Elect. Comm. in Probab. **10** (2005), 235–243

ELECTRONIC COMMUNICATIONS in PROBABILITY

MOTION OF A RIGID BODY UNDER RANDOM PERTURBATION

MING LIAO[1]
Department of Mathematics, Auburn University, Auburn, AL 36849, USA.
email: liaomin@auburn.edu

LONGMIN WANG
Department of Mathematics, Nankai University, Tianjin, China.
email: wanglm@mail.nankai.edu.cn

Submitted 14 December 2004, *accepted in final form* 21 November 2005

AMS 2000 Subject classification: 58J65, 60H10
Keywords: rigid body, stochastic differential equation, rotation group, compact Lie group

Abstract
We use stochastic analysis to study the random motion of a rigid body under a white noise perturbation. We obtain a formula for the angular velocity in an average sense and discuss the stability near a principle axis.

1 Introduction

In this paper, we study the random motion of a rigid body under a white noise perturbation. In an earlier paper [4] by the first author, the random motion of a rigid body is studied using a stochastic differential equation on the rotation group $SO(3)$ driven by a standard Brownian motion, which corresponds to an isotropic white noise perturbation. In the present paper, we will allow the driving Brownian motion to be non-standard, that is, there may be correlation among different components and the Brownian motion may be degenerate. This will allow us to consider the motion of a rigid body when the random perturbation is not uniform in different directions. In particular, the perturbation may be concentrated around a single axis. We will in fact work on a general compact Lie group G without much additional effort, but will give interpretation only for $G = SO(3)$.

The rotation group $G = SO(3)$ is the group of 3×3 orthogonal matrices of determinant one. Its Lie algebra $\mathfrak{g}$, the tangent space T_eG of G at the identity element e, is the space $o(3)$ of 3×3 skew-symmetric matrices with Lie bracket $[X, Y] = XY - YX$. For $g \in G$, let gX and Xg denote respectively the elements in T_gG obtained from $X \in \mathfrak{g}$ by the left and the right translations. The adjoint action Ad of G on its Lie algebra $\mathfrak{g}$ is defined by $\mathrm{Ad}(g)X = gXg^{-1}$. The canonical inner product on $\mathfrak{g}$, defined by $\langle X, Y\rangle = (1/2)\mathrm{Trace}(XY')$, where Y' is the matrix transpose of Y, is $\mathrm{Ad}(G)$-invariant in the sense that $\langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y\rangle = \langle X, Y\rangle$ for

[1]THE PAPER WAS COMPLETED AT AND WAS SUPPORTED BY NANKAI UNIVERSITY

any $g \in G$ and $X, Y \in \mathfrak{g}$. The space $o(3)$ may be identified with $\mathbb{R}^3$ with the inner products preserved such that any $X \in o(3)$ is a rotation in $\mathbb{R}^3$ in the sense that e^X rotates $\mathbb{R}^3$ about the axis through X (regarded as a vector in $\mathbb{R}^3$) by an angle $\|X\| = \langle X, X\rangle^{1/2}$.
The motion of a free rigid body fixed at the origin o of $\mathbb{R}^3$ may be described by a smooth path $g(t)$ in $G = SO(3)$ with $g(0) = e$. The angular velocity $\boldsymbol{\omega}$ in the space and the angular velocity $\mathbf{\Omega}$ in the body, at time t and with respect to o, are the elements of $\mathfrak{g} = o(3)$ (or vectors in $\mathbb{R}^3$) defined by $(d/dt)g(t) = \boldsymbol{\omega} g(t) = g(t)\mathbf{\Omega}$. They are respectively the angular velocities viewed by a person not moving with the body and by a person moving with the body. Note that they are related by $\boldsymbol{\omega} = \mathrm{Ad}(g(t))\mathbf{\Omega}$.
The angular momenta $\mathbf{m}$ in the space and $\mathbf{M}$ in the body are elements in $\mathfrak{g} = o(3)$ (or vectors in $\mathbb{R}^3$) related by $\mathbf{m} = \mathrm{Ad}(g(t))\mathbf{M}$. The inertia operator L of the body respect to o relates $\mathbf{M}$ to $\mathbf{\Omega}$ as $\mathbf{M} = L\mathbf{\Omega}$. This is a symmetric and positive definite linear operator: $\mathfrak{g} \to \mathfrak{g}$ in the sense that $\langle LX, Y\rangle = \langle X, LY\rangle$ for $X, Y \in \mathfrak{g}$ and $\langle LX, X\rangle > 0$ for nonzero X. Let $\Lambda = L^{-1}$. For a free rigid body, the angular momentum $\mathbf{m}$ is a constant. We have $(d/dt)g(t) = g(t)\mathbf{\Omega} = g(t)\Lambda(\mathbf{M}) = g(t)\Lambda[\mathrm{Ad}(g(t)^{-1})\mathbf{m}]$. Thus, the motion of a free rigid body may be determined by the following differential equation:

$$dg(t) = g(t)\Lambda[\mathrm{Ad}(g(t)^{-1})\mathbf{m}dt]. \tag{1}$$

The reader is referred to [1] for more details on the dynamical theory of a rigid body.
To model a random perturbation that causes a continuous change in the position of the rigid body, and has independent and stationary effects over non-overlapping time intervals, we may replace $\mathbf{m}\,dt$ in (1) by $X_0 dt + \sum_{i=1}^3 E_i \circ dB_t^i$ for a 3-dimensional Brownian motion $B_t = (B_t^1, B_t^2, B_t^3)$, where $\{E_1, E_2, E_3\}$ is an orthonormal basis of $\mathfrak{g}$ and $\circ d$ denotes the Stratonovich stochastic differential. As mentioned earlier, we will allow the driving Brownian motion B_t to be non-standard, that is, its covariance matrix $\{a_{ij}\}$, with $a_{ij} = \mathrm{cov}(B_1^i, B_1^j)$, is not necessarily a diagonal matrix and may be degenerate. Thus, with $n = 3$, the motion of the rigid body is a stochastic process g_t in G determined by the stochastic differential equation

$$dg_t = \sum_{i=1}^n g_t\Lambda[\mathrm{Ad}(g_t^{-1})E_i \circ dB_t^i] + g_t\Lambda[\mathrm{Ad}(g_t^{-1})X_0 dt] \tag{2}$$

with the initial condition $g_0 = e$. The process g_t is the motion of the rigid body with an initial angular momentum X_0 under the perturbation of the Brownian motion B_t. As for a free rigid body, we will call $X_t = \mathrm{Ad}(g_t^{-1})X_0$ the angular momentum of the body at time t.
We will work under a more general framework. Let G be an arbitrary compact connected Lie group with Lie algebra $\mathfrak{g}$. Fix an $\mathrm{Ad}(G)$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ and let $\{E_1, E_2, \ldots, E_n\}$ be an orthonormal basis of $\mathfrak{g}$. Let g_t be the process in G determined by the stochastic differential equation (2) with the initial condition $g_0 = e$, where Λ: $\mathfrak{g} \to \mathfrak{g}$ is a symmetric and positive definite linear map, and $B_t = (B_t^1, B_t^2, \ldots, B_t^n)$ is an n-dimensional Brownian motion with an arbitrary covariance matrix $\{a_{ij}\}$.
Since the path of the process g_t is almost surely nowhere differentiable, the angular velocity cannot be defined in the usual sense, but may be defined in an average sense as follows. Write $g_{t+s} = g_t \exp(Z_s^t)$ for some $Z_s^t \in \mathfrak{g}$ with $Z_0^t = 0$. Such a Z_s^t always exists and is unique when Z_s^t is sufficiently close to 0, and in the case of $G = SO(3)$, it is the amount of rotation viewed in the body from time t to time $t + s$. Let

$$L_t = \lim_{s\to 0} \frac{1}{s} E(Z_s^t \mid \mathcal{F}_t), \tag{3}$$

where $\mathcal{F}_t$ is the σ-algebra generated by the Brownian motion up to time t, provided the limit in (3) exists almost surely. For $G = SO(3)$, $E(Z_s^t \mid \mathcal{F}_t)$ is the average amount of rotation from t to $t+s$, thus L_t may be called the average angular velocity in the body at time t.

We will obtain an explicit formula for L_t in Section 2. Unlike in the special case of isotropic perturbation in [4], L_t may not vanish when $X_0 = 0$. In particular, we have the following interesting consequence: If the random perturbation is concentrated around a fixed axis, then the average random rotation will be around an axis that is orthogonal to the axis of perturbation. This result appears to be more significant than the one in [4] and the computation is also more complicated because the stochastic differential equation cannot be simplified due to the non-uniform perturbation. The reader is referred to a standard reference such as [3] for the stochastic calculus used in this paper.

In Section 3, we discuss the stability of a random rigid body, motivated by the stability of a free rigid body rotating near a principle axis, an axis along the direction of an eigenvector of Λ. When X_0 is along a principle axis and the perturbation is concentrate around X_0, the body will rotate randomly around X_0. For a general perturbation, because the stationary measure of the process g_t can be shown to be the Haar measure on G, the body does not have a long run tendency to be rotating near a principle axis. However, the stability is reflected in the mean time it takes for the angular momentum $X_t = \mathrm{Ad}(g_t^{-1})X_0$ to deviate a fixed distance from the principle axis. It will be shown that the mean time converges to infinity as the part of perturbation not around X_0 tends to zero and we will obtain a lower bound for the mean time which provides a quantitative control for this convergence. The proof is based on estimating the mean time when the energy is changed by $\delta > 0$.

2 Average angular velocity

We will derive a stochastic differential equation satisfied by Z_s^t as a process in time s. To simplify the notation, let

$$X_t = \mathrm{Ad}(g_t^{-1})X_0 \qquad \text{and} \qquad E_{t,i} = \mathrm{Ad}(g_t^{-1})E_i. \tag{4}$$

For simplicity, $\Lambda(X)$ may be written as ΛX for $X \in \mathfrak{g}$. Then (2) may be written as

$$dg_t = \sum_{i=1}^{n} g_t \Lambda E_{t,i} \circ dB_t^i + g_t \Lambda X_t dt. \tag{5}$$

Using the Stratonovich calculus, we may write $(\circ dg_t)g_t^{-1} + g_t \circ dg_t^{-1} = d(g_t g_t^{-1}) = 0$, from which we obtain

$$dg_t^{-1} = -\sum_{i=1}^{n} \Lambda(E_{t,i}) g_t^{-1} \circ dB_t^i - \Lambda(X_t) g_t^{-1} dt, \tag{6}$$

and for $X \in \mathfrak{g}$,

$$\begin{aligned} d\mathrm{Ad}(g_t^{-1})X &= (\circ dg_t^{-1})X g_t + g_t^{-1} X \circ dg_t \\ &= \sum_{i=1}^{n} [\mathrm{Ad}(g_t^{-1})X,\, \Lambda E_{t,i}] \circ dB_t^i + [\mathrm{Ad}(g_t^{-1})X,\, \Lambda X_t] dt. \end{aligned} \tag{7}$$

The differential of the exponential map at $X \in \mathfrak{g}$ is a linear map $D\exp(X)\colon T_X\mathfrak{g} \to T_{\exp(X)}G$. By Theorem 1.7 in [2, chapter II], if $Y \in \mathfrak{g}$ is regarded as an element of $T_X\mathfrak{g}$, then

$$D\exp(X)Y = e^X \Gamma(X) Y, \tag{8}$$

where

$$\Gamma(X)=\frac{\mathrm{id}_{\mathfrak{g}}-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}=\mathrm{id}_{\mathfrak{g}}-\frac{1}{2!}\mathrm{ad}(X)+\frac{1}{3!}\mathrm{ad}(X)^2-\cdots \tag{9}$$

and $\mathrm{ad}(X)\colon \mathfrak{g}\to\mathfrak{g}$ is defined by $\mathrm{ad}(X)Y=[X,Y]$ (Lie bracket). The linear map $\Gamma(X)\colon \mathfrak{g}\to\mathfrak{g}$ is invertible when $\|X\|$ is sufficiently small.

Fix $t\in\mathbb{R}_+$ and apply the Stratonovich stochastic differentiation to $g_{t+s}=g_t\exp(Z_s^t)$ regarding both sides as processes in time s, we obtain $dg_{t+s}=g_t\exp(Z_s^t)\Gamma(Z_s^t)\circ dZ_s^t$. On the other hand, $dg_{t+s}=\sum_{i=1}^n g_{t+s}\Lambda E_{t+s,i}\circ dB_{t+s}^i+g_{t+s}\Lambda X_{t+s}ds$. This implies that

$$\begin{aligned}dZ_s^t &= \sum_{i=1}^n \Gamma(Z_s^t)^{-1}\Lambda E_{t+s,i}\circ dB_{t+s}^i+\Gamma(Z_s^t)^{-1}\Lambda X_{t+s}ds\\
&= \sum_{i=1}^n \Gamma(Z_s^t)^{-1}\Lambda E_{t+s,i}\, dB_{t+s}^i+\Gamma(Z_s^t)^{-1}\Lambda X_{t+s}ds\\
&\quad +\frac{1}{2}\sum_{i=1}^n \Gamma(Z_s^t)^{-1}\Lambda(dE_{t+s,i})\cdot dB_{t+s}^i+\frac{1}{2}\sum_{i=1}^n D\Gamma(Z_s^t)^{-1}(dZ_s^t,\Lambda E_{t+s,i})\cdot dB_{t+s}^i,\end{aligned} \tag{10}$$

where dB_{t+s}^i without the circle $\circ$ is the Itô stochastic differential as usual, and for any two continuous semi-martingales X_t and Y_t, $dX_t\cdot dY_t$ denotes the differential of the associated quadratic covariance process. Moreover, if V is a finite dimensional linear space, $L(V)$ is the space of linear endomorphisms on V and $F\colon V\to L(V)$ is a smooth map, then for $x,y,z\in V$, $DF(x)(z,y)$ denotes the derivative $(d/dt)F(x+tz)y|_{t=0}$. We obtain, by (7),

$$\begin{aligned}dZ_s^t &= \sum_{i=1}^n \Gamma(Z_s^t)^{-1}\Lambda E_{t+s,i}\, dB_{t+s}^i+\Gamma(Z_s^t)^{-1}\Lambda X_{t+s}ds+\frac{1}{2}\sum_{i,j=1}^n \Gamma(Z_s^t)^{-1}\Lambda[E_{t+s,i},\,\Lambda E_{t+s,j}]a_{ij}ds\\
&\quad +\frac{1}{2}\sum_{i,j=1}^n D\Gamma(Z_s^t)^{-1}(\Gamma(Z_s^t)^{-1}\Lambda E_{t+s,i},\,\Lambda E_{t+s,j})a_{ij}ds.\end{aligned} \tag{11}$$

Because $Z_0^t=0$, Z_s^t is equal to the integral $\int_0^s$ of the right hand side of (11).

Note that $Z_s^t=\eta(g_t^{-1}g_{t+s})$ for a Borel measurable function $\eta\colon G\to\mathfrak{g}$ with $\exp\circ\eta=\mathrm{id}_G$ that is smooth only in a neighborhood U of e. Therefore, Z_s^t may not be a semi-martingale for all time s and the usual rule of the stochastic calculus that has been used here may not be applied. Moreover, $\Gamma(Z)$ is invertible only when $\|Z\|$ is sufficiently small and hence $\Gamma(Z_s^t)^{-1}$ in (11) may not be meaningful. However, there is a constant $c>0$ such that if $\|Z\|<c$, then $\Gamma(Z)$ is invertible and $e^Z\in U$. It follows that Z_s^t is a semi-martingale and $\Gamma(Z_s^t)$ is invertible for $s<\tau$, where τ is the first time s when $\|Z_s^t\|\ge c$. We see that (11) is valid for $s<\tau$.

We may modify the $\Gamma(Z)$ for $\|Z\|\ge c$ so that $\Gamma(Z)$ is an invertible linear operator for all $Z\in\mathfrak{g}$, both $\Gamma(Z)$ and $\Gamma(Z)^{-1}$ are smooth in Z, and $\Gamma(Z)=\mathrm{id}_{\mathfrak{g}}$ when $\|Z\|$ is sufficiently large. Then the operator norms of $\Gamma(Z)$ and $\Gamma(Z)^{-1}$, and their derivatives, are bounded in Z. With Γ thus modified, all the integrands in the stochastic differential equation (11) are bounded and hence we can solve it to obtain a process $\tilde{Z}_s^t$ in $\mathfrak{g}$ with $\tilde{Z}_0^t=0$. Then $Z_s^t=\tilde{Z}_s^t$ for $s<\tau$. We claim that

$$\lim_{s\to 0}\frac{1}{s}E(Z_s^t\mid\mathcal{F}_t)=\lim_{s\to 0}\frac{1}{s}E(\tilde{Z}_s^t\mid\mathcal{F}_t) \tag{12}$$

provided the limit on the right hand side exists. To prove (12), noting that Z_s^t is bounded but

$\tilde{Z}^t_s$ may not be so, we need to prove

$$\lim_{s\to 0}\frac{1}{s}P(s\geq\tau)=\lim_{s\to 0}\frac{1}{s}P(\sup_{0\leq u\leq s}\|\tilde{Z}^t_u\|\geq c)=0 \quad \text{and} \quad \lim_{s\to 0}\frac{1}{s}E[\|\tilde{Z}^t_s\|;\, s\geq\tau]=0,$$

where $E[Z;\,B]=\int_B ZdP$. We will only prove the second equation here because the proof of the first is similar. Since all the integrands on the right hand side of (11), with Z^t_s replaced by $\tilde{Z}^t_s$, are bounded, and except the first term, the integrals $\int_0^s$ of the three other terms tend to 0 as $s\to 0$, hence, it suffices to show $\frac{1}{s}E(M^*_s;\, M^*_s>c)\to 0$ as $s\to 0$ for any constant $c>0$, where $M_s=\int_0^s H_u\,dB_u$, H_u is a bounded continuous adapted process, B_u is a standard Brownian motion, and $M^*_s=\sup_{0\leq u\leq s}|M_u|$. By a well known moment inequality for martingales, see III.Theorem 3.1 in [3], $E[(M^*_s)^4]\leq bE[\langle M,M\rangle^2_s]=bE[(\int_0^s H^2_u du)^2]$ for some constant $b>0$. Then

$$\frac{1}{s}E[M^*_s;\, M^*_s>c]\leq\frac{1}{c^3 s}E[(M^*_s)^4]\leq\frac{b}{c^3 s}E[(\int_0^s H^2_u du)^2]\to 0 \text{ as } s\to 0.$$

This proves (12). Therefore, to determine the limit L_t in (3), we may replace Z^t_s by $\tilde{Z}^t_s$ which satisfies (11) for all s. For simplicity, we will write Z^t_s for $\tilde{Z}^t_s$ and we may assume (11) holds for all s in the following.

By (9),

$$\begin{aligned}\Gamma(Z+Y)X &= X-(1/2!)[Z+Y,X]+(1/3!)[Z+Y,[Z+Y,X]]+\ldots\\ &= \Gamma(Z)X+\Gamma(Y)X+O(\|Y\|\cdot\|Z\|).\end{aligned}$$

It follows that

$$D\Gamma(Z)(Y,X)=\frac{d}{dt}\Gamma(Z+tY)X\mid_{t=0}=-\frac{1}{2}[Y,X]+O(\|Z\|). \tag{13}$$

In particular, $D\Gamma(Z)(X,X)=O(\|Z\|)$. Differentiating the identity $\Gamma(Z)\Gamma(Z)^{-1}X=X$ with respect to Z yields $D\Gamma(Z)(Y,\Gamma(Z)^{-1}X)+\Gamma(Z)D\Gamma(Z)^{-1}(Y,X)=0$ and hence

$$D\Gamma(Z)^{-1}(Y,X)=-\Gamma(Z)^{-1}D\Gamma(Z)(Y,\Gamma(Z)^{-1}X). \tag{14}$$

Then

$$\begin{aligned}&\sum_{i,j=1}^n D\Gamma(Z^t_u)^{-1}(\Gamma(Z^t_u)^{-1}\Lambda E_{t+u,i},\,\Lambda E_{t+u,j})a_{ij}\\ =\;& -\sum_{i,j=1}^n \Gamma(Z^t_u)^{-1}D\Gamma(Z^t_u)(\Gamma(Z^t_u)^{-1}\Lambda E_{t+u,i},\,\Gamma(Z^t_u)^{-1}\Lambda E_{t+u,j})a_{ij}\\ =\;& \frac{1}{2}\sum_{i,j=1}^n \Gamma(Z^t_u)^{-1}\{[\Gamma(Z^t_u)^{-1}\Lambda E_{t+u,i},\,\Gamma(Z^t_u)^{-1}\Lambda E_{t+u,j}]+O(\|Z^t_u\|)\}a_{ij}\\ =\;& \Gamma(Z^t_u)^{-1}O(\|Z^t_u\|)=O(\|Z^t_u\|).\end{aligned}$$

Therefore, the integral of the last term on the right hand side of (11) is equal to $\int_0^s O(\|Z^t_u\|)du$. The integral of the first term is an Itô integral, and because its integrand is bounded, it vanishes after taking the expectation. It follows that

$$\begin{aligned}&\frac{1}{s}E(Z^t_s\mid\mathcal{F}_t)=E[\frac{1}{s}\int_0^s\Gamma(Z^t_u)^{-1}\Lambda X_{t+u}du\mid\mathcal{F}_t]\\ &+E\{\frac{1}{s}\int_0^s\frac{1}{2}\sum_{i,j=1}^n\Gamma(Z^t_u)^{-1}\Lambda[E_{t+u,i},\,\Lambda E_{t+u,j}]a_{ij}du\mid\mathcal{F}_t\}+E[\frac{1}{s}\int_0^s O(\|Z^t_u\|)du\mid\mathcal{F}_t],\end{aligned}$$

which converges to $\Lambda X_t + \frac{1}{2}\sum_{i,j=1}^n \Lambda[E_{t,i}, \Lambda E_{t,j}]a_{ij}$ as $s \to 0$. The following result is proved.

Theorem 1 *The average angular velocity in the body at time t, $L_t = \lim_{s\to 0} E(Z_s^t \mid \mathcal{F}_t)$, exists almost surely and is given by*

$$L_t = \Lambda \mathrm{Ad}(g_t^{-1})X_0 + \frac{1}{2}\sum_{i,j=1}^n a_{ij}\Lambda[\mathrm{Ad}(g_t^{-1})E_i,\, \Lambda \mathrm{Ad}(g_t^{-1})E_j]. \tag{15}$$

Let L_t^B be the second term on the right hand side of (15), which may be regarded as the contribution to the average angular velocity L_t from the Brownian motion B_t, whereas the first term is the contribution from the initial angular momentum X_0. As mentioned earlier, by the result in [4], $L_t^B = 0$ if $a_{ij} = \varepsilon\delta_{ij}$ for some $\varepsilon > 0$. We can verify this directly as follows. Because $\mathrm{Ad}(g)$ is an orthogonal transformation on $\mathfrak{g}$ and the matrix $\{\Lambda_{ij}\}$ representing Λ is symmetric,

$$L_t^B = \frac{\varepsilon}{2}\sum_{i=1}^n \Lambda[\mathrm{Ad}(g_t^{-1})E_i,\, \Lambda\mathrm{Ad}(g_t^{-1})E_i] = \frac{\varepsilon}{2}\sum_{i=1}^n \Lambda[E_i, \Lambda E_i] = \frac{\varepsilon}{2}\Lambda\sum_{i,j=1}^n \Lambda_{ij}[E_i, E_j] = 0.$$

Thus, $L_t^B = 0$.

Recall that the covariance matrix $\{a_{ij}\}$ is allowed to be degenerate. We now consider the case when $\{a_{ij}\}$ has only one nonzero eigenvalue $\varepsilon > 0$ (counting multiplicity). By choosing the orthonormal basis $\{E_1, \ldots, E_n\}$ of $\mathfrak{g}$ properly, we may assume: $a_{11} = \varepsilon$ and all other $a_{ij} = 0$. Then $L_t^B = (1/2)\Lambda[\mathrm{Ad}(g_t^{-1})E_1, \Lambda\mathrm{Ad}(g_t^{-1})E_1]$. Because $\langle\Lambda[X, \Lambda X], X\rangle = \langle[X, \Lambda X], \Lambda X\rangle = -\langle X, [\Lambda X, \Lambda X]\rangle = 0$,

$$\langle L_t^B,\, \mathrm{Ad}(g_t^{-1})E_1\rangle = 0, \tag{16}$$

that is, L_t^B is orthogonal to $\mathrm{Ad}(g_t^{-1})E_1$. Note that $L_t^B = 0$ implies $[E_1, \Lambda E_1] = 0$. On the other hand, if $[E_1, \Lambda E_1] = 0$, then $g_t = \exp(t\Lambda E_1)$ is a solution of (2) in the present case and $\mathrm{Ad}(g_t^{-1})E_1 = E_1$, which implies $L_t^B = 0$. To summarize, we have the following result.

Theorem 2 *Assume $a_{11} > 0$ and all other $a_{ij} = 0$ (this means that the perturbation is concentrated around E_1). Then L_t^B is orthogonal to $\mathrm{Ad}(g_t^{-1})E_1$. Moreover, $L_t^B = 0$ if and only if $[E_1, \Lambda E_1] = 0$.*

Note that for $G = SO(3)$, $[E_1, \Lambda E_1] = 0$ if and only if ΛE_1 is proportional to E_1, therefore, $L_t^B = 0$ if and only if E_1 is an eigenvector of Λ.

3 Stability

By (2), g_t is a diffusion process in G with generator given by $L = (1/2)\sum_{i,j=1}^n a_{ij}U_iU_i + U_0$, where $U_i(g) = g[\Lambda\mathrm{Ad}(g^{-1})E_i]$ and $U_0 = g[\Lambda\mathrm{Ad}(g^{-1})X_0]$ for $g \in G$. Let P_t be its transition semigroup. A probability measure μ on G is called a stationary measure of the diffusion process g_t if $\int_G \mu(dx)P_t(x,\cdot) = \mu$ for any $t > 0$. Thus, if the process g_t is started with a stationary measure as the initial distribution, then it will have the same distribution at all time t. It is easy to show that the normalized Haar measure ρ on G is a stationary measure if and only if $L^*1 = 0$, where L^* is the operator dual to L under ρ, that is, $\int (L\phi)\psi d\rho = \int \phi(L^*\psi)d\rho$ for $\phi, \psi \in C^\infty(G)$. The proof of Proposition 2 in [4] shows that $U_i^* = -U_i$ for $0 \le i \le n$. It follows that $L^*1 = 0$ and hence ρ is a stationary measure of g_t.

The perturbation is called non-degenerate if the matrix $\{a_{ij}\}$ is so. In this case, ρ is the unique stationary measure of g_t and hence, by the compactness of G, the distribution of g_t converges to ρ weakly as $t \to \infty$. For a degenerate perturbation, there may be other stationary measures and the distribution of g_t may not converge to ρ.

As an example, assume the perturbation is concentrated around X_0. Without loss of generality, we may assume $X_0 = bE_1$ for some constant b, $a_{11} > 0$ and all other $a_{ij} = 0$. The equation (2) now becomes $dg_t = g_t \Lambda \mathrm{Ad}(g_t^{-1}) E_1 \circ d(B_t^1 + bt)$. It is easy to show by stochastic calculus that the process $X_t = \mathrm{Ad}(g_t^{-1}) X_0$ in $\mathfrak{g}$ preserves the energy function

$$T(X) = \langle X, \Lambda X \rangle. \tag{17}$$

Moreover, by the Itô formula, the process g_t is supported by the path of a free rigid body in the sense that $g_t = \Phi(B_t^1 + bt)$, where $\Phi(t)$ is the solution of the ordinary differential equation (1) with $\mathbf{m} = E_1$ and $g(0) = e$. In the case of $G = SO(3)$, the angular momentum $\mathrm{Ad}(\Phi(t)^{-1}) X_0$ traces out a closed curve in $\mathfrak{g} = o(3) \equiv \mathbb{R}^3$ and $\Phi(t)$ is supported by a two-dimensional torus in $G = SO(3)$ (a sub-manifold, but not a subgroup), see the discussion in Section 29 of [1] for more details. When X_0 is an eigenvector of Λ, the body rotates randomly around X_0 and the torus reduces to a simple closed curve in $G = SO(3)$.

The rotating axis or the angular momentum of a free rigid body is stable near a principle axis associated to the largest or the smallest eigenvalue of Λ in the sense that if it starts near such an axis, it will remain close forever. For a random rigid body, the angular momentum $X_t = \mathrm{Ad}(g_t^{-1}) X_0$ is a process in the sphere S_M of radius $M = \|X_0\|$ centered at the origin in $\mathfrak{g}$. If the perturbation is non-degenerate, then g_t converges in distribution to ρ and hence X_t converges in distribution to the uniform distribution on S_M. This means that X_t does not have a long time tendency to be near a principle axis. However, assuming X_0 is along a principle axis, the stability around this axis is reflected in the mean time $E(\tau_\delta)$ when X_t deviates from X_0 by a fixed distance. By the discussion in the previous paragraph, if the perturbation is concentrated around X_0, then $X_t = X_0$ for all $t > 0$ and hence $E(\tau_\delta) = \infty$. We will show that in general $E(\tau_\delta) \to \infty$ when the part of the perturbation not around X_0 tends to zero and we will obtain a lower bound for $E(\tau_\delta)$ which provides a quantitative control for this convergence.

Assume $X_0 \neq 0$ is an eigenvector of Λ associated to either the largest or the smallest eigenvalue. Without loss of generality, we may assume $X_0 = ME_1$ with $M = \|X_0\|$. By (7), the symmetry of Λ and the orthogonality of $[X_t, \Lambda X_t] = M[X_t, \Lambda E_{t,1}]$ to ΛX_t,

$$\begin{aligned} dT(X_t) &= 2\sum_{i=2}^{n} \langle [X_t, \Lambda E_{t,i}],\, \Lambda X_t \rangle \circ dB_t^i = 2M^2 \sum_{i=2}^{n} \langle [E_{t,1}, \Lambda E_{t,i}],\, \Lambda E_{t,1} \rangle \circ dB_t^i \\ &= 2M^2 \sum_{i=2}^{n} \langle [E_{t,1}, \Lambda E_{t,i}],\, \Lambda E_{t,1} \rangle dB_t^i + M^2 \sum_{i=2}^{n} \sum_{j=1}^{n} a_{ij} F_{ij}(g_t) dt, \end{aligned} \tag{18}$$

where

$$\begin{aligned} F_{ij}(g) = &\langle [[\mathrm{Ad}(g^{-1})E_1, \Lambda \mathrm{Ad}(g^{-1})E_j], \Lambda \mathrm{Ad}(g^{-1})E_i],\, \Lambda \mathrm{Ad}(g^{-1})E_1 \rangle \\ &+ \langle [\mathrm{Ad}(g^{-1})E_1, \Lambda [\mathrm{Ad}(g^{-1})E_i, \Lambda \mathrm{Ad}(g^{-1})E_j]],\, \Lambda \mathrm{Ad}(g^{-1})E_1 \rangle \\ &+ \langle [\mathrm{Ad}(g^{-1})E_1, \Lambda \mathrm{Ad}(g^{-1})E_i],\, \Lambda [\mathrm{Ad}(g^{-1})E_1, \Lambda \mathrm{Ad}(g^{-1})E_j] \rangle. \end{aligned} \tag{19}$$

For $\delta > 0$, let τ_δ be the first time the change in X_t causes an energy change of δ, that is,

$$\tau_\delta = \inf\{t > 0;\ \ |T(X_0) - T(X_t)| = \delta\}. \tag{20}$$

By the usual convention, the inf of an empty set is defined to be ∞.
Since first term on the right hand side of (18) is the Itô stochastic differential of a martingale, if $E(\tau_\delta) < \infty$, then

$$\pm\delta = E[T(X_0) - T(X_{\tau_\delta})] = M^2 E[\int_0^{\tau_\delta} \sum_{i=2}^n \sum_{j=1}^n a_{ij} F_{ij}(g_t) dt]. \tag{21}$$

Recall $M = \|X_0\|$. Assume $M > 0$. Let $C = \sup\{|F_{ij}(g)|;\ g \in G \text{ and } 1 \le i, j \le n\}$. By (21), $\delta \le M^2 C E(\tau_\delta) \sum_{i=2}^n \sum_{j=1}^n |a_{ij}|$. It follows that

$$E(\tau_\delta) \ge \frac{\delta}{M^2 C[\sum_{i=2}^n \sum_{j=1}^n |a_{ij}|]}. \tag{22}$$

By (22) and noting $|a_{ij}| \le \sqrt{a_{ii}}\sqrt{a_{jj}}$, we obtain the following result.

Theorem 3 *Assume $X_0 \neq 0$ is an eigenvector of Λ associated to either the largest or the smallest eigenvalue. Let τ_δ be the first time when the energy changes by $\delta > 0$, defined by (20). Then $E(\tau_\delta) \to \infty$ as $a_{ii} \to 0$ for $2 \le i \le n$ and a_{11} remains bounded.*

Note that if in Theorem 3 the extreme eigenvalue is simple, then for any neighborhood U of X_0 in $\mathfrak{g}$, the first exit time τ_U of X_t from U is larger than τ_δ for a sufficiently small $\delta > 0$. Consequently, $E(\tau_U) \to \infty$ as $a_{ii} \to 0$ for $2 \le i \le n$ and a_{11} remains bounded.
We may replace C in (22) by a larger constant for which an explicit expression may be obtained. As before, we will assume the orthonormal basis $\{E_1, \ldots, E_n\}$ of $\mathfrak{g}$ is chosen so that $X_0 = ME_1$ is an eigenvector of Λ associated to either the largest or the smallest eigenvalue λ_1. We may assume $E_2, \ldots, E_n$ are also eigenvectors of Λ associated to eigenvalues $\lambda_2, \ldots, \lambda_n$. Let C_{ij}^k be the structure constants of the Lie group G under this basis given by

$$[E_i, E_j] = \sum_{k=1}^n C_{ij}^k E_k. \tag{23}$$

We have $C_{ij}^k = -C_{ji}^k$, and because the basis is orthonormal, $C_{ij}^k = -C_{ik}^j$. Let $\{b_{ij}(g)\}$ be the orthogonal matrix defined by $\mathrm{Ad}(g^{-1})E_i = \sum_{j=1}^n b_{ij}(g)E_j$. By (19),

$$\begin{aligned} F_{ij} &= \sum_{p,q,r,s=1}^n b_{ip}b_{jq}b_{1r}b_{1s}\{\langle[[[E_r, \Lambda E_q], \Lambda E_p],\ \Lambda E_s]\rangle + \langle[E_r, \Lambda[E_p, \Lambda E_q]],\ \Lambda E_s\rangle \\ &\qquad + \langle[E_r, \Lambda E_p],\ \Lambda[E_s, \Lambda E_q]\rangle\} \\ &= \sum_{p,q,r,s} \sum_k b_{ip}b_{jq}b_{1r}b_{1s}\{C_{rq}^k C_{kp}^s \lambda_p\lambda_q\lambda_s + C_{pq}^k C_{rk}^s \lambda_q\lambda_k\lambda_s + C_{rp}^k C_{sq}^k \lambda_p\lambda_q\lambda_k\} \end{aligned} \tag{24}$$

By the Schwartz inequality and the fact that $\sum_j b_{ij}^2 = 1$,

$$\begin{aligned} |F_{ij}| &\le (3n)^{1/2}\{\sum_{p,q,r,s,k=1}^n [(C_{rq}^k C_{kp}^s)^2 \lambda_p^2\lambda_q^2\lambda_s^2 + (C_{pq}^k C_{rk}^s)^2 \lambda_q^2\lambda_k^2\lambda_s^2 + (C_{rp}^k C_{sq}^k)^2 \lambda_p^2\lambda_q^2\lambda_k^2]\}^{1/2}. \\ &= \sqrt{3n \sum_{p,q,r,s,k=1}^n (C_{pr}^k C_{qs}^k)^2 \lambda_p^2\lambda_q^2(\lambda_s^2 + 2\lambda_k^2)}. \end{aligned} \tag{25}$$

Let C' be the expression in (25). Then the inequality (22) holds with C replaced by C'.
In the case of $G = SO(3)$ equipped with the inner product $\langle X, Y\rangle = (1/2)\mathrm{Trace}(XY')$ on $\mathfrak{g} = o(3)$, the structure constants under any orthonormal basis satisfy $C_{ij}^k = \pm 1$ if i, j, k are distinct and $C_{ij}^k = 0$ otherwise. Then

$$C' = 6\sqrt{\lambda_1^4(\lambda_2^2+\lambda_3^2)+\lambda_2^4(\lambda_1^2+\lambda_3^2)+\lambda_3^4(\lambda_1^2+\lambda_2^2)+3\lambda_1^2\lambda_2^2\lambda_3^2}. \tag{26}$$

References

[1] V.I. Arnold. *Mathematical methods of classical mechanics.* (1978) Springer-Verlag.

[2] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces.* (1978) Academic Press.

[3] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes, 2nd ed.* (1989) North-Holland-Kodansha.

[4] M. Liao. Random motion of a rigid body. *J. of Theor. Prob.* **10** (1997), 201 - 211.