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SOME NOTES ON TOPOLOGICAL RECURRENCE

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Abstract
We review the concept of topological recurrence for weak Feller Markov chains on compact state spaces and explore the implications of this concept for the ergodicity of the processes. We also prove some conditions for existence and uniqueness of invariant measures of certain types. Examples are given from the class of iterated function systems on the real line.

1 Introduction

Harris recurrence is a well-known condition that can be used to prove ergodicity of Markov chains on general state spaces. There are, however, many families of Markov chains for which Harris recurrence cannot hold. Notably, among these we have the class of iterated function systems consisting of a finite number of maps. For these it would be highly desirable to find a weaker condition for ergodicity.

There is a large body of literature concerning iterated function systems that have contractivity properties, see for instance [6] for a survey. Contractive systems, possibly under some additional conditions, converge weakly to a unique invariant measure. However, this seems unnecessarily restrictive. There are, for instance, powerful ergodic theorems that can be proved without using the fact that the process is contractive [3]. One gets a feeling that there is some larger class of processes for which we should be able to prove unique ergodicity.

A very tempting candidate for such a class would be the class of topologically recurrent processes. In this article, however, we will show that even very simple iterated function systems can have several invariant measures and still be topologically recurrent. Despite this, the class has many interesting properties and proves to be a good source of examples. For instance, in the light of results in [9], we can view the counterexamples in this article as relatively simple examples of topologically irreducible weak Feller non-e-chains.

In order not to be too pessimistic, we also give some conditions and examples to aid in understanding what needs to be done in order to prove ergodicity in a more general setting. Finally, we prove unique ergodicity for a simple class of expanding iterated function systems.

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Note that we only consider compact state spaces here or, more specifically, chains on a compact real interval.

2 Definitions

Let $(S, \mathcal{B}(S))$ be a compact topological space equipped with its Borel sets and let $\mathcal{M}$ denote the set of non-negative finite measures on S. Let $\mathcal{P} \subset \mathcal{M}$ denote the set of probability measures. Now let $\{f_k\}_{k=1}^N$ be a finite set of measurable functions from S to itself. Let ω_0, ω_1, ... be a sequence of i.i.d. discrete random variables taking values in the set $\{1, 2, \ldots, N\}$. We define $p_k := \mathbb{P}(\omega_i = k)$.
We are interested in the processes that, given an initial distribution $X_0 \sim \mu_0$ independent of ω_i, are defined recursively by

$$X_{n+1} := f_{\omega_n}(X_n),\ n = 0, 1, \ldots \tag{1}$$

These are Markov chains in discrete time on a general state space and are commonly referred to as **iterated function systems**.
The Markov operator associated with this process is given by

$$Th = \sum_{k=1}^{N} p_k h(f_k), \tag{2}$$

operating on bounded measurable functions from S to $\mathbb{R}$. If μ is a probability measure on $(S, \mathcal{B}(S))$, we can also define the adjoint operator $T^\star$ operating on measures as

$$T^\star \mu = \sum_{k=1}^{N} p_k \mu(f_k^{-1}). \tag{3}$$

In fact, this operator describes the time evolution of the process one step forward in time starting from the distribution given by the measure μ. An **invariant measure** for a given iterated function system is a probability measure μ satisfying $\mu = T^\star\mu$. If the functions f_k are assumed to be continuous, the corresponding Markov chain will have the weak Feller property. Therefore, since S is compact, there will always exist at least one invariant probability measure in that case, as is shown for instance by Meyn and Tweedie in [9].
A Markov chain is said to be **topologically recurrent**, if for any open set $\mathcal{O}$ and starting point $x \in S$

$$\mathbb{P}\left(\sum_{n=1}^{\infty} 1_{\mathcal{O}}(X_n(x)) = +\infty\right) = 1.$$

For Feller chains on compact spaces this is equivalent to the condition that for every $x \in S$

$$\mathbb{P}(X_n(x) \in \mathcal{O} \text{ for some } n) > 0,$$

see for instance [10].

3 Some initial results

It is relatively easy to find iterated function systems, or more generally Markov chains, that are topologically recurrent. The fact that a process is topologically recurrent also implies that

it has some nice properties. For instance, we cannot have more than one set that is both topologically and stochastically closed. There are also no invariant measures supported on a finite number of points.
We begin here by giving a simple class of iterated function systems that are topologically recurrent. This result will be needed for our examples later on.

Lemma 1 *Consider an iterated function system on* $[0,1]$ *containing, with positive probability, at least the following functions:*

$$f_i(x) = \frac{1}{n} g_i(x) + \frac{i}{n},\ \ i = 0, 1, \ldots, n-1,$$

where $g_i(x)$ *is either* x *or* $1-x$ *for any* i *and* $n > 1$ *is fixed. Then that iterated function system is topologically recurrent.*

Proof: Consider a system of the above form, started from an arbitrary starting point x. We claim that by iterating a combination of the functions f_i, the system can reach any open set. Any open set must contain an interval of the form $[\frac{m}{n^k}, \frac{m+1}{n^k}[$ for some $m, k \in \mathbb{N}$. It is therefore sufficient to show that the first k digits of the base n expansion of the state can be made arbitrary.

We start by making sure that $x \in]0,1[$. This can be done in at most two iterations using the f_i. After that, the first digit can clearly be made arbitrary by picking the function f_i corresponding to the digit i. Assume now that the first $n-1$ digits of the state can be made arbitrary by iterating the functions f_i. Suppose we want to construct a state with the first n digits $y_1 \ldots y_n$. If $g_{y_1}(x) = x$, simply construct first $y_2 \ldots y_n$ and apply $f_{y_1}(x)$ to obtain the n digits $y_1 y_2 \ldots y_n$. In the other case, when $g_{y_1}(x) = 1 - x$, construct the $n-1$ digits $(n-1-y_2)\ldots(n-1-y_n)$. Applying $g_{y_1}(x)$ gives the sequence $y_2 \ldots y_n$, which means that applying $f_{y_1}(x)$ gives the desired n digit sequence. It follows by induction that any sequence of initial digits in base n can be constructed. □

4 A first example

Consider the following iterated function system on the interval $[0,1]$:

$$\begin{array}{lll} f_1(x) = \frac{1}{2}x, & & \text{w.p. } \frac{p}{2} \\ f_2(x) = 1 - \frac{1}{2}x, & & \text{w.p. } \frac{p}{2} \\ f_3(x) = \left\{ \begin{array}{ll} 2x, & x \le \frac{1}{2} \\ 2(1-x), & x > \frac{1}{2} \end{array} \right. & & \text{w.p. } 1-p \end{array} \tag{4}$$

Theorem 1 *Consider* (4) *and let* $0 < p < \frac{1}{2}$. *The iterated function system is topologically recurrent, but admits more than one invariant measure.*

Remark 1 *The intuition behind this example is that* f_1 *and* f_2 *correspond to a right-shift on a symbol space. We also have to flip some bits to make sure that our maps are continuous, but the intuition still holds. The map* f_3 *on the other hand corresponds to a left-shift. So, since* $p < \frac{1}{2}$, *we shift left on average, meaning that the behaviour of* f_3 *dominates, giving dependence on the initial distribution. The occasional sequences of right-shifts make sure that we have topological recurrence.*

Proof: Since all the functions f_i are continuous, the process is clearly weak Feller and by Lemma 1, it is topologically recurrent.

To prove the non-uniqueness result, first note that Lebesgue measure is invariant under the iterated function system consisting of f_1 and f_2 with equal probability as well as under the dynamical system f_3. Therefore it is invariant under the system (4).

On the other hand, we can construct a discrete invariant measure by using the fact that f_3 is the inverse function of both f_1 and f_2. If we start the process from $x = 0$, which is a fixed point of f_3, the reachable states will form a tree structure, as shown in Figure 1. The functions f_1 and f_2 always take the process deeper into the tree, except at the root $x = 0$ where f_1 has a fixed point. The function f_3 always returns the process towards the root.

The random variable Y_n, corresponding to the distance from the current state to the root node, is then a random walk with transition probabilities:

$$\begin{cases} \mathbb{P}(Y_{n+1} = Y_n + 1 \mid Y_n = 0) & = \frac{p}{2} \\ \mathbb{P}(Y_{n+1} = Y_n \mid Y_n = 0) & = 1 - \frac{p}{2} \\ \mathbb{P}(Y_{n+1} = Y_n + 1 \mid Y_n > 0) & = p \\ \mathbb{P}(Y_{n+1} = Y_n - 1 \mid Y_n > 0) & = 1 - p \end{cases}$$

It is easily seen from the standard theory of Markov chains on denumerable state spaces that the expected return time to the origin is finite. This means that the process has an invariant measure with positive mass in the origin. □

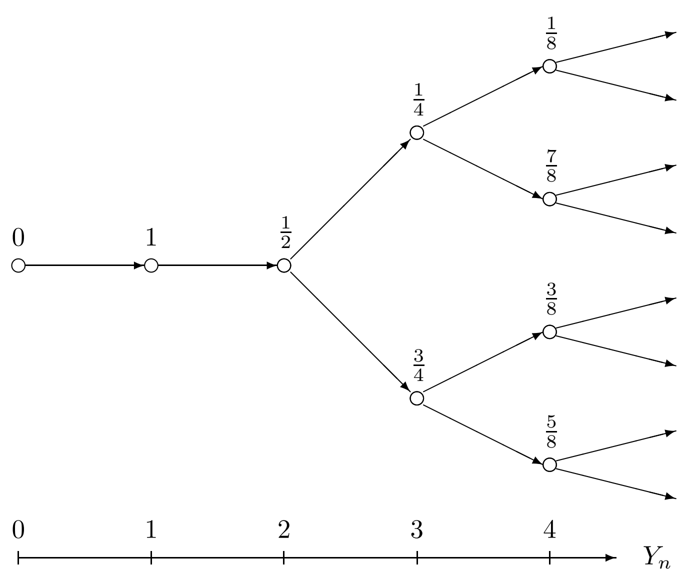

Figure 1: The tree structure in Example 1

Remark 2 *If $p = 0$, the system (4) will simply be the tent-map, which is well known in dynamical systems. If $p > \frac{1}{2}$, the iterated function system will be contractive with Lebesgue* *measure as its only invariant measure. This follows for instance from Corollary 2.2 of [1]. The invariant measures when $p = 0.4$, as constructed in the above theorem, are depicted in Figure 2. Also note that this system is either expanding or contractive depending on the choice of p, but it is always topologically recurrent. This means that topological recurrence is not directly tied to whether the system is contractive or expanding.*

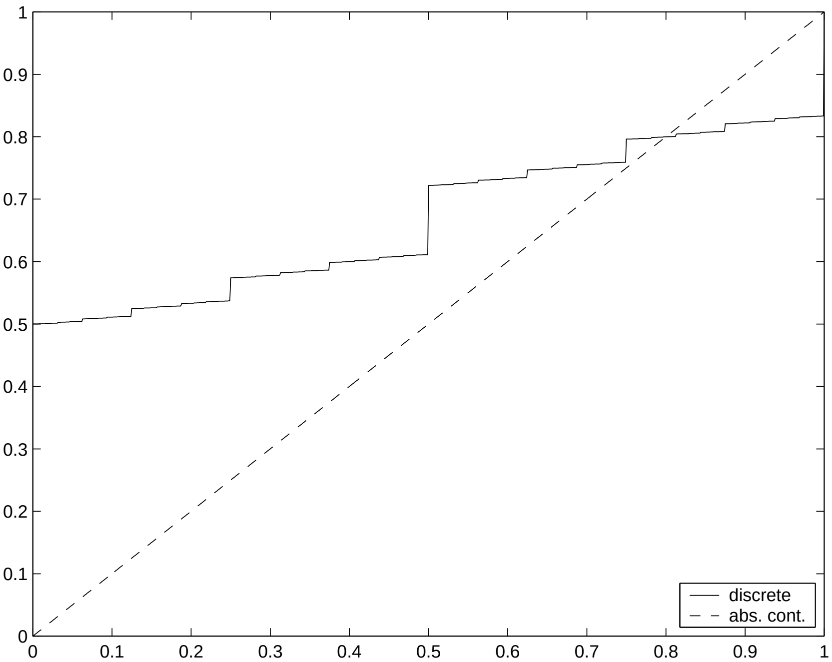

Figure 2: Invariant measures in Example 1 ($p = 0.4$)

5 Laws of pure type

In the upcoming sections we will need some law of pure types results that will be proved here. A Markov chain on $\mathbb{R}^n$ is said to obey a **law of pure types**, if μ being an invariant measures implies that the discrete, Lebesgue singular and absolutely continuous parts of μ are also invariant. In particular, if μ is a unique invariant measure for a process obeying a law of pure types, then μ is necessarily purely discrete, purely singular or absolutely continuous. Of course, one can also obtain similar results for other reference measures than Lebesgue measure. The main theorem of this section is adapted from [7]. The fact that this holds for a large class of iterated function systems is also noted in passing in [1]. Although more general results can be obtained, we will carry out the arguments only for $\mathbb{R}$ in order to keep things short. This is also what we need for our examples.

We write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ if each measure in $\mathcal{M}$ can be uniquely decomposed into a sum of two measures belonging to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. The measures in $\mathcal{M}_1$ and $\mathcal{M}_2$ are assumed to be mutually singular.

Theorem 2 (Dubins and Freedman) *Let L be a linear operator on $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $L\mathcal{M}_1 \subseteq \mathcal{M}_1$ and $L\mu(S) = \mu(S)$ for any $\mu \in \mathcal{M}$. Then if $L\mu = \mu$, where $\mu = \mu_1 + \mu_2$ with $\mu_1 \in \mathcal{M}_1$ and $\mu_2 \in \mathcal{M}_2$, we have that $L\mu_1 = \mu_1$ and $L\mu_2 = \mu_2$.*

Proof: Clearly $\mu = \mu_1 + \mu_2 = L(\mu_1 + \mu_2) = L\mu_1 + L\mu_2 \geq L\mu_1$, but since $\mu_2 \perp \mu_1$ and $\mu_2 \perp L\mu_1$ it must hold that $\mu_1 \geq L\mu_1$, i.e.

$$\forall A \in \mathcal{B}(S): \ \mu_1(A) \geq L\mu_1(A). \tag{5}$$

But we know that $\mu_1(S) = L\mu_1(S)$, which implies that the measures must be equal.

For assume that $\mu_1 \neq L\mu_1$. Then there is an $A \in \mathcal{B}(S)$ such that $\mu_1(A) > L\mu_1(A)$. But then $\mu_1(S \setminus A) < L\mu_1(S \setminus A)$, which contradicts (5). □

In order to proceed, we note the well-known fact (cf. Lebesgue decomposition) that any finite measure on $\mathbb{R}$ can be uniquely decomposed into a part with Lebesgue density, a part which is continuous but Lebesgue singular, and a discrete part.

Corollary 1 *For an iterated function system with at most countably many maps, the discrete part of an invariant measure is always invariant.*

Proof: It suffices to show that the Markov operator $T^\star$ preserves the set of measures with all mass in a countable set of points and apply Theorem 2. But if μ_n is concentrated on $\{x_i\}_{i\geq 0}$, then μ_{n+1} is concentrated on $\{f_j(x_i)\}_{i,j\geq 0}$, which is clearly countable. □

Corollary 2 *For an iterated function system with a countable number of uniformly continuous functions, the Lebesgue singular part of an invariant measure is always invariant.*

Proof: We need to show that $T^\star$ preserves the set of Lebesgue singular measures and apply Theorem 2. We can without loss of generality consider the case of an iterated function system consisting of only a single map f, since a countable sum of singular measures is singular.

Suppose that μ_n is concentrated on the set $A \in \mathcal{B}(S)$, where $\mathrm{Leb}(A) = 0$. By regularity there exists a sequence of open sets $O_n \downarrow A$ with $\mathrm{Leb}(O_n) \to 0$ as $n \to \infty$.

Let $\Delta(x)$ denote the modulus of uniform continuity of the map f. Since the O_n can be represented as a disjoint union $O_n = \cup_i]a_{n,i}, b_{n,i}[$, we have

$$\mathrm{Leb}(f(O_n)) \leq \sum_i \mathrm{Leb}(f(]a_{n,i}, b_{n,i}[))$$

$$\leq \Delta(\sup_i |a_{n,i} - b_{n,i}|) \sum_i \mathrm{Leb}(]a_{n,i}, b_{n,i}[)$$

$$= \Delta(\sup_i |a_{n,i} - b_{n,i}|)\mathrm{Leb}(O_n) \to 0.$$

In other words μ_{n+1} is concentrated on a set of zero Lebesgue measure as well. □

6 A condition for singular invariant measures

In the particular case of a uniformly expanding iterated function system, as in the example in Section 4, it is known that there always exists an invariant measure which is absolutely continuous with respect to Lebesgue measure. In fact, there is a result in [5], which says that if the system is expanding on average in a certain sense, then the set of invariant densities is spanned by a finite number of invariant densities of bounded variation. This means, since ergodic measures are mutually singular, that if such a system is topologically recurrent, then it can have only one invariant density. The theorem applies for instance if all the functions in the iterated function system are piecewise linear and if the system is expanding on average.

Since the exclusion of measures that are singular with respect to Lebesgue measure would suffice to prove ergodicity, we will give a necessary and sufficient condition for this, in the case of weak Feller Markov chains on a compact state space S.

Theorem 3 *Let $X_n(x)$ be a weak Feller Markov chain on a compact state space S. There is an invariant measure μ with a positive singular part if and only if there exist a starting point x_0 in S and $K \subset S$, where K is compact and of zero Lebesgue measure, such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(X_k(x_0) \in K) > 0. \tag{6}$$

Proof: (if part) Let $\mu_0 = \delta_{x_0}$, $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} T^{\star k}\mu_0$. Then (6) gives that

$$\lim_{n\to\infty} \mu_n(K) > 0.$$

Now choose a weakly convergent subsequence n_i, such that for some $\widehat{\mu}$, $\mu_{n_i} \xrightarrow{\mathrm{w}} \widehat{\mu}$ as $i \to \infty$. This is possible since the state space is compact. It is also clear, by a standard argument, that $\widehat{\mu}$ is invariant. Finally by the Portmanteau theorem [2], since K is topologically closed, we get that

$$\widehat{\mu}(K) \geq \limsup_{i\to\infty} \mu_{n_i}(K) > 0.$$

(only if part) Let K be a compact set of zero Lebesgue measure such that $\mu(K) > 0$, where μ has a positive singular part. Such a set exists by the regularity of the measure. Recall that according to the individual ergodic theorem [12], if $f \in L_1(\mu)$, then the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}\, f(X_k(x)) = f^\star(x) \tag{7}$$

exists μ-almost everywhere, where $\int f d\mu = \int f^\star d\mu$. Now, let $f(x) = 1_K(x)$ and let N be such that $\mu(N) = 0$ and (7) is convergent for $x \in S \setminus N$. We then have that $\int f^\star 1_{S\setminus N} d\mu = \int f^\star d\mu = \mu(K) > 0$, so that there has to exist at least one $x_0 \in S$ such that (6) holds. $\square$

If the process satisfies a law of pure types, then we can strengthen the result as follows.

Corollary 3 *Let $X_n(x)$ be a weak Feller Markov chain on a compact state space S, starting from $x \in S$. Assume that it satisfies a law of pure types and let $0 < \epsilon \leq 1$ be arbitrary. Then there exists an invariant measure μ with a positive singular part if and only if there exist x_0 in S and $K \subset S$, where K is compact and of zero Lebesgue measure, such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(X_k(x_0) \in K) > 1 - \epsilon. \tag{8}$$

Proof: The (if)-part of the proof is almost identical to that of Theorem 3. For the (only if)-part we can start out with a purely singular invariant measure and choose K so that $\mu(K) > 1 - \epsilon$ and continue as in the proof of Theorem 3. □

Corollary 4 *If there exists an initial distribution so that* (6) *holds, then there exists an invariant measure with a positive singular part.*

Proof: We can easily replace δ_{x_0} with our initial distribution μ_0 in the (if)-part of the proof of Theorem 3. □

7 A second example

In our first example, one of the invariant measures was discrete. One could think that this is a pathology, so that all examples of non-ergodicity in topologically recurrent processes are such that all but one of the measures are discrete. This is not the case, as is shown by the following example.
Consider the following small modification of the iterated function system (4):

$$\begin{array}{lll} f_1(x) = \frac{1}{3}x, & & \text{w.p. } \frac{p}{3} \\ f_2(x) = \frac{2}{3} - \frac{1}{3}x, & & \text{w.p. } \frac{p}{3} \\ f_3(x) = \frac{2}{3} + \frac{1}{3}x, & & \text{w.p. } \frac{p}{3} \\ f_4(x) = \begin{cases} 3x, & x \leq \frac{1}{3} \\ 2 - 3x, & \frac{1}{3} < x \leq \frac{2}{3} \\ -2 + 3x, & x > \frac{2}{3} \end{cases} & & \text{w.p. } 1 - p \end{array} \tag{9}$$

Theorem 4 *Consider* (9) *and let $0 < p < \frac{1}{2}$. The process is weak Feller and topologically recurrent, but it admits at least three invariant measures that are discrete, singular and absolutely continuous respectively.*

Proof: The Feller property is obvious and the topological recurrence follows from Lemma 1.

The existence of discrete and absolutely continuous invariant measures is proved as in Theorem 1 above.

To show that we also have a singular invariant measure, we start the process from the uniform distribution on the middle-third Cantor set. Observe that $f_4 \circ f_i \equiv x$ for $i = 1, 2, 3$ and that the mapping f_4 preserves the initial distribution.

Similarly to the previous examples we consider the random walk $Y_{n+1} = (Y_n + \xi_{n+1})^+$, where $\mathbb{P}(\xi = +1) = p = 1 - \mathbb{P}(\xi = -1)$, started from the origin. This random walk is aperiodic and positively recurrent, so that $\lim_{n\to\infty} \mathbb{P}(Y_n = 0) = y_0 > 0$. Now, a step upwards for this walk, corresponds to choosing f_i, $i = 1, 2, 3$ and a step downwards corresponds to choosing f_4. In this setting, if $Y_n = 0$, the corresponding moves for the iterated function system would leave the initial distribution invariant, since all applications of f_1, f_2 and f_3 cancel out. This implies that if μ_n is the distribution of the process after n steps and if K is the Cantor set, then

$$\lim_{n\to\infty} \mu_n(K) \geq y_0 > 0.$$

By using Corollary 4 together with the fact that the process satisfies the law of pure types, we can conclude that there also exists a continuous invariant measure that is singular with respect to Lebesgue measure. □

The distribution functions of the invariant measures as constructed in Theorem 4 are depicted in Figure 3 for the case $p = 0.25$.

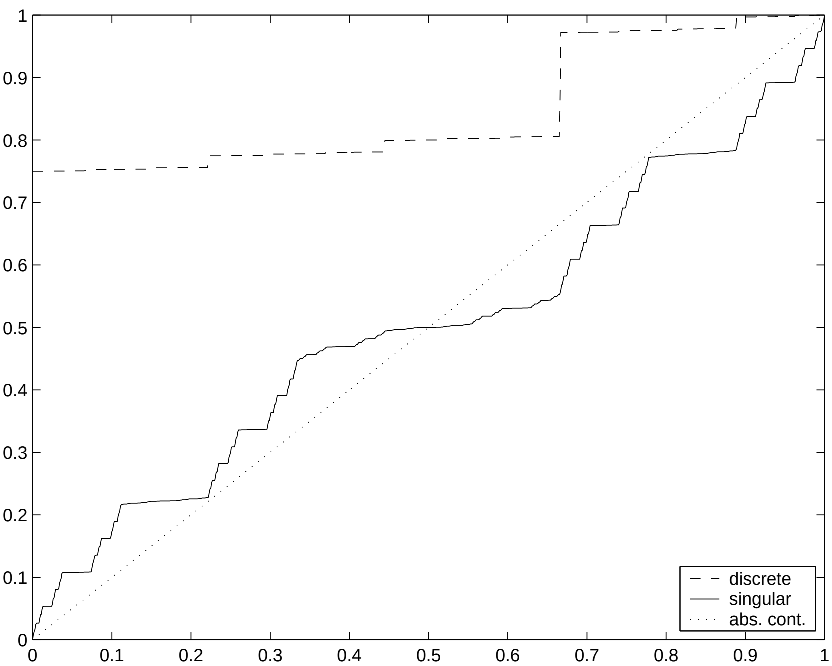

Figure 3: Invariant measures in Example 2 ($p = 0.25$)

8 A third example

In order to find some more positive results we will finally give an example of an expanding topologically recurrent system that is uniquely ergodic. Consider the iterated function system of circle maps on $[0, 1)$ as follows:

$$\begin{aligned} f_1(x) &= x + \alpha \quad (\text{mod } 1) \quad \text{w.p. } p \\ f_2(x) &= 2x \quad\quad\; (\text{mod } 1) \quad \text{w.p. } 1 - p \end{aligned} \tag{10}$$

where $\alpha \in [0,1)$ is irrational and $0 < p < 1$. It is clearly expanding. For our proof, we will need the following general Lemma that can be found in the Appendix of [4]. It has been previously used in the context of perfect simulation [4, 8].

Lemma 2 *If μ is an invariant measure for a stochastic process with Markov operator $T^\star = pT_1^\star + (1-p)T_2^\star$, then*

$$\sum_{k=0}^{\infty}(1-p)p^k T_1^{\star k} T_2^\star \mu = \mu.$$

Proof: We first show by induction that

$$p^{n+1}T_1^{\star(n+1)}\mu + \sum_{k=0}^{n}(1-p)p^k T_1^{\star k} T_2^\star \mu = \mu. \tag{11}$$

For $n = 0$, we simply get $pT_1^\star\mu + (1-p)T_2^\star\mu = \mu$, which is satisfied since μ is an invariant measure. Assume now that

$$p^n T_1^{\star n}\mu + \sum_{k=0}^{n-1}(1-p)p^k T_1^{\star k} T_2^\star \mu = \mu.$$

Then it follows that

$$\begin{aligned} p^{n+1}T_1^{\star(n+1)}\mu &+ \sum_{k=0}^{n}(1-p)p^k T_1^{\star k} T_2^\star \mu \\ &= p^{n+1}T_1^{\star(n+1)}\mu + (1-p)p^n T_1^{\star n} T_2^\star\mu + \mu - p^n T_1^{\star n}\mu \\ &= p^n T_1^{\star n}\left(pT_1^\star\mu + (1-p)T_2^\star\mu - \mu\right) + \mu = \mu, \end{aligned}$$

which proves (11). The result now follows by taking the limit as $n \to \infty$. □

Theorem 5 *The system* (10) *is topologically recurrent and uniquely ergodic with the uniform distribution as its invariant distribution.*

Proof: The fact that (10) is topologically recurrent follows from the classical Weyl equidistribution theorem [11] for rotation maps.

Appealing to Lemma 2 it is sufficient to show ergodicity for the process Y_n with transition probabilities

$$\mathbb{P}(Y_{n+1} = 2Y_n + k\alpha \ (\text{mod } 1)\ |Y_n) = (1-p)p^k.$$

Now consider the characteristic function $M_n(m) = \mathbb{E}\,(e^{2\pi i m Y_n})$. We find that

$$M_{n+1}(m) = M_n(2m) \cdot F(m),$$

where

$$F(m) = (1-p)\sum_{k=0}^{\infty} p^k e^{2\pi i m k\alpha} = \frac{1-p}{1-pe^{2\pi i m\alpha}}.$$

Using this, we obtain the following explicit formula:

$$M_n(m) = M_0(2^n m)\prod_{k=0}^{n-1} F(2^k m) = e^{2\pi i 2^n m x_0}\prod_{k=0}^{n-1}\frac{1-p}{1-pe^{2\pi i(2^k m\alpha)}}.$$

The product in the above expression can be shown to vanish as $n \to \infty$ for $m \neq 0$. The reason for this is that every factor is at most of absolute value one, but there will be an infinite number of factors of absolute value at most $1-p$. To see this observe that the k:th factor will depend on the binary expansion of $m\alpha$, which is irrational. This in turn means that we will have an infinite number of cases where the initial digits of $2^k m\alpha \pmod 1$ in base two are either 01 or 10 meaning that $2^k m\alpha \in [\frac{1}{4}, \frac{3}{4}]$. From this it follows that $M_n(m) \to 0$ for $m \neq 0$, which implies that Y_n converges in distribution to the uniform distribution on the unit circle.

Since any invariant distribution of X_n is an invariant distribution of Y_n, we conclude that the unique invariant measure for X_n is given by the uniform distribution. □

In other words, the iterated function system (10) is uniquely ergodic, and by a result in [3] we can even conclude that simulated trajectories converge weakly to the uniform distribution for any starting point. In practical applications, however, rounding errors may very well destroy this property and it is beyond the scope of this article to analyze that aspect of the problem.

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