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MEASURE CONCENTRATION FOR STABLE LAWS WITH INDEX CLOSE TO 2

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Abstract
We give upper bounds for the probability $\mathbb{P}(|f(X)-Ef(X)|>x)$, where X is a stable random variable with index close to 2 and f is a Lipschitz function. While the optimal upper bound is known to be of order $1/x^\alpha$ for large x, we establish, for smaller x, an upper bound of order $\exp(-x^\alpha/2)$, which relates the result to the gaussian concentration.

1 Statement of the result

Let X be an α-stable random variable on $\mathbb{R}^d$, $0<\alpha<2$, with Lévy measure ν given by

$$\nu(B)=\int_{S^{d-1}}\lambda(d\xi)\int_0^{+\infty}\mathbf{1}_B(r\xi)\frac{dr}{r^{1+\alpha}}, \tag{1}$$

for any Borel set $B\in\mathcal{B}(\mathbb{R}^d)$. Here λ, which is called the spherical component of ν, is a finite positive measure on S^{d-1}, the unit sphere of $\mathbb{R}^d$ (see [5]). The following concentration result is established in [3]:

Theorem 1 ([3]) *Let X be an α-stable random variable, $\alpha>3/2$, with Lévy measure given by* (1)*. Set $L=\lambda(S^{d-1})$ and $M=1/(2-\alpha)$. Then if $f:\mathbb{R}^d\to\mathbb{R}$ is a Lipschitz function such that $\|f\|_{\mathrm{Lip}}\le 1$,*

$$P(f(X)-Ef(X)\ge x)\le\frac{(1+8e^2)L}{x^\alpha}, \tag{2}$$

for every x satisfying

$$x^\alpha\ge 4LM\log M\log(1+2M\log M).$$

For α close to 2, this roughly tells us that the natural (and optimal, up to a multiplicative constant) upper bound L/x^α holds for x^α of order $LM(logM)^2$. On the other hand, suppose that X is a 1–dimensional, stable random variable and let $Y^{(1)}$ be the infinitely divisible vector whose Lévy measure is the Lévy measure of X truncated at 1. Then it is easy to check that $var(Y^{(1)})=LM$. This clearly indicates that one cannot hope to obtain any interesting inequality if x^2 is much smaller than LM. In fact, when x^α is of order LM, another result in [3] gives an upper bound of order cLM/x^α. However, comparing this with the bound cL/x^α of Theorem 1, we see that there is an important discrepancy when M is large, and so it is natural to investigate the case when x^α lies in the range $[LM, LM(\log M)^2]$ for large M. Here is our result:

Theorem 2 *Using the same notations as in Theorem 1, we have:*
(i) Let $a < 1$ and $a', \varepsilon > 0$. Then if M is sufficiently large, for every x of the form $x^\alpha = bLM$ with $a' < b < a \log M$,

$$P(f(X) - Ef(X) \geq x) \leq (1+\varepsilon)e^{-b/2}. \tag{3}$$

(ii) Let $a > 2$, $\varepsilon > 0$. Then if M is sufficiently large, for every x such that $x^\alpha > aLM \log M$,

$$P(f(X) - Ef(X) \geq x) \leq \left[\frac{1}{\alpha} + (2+\varepsilon)\exp\left(1 + \frac{(1+\varepsilon)LM(\log M)^2}{2x^\alpha}\right)\right]\frac{L}{x^\alpha}.$$

As a consequence of (i), let $X^{(\alpha)}$ be the stable law whose Lévy measure ν is the uniform measure on S^{d-1} with total mass $1/M$. Then since $LM = 1$, (3) can be rewritten as

$$P(f(X^{(\alpha)}) - Ef(X^{(\alpha)}) \geq x) \leq (1+\varepsilon)e^{-x^\alpha/2} \tag{4}$$

for x smaller than $(\log M)^{1/\alpha}$. When $\alpha \to 2$, $X^{(\alpha)}$ converges in distribution to a standard gaussian variable X', for which we have the following classical bound [1, 6], valid for all $x > 0$:

$$P(f(X') - Ef(X') \geq x) \leq e^{-x^2/2}$$

So we see that (4) recovers the result for the gaussian concentration.
Remark that (ii) slightly improves Theorem 1 when the index α is close to 2 and x^α is of order $LM(\log M)^2$.
To some extent, the existence of two regimes (i) and (ii), depending on the order of magnitude of x with regard to $(LM \log M)^{1/\alpha}$, is reminiscent of the famous Talagrand inequality:

$$P(f(U) - Ef(U) \geq x) \leq \exp(-\inf(x/a, x^2/b))$$

where U is an infinitely divisible random variable with Lévy measure given by

$$\nu(dx_1 \dots dx_k) = 2^{-k}e^{-(|x_1|+\dots+|x_k|)}dx_1 \dots dx_k,$$

and f is a Lipschitz function, a and b being related to the L^1 and L^2 norm of f, respectively (see [7] for a precise statement). We now proceed to the proof of Theorem 2.

2 Proof of the result

The proof essentially follows the lines of the proof to be found in [3], where the case $x^\alpha < LM(\log M)^2$ had been overlooked. We write $X = Y^{(R)} + Z^{(R)}$, where $Y^{(R)}$, $Z^{(R)}$ are two independent, infinitely divisible random variables whose Lévy measures are the Lévy measure of X truncated, above and below respectively, at $R > 0$. We have

$$P(f(X) - Ef(X) \geq x) \leq P(f(Y^{(R)}) - Ef(X) \geq x) + P(Z^{(R)} \neq 0). \tag{5}$$

Since $Z^{(R)}$ is a compound Poisson process, it is easy to check that

$$P(Z^{(R)} \neq 0) \leq \frac{L}{\alpha R^{\alpha}}. \tag{6}$$

On the other hand,

$$P(f(Y^{(R)}) - Ef(X) \geq x) \leq P(f(Y^{(R)}) - Ef(Y^{(R)}) \geq x')$$

with

$$x' = x - |Ef(X) - Ef(Y^{(R)})|.$$

Thus we have to compare $Ef(X)$ and $Ef(Y^{(R)})$. For large R, these two quantities are very close, since

$$|Ef(X) - Ef(Y^{(R)})| \leq \frac{LR^{1-\alpha}}{\alpha - 1}. \tag{7}$$

Given x, we choose R so that

$$R = x - \frac{LR^{1-\alpha}}{\alpha - 1}, \tag{8}$$

which entails that $x' \leq R$. Therefore we can write

$$P(f(Y^{(R)}) - Ef(X) \geq x) \leq P(f(Y^{(R)}) - Ef(Y^{(R)}) \geq R),$$

Let b be the real such that $x^{\alpha} = bLM$. Let b' be such that $R^{\alpha} = b'LM$, which, according to (8), entails

$$(b'LM)^{1/\alpha} = (bLM)^{1/\alpha} - \frac{L}{\alpha - 1}(b'LM)^{(1-\alpha)/\alpha}$$

or, equivalently,

$$b'\left(1 + \frac{1}{(\alpha - 1)Mb'}\right)^{\alpha} = b. \tag{9}$$

When M is large, b' can be made arbitrarily close to b. To estimate quantities of the type $P(f(Y^{(R)}) - Ef(Y^{(R)}) \geq y)$, we use Theorem 1 in [2], which states that

$$P(f(Y^{(R)}) - Ef(Y^{(R)}) \geq y) \leq \exp\left(-\int_0^y h_R^{-1}(s)ds\right), \tag{10}$$

where h_R^{-1} is the inverse of the function

$$h_R(s) = \int_{\|u\| \leq R} \|u\|(e^{s\|u\|} - 1)\nu(du).$$

Using the fact that for $s \in (0, R)$,

$$e^{sy} - 1 \leq sy + \frac{e^{sR} - 1 - sR}{R^2}y^2,$$

we get the following upper bound for $h_R(s)$:

$$h_R(s) \leq \left(\frac{MLR^{2-\alpha}}{3 - \alpha}\right)s + \left(\frac{LR^{1-\alpha}}{3 - \alpha}\right)(e^{sR} - 1). \tag{11}$$

See [3] for details of computations. The idea is to compare the two terms in the right-hand side of (11). Typically, for small s, the first term is dominant while for large s, the second term is dominant.

Let us first prove (i). Fix $\varepsilon, a' > 0$ and $a < 1$. If $\delta, s, R > 0$ are three reals satisfying the inequality

$$\frac{e^{sR}-1}{sR} \leq \delta M, \tag{12}$$

then

$$\left(\frac{LR^{1-\alpha}}{3-\alpha}\right)(e^{sR}-1) \leq \left(\frac{\delta LMR^{2-\alpha}}{3-\alpha}\right) s$$

and so

$$h_R(s) \leq \left(\frac{(1+\delta)LMR^{2-\alpha}}{3-\alpha}\right) s.$$

As a consequence, if y is such that the real $s = s(y)$ defined by

$$s(y) = \frac{(3-\alpha)y}{(1+\delta)LMR^{2-\alpha}}$$

satisfies (12), then

$$h_R^{-1}(y) \geq \frac{(3-\alpha)y}{(1+\delta)LMR^{2-\alpha}}. \tag{13}$$

It is clear that if $s(y)$ satisfies (12), then for every $0 < y' < y$, $s(y')$ also satisfies (12) with the same reals δ and R. Therefore one can integrate (13) and one has:

$$\int_0^y h_R^{-1}(t)dt \geq \frac{(3-\alpha)y^2}{2(1+\delta)LMR^{2-\alpha}} \tag{14}$$

whenever $s(y)$ satisfies (12). If y has the form $y^\alpha = ALM/(3-\alpha)$ with $A/(3-\alpha) < a \log M$ and if we take $R = y$, Condition (12) becomes

$$\frac{(1+\delta)[\exp(A/(1+\delta)) - 1]}{A} \leq \delta M.$$

For M sufficiently large, this holds whenever

$$\frac{(1+\delta)e^A}{A} \leq \delta M. \tag{15}$$

Set

$$\delta = \delta(A) = \frac{e^A}{AM - e^A}.$$

Given $a' > 0$, if M is large enough, $\delta(A) > 0$ for every A such that $a'/2 < A < \log M$, and thus (15) is fulfilled. In that case, since we take $R = y$, (14) becomes

$$\int_0^R h_R^{-1}(t)dt \geq \frac{A}{2(1+\delta)}.$$

Using the expression of δ,

$$\exp\left(-\int_0^R h_R^{-1}(t)dt\right) \le e^{-A/2}\exp\left(\frac{e^A}{2M}\right).$$

Put $b' = A/(3-\alpha)$, so that $R^\alpha = b'LM$. Then the last inequality becomes

$$\exp\left(-\int_0^R h_R^{-1}(t)dt\right) \le e^{-b'/2}\exp\left(\frac{e^{b'/(3-\alpha)}}{2M} + \frac{b'}{2M(3-\alpha)}\right). \tag{16}$$

For M large enough, this quantity is bounded by $(1+\varepsilon/4)e^{-b'/2}$. To sum up, given $\varepsilon > 0$ and $a' > 0$, if M is large enough, then for every b' satisfying $a'/2 < b' < \log M$, writing $R^\alpha = b'LM$, we have

$$P((f(Y^{(R)}) - Ef(Y^{(R)}) \ge R) \le (1+\varepsilon/4)e^{-b'/2}. \tag{17}$$

Remark that given $a' > 0$ and $a < 1$, if $a' < b < a\log M$, then taking b' as defined by (9), we have $a'/2 < b' < \log M$ for M large enough and we can apply (17). Hence if x has the form $x^\alpha = bLM$ with $a' < b < a\log M$, setting $R^\alpha = b'LM$, we have for M large enough,

$$P((f(Y^{(R)}) - Ef(Y^{(R)}) \ge R) \le (1+\varepsilon/4)e^{-b'/2} \le (1+\varepsilon/2)e^{-b/2}.$$

This provides an upper bound for the first term of the right-hand side of (5).
To bound the second term of the right-hand side of (5), recall (6) and remark that choosing $R^\alpha = b'LM$,

$$\frac{L}{\alpha R^\alpha} = \frac{1}{b'M}.$$

Given $a' > 0$ and $a < 1$, if b satisfies $a' < b < a\log M$, then for M large enough, using again (9),

$$\frac{1}{b'M} < \frac{\varepsilon}{2}e^{-b/2}.$$

This concludes the proof of (i).

To prove (ii), we shall decompose the integral (10). Fix $a > 2$, take x of the form $x^\alpha = bLM\log M$ with $b \ge a$ and let $R = (b'LM\log M)^{1/\alpha}$ with b' given by (9). First let

$$u_0 = \frac{(1-\varepsilon)LM\log M}{(3-\alpha)R^{\alpha-1}}.$$

Then for M large enough, the same arguments as for (14) give

$$\int_0^{u_0} h_R^{-1}(t)dt \ge \frac{(3-\alpha)u_0^2}{2(1+\varepsilon')LMR^{2-\alpha}} \ge \frac{(1-\varepsilon'')\log M}{2b'}. \tag{18}$$

On the other hand, for M large enough, if $sR \ge \log M + \log\log M$,

$$\frac{e^{sR}-1}{sR} \ge \frac{M}{1+\varepsilon}.$$

Hence using (11), we have

$$h_R^{-1}(u) \geq \frac{1}{R}\log\left(1+\frac{(3-\alpha)u}{(2+\varepsilon)LR^{1-\alpha}}\right) \tag{19}$$

for every $u > u_1$, where

$$u_1 = \frac{(2+\varepsilon)LM\log M}{(3-\alpha)R^{\alpha-1}}.$$

Now let $R = (b'LM\log M)^{1/\alpha}$ with b' given by (9). Then for M sufficiently large, $R > u_1$. In that case, we can integrate (19) and this gives

$$\int_{u_1}^{R} h_R^{-1}(t)dt \geq \left[\left(1-\frac{1}{cR}\right)\log(1+cR)-1\right] - \left[\left(\frac{u_1}{R}-\frac{1}{cR}\right)\log(1+cu_1)-\frac{u_1}{R}\right]$$

where we denote

$$c = \frac{(3-\alpha)R^{\alpha-1}}{(2+\varepsilon)L}.$$

For M large enough, this leads to

$$\exp\left(-\int_{u_1}^{R} h_R^{-1}(t)dt\right) \leq \frac{(2+\varepsilon')eL}{R^\alpha}\exp\left(\frac{(2+\varepsilon')[\log(M\log M)-1]}{b'}\right). \tag{20}$$

Finally, since h_R^{-1} is increasing,

$$\int_{u_0}^{u_1} h_R^{-1}(t)dt \geq (u_1-u_0)h_R^{-1}(u_0) \geq \frac{(1-\varepsilon)\log M}{b'}$$

Together with (18),(20), (6) and (9), this yields (ii).

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