{E}[\exp(-\lambda T_t)] = \exp(-t\phi(\lambda)), \quad \lambda > 0. \quad (2.1)$$

The function ϕ has a representation

$$\phi(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt). \quad (2.2)$$

Here $a, b \geq 0$, and μ is a σ -finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1) \mu(dt) < \infty.$$

The constant a is called the killing rate, b the drift, and μ the Lévy measure of the subordinator T .

Recall that a C^∞ function $\phi : (0, \infty) \rightarrow [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \leq 0$ for every $n \in \mathbb{N}$. It is well known that a function $\phi : (0, \infty) \rightarrow \mathbb{R}$ is a Bernstein function if and only if it has the representation given by (2.2).

The potential measure of the subordinator T is defined by

$$U(A) = \mathbb{E} \int_0^\infty 1_{(T_t \in A)} dt, \quad (2.3)$$

and its Laplace transform is given by

$$\mathcal{L}U(\lambda) = \int_0^\infty e^{-\lambda t} dU(t) = \mathbb{E} \int_0^\infty \exp(-\lambda T_t) dt = \frac{1}{\phi(\lambda)}. \quad (2.4)$$

It is well known that if the drift b is strictly positive, then the potential measure U is absolutely continuous with a density $u : (0, \infty) \rightarrow \mathbb{R}$ that is continuous and positive, and $u(0+) = 1/b$ (e.g., [1], p.79). Moreover, for every $t > 0$, $u(t) \leq u(0+)$ (e.g., [10]). In order to establish the main results of this paper we will need the existence of a decreasing potential density for subordinators not necessarily having a strictly positive drift. A large class of such subordinators, called special subordinators, was introduced and studied in [13]. We recall the definition below.

Definition 2.1 A Bernstein function ϕ is called a special Bernstein function if the function $\lambda/\phi(\lambda)$ is also a Bernstein function. A subordinator T is called a special subordinator if its Laplace exponent ϕ is a special Bernstein function.

The family of special Bernstein functions is very large, and it contains in particular the family of complete Bernstein functions. Recall that a function $\phi : (0, \infty) \rightarrow \mathbb{R}$ is called a complete Bernstein function if there exists a Bernstein function η such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\eta(\lambda), \quad \lambda > 0.$$

The family of complete Bernstein functions is also very large and it contains the following well known Bernstein functions: (i) λ^α , $\alpha \in (0, 1]$; (ii) $(\lambda + 1)^\alpha - 1$, $\lambda \in (0, 1)$, and (iii) $\ln(1 + \lambda)$. It is known (see [8], for instance) that every complete Bernstein function is a special Bernstein function and that a Bernstein function ϕ of the form (2.2) is a complete Bernstein function if and only if its integral part

$$\phi_1(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu(dt), \quad \lambda > 0 \tag{2.5}$$

is a complete Bernstein function.

One of the main results about special subordinators proved in [13] is the following: If T is a special subordinator such that $b > 0$ or $\mu((0, \infty)) = \infty$, then the potential measure U of T has a decreasing density u . For other results about special subordinators, as well as for many examples, we refer the reader to [13].

Suppose that $X = (X_t : t \geq 0)$ is a Brownian motion in \mathbb{R}^d with generator Δ . Suppose that D is a bounded domain in \mathbb{R}^d and that X^D is the killed Brownian motion in D . Suppose that $T = (T_t : t \geq 0)$ is a subordinator independent of X . The process $Y^D = (Y_t^D : t \geq 0)$ defined by $Y_t^D = X^D(T_t)$ is called a subordinate killed Brownian motion.

Let $p^D(t, x, y)$, $t \geq 0$, $x, y \in D$, denote the transition density of X^D and $(P_t^D : t \geq 0)$ the transition semigroup of X^D . It is well known that the potential operator of the subordinate process Y^D has a density U^D given by the formula

$$U^D(x, y) = \int_0^\infty p^D(t, x, y) U(dt). \tag{2.6}$$

We call $U^D(x, y)$ the Green function of Y^D . In case the potential measure U of the subordinator T has a density u , we may write

$$U^D(x, y) = \int_0^\infty p^D(t, x, y) u(t) dt. \tag{2.7}$$

Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form corresponding to Y^D , then $H^1(D) \subset \mathcal{F}$. When the drift b of T is positive, we have $\mathcal{F} = H^1(D)$ and for $u \in \mathcal{F}$,

$$\mathcal{E}(u, u) = b \int_D |\nabla u|^2(x) dx + \int_{D \times D} (u(x) - u(y))^2 J^D(x, y) dx dy + \int_D u^2(x) \kappa^D(x) dx,$$

where

$$J^D(x, y) = \frac{1}{2} \int_0^\infty p^D(s, x, y) \mu(ds),$$

and

$$\kappa^D(x) = a + \int_0^\infty (1 - P_s^D 1_D(x)) \mu(ds).$$

When T has no drift,

$$\mathcal{F} = \{u \in L^2(D) : \int_0^\infty (u - P_s^D u, u) \mu(ds) < \infty\} \quad (2.8)$$

and for any $u \in \mathcal{F}$,

$$\mathcal{E}(u, u) = \int_{D \times D} (u(x) - u(y))^2 J^D(x, y) dx dy + \int_D u^2(x) \kappa^D(x) dx,$$

with J^D and κ^D given above. The functions J^D and κ^D are called the jumping function and killing functions of Y^D respectively. For the above facts on the Dirichlet form of Y^D , please see [9].

We recall now the definition of a $C^{1,1}$ domain. A bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, is called a bounded $C^{1,1}$ domain if there exist positive constants r_0 and M with the following property: for every $z \in \partial D$ and every $r \in (0, r_0]$, there exist a function $\Gamma_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying the condition $|\nabla \Gamma_z(\xi) - \nabla \Gamma_z(\eta)| \leq M|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}^{d-1}$, and an orthonormal coordinate system CS_z such that if $y = (y_1, \dots, y_d)$ in CS_z coordinates, then

$$B(z, r) \cap D = B(z, r) \cap \{y : y_d > \Gamma_z(y_1, \dots, y_{d-1})\}.$$

When we speak of a bounded $C^{1,1}$ domain in \mathbb{R} we mean a finite open interval.

3 Main results

In this section we will always assume that D is a bounded $C^{1,1}$ domain in \mathbb{R}^d . For any $x \in D$, we use $\rho(x)$ to denote the distance between x and ∂D .

In this section we will establish sharp estimates on the Green function and jumping function of Y^D following the method of [11]. Our basic information is the Laplace exponent ϕ of the subordinator T and our basic assumption will be the asymptotic behavior of the Laplace exponent $\phi(\lambda)$ at infinity. We assume that ϕ has the representation (2.2).

For the Green function estimates, we will consider two cases: In the first case we will consider special subordinators T with Laplace exponent ϕ satisfying $\phi(\lambda) \sim \gamma^{-1} \lambda^{\alpha/2}$ as $\lambda \rightarrow \infty$ for $\alpha \in (0, 2)$ and a positive constant γ . Note that in this case the drift of the subordinator must be zero. In the second case, the drift b is strictly positive, but we do not assume that the subordinator T is special. Note that the drift b is strictly positive if and only if $\phi(\lambda) \sim b\lambda$ as $\lambda \rightarrow \infty$. We put these two cases into the following assumption:

Assumption A: *The Laplace exponent ϕ of T satisfies*

$$\phi(\lambda) \sim \gamma^{-1} \lambda^{\alpha/2}, \quad \lambda \rightarrow \infty, \quad (3.1)$$

for some $\alpha \in (0, 2]$, and in the case $\alpha \in (0, 2)$ we always assume that ϕ is a special Bernstein function.

Note that in the case $\alpha = 2$, we have that $\gamma^{-1} = b$, the drift of the subordinator.

Assumption A implies that, in the case $\alpha \in (0, 2)$, the Lévy measure μ of T must satisfy $\mu((0, \infty)) = \infty$. So under the Assumption A, the subordinator T has a potential density $u(t)$.

In the case $\alpha \in (0, 2)$, it follows from (2.4) and Assumption A by use of the Tauberian theorem and the monotone density theorem, that

$$u(t) \sim \frac{\gamma}{\Gamma(\alpha/2)} t^{\alpha/2-1}, \quad t \rightarrow 0+.$$

Therefore, for each $A > 0$, there exists a positive constant $c_1 = c_1(A)$ such that

$$\frac{u(t)}{t^{\alpha/2-1}} \geq c_1, \quad 0 < t \leq A. \quad (3.2)$$

Moreover, there exists a positive constant c_2 such that

$$\frac{u(t)}{t^{\alpha/2-1}} \leq \begin{cases} c_2, & t < 1 \\ c_2 t^{-\alpha/2+1}, & t \geq 1 \end{cases} \quad (3.3)$$

Note also that both (3.2) and (3.3) are valid for the case of a strictly positive drift (i.e., $\alpha = 2$). For the jumping kernel estimates we will need the following assumption.

Assumption B: *The Laplace exponent ϕ of T is a complete Bernstein function and satisfies one of the following two conditions: (i) The drift b is positive and the integral part ϕ_1 of ϕ has the following asymptotic behavior*

$$\phi_1(\lambda) \sim \gamma^{-1} \lambda^{\beta/2}, \quad \lambda \rightarrow \infty, \quad (3.4)$$

for some $\beta \in (0, 2)$. (ii) $b = 0$ and ϕ has the following asymptotic behavior

$$\phi(\lambda) \sim \gamma^{-1} \lambda^{\beta/2}, \quad \lambda \rightarrow \infty, \quad (3.5)$$

for some $\beta \in (0, 2)$.

It is known (see [8] for instance) that the Lévy measure of any complete Bernstein function has a completely monotone density. The asymptotic relations (3.4) and (3.5) imply that the Lévy measure μ of T must satisfy $\mu((0, \infty)) = \infty$. Corollary 2.4 of [13] implies that the Lévy measure ν of the complete Bernstein function $\lambda/\phi_1(\lambda)$ in case (i), and of the complete Bernstein function $\lambda/\phi(\lambda)$ in case (ii), satisfies $\nu((0, \infty)) = \infty$. Hence the potential measure of the subordinator with Laplace exponent $\lambda/\phi_1(\lambda)$ in case (i), and with Laplace exponent $\lambda/\phi(\lambda)$ in case (ii), has a decreasing density v . Again from Corollary 2.4 of [13] we know that the Lévy measure μ of ϕ and the potential density v are related as follows

$$v(t) = \tilde{a} + \mu((t, \infty)), \quad t > 0,$$

with $\tilde{a} = 0$ in case (i) and $\tilde{a} = a$ in case (ii). It follows from (3.4), (3.5), the Tauberian theorem and the monotone density theorem that

$$v(t) \sim \frac{1}{\gamma \Gamma(1 - \beta/2)} t^{-\beta/2}, \quad t \rightarrow 0+,$$

hence we have

$$\mu((t, \infty)) \sim \frac{1}{\gamma \Gamma(1 - \beta/2)} t^{-\beta/2}, \quad t \rightarrow 0+.$$

Using the monotone density theorem again we get that

$$\mu(t) \sim \frac{\beta}{2\gamma \Gamma(1 - \beta/2)} t^{-\beta/2-1}, \quad t \rightarrow 0+, \quad (3.6)$$

where $\mu(t)$ denotes the density of the measure μ . Therefore, for each $A > 0$, there exists a positive constant $c_3 = c_3(A)$ such that

$$\frac{\mu(t)}{t^{-\beta/2-1}} \geq c_3, \quad 0 < t \leq A. \quad (3.7)$$

Moreover, there exists a positive constant c_4 such that

$$\frac{\mu(t)}{t^{-\beta/2-1}} \leq \begin{cases} c_4, & t < 1 \\ c_4 t^{\beta/2+1}, & t \geq 1 \end{cases} \quad (3.8)$$

The sharp estimates on the Green function of Y^D are included in the following result.

Theorem 3.1 *Suppose that D is a bounded $C^{1,1}$ domain in \mathbb{R}^d and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption A. If $d > \alpha$, then there exist positive constants $C_1 \leq C_2$ such that for all $x, y \in D$*

$$C_1 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}} \leq U^D(x, y) \leq C_2 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d-\alpha}}.$$

Proof. Upper bound. It is known (see [5] and [12]) that there exists a positive constant c_5 such that for all $t > 0$ and any $x, y \in D$,

$$p^D(t, x, y) \leq c_5 t^{-d/2-1} \rho(x)\rho(y) \exp\left(-\frac{|x-y|^2}{6t}\right). \quad (3.9)$$

Hence by the formula for U^D ,

$$\begin{aligned} U^D(x, y) &\leq c_5 \rho(x)\rho(y) \int_0^\infty t^{-d/2-1} e^{-|x-y|^2/6t} u(t) dt \\ &= c_6 \rho(x)\rho(y) |x-y|^{-d} \int_0^\infty s^{d/2-1} e^{-s} u\left(\frac{|x-y|^2}{6s}\right) ds \\ &= c_7 \rho(x)\rho(y) |x-y|^{-d-2+\alpha} \int_0^\infty s^{d/2-\alpha/2} e^{-s} \frac{u(|x-y|^2/6s)}{(|x-y|^2/6s)^{\alpha/2-1}} ds. \end{aligned}$$

For $\alpha = 2$, the last integral is clearly bounded by a positive constant. For $0 < \alpha < 2$, we estimate the integral by use of (3.3):

$$\begin{aligned} &\int_0^\infty s^{d/2-\alpha/2} e^{-s} \frac{u(|x-y|^2/6s)}{(|x-y|^2/6s)^{\alpha/2-1}} ds = \int_0^{|x-y|^2/6} + \int_{|x-y|^2/6}^\infty \\ &\leq c_2 \int_0^{|x-y|^2/6} s^{d/2-\alpha/2} e^{-s} (|x-y|^2/6s)^{-\alpha/2+1} ds + c_2 \int_{|x-y|^2/6}^\infty s^{d/2-\alpha/2} e^{-s} ds \\ &\leq c_8 \left(|x-y|^{-\alpha+2} \int_0^{|x-y|^2/6} s^{d/2-1} e^{-s} ds + \int_{|x-y|^2/6}^\infty s^{d/2-\alpha/2} e^{-s} ds \right) \\ &\leq c_9 \left(\int_0^\infty s^{d/2-1} e^{-s} ds + \int_0^\infty s^{d/2-\alpha/2} e^{-s} ds \right) = c_{10}. \end{aligned}$$

Here we used that $|x-y| \leq \text{diam}(D)$. Hence we have shown that there exists a positive constant c_{11} such that for all $x, y \in D$,

$$U^D(x, y) \leq c_{11} \frac{\rho(x)\rho(y)}{|x-y|^{d+2-\alpha}}. \quad (3.10)$$

Let $p(t, x, y)$, $t \geq 0$, $x, y \in \mathbb{R}^d$, be the transition density of the Brownian motion X in \mathbb{R}^d . Then $p^D(t, x, y) \leq p(t, x, y)$, implying

$$U^D(x, y) \leq \int_0^\infty p(t, x, y)u(t) dt.$$

A similar (but easier) argument to the one in the paragraph above shows that there is a constant $c_{12} > 0$ such that

$$\int_0^\infty p(t, x, y)u(t) dt \leq c_{12}|x - y|^{\alpha-d}, \quad x, y \in D.$$

Therefore,

$$U^D(x, y) \leq c_{12}|x - y|^{\alpha-d}, \quad x, y \in D. \quad (3.11)$$

Combining (3.10) and (3.11) we get the upper bound of the theorem.

Lower bound. It was proved in [14] and [11] that for any $A > 0$, there exist positive constants c_{13} and c_{14} such that for any $t \in (0, A]$ and any $x, y \in D$,

$$p^D(t, x, y) \geq c_{13} \left(\frac{\rho(x)\rho(y)}{t} \wedge 1 \right) t^{-d/2} \exp\left(-\frac{c_{14}|x - y|^2}{t}\right). \quad (3.12)$$

Hence by the formula for U^D ,

$$U^D(x, y) \geq c_{13} \int_0^A \left(\frac{\rho(x)\rho(y)}{t} \wedge 1 \right) t^{-d/2} \exp\{-c_{14}|x - y|^2/t\}u(t) dt.$$

Assume $x \neq y$. Let R be the diameter of D and assume that A has been chosen so that $A = R^2$. Then for any $x, y \in D$, $\rho(x)\rho(y) < R^2 = A$. The lower bound is proved by considering two separate cases:

(i) $\frac{|x - y|^2}{\rho(x)\rho(y)} < \frac{2R^2}{A}$. In this case we have:

$$\begin{aligned} U^D(x, y) &\geq c_{13} \int_0^{\rho(x)\rho(y)} \left(\frac{\rho(x)\rho(y)}{t} \wedge 1 \right) t^{-d/2} \exp\{-c_{14}|x - y|^2/t\}u(t) dt \\ &\geq c_{15}|x - y|^{-d+2} \int_{\frac{c_{14}|x - y|^2}{\rho(x)\rho(y)}}^\infty s^{d/2-2} e^{-s} u(c_{14}|x - y|^2/s) ds \\ &\geq c_{15}|x - y|^{-d+2} \int_{\frac{2c_{14}R^2}{A}}^\infty s^{d/2-2} e^{-s} u(c_{14}|x - y|^2/s) ds \\ &= c_{16}|x - y|^{-d+\alpha} \int_{\frac{2c_{14}R^2}{A}}^\infty s^{d/2-\alpha/2-1} e^{-s} \frac{u(c_{14}|x - y|^2/s)}{(c_{14}|x - y|^2/s)^{\alpha/2-1}} ds. \end{aligned}$$

For $s > 2c_{14}R^2/A$, we have that $c_{14}|x - y|^2/s \leq A/2$, hence we can estimate the fraction in the above integral by c_1 , see (3.2). Hence,

$$U^D(x, y) \geq c_{17}|x - y|^{-d+\alpha}.$$

(ii) $\frac{|x-y|^2}{\rho(x)\rho(y)} \geq \frac{2R^2}{A}$. In this case we have:

$$\begin{aligned}
U^D(x, y) &\geq c_{13}\rho(x)\rho(y) \int_{\rho(x)\rho(y)}^A t^{-d/2-1} \exp\{-c_{14}|x-y|^2/t\} u(t) dt \\
&= c_{18}\rho(x)\rho(y)|x-y|^{-d} \int_{\frac{c_{14}|x-y|^2}{A}}^{\frac{c_{14}|x-y|^2}{\rho(x)\rho(y)}} s^{d/2-1} e^{-s} u(c_{14}|x-y|^2/s) ds \\
&= c_{19}\rho(x)\rho(y)|x-y|^{-d+\alpha-2} \int_{\frac{c_{14}|x-y|^2}{A}}^{\frac{c_{14}|x-y|^2}{\rho(x)\rho(y)}} s^{-d/2-\alpha/2} e^{-s} \frac{u(c_{14}|x-y|^2/s)}{(c_{14}|x-y|^2/s)^{\alpha/2-1}} ds \\
&\geq c_{19}\rho(x)\rho(y)|x-y|^{-d+\alpha-2} \int_{\frac{c_{14}R^2}{A}}^{\frac{2c_{14}R^2}{A}} s^{-d/2-\alpha/2} e^{-s} \frac{u(c_{14}|x-y|^2/s)}{(c_{14}|x-y|^2/s)^{\alpha/2-1}} ds.
\end{aligned}$$

Again, if $s > c_{14}R^2/A \geq c_{14}|x-y|^2/A$, then $c_{14}|x-y|^2/A < s$, and we use (3.2) to estimate the fraction above by c_1 . Hence,

$$U^D(x, y) \geq c_{20} \frac{\rho(x)\rho(y)}{|x-y|^{d-\alpha+2}}.$$

Combining the two cases above we arrive at the lower bound of the theorem. \square

Remark 3.2 Note that the theorem above implies that when b is positive, the Green function U^D of Y^D is comparable to the Green function of X^D .

The sharp estimates on the jumping function of Y^D are included in the following result.

Theorem 3.3 Suppose that D is a bounded $C^{1,1}$ domain in \mathbb{R}^d and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption B. Then there exist positive constants $C_3 \leq C_4$ such that for all $x, y \in D$

$$C_3 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+\beta}} \leq J^D(x, y) \leq C_4 \left(\frac{\rho(x)\rho(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{d+\beta}}.$$

Proof. The proof of this theorem is the same as that of Theorem 3.1, the only differences are that we use Assumption B, the formula for J , (3.7) and (3.8) instead of Assumption A, the formula for U^D , (3.2) and (3.3). We omit the details. \square

Using arguments similar to that of Proposition 3.2 of [12], we can get the following estimates on the killing function of Y^D .

Theorem 3.4 Suppose that D is a bounded $C^{1,1}$ domain in \mathbb{R}^d and that the subordinator $T = (T_t : t \geq 0)$ satisfies Assumption B. Then there exist positive constants $C_5 \leq C_6$ such that

$$C_5(\rho(x))^{-\beta} \leq \kappa^D(x) \leq C_6(\rho(x))^{-\beta}, \quad x \in D.$$

Proof. Let $Z = (Z_t : t \geq 0)$ be the subordinate Brownian motion defined by $Z_t = X(T_t)$. Clearly, the killing function $\kappa(x)$ of Z is given by $\kappa(x) = a$. It follows from [9] that the jumping function $J(x, y)$ of this process is given by

$$J(x, y) = \frac{1}{2} \int_0^\infty p(t, x, y) \mu(t) dt, \quad (3.13)$$

where $p(t, x, y)$ is the transition density of X . Let $Z^D = (Z_t^D, \mathbb{P}_x)$ be the process Z killed upon exiting D . Then it is known (see [6], for instance) that the killing function $\tilde{\kappa}^D(x)$ of Z^D is given by

$$\tilde{\kappa}^D(x) = a + 2 \int_{D^c} J(x, y) dy, \quad x \in D.$$

Now using (3.6), (3.13) and an argument similar to that of Theorem 3.1 of [10] we get that

$$J(x, y) \sim c_{21} |x - y|^{-d-\beta}, \quad |x - y| \rightarrow 0$$

for some constant $c_{21} > 0$. From this we immediately get that there exist constants $0 < c_{22} < c_{23}$ such that

$$c_{22}(\rho(x))^{-\beta} \leq \tilde{\kappa}^D(x) \leq c_{23}(\rho(x))^{-\beta}, \quad x \in D.$$

Repeating the argument of Proposition 3.2 of [12] we get that there exist constants $0 < c_{24} < c_{25}$ such that

$$c_{24} \tilde{\kappa}^D(x) \leq \kappa^D(x) \leq c_{25} \tilde{\kappa}^D(x), \quad x \in D.$$

The proof is now finished. \square

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